Stable Sunspot Equilibria in a Cash-in-Advance Economy*

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Abstract

We develop a monetary model with flexible supply of labor, cash in advance constraints and government spending financed by seignorage. This model has two regimes. One regime is conventional with two steady states. The other regime has a unique steady state which can be determinate or indeterminate. In the latter case there exist sunspot equilibria which are stable under adaptive learning, taking the form of noisy finite state Markov processes at resonant frequencies. For a range of parameter values, a sufficient reduction in government purchases will eliminate these equilibria.

JEL classifications: C62, D83, D84, E31, E32

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1 Introduction

There has recently been considerable interest in the issue of indeterminacy in both theoretical and applied macroeconomics. The distinctive feature of

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indeterminacy is that there are multiple well-behaved rational expectations (RE) solutions to the model. These can take various forms, including a dependence on extraneous random variables (called "sunspots"). Such solutions correspond to self-fulfilling prophecies and have been offered as an explanation of the business cycle. Much of this recent research has focused on stationary sunspot equilibria (SSEs) near an indeterminate steady state, a possibility that has been examined in both extensions of Real Business Cycle models, incorporating externalities, and in a variety of monetary models.[1] A detailed survey of this literature is provided by (Benhabib and Farmer 1999), which gives extensive references.

A crucial related issue is the learnability of such solutions. Suppose agents are not assumed to have rational expectations *a priori* but instead make forecasts using a perceived law of motion with parameters that they update over time using an adaptive learning rule, such as least squares. Can such learning rules lead agents eventually to coordinate on an SSE? That it is indeed possible for sunspot solutions to be learned by agents was demonstrated by (Woodford 1990) in the context of the Overlapping Generations model of money. In general it can be shown that the possibility of such coordination depends on stability conditions for the SSE, as will be discussed further below. It is also easy to develop examples in which SSEs are not stable under learning. For an extensive treatment of adaptive learning in macroeconomics, see (Evans and Honkapohja 2001c).

General conditions for stability under learning have been obtained for a wide variety of models and solutions. These include several different types of SSEs in both linear and nonlinear models. Again, for a full discussion, see (Evans and Honkapohja 2001c), who show that "expectational stability" (E-stability) conditions typically govern the local learnability of SSEs, as well as other RE solutions. In the context of an endogenous growth model, an example of learnable SSEs are the "growth cycles" studied in (Evans, Honkapohja, and Romer 1998).[2] These SSEs fluctuate between neighborhoods of two distinct steady states, so that nonlinearity of the model is a crucial element in the model.

Stability results under learning for the case of SSEs in a neighborhood of a single indeterminate steady state have remained incomplete, despite

[1] In nonlinear models SSEs can also exist near multiple distinct steady states or rational deterministic cycles.

[2] For another example of a learnable "animal spirits" equilibrium, see (Howitt and McAfee 1992).

the above noted prominence of such equilibria in applied macroeconomics. However, new work has completed the stability results for this case in the context of one-step forward-looking univariate models. These results are given in (Evans and Honkapohja 2001a) and (Evans and Honkapohja 2001b), and for the models studied they show that only a subset of indeterminate steady states can have E-stable SSEs in their neighborhood. Moreover, these E-stable SSEs take the particular form of a dependence on finite state Markov processes.

We take up these issues in a standard type of monetary model. We use a model with representative agents and money demand arising from cash-in-advance constraints. Labor supply is flexible and the government finances its consumption purchases using seignorage. It is well known that cash-in-advance can give rise to indeterminacies.[3] However, a systematic study of the stability under learning of the corresponding SSEs has not previously been undertaken.

Our primary aims are to demonstrate that standard cash-in-advance monetary models can have sunspot equilibria, in a neighborhood of an indeterminate steady state, which are stable under simple adaptive learning rules, and to characterize the subset of SSEs that are stable under learning.[4] A secondary aim is to demonstrate how to apply the requisite techniques to models that are sufficiently complex to require numerical evaluation.

Our findings are both sharp and somewhat unexpected. The model has two regimes and the results depend on the regime. For one part of the parameter domain the model has two steady states. One of them, the one associated with low inflation, is determinate and stable under learning, while the other is indeterminate but unstable under learning. In this regime, none of the SSEs near the indeterminate steady state are learnable. However, for another part of the parameter domain there is a unique steady state that can be either determinate or indeterminate. If the steady state is indeterminate, then there are nearby Markov chain sunspot equilibria that are stable under learning. In a neighborhood of this steady state, there also exist stationary vector autoregressive (VAR) solutions, depending on extraneous sunspot noise, but these are not stable under learning. These results show that requiring stability under learning redirects the focus of analysis to a particular

[3] See (Woodford 1994) for an extensive discussion of indeterminacy in cash-in-advance models.

[4] (Packalén 1999) and (Weder 2001) consider learnability of SSEs in variants of real business cycle models.

type of indeterminate steady state and to particular SSEs near that steady state.

When stable SSEs do exist, this raises the issue of whether economic policy can be used to avoid them, either by eliminating the existence of SSEs or by rendering them unstable. In our model the natural policy variable is the level of government spending financed by seignorage, and we find that changes in the level of government purchases can indeed be effective. For a range of parameter values in which the steady state has nearby stable Markov sunspot equilibria, lowering government purchases sufficiently can render the steady state determinate, making the nearby sunspot solutions disappear.

2 The Model

We consider an infinite-horizon representative agent economy. There are two types of consumption goods, cash and credit goods. (Cash goods must be paid for by cash at hand.) There is also a flexible labor supply, one unit of which produces one unit of either consumption good. The unit endowment of time is split between leisure and labor. Both consumption goods are perishable and there are no capital goods.

Let the utility function be

$$U_t = E_t \sum_{s=t}^{\infty} \mathfrak{B}^{s-t} \left[\frac{(c_s^1)^{1-\sigma}}{1-\sigma} + \frac{(c_s^2)^{1-\sigma}}{1-\sigma} + \alpha \frac{(1-n_s)^{1-\sigma}}{1-\sigma} \right]$$

where c_s^1, c_s^2 and n_s denote cash goods, credit goods and labor supply, respectively. $\mathfrak{B}$ is the discount factor and $\sigma > 0$. The (sub-)utilities of cash goods, credit goods and leisure are assumed identical in order to facilitate investigation and presentation of numerical results. α is the relative weight placed on leisure.

The household budget constraint is

$$M_{s+1} + B_{s+1} = p_s(n_s - c_s^1 - c_s^2) + M_s + I_s B_s. \tag{1}$$

Here M_{s+1} and B_{s+1} denote the stocks of money and bonds at the beginning of period $s+1$. I_s is the nominal one-period interest rate factor on risk-free bonds earned during period s and known at the end of period $s-1$. p_s is the price of goods and labor in period s. The cash-in-advance constraint (CA constraint) takes the form

$$p_s c_s^1 = M_s. \tag{2}$$

We will focus on the case in which bonds are in zero net supply, but they must be included at this stage in order to derive the household optimization conditions.

Defining

$$m_{s+1} = M_{s+1}/p_s \text{ and } \pi_s = p_s/p_{s-1},$$

we can write the first order conditions as

$$(c_t^2)^{-\sigma} = \mathfrak{B} E_t^*[\pi_{t+1}^{-1}(c_{t+1}^1)^{-\sigma}] \tag{3}$$

$$(c_t^2)^{-\sigma} = \mathfrak{B} E_t^*[I_{t+1}\pi_{t+1}^{-1}(c_{t+1}^2)^{-\sigma}] \tag{4}$$

$$(c_t^2)^{-\sigma} = \alpha(1 - n_t)^{-\sigma}. \tag{5}$$

Here E_t^* denotes the expectations of the household, conditional on time t information, where we use the notation E_t^* to indicate that the expectations are not necessarily assumed to be fully rational, due to adaptive learning. When rational expectations are assumed we will use the notation E_t.

The market clearing condition is

$$n_t = c_t^1 + c_t^2 + g_t, \tag{6}$$

where g_t denotes government spending on goods. We assume that g_t is an *iid* random variable with small bounded support around the mean $g > 0$. Note that the CA condition can be written in the form $m_{t+1} = c_{t+1}^1 \pi_{t+1}$.

There is also a government finance constraint taking the form

$$B_{t+1} + M_{t+1} = p_t g_t + I_t B_t + M_t.$$

For simplicity we ignore taxes, but the model could easily be modified to include fixed lump-sum taxes. If bonds are not held in positive net amount in equilibrium, then this constraint yields the familiar seignorage equation

$$\pi_t = \frac{m_t}{m_{t+1} - g_t}. \tag{7}$$

Household optimization, market clearing and the CA constraint lead to the equations

$$n_t = 1 - \alpha^{1/\sigma}\mathfrak{B}^{-1/\sigma}\{E_t^*[\pi_{t+1}^{-1}(c_{t+1}^1)^{-\sigma}]\}^{-1/\sigma} \quad (8)$$
$$m_{t+1} = (1+\alpha^{-1/\sigma})n_t - \alpha^{-1/\sigma} \quad (9)$$
$$c_t^1 = m_{t+1} - g_t \quad (10)$$
$$c_t^2 = n_t - c_t^1 - g_t, \quad (11)$$
$$I_{t+1} = (c_t^2)^{-\sigma}\mathfrak{B}^{-1}\{E_t^*[\pi_{t+1}^{-1}(c_{t+1}^1)^{-\sigma}]\}^{-1} \quad (12)$$

Note that $m_{t+1} = M_{t+1}/p_t$, the real money stock carried forward from period t to period $t+1$, is determined at time t. Similarly, I_{t+1} is determined and known in period t. Equations (8)-(12), together with (7), give the temporary equilibrium equations determining $\pi_t, n_t, m_{t+1}, c_t^1, c_t^2$ and I_{t+1} as functions of time t expectations, the exogenous government spending shock g_t, and the previous period's real money stock m_t.

We note that the labor supply response in this model is entirely standard. It can be shown that, under perfect foresight, dynamic labor supply is characterized by

$$\frac{1-n_t}{1-n_{t+1}} = \mathfrak{B}R_{t+1}^{-1/\sigma},$$

where $R_{t+1} = I_{t+1}/\pi_{t+1}$. Thus increases in the real interest rate factor R_{t+1} lead to increases in current labor supply n_t for any value of σ.

3 Linearized Model

Our first step is to determine the possible steady states and then to linearize the model around the steady states.

3.1 Nonstochastic Steady States

We begin by determining the non-stochastic perfect foresight steady states that are possible when $g_t = g$ is constant and nonstochastic. Denoting steady state values by bars over the variables, (8)-(10) and (7) imply

$$\begin{aligned}\bar{n} &= 1-\alpha^{1/\sigma}\mathfrak{B}^{-1/\sigma}\bar{\pi}^{1/\sigma}\bar{c}^{1} \\ \bar{m} &= (1+\alpha^{-1/\sigma})\bar{n}-\alpha^{-1/\sigma} \\ \bar{c}^{1} &= \bar{m}-g \\ \bar{\pi} &= \bar{m}/(\bar{m}-g).\end{aligned}$$

These equations can be reduced to a single equation in the steady state inflation rate $\bar{\pi}$,

$$(1-g)\bar{\pi} = 1 + gA\bar{\pi}^{1/\sigma}, \tag{13}$$

where $A = (1+\alpha^{1/\sigma})\mathfrak{B}^{-1/\sigma} > 0$.

For $g = 0$ there is a unique steady state $\pi = 1$. For $g > 0$, it can be seen that the model has two regimes, depending on σ. If $\sigma < 1$, then the right-hand side of (13) is a convex function, while the left-hand side defines a straight line. When $\sigma < 1$ there are therefore two cases. Provided $g > 0$ is below a threshold value, depending on α, σ and $\mathfrak{B}$, there are two distinct steady states, $1 < \bar{\pi}_L < \bar{\pi}_H$, while if g exceeds this threshold there are no perfect foresight steady states. Below this threshold value, increases in g raise $\bar{\pi}_L$ and lower $\bar{\pi}_H$. The $\sigma < 1$ regime is standard in seignorage models. However, this model also has a less familiar regime that arises when $\sigma > 1$. In this case the right hand side is concave, and provided $0 < g < 1$, which we assume throughout the paper, there is a unique steady state $\bar{\pi}$. In this regime increases in g raise $\bar{\pi}$. Figures 1 and 2 illustrate the two cases $\sigma < 1$ and $\sigma > 1$.

3.2 Linearization

Linearizing the model around a steady state, the system can be reduced to two dynamic equations in the endogenous variables n_t and π_t. Let $g_t = g + u_t$, where u_t is now assumed to be white noise. The linearized model takes the form

$$n_t = \delta_0 + \delta_n E_t^* n_{t+1} + \delta_\pi E_t^* \pi_{t+1}, \tag{14}$$

and

$$\pi_t = \beta_{00} + \beta_{n0} E_{t-1}^* n_t + \beta_{\pi 0} E_{t-1}^* \pi_t + \beta_{n1} E_t^* n_{t+1} + \beta_{\pi 1} E_t^* \pi_{t+1} + \beta_g u_t. \tag{15}$$

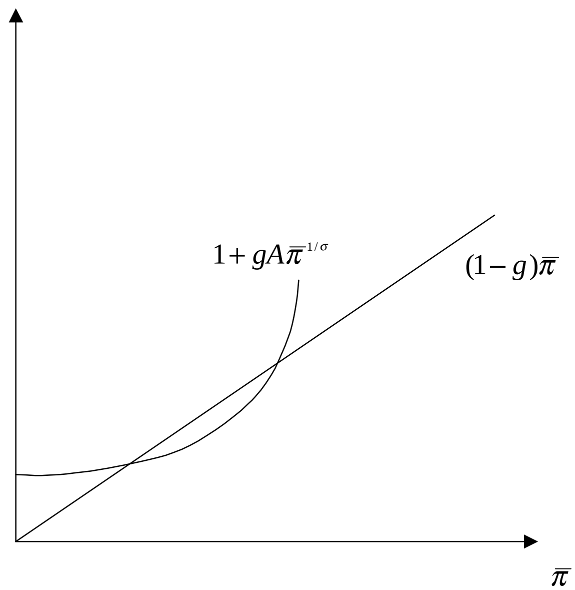

Figure 1: $\sigma < 1$

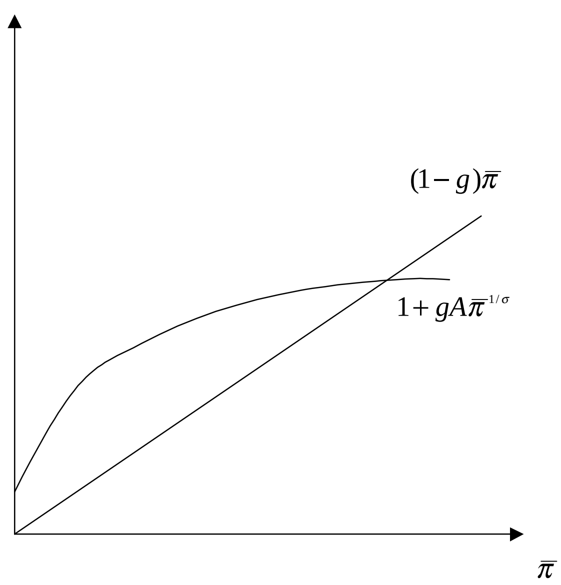

Figure 2: $\sigma > 1$

The coefficients of the equations are given by

$$\begin{aligned}
\delta_n &= -(1-\bar{m})(\bar{m}-g)^{-1} \\
\delta_\pi &= -\sigma^{-1}\bar{\pi}^{-1}(1+\alpha^{-1/\sigma})^{-1}(1-\bar{m}) \\
\beta_{n0} &= -(1+\alpha^{-1/\sigma})(1-\bar{m})(\bar{m}-g)^{-2} \\
\beta_{\pi 0} &= -\sigma^{-1}(1-\bar{m})\bar{m}^{-1} \\
\beta_{n1} &= -\bar{\pi}\beta_{n0} \\
\beta_{\pi 1} &= -\bar{\pi}\beta_{\pi 0}.
\end{aligned} \tag{16}$$

Further details are given in the Appendix.

The linearized reduced form (14)-(15) can be written as

$$y_t = \xi + J_0 E^*_{t-1} y_t + J_1 E^*_t y_{t+1} + K u_t, \tag{17}$$

where $y_t = (n_t, \pi_t)'$. The coefficient matrices are given by

$$J_0 = \begin{pmatrix} 0 & 0 \\ \beta_{n0} & \beta_{\pi 0} \end{pmatrix},\ J_1 = \begin{pmatrix} \delta_n & \delta_\pi \\ \beta_{n1} & \beta_{\pi 1} \end{pmatrix} \text{ and } K = \begin{pmatrix} 0 \\ \beta_g \end{pmatrix}. \tag{18}$$

Moreover, $\xi' = (\delta_0, \beta_{00})$.[5] We next discuss the possible RE solutions to (17).

3.3 Noisy Steady State and VAR Solutions

It is straightforward to see that the reduced form (17) has RE solutions taking the form of "noisy steady states,"

$$y_t = \bar{a} + K u_t, \text{ where } \bar{a} = (I - J_0 - J_1)^{-1}\xi. \tag{19}$$

These solutions are often called minimal state variable (MSV) solutions[6] and are the solutions most typically adopted in applied work.

If the steady state, around which we have linearized the model, is determinate then, as is well known, this is the unique stationary solution near the steady state. See, for example, (Blanchard and Kahn 1980), (Farmer 1999) and (Evans and Honkapohja 2001c). On the other hand, if the steady state is indeterminate there exist SSEs in a neighborhood of that steady state.[7]

[5] $\xi = 0$ if the model is expressed in terms of deviations from the steady state. The specific value of β_g will not be needed.

[6] See (McCallum 1983).

[7] The terminology "regular" and "irregular" is often used synonymously with "determinate" and "indeterminate."

The possibility of modeling business cycle fluctuations as SSEs has been emphasized by (Cass and Shell 1983), (Azariadis 1981), (Farmer 1999) and (Guesnerie and Woodford 1992).

Standard procedures yield the following result (e.g. see Chapter 10 of (Evans and Honkapohja 2001c)):

Proposition 1 *The linearized model (17) is indeterminate if and only if at least one of the eigenvalues of* $(I-J_0)^{-1}J_1$ *lies outside the unit circle. This holds near the steady state if and only if*

$$|1-\beta_{\pi 0}| < |\delta_n - \delta_n\beta_{\pi 0} + \delta_\pi\beta_{n0} + \beta_{\pi 1}|\,.$$

In this and other propositions we are excluding non-generic knife-edge cases in which an eigenvalue lies on the boundary of the indeterminacy or stability condition.

Given the specific form of the matrices (18), with coefficients given in (16), it can be shown that J_1 is singular and thus that $(I-J_0)^{-1}J_1$ has a zero eigenvalue. The steady state is therefore indeterminate if and only if the other root has absolute value greater than one, leading to the condition given. Details are provided in the Appendix.

A familiar feature of the seignorage model is that in the case of two steady states the high inflation equilibrium is indeterminate. As we will see, this result also holds in our model. In addition, it can be shown that for some parameter values we have indeterminacy in the regime $\sigma > 1$ when there is a single steady state. In fact we have:

Corollary 2 *In the case* $\sigma > 1$ *the steady state is indeterminate when* σ *is sufficiently large.*

This result can be seen by noting that $\lim_{\sigma\to\infty}\bar{m} = m^{\infty}$ and $\lim_{\sigma\to\infty}\bar{\pi} = \pi^{\infty}$ are finite, and that we have $\beta_{\pi 0}\to 0-$, $\beta_{\pi 1}\to 0+$, $\beta_{n0}\to 2(1-m^{\infty})(m^{\infty}-g)^{-2}$, $\beta_{n1}\to -\pi^{\infty}2(1-m^{\infty})(m^{\infty}-g)^{2}$, $\delta_\pi \to 0-$ and $\delta_n \to -(1-m^{\infty})(m^{\infty}-g)^{-2}$. With these values the condition in the proposition is satisfied.

We next consider the form of SSEs in cases of indeterminate steady states. Using the method of undetermined coefficients one can show that there exist stochastically stationary solutions of the form

$$y_t = a + by_{t-1} + c_0u_t + c_1u_{t-1} + d_0\eta_t + d_1\eta_{t-1}, \tag{20}$$

where η_t, the sunspot variable, is an arbitrary (observable) exogenous variable satisfying $E_t\eta_{t+1} = 0$. For convenience we will refer to these as vector autoregressive (VAR) SSEs, though they also include moving average dependencies on both the intrinsic and extrinsic disturbances. This is the type of solution that is emphasized by much of the applied indeterminacy literature, see e.g. (Benhabib and Farmer 1999). For the case at hand, when the steady state is indeterminate, there are VAR SSEs in which b has one eigenvalue of 0 and one root equal to the inverse of the nonzero root of $(I - J_0)^{-1}J_1$.[8]

Although the VAR solutions are the form of SSEs that have recently received the most attention, the literature has also drawn attention to the existence of solutions generated by finite state Markov solutions. We next show that this type of solution can exists in the monetary model that we have developed.

4 Markov Sunspot Solutions

When a steady state is indeterminate it can be anticipated from the theoretical literature, see e.g. the survey paper by (Chiappori and Guesnerie 1991), that there will also exist SSEs around the steady state for which the sunspot process is a Markov chain with a finite number of states. We will call such solutions Markov SSEs to distinguish them from the VAR SSEs discussed above.[9]

For simplicity, we focus on SSEs driven by a 2-state Markov chain. Thus assume that s_t is a two state exogenous process, taking values $s_t = 1$ or $s_t = 2$. The transition probabilities are p_{ij}, $j = 1, 2$, so that $p_{12} = 1 - p_{11}$ and $p_{21} = 1 - p_{22}$. We look for solutions of the form:

$$n_t = n(j) \text{ and } \pi_t = \pi(i, j) + \beta_g u_t \text{ if } s_{t-1} = i \text{ and } s_t = j,$$

for $i, j = 1, 2$. To satisfy (14)-(15) under RE the values of $n(j)$ and $\pi(i, j)$ we must have

$$\begin{aligned} n(j) &= \delta_n(p_{j1}n(1) + p_{j2}n(2)) + \delta_\pi(p_{j1}\pi(j, 1) + p_{j2}\pi(j, 2)), \\ \pi(i, j) &= \beta_{n0}(p_{i1}n(1) + p_{i2}n(2)) + \beta_{\pi 0}(p_{i1}\pi(i, 1) + p_{i2}\pi(i, 2)) + \\ &\quad \beta_{n1}(p_{j1}n(1) + p_{j2}n(2)) + \beta_{\pi 1}(p_{j1}\pi(j, 1) + p_{j2}\pi(j, 2)). \end{aligned} \tag{21}$$

[8] See the Appendix for a brief discussion and references.

[9] (Evans and Honkapohja 2001b) examine the relation between these two types of SSEs and their E-stability properties in the basic one-step forward looking model.

This can be rewritten in the form

$$\theta = \mathcal{T}\theta, \tag{22}$$

where

$$\theta' = (n(1), n(2), \pi(1,1), \pi(1,2), \pi(2,1), \pi(2,2))$$

and the matrix $\mathcal{T}$ is obtainable from equations (21). A Markov SSE θ exists if there exist $0 < p_{11} < 1$ and $0 < p_{22} < 1$ and $\theta \neq 0$ for which θ satisfies the equation (22).

Note that if an SSE exists then, in our linearized model, $k\theta$ is also an SSE for any real k for the same transition probabilities, so that the "size" of the sunspot fluctuations is indeterminate. Formally, SSEs exist if and only if $\mathcal{T} - I$ is singular for some $0 < p_{11}, p_{22} < 1$. Noting that $\mathcal{T}$ depends on p_{11} and p_{22}, we solve the equation

$$\det(\mathcal{T}(p_{11}, p_{22})) = 0$$

that gives the required relationship

$$p_{11} = f(p_{22}). \tag{23}$$

Based on previous studies of simpler forward-looking models, see in particular (Chiappori, Geoffard, and Guesnerie 1992), we hypothesize that there exist Markov SSEs if and only if the steady state is indeterminate. Numerical support for this proposition is given below.

In linearized models such as the current one, SSEs exist only for very particular transition probabilities. That is, for arbitrary transition probabilities the matrix $\mathcal{T}$ is nonsingular, so that the equation $\theta = \mathcal{T}\theta$ has only the trivial solution $\theta = 0$, which corresponds to the steady state. The condition (23) can be thought of as a *resonant frequency* condition that makes it possible for the excitation of the SSE.[10] As indicated above, corresponding to a probability pair (p_{11}, p_{22}) satisfying (23) is a 1-dimensional continuum of values for θ that constitute SSEs.

[10] This terminology is suggested by (Evans and Honkapohja 2001b) in which such Markov SSEs are theoretically studied for the basic one-step forward looking linear model.

5 Stability Under Learning

We now take up the question of stability of the RE solutions under adaptive learning rules. In the case of indeterminate steady states we separately assess each of the three types of solution for their stability under learning.

The starting point for analysis of learning is the temporary equilibrium in the model. If agents optimize using subjective (but possibly nonrational) probability distributions over future variables, optimal behavior is characterized by first-order necessary conditions that can be written as a sequence of Euler equations involving subjective expectations over the entire future. The Euler equation for the current period is assumed to be the behavioral rule giving the current decision as a function of the expected state next period. To complete the description of the agents' behavior we must supplement this Euler equation with a rule for forecasting the required state variables next period. The parameters of the forecast functions are updated using a standard adaptive learning rule such as least squares.

Least squares and related learning dynamics have been widely studied and shown to converge to the usually employed REE in many standard models. This is true of the stationary solutions of, for example, the Cagan model of inflation, the Sargent-Wallace IS-LM-PC model, the Samuelson overlapping generations model and the real business cycle model.[11]

In modeling learning the private agents are assumed to have perceptions about the (in general stochastic) equilibrium process of the economy. This is usually called the perceived law of motion (PLM) and depends on parameters that are updated as new data become available over time. At each period t, agents form expectations by making forecasts using the estimated PLM. This leads to a temporary equilibrium, called the actual law of motion (ALM), which provides the agents a new data point of the key variables. Estimated parameters are updated in each period according to least squares and the new data. The issue of interest is the stability under learning of some rational expectations solution, i.e. whether the estimated parameters of the PLM converge to REE values over time.

It is well known that, for a wide range of models, stability under adaptive learning is governed by E-stability conditions, see the (Evans and Honkapohja 2001c) book for an extensive discussion of these concepts and analytical tech-

[11]Recent overviews of the literature are provided e.g. in (Evans and Honkapohja 1999) and (Evans and Honkapohja 2001c). (Bray and Savin 1986) and (Marcet and Sargent 1989b) are key early papers on adaptive learning.

niques. The E-stability conditions are developed as follows. For given values of the parameters of the PLM one computes the resulting ALM, and E-stability is then determined by a differential equation in notional time in which the parameters adjust in the direction of the ALM parameter values. We now illustrate these steps in detail for the case of steady state REE and then just sketch the procedure and provide the E-stability conditions in the other cases.

Consider the noisy steady state solutions (19). These solutions exist whether or not a steady state is indeterminate. Agents have a PLM of the form

$$y_t = a + Ku_t.$$

Note that it is of the same form as the steady state REE, but in general the value of a differs from $\bar{a}$ given in (19). Under adaptive learning agents estimate a as the sample mean of past y_t, i.e.[12]

$$a_t = t^{-1} \sum_{\ell=1}^{t} y_{t-\ell}.$$

The temporary equilibrium is then given by (17) with $E^*_{t-1} y_t = a_{t-1}$ and $E^*_t y_{t+1} = a_t$. The question of interest is whether for this system $a_t \to \bar{a}$ as $t \to \infty$. The answer is that convergence is governed by E-stability.

To determine E-stability one assumes expectations $E_{t-1} y_t = E_t y_{t+1} = a$, based on the above PLM, for an arbitrary a (intuitively, a_t evolves asymptotically slowly under adaptive learning). Substituting these into (17) the implied ALM takes the form

$$y_t = \xi + (J_0 + J_1)a + Ku_t.$$

This gives rise to a mapping $T(a) = \xi + (J_0 + J_1)a$, and E-stability is defined as the local asymptotic stability of the fixed point $\bar{a}$ of the differential equation

$$\frac{da}{d\tau} = T(a) - a,$$

[12] For this simple set-up, least squares estimation of an unknown constant amounts to computing the sample mean from past data. We are assuming that the current value of y_t is unavailable when a is estimated at t.

where τ is virtual or notional time. (Note that this is simply a partial adjustment formula in the virtual time.) Exploiting the linearity of the T-map in this case and using the techniques in (Evans and Honkapohja 2001c) we establish:

Proposition 3 *The noisy steady state is stable under learning if and only if it is E-stable. The E-stability condition is that all eigenvalues of $J_0 + J_1$ have real parts less than one. This is equivalent to the conditions*

$$\delta_n + \beta_{\pi 0} + \beta_{\pi 1} < 2 \text{ and } \delta_n - \delta_n \beta_{\pi 0} + \delta_\pi \beta_{n0} + \beta_{\pi 1} < 1 - \beta_{\pi 0}.$$

See the Appendix for the derivation of the stated condition. We remark that this E-stability condition is not the same as the determinacy condition.

Consider next the VAR SSEs. Now agents are assumed to estimate the coefficients of their PLM (20) using recursive least squares, and to make forecasts based on the estimated PLM. E-stability of such solutions is examined by constructing the mapping from the PLM to the ALM. In the Appendix we outline the arguments establishing that necessary conditions for E-stability are given by

Proposition 4 *Necessary E-stability conditions for the VAR SSEs are that all the eigenvalues of the matrices*

$$J_0 + J_1(I + b) \text{ and } b' \otimes J_1 + I \otimes (J_0 + J_1 b)$$

have real parts less than one.

We remark that this condition can be readily evaluated numerically.[13]

Finally we consider the Markov SSEs. The possibility of convergence of adaptive learning to Markov sunspot solutions was first shown by (Woodford 1990), in the context of an overlapping generations model. (See also (Evans 1989)). Local stability conditions for Markov SSEs in simple forward looking models were developed in (Evans and Honkapohja 1994). For Markov SSEs near an indeterminate steady state, additional results have recently been

[13]We note that, in these results, the agents are assumed to use only data up to the preceding period $t-1$ for their parameter estimates for period t. If instead contemporaneous data can be used, the stability condition is sometimes though not always altered, see (Van Zandt and Lettau 2001) and Section 3.4 of (Evans and Honkapohja 2001c) for discussions of this issue.

obtained in (Evans and Honkapohja 2001b) and (Evans and Honkapohja 2001a). We extend these techniques to model learning in the monetary model of this paper.

Suppose that agents observe a sunspot s_t satisfying the resonant frequency condition and that they consider conditioning their actions on the values of the sunspot. A simple learning rule is that agents compute state contingent averages. Thus at any time t they estimate the value of $n(j)$ as the mean value that $n_{t-\ell}$ has taken in state $s_{t-\ell} = j$, for $\ell = 1, \ldots, t$. Similarly $\pi(i, j)$ is estimated as the mean value $\pi_{t-\ell}$ has taken whenever $s_{t-\ell-1} = i$ and $s_{t-\ell} = j$. Agents then make forecasts using these estimates and the transition probabilities for the observed sunspot (which can also be estimated if they are unknown).[14]

Based on previous research we expect that the stability of SSEs under adaptive learning depends upon the corresponding E-stability condition and we now examine E-stability of resonant frequency Markov SSEs. The definition of E-stability is easily formulated as follows. The T-mapping that maps the PLM to the corresponding ALM (actual law of motion) is here linear and given by the matrix $\mathcal{T}$ in (22). E-stability is determined by the stability of the differential equation

$$d\theta/d\tau = \mathcal{T}\theta - \theta. \tag{24}$$

This leads to the following result:[15]

Proposition 5 *Necessary and sufficient conditions for a Markov SSE to be E-stable are that $\mathcal{T} - I$ has one zero root and that all other roots of $\mathcal{T} - I$ have negative real parts.*

We end this section with two remarks. First, it is possible in principle for a steady state to be stable under learning and at the same time for nearby SSEs to be stable under learning. When this occurs, the solution to which

[14] An alternative learning procedure would be to estimate the expectations of n_{t+1} and π_{t+1} conditional on the current sunspot state s_t. Because it is simpler to simulate, we later adopt this alternative scheme when examining the nonlinear model numerically. The stability properties of the two approaches appear to be identical.

[15] Note that $\mathcal{T} - I$ always has a zero eigenvalue as a result of the resonant frequency condition (23). The necessary condition was proved in Chapter 10 of (Evans and Honkapohja 2001c). Sufficiency follows from the mathematical Lemma in the Appendix of (Honkapohja and Mitra 2001).

the economy converges depends on the form of the PLM, i.e. on whether or not agents in their learning allow for a possible dependence on sunspots. Second, the results of this section are subject to the qualification that our linearized monetary model has been derived from an underlying nonlinear model. Therefore, we will also investigate the existence and stability of SSEs directly for the nonlinear model using numerical methods and simulations.

6 Numerical Results

Using the theoretical results in the preceding sections one can numerically investigate the existence and stability under learning of the various types of solution for the different regimes. The underlying structural parameters are the utility function parameters $\mathfrak{B}, \alpha, \sigma$ together with the mean level of government spending g. To keep the numerical search manageable, we fix the discount rate at $\mathfrak{B} = 0.95$ and $\alpha = 1$. We then consider a grid of possible values for the utility parameter $\sigma > 0$ and government spending $g > 0$. Recall that in the regime $\sigma < 1$ there are two (noisy) steady states, provided that $g > 0$ is less than the critical value (which will be assumed hereafter), while in the regime $\sigma > 1$ there is a single (noisy) steady state. We have the following numerical results:[16]

1. If $\sigma < 1$ the low inflation steady state π_L is determinate and the high inflation steady state π_H is indeterminate. If $\sigma > 1$ the steady state can be determinate or indeterminate. Both VAR and Markov SSEs exist near indeterminate steady states.

2. If $\sigma < 1$ the determinate steady state solution π_L is always stable under learning and the indeterminate steady state solution π_H is always unstable under learning. If $\sigma \geq 1$ the steady state is always stable under learning whether it is determinate or indeterminate.

3. The VAR SSEs are never stable under least squares learning.

4. When $\sigma < 1$, Markov SSEs near π_H are not stable under learning (one of the roots of $T - I$ is positive at the resonant transition probabilities).

5. When $\sigma > 1$ and the steady state is indeterminate, Markov SSEs exist and are stable under learning.

[16] The numerical routines are available from the first author upon request.

These numerical results were also confirmed for a range of values of $\alpha > 0$.

Here is a numerical example of a Markov SSE satisfying the resonant frequency condition.

Example: Suppose that $\mathfrak{B} = 0.95$, $\alpha = 1, \sigma = 3.5$ and $g = .19$.[17] The steady state is indeterminate: $(I - J_0)^{-1} J_1$ has one root of 0 and one root of -1.2408. There are Markov SSEs with, for example, $p_{11} = 0.1041$ and $p_{22} = .09$, and these SSEs are stable under learning.

We now return to the original nonlinear system to discuss further the Markov SSEs. There are several important differences between the nonlinear system (7)-(10) and the linearized system (14)-(16) with respect to the Markov SSEs. In the linearized system the resonant frequency condition (23) must be satisfied exactly and the "size" of the SSE is indeterminate, as earlier discussed. In the exact nonlinear system there are Markov SSEs for transition probabilities close to the resonant frequency condition and the value of θ is in part determined by these probabilities. This issue is fully analyzed for the univariate forward looking model in (Evans and Honkapohja 2001a). Thus, although the linearized model is convenient for obtaining existence and stability conditions for Markov SSEs, it is important to establish further details using the nonlinear model.

Consider, therefore, the learning dynamics in the original nonlinear model (7)-(10). The key variable that agents must forecast is $X_{t+1} = \pi_{t+1}^{-1}(c_{t+1}^1)^{-\sigma}$. Because the sunspot variable is assumed to be first-order Markov, the conditional expectation of this variable depends only on the current state. A simple learning rule is thus to estimate the mean value of X_{t+1} conditional on the current sunspot state at t, e.g. by state contingent averaging:

$$\hat{X}_{i,t} = (\#N_i(t))^{-1} \sum_{\substack{1 \le \ell \le t, \\ s_{t-\ell-1}=i}} X_{t-\ell},$$

$i = 1, 2$, where $\#N_i(t)$ denotes the number of data points in which $s_{t-\ell-1} = i$ for $1 \le \ell \le t$. Thus agents at time t estimate the mean value that X_{t+1} will take, next period, as $\hat{X}_{i,t}$, when the sunspot in period t is in state $s_t = i$, $i = 1, 2$. Accordingly they form expectations at t as $E_t^* X_{t+1} = \hat{X}_{i,t}$. Over time the estimates $\hat{X}_{i,t}$ are revised in accordance with observed values of

[17] We have not tried to obtain the parameter values by calibrating the model to data, since our main goal is to illustrate numerically the different cases that can arise.

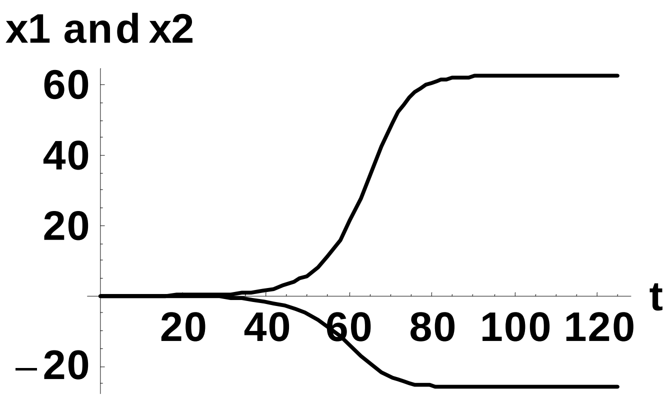

Figure 3: Convergence of adaptive learning to sunspot solution

X_t following each of the two different states. Under adaptive learning the model consists of these learning dynamics together with the equations (7)-(12). From the numerical results for the linearized model, we anticipate convergence to a stationary sunspot solution, for transition probabilities close to satisfying the resonant frequency condition, when $\sigma > 1$ and the steady state is indeterminate.

We have simulated the nonlinear system and the corresponding E-stability differential equation using the parameter values from the above example. Figure 3 illustrates convergence to the sunspot equilibrium for the choice of transition probabilities $p_{11} = 0.07$ and $p_{22} = 0.05$ and initial conditions near the steady state. (The vertical axis shows deviations of $\hat{X}_1$ and $\hat{X}_2$ from the steady state value of $\bar{X}$ and the initial deviations were $\hat{X}_1 = \bar{X} + 0.01$ and $\hat{X}_2 = \bar{X} - 0.01$). The simulation clearly shows convergence to a Markov SSE. In this SSE the ratio of output in the two states is $n_1/n_2 = 1.11$ and expected inflation $E_t^* \pi_{t+1}$ in the two states are 1.20 and 2.86.

The results of this section show that, in the case in which $\sigma > 1$ and the steady state is indeterminate, endogenous fluctuations due to expectational indeterminacy are a real concern. For exogenous sunspots near the resonant frequency, rational expectation SSEs exist and are stable under simple learning rules. These considerations raise the question of whether policy is able to avoid such expectational volatility.

7 Changes in Policy

We have seen that in this model there are two steady states, π_L and π_H when $\sigma < 1$ and $g > 0$. There has been much discussion of the issue of whether the economy might converge to the indeterminate steady state π_H in related seignorage models. While we do take seriously the potential economic instability of the economy due to the multiplicity of equilibria in this case, we believe that the primary concern in this case is divergent paths (with π_t increasing beyond π_H to unsustainable levels) if for some reason π_t escapes from the basin of attraction of π_L. Reductions in g tend to stabilize the economy in this case, making convergence to π_L more likely. Fiscal constraints on deficits and debt can also play an important role. The stability results of this and other papers suggest that π_H and SSEs near π_H are not locally stable under learning, though divergent paths are a concern.[18]

However, a new case appears in our model of seignorage finance. In the case $\sigma > 1$ there is a unique steady state that can be determinate or indeterminate and for σ sufficiently large it will necessarily be indeterminate. For values of $\sigma > 1$ the (noisy) steady state is stable under steady state learning. Furthermore, if the steady state is indeterminate and agents condition their actions on an exogenous sunspot near the resonant frequency, then they will converge to a noisy Markov SSE. Can policy help avoid the endogenous fluctuations which can arise in this case?

That this is indeed possible can be seen by continuing the example from the previous section. Simulations show that as g is reduced the amplitude of the fluctuations becomes smaller, and that if g is reduced sufficiently there will instead be convergence to the steady state, even though in their learning agents allow for the possibility of a dependence on the sunspot state. Figure 4 simulates the E-stability differential equation for the nonlinear model using the same structural parameter values and the transition probabilities for the Markov sunspot variable, but with g reduced to $g = 0.05$. (Initial values are taken as $\hat{X}_1, \hat{X}_2 = \bar{X} \pm 1$. Clearly there is now convergence to the steady state. In this steady state $\pi = 1.16$ and output fluctuations are eliminated.

Using the theoretical results of earlier sections, we can numerically investigate this issue more generally. For the results given in Table 1, we set $\alpha = 1$ and again fix the discount rate at $\mathfrak{B} = 0.95$. For each value of σ

[18]The standard seignorage model under learning was first studied by (Marcet and Sargent 1989a). The role of fiscal constraints is discussed in (Evans, Honkapohja, and Marimon 2001).

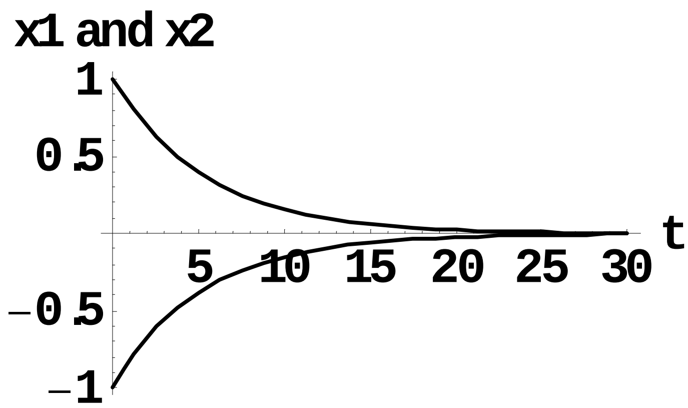

Figure 4: Convergence af adaptive learning to steady state

we examine the implications of different choices of g. Table 1 shows that for a substantial range of values for σ, endogenous fluctuations depending on extraneous sunspot variables can be avoided by decreasing government purchases g sufficiently.

TABLE 1: Critical values of g for $\alpha = 1$.

σ	1.1	1.5	2.0	2.5	3.0	3.5	3.7	3.9	≥ 4
g	0.359	0.341	0.267	0.190	0.118	0.053	0.029	0.006	0
π	16.89	4.39	2.56	1.84	1.44	1.175	1.091	1.018	1

For each value of σ reported in Table 1, a critical value of g is reported, together with the associated value of the steady state inflation rate. At the stated or larger values of g the steady state is indeterminate, there exist Markov SSEs, and these SSEs are stable under adaptive learning. For lower values of g the steady state becomes determinate and SSEs near the steady state no longer exist. Note that for $\sigma \geq 4$ any positive value of g is consistent with stable Markov SSEs. However, for any σ in the range $1 < \sigma < 4$ stable Markov SSEs exist for high values of g but not for sufficiently low values. In this region a reduction in g can bring about a double benefit by both reducing steady state inflation and eliminating SSEs.[19]

[19] These results are quantitatively, but not qualitatively, sensitive to the choice of α. For example, for $\alpha < 1$ determinacy can be obtained, with sufficiently low g, for values of σ less than an upper limit that now exceeds 4.

In the model of this paper we have focussed on the seignorage case in which government purchases are financed entirely by printing money. Seignorage models have been used most commonly as a potential explanation of hyperinflation, and our numerical example indeed emphasized the possibility of endogenous fluctuations arising at high levels of inflation when the level of seignorage is large. It is immediate from Table 1, however, that stable Markov SSEs near indeterminate steady states can also arise at low levels of inflation. We have made no attempt to calibrate the model to actual economies, and this would only be more appropriate with more elaborate versions of the model. However, the theoretical results of this paper show that monetary models of this type indeed have the power to explain business cycle fluctuations.

8 Conclusions

Indeterminacy of equilibria has been a major issue in both business cycle analysis and monetary economics. Most of the applied research has examined the question of the existence of self-fulfilling fluctuations in a neighborhood of an indeterminate steady state. This paper has imposed the additional discipline of asking whether there exist such RE solutions which are stable under adaptive learning dynamics. If solutions are not learnable, they may well be just theoretical artifacts. On the other hand, if they are stable under learning dynamics, then agents could plausibly coordinate on such solutions.

We have examined these issues in the context of a standard infinite horizon representative agent framework in which money demand is generated by cash in advance constraints. The model has allowed for both cash and credit goods as well as for a flexible labor supply with exogenous government spending financed by seignorage.

Using numerical techniques, we have shown that, for some regions of the parameter space, there do exist learnable sunspot equilibria in a neighborhood of an indeterminate steady state. The learnable SSEs take a particular form, namely the sunspot process is a finite state Markov chain with transition probabilities close to the resonant frequency property. The other types of SSEs are not learnable. In our model, the case of learnable SSEs arises for the regime in which there is a single steady state. In contrast, in the often studied regime with two steady states the SSEs near the indeterminate steady state are not learnable.

The existence of learnable non-fundamental equilibria is a key question for the endogenous fluctuations approach in monetary economics. Finding such solutions in models with indeterminacies is an additional desideratum which narrows the set of acceptable equilibria. However, the results in this paper indicate the potential of this line of thought for macroeconomics.

Appendix: Technical Details

Eliminating m_{t+1} and c_t^1 from (7) and (8)-(10) and linearizing leads to

$$n_t = -\sigma^{-1}\bar{\pi}^{-1}(1-\bar{n})E_t^*\pi_{t+1} - (1+\alpha^{-1/\sigma})(1-\bar{n})(\bar{c}^1)^{-1}E_t^* n_{t+1}, \text{ and}$$

$$\begin{aligned}
\pi_t &= \left(\frac{1}{\bar{m}-g}\right)(1+\alpha^{1/\sigma})\mathfrak{B}^{-1/\sigma}[-\sigma^{-1}\bar{\pi}^{(1-\sigma)/\sigma}(\bar{c}^1)E_{t-1}^*\pi_t - \bar{\pi}^{1/\sigma}(1+\alpha^{-1/\sigma})E_{t-1}^* n_t] \\
&\quad -\left(\frac{\bar{m}}{(\bar{m}-g)^2}\right)\times \\
&\quad \{(1+\alpha^{1/\sigma})\mathfrak{B}^{-1/\sigma}[-\sigma^{-1}\bar{\pi}^{(1-\sigma)/\sigma}(\bar{c}^1)E_t^*\pi_{t+1} - \bar{\pi}^{1/\sigma}(1+\alpha^{-1/\sigma})E_t^* n_{t+1}] - g_t\}.
\end{aligned}$$

Using the steady state relationships yields (14) and (15) with parameter values (16).

To see that J_1 is singular, note that

$$J_1 = (1-\bar{m})\begin{pmatrix} -(\bar{m}-g)^{-1} & -\sigma^{-1}\bar{\pi}^{-1}(1+\alpha^{-1/\sigma})^{-1} \\ \bar{\pi}(1+\alpha^{-1/\sigma})(\bar{m}-g)^{-2} & \bar{\pi}\sigma\bar{m}^{-1} \end{pmatrix}.$$

Thus $\det(J_1) = (1-\bar{m})\sigma^{-1}(\bar{m}-g)^{-1}\left(-\bar{\pi}\bar{m}^{-1} + (\bar{m}-g)^{-1}\right) = 0$ using $\bar{\pi} = \bar{m}/(\bar{m}-g)$.

Proof of Proposition 1: The model has no predetermined variables. Therefore indeterminacy prevails if at least one of the eigenvalues of $\Omega = (I-J_0)^{-1}J_1$ lies outside the unit circle., and otherwise the steady state is determinate. The matrix is given by

$$\begin{aligned}
\Omega &= \begin{pmatrix} 1 & 0 \\ -\beta_{n0} & 1-\beta_{\pi 0} \end{pmatrix}^{-1} \begin{pmatrix} \delta_n & \delta_\pi \\ \beta_{n1} & \beta_{\pi 1} \end{pmatrix} \\
&= \begin{pmatrix} \delta_n & \delta_\pi \\ \frac{\delta_n\beta_{n0}+\beta_{n1}}{1-\beta_{\pi 0}} & \frac{\delta_\pi\beta_{n0}+\beta_{\pi 1}}{1-\beta_{\pi 0}} \end{pmatrix}.
\end{aligned}$$

Because J_1 is singular, $\det(\Omega) = 0$ and the conditions for determinacy simplify to the single condition $|tr(\Omega)| < 1$. Since

$$tr(\Omega) = \frac{\delta_n - \delta_n \beta_{\pi 0} + \delta_\pi \beta_{n0} + \beta_{\pi 1}}{1 - \beta_{\pi 0}},$$

the indeterminacy condition $|tr(\Omega)| > 1$ is just the condition given in the statement of the proposition.

Proof of Proposition 3: The method of proof for Proposition 11.3 in (Evans and Honkapohja 2001c), which deals with the same situation of mixed datings of expectations, can be easily modified to the multivariate linear setting in this paper. Compare also Chapter 10 of (Evans and Honkapohja 2001c). Thus E-stability is equivalent to stability under learning.

The E-stability condition is that all the eigenvalues of $J_0 + J_1$ have real parts less than one, or equivalently that the eigenvalues of $J_0 + J_1 - I$ have real parts less than zero. For 2×2 matrices this condition can be written as

$$tr(J_0 + J_1) < 2 \text{ and } \det(J_0 + J_1 - I) > 0.$$

Given the form of the matrices in (18) we have

$$J_0 + J_1 = \begin{pmatrix} \delta_n & \delta_\pi \\ \beta_{n0} + \beta_{n1} & \beta_{\pi 0} + \beta_{\pi 1} \end{pmatrix}.$$

Using $\delta_n \beta_{\pi 1} - \delta_\pi \beta_{n1} = 0$ from $\det(J_1) = 0$ yields the condition stated in the proposition.

Proof of Proposition 4: For the VAR solutions one starts with PLM of the form (20) and computes that, under the PLM

$$\begin{aligned} E^*_{t-1} y_t &= a + b y_{t-1} + c_1 u_{t-1} + d_1 \eta_{t-1} \\ E^*_t y_{t+1} &= a + b(a + b y_{t-1} + c_1 u_{t-1} + d_1 \eta_{t-1}) + c_1 u_t + d_1 \eta_t, \end{aligned}$$

where it is assumed that the current values u_t, η_t of the exogenous shocks, but not the endogenous variable y_t are in the information set for period t.[20]

[20] This assumption is commonly made, see Chapter 10 of (Evans and Honkapohja 2001c) for a discussion.

Substituting these into the linear model (17) yields the mapping from PLM to ALM, which, component by component, is given by

$$\begin{aligned}
a &\to \xi + J_0 a + J_1(1+b)a \\
b &\to J_0 b + J_1 b^2 \\
c_0 &\to K + J_1 c_1 \\
c_1 &\to J_0 c_1 + J_1 b c_1 \\
d_0 &\to J_1 d_1 \\
d_1 &\to J_0 d_1 + J_1 b d_1.
\end{aligned}$$

The matrix quadratic in b in general has multiple solutions. The noisy steady state corresponds to the solution $b = 0$, and in the determinate case this is the unique stationary solution. Following the methodology of (Blanchard and Kahn 1980), (Farmer 1999) and (Evans and Honkapohja 2001c), Section 10.8, a solution $b \neq 0$ can be computed in the indeterminate case in which $(I - J_0)^{-1}J_1$ has a root outside the unit circle. Because $(I - J_0)^{-1}J_1$ always has one root inside the unit circle (specifically 0), imposing the condition that the solution be nonexplosive leads to a linear restriction between n_t, π_t and exogenous innovations dated t and $t-1$. The solution $b \neq 0$ has one eigenvalue of 0, corresponding to this linear restriction, and an eigenvalue equal to the inverse of the nonzero root of $(I - J_0)^{-1}J_1$. The necessary E-stability conditions in Proposition 4 are the local stability condition of the differential equations for the a and b components of the E-stability differential equation. Their formal derivation is analogous to those in Section 10.2 of (Evans and Honkapohja 2001c).

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