Asset Return Dynamics and Learning*

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Abstract

This paper advocates a theory of expectation formation that incorporates many of the central motivations of behavioral finance theory while retaining much of the discipline of the rational expectations approach. We provide a framework in which agents, in an asset pricing model, underparameterize their forecasting model in a spirit similar to Hong, Stein, and Yu (2005) and Barberis, Shleifer, and Vishny (1998), except that the parameters of the forecasting model, and the choice of predictor, are determined jointly in equilibrium. We show that multiple equilibria can exist even if agents choose only models that maximize (risk-adjusted) expected profits. A real-time learning formulation yields endogenous switching between equilibria. We demonstrate that a real-time learning version of the model, calibrated to U.S. stock data, is capable of reproducing many of the salient empirical regularities in excess return dynamics such as under/overreaction, persistence, and volatility clustering.

JEL Classifications: G12; G14; D82; D83

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1 Introduction

There is, by now, an established literature that studies financial market anomalies such as excess volatility and predictability of long-run excess returns. (See Lettau

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and Ludvigson (2005) for a recent discussion.) Additionally, a more recent strand of literature has developed that studies the relationship between the collapse of internet stock prices and the increase in publicly tradeable asset shares. (See Cochrane (2005), Ofek and Richardson (2003), and Hong, Scheinkman, and Xiong (2005)). Despite the extensive scope of this research, an important open issue is the implications of asset share supply dynamics for the properties of long-run excess returns. This paper develops a model of bounded rationality that is able to capture many of the salient empirical regularities.

One popular viewpoint is that empirically observed long-run excess returns cannot be explained by a standard Rational Expectations (RE) model. An explosion of research proposes alternative theoretical foundations for the empirical findings.[1] An offshoot of this literature looks beyond RE and formulates behavioral or boundedly rational channels through which these anomalies might arise (e.g., Barberis, Shleifer, and Vishny (1998), Hong and Stein (1999), Hong, Stein, and Yu (2005), and Lansing (2006)). Bounded rationality, of course, is not only of interest to financial economists. In macroeconomics there is a broad literature that replaces full rationality with agents who behave as econometricians; that is, by agents who estimate and select their models in real-time. (See, for example, Marcet and Sargent (1989), Evans and Honkapohja (2001), and Sargent (1999).)

While similar in spirit, these two approaches differ in the degree to which agents' expectations differ from rational expectations. For example, in Marcet and Sargent (1989) and Evans and Honkapohja (2001) agents typically have correctly specified reduced-form models but update their parameter estimates in real-time. In many models, these expectations converge to rational expectations. In Sargent (1999) and Williams (2004), agents may have misspecified econometric models but within the context of their subjective model they are unable to detect their misspecification. In Branch and Evans (2006a), computational and cognitive limitations force agents to underparameterize their forecasting models. These self-referential models restrict beliefs and the nature of misspecification to be determined in equilibrium.

In this paper, we apply the econometric misspecification approach, employed frequently in macroeconomics, to asset pricing questions. We develop our results in the context of an asset pricing model, with downward sloping demand for the risky asset, in which the stock price depends on expected future returns and on an exogenous process for share supply. Our modeling of share supply is meant to proxy for asset float, as discussed in Cochrane (2005) and Hong, Scheinkman, and Xiong (2005). We are motivated, in part, by Hong, et al. who demonstrate strong empirical implications from a model of heterogeneous expectations, increasing supply of shares, and short-sales constraints. Following the approach of Branch and Evans (2006a), we assume that agents underparameterize their forecasting model for price: agents

[1] An important counter viewpoint is provided by Fama and French (1996).

perceive price as depending on dividends or share supply, but not both.[2] This simple framework is meant to stand in for a more complex environment in which traders face computational limitations that force them to choose parsimonious trading strategies. We assume that agents only choose those models, or trading strategies, which yield the highest (risk-adjusted) trading profits. Within the class of underparameterized models, the key condition restricting beliefs is that expectations must satisfy a least-squares orthogonality condition. Agents' forecasting models are statistically optimal in the sense that their forecast errors are orthogonal to their predictor. We further restrict the set of admissible models by assuming agents only choose those models that maximize risk-adjusted trading profits.

While our approach to bounded rationality retains many of the more disciplined features of the rational expectations hypothesis, there are important deviations. Most importantly, exploitable trading profits exist that are not arbitraged away. We do not directly model why arbitrage fails in this model other than to point to recent work on the effect of short sales constraints by Cochrane (2005), Lamont and Thaler (2003), and Ofek and Richardson (2003). The short sales constraints studied in these papers matter for why expected increases in future supplies of a stock are not arbitraged away through short sales. The motivation for these studies was the large increase in the supply of shares at the beginning of 2000 as a number of dot-com IPO's lock-ups expired. This increase in supply was not priced into the market at the time of the IPO. We view our model as providing an equilibrium foundation to the empirical findings of Ofek and Richardson, among others, in the presence of short sales constraints.

Previous models taking a behavioralist perspective often address empirical puzzles: overreaction to 'news' about dividends, excess trading, long-run predictability, and volatile long-run excess returns. For example, Cutler, Poterba, and Summers (1991) find that stock returns in many countries are positively autocorrelated at short horizons and negatively autocorrelated over longer horizons. Cutler, et al. interpret this as evidence that there is initially underreaction to news and then overreaction over time.[3] Debondt and Thaler (1985) found that stocks receiving five years of good earnings news will underperform those with a five year period of relatively bad earnings. Similar cross-sectional evidence of underreaction is provided in Bernard (1992). Fama and French (1996) argue that these findings can be explained in part by properly defining risk, so that those stocks that are underreacting to news are actually less risky.

Many financial economists, however, embrace bounded rationality as a way of explaining the existence of multiple trading strategies, heterogeneity in expectations

[2]One could generalize the model further by assuming dividends and asset float follow multivariate stochastic processes with high order lags, and agents are restricted to underparameterize in at least one dimension. The main qualitative findings of this paper would extend to this more general formulation.

[3]This issue is discussed extensively in Cochrane (2001).

and preferences, volatility and under/over reaction to economic news. A gap exists in the literature that we seek to fill by studying the existence of these multiple trading strategies, and evolution over time, as an equilibrium phenomenon. To these ends, this paper makes a number of contributions. We demonstrate that underparameterization and misspecification equilibria can arise in a simple asset pricing model with linear demand. Moreover, depending on the deep parameters of the model, there may exist multiple misspecification equilibria in asset prices. Prices are partially revealing in these equilibria if the forecasting model variables are correlated with the omitted information. This follows from the least squares orthogonality condition, and is a feature that does not depend on the number and nature of misspecification equilibria in the model.

Our approach is closely related to a number of finance papers that also assume financial market participants underparameterize their forecasting models. For example, Hong, Stein, and Yu (2005) assume that dividends are driven by two exogenous processes and agents can only condition their expectations on one part of the process. In Barberis, Shleifer, and Vishny (1998) dividends follow a multi-layered Markov chain. It is multi-layered in the following sense: dividends can be either high or low, with transition probabilities associated with switching between states; there is also a Markov process switching between high and low probabilities of switching between states. The Barberis, et al. approach is meant to proxy for a simple model in which there are two different Markov processes governing dividends, one with high persistence and one with low persistence. Agents, though, only believe in one of the two models; hence the underparameterization.

These models, however, do not fully exploit the self-referential nature of asset pricing models. Instead, they appeal to behavioral and psychological explanations. While behavioral approaches are interesting and important there is still an open question of whether one can address these financial market anomalies and still assert the kind of discipline imposed by rational expectations. In a rational expectations model, the self-referential feature of the model requires that both the forecasts generated from the model and the market outcomes be jointly determined. In the approach presented in this paper we assume that agents underparameterize their forecasting model. However, we require that their beliefs, the form of their underparameterization, and the asset prices all be equilibrium outcomes. In a sense, our approach generalizes Hong, Stein, and Yu (2005) and Barberis, Shleifer and Vishny (1998) by interacting agents' deviations from full rationality with the economic environment.

The equilibrium in the model described so far implies that trading strategies and expectations are time-invariant. We also consider a real-time learning and dynamic predictor selection version in order to study the model's ability to capture empirical regularities in excess return dynamics. In this extension of the basic model, agents update in real-time their parameter estimates and a geometric average of past trading

profits. They then decide on their predictor and holdings of the risky asset conditional on these real-time estimates. We demonstrate that, with the model calibrated to U.S. stock data, the model implies a similar autocorrelation pattern as Cutler, Poterba, and Summers (1991). Moreover, the approach presented here is able to match the pattern of persistence and volatility clustering found in Guidolin and Timmermann (2005a).

In addition to the theoretical implications, we argue that the model's ability to capture many of the salient features of excess returns dynamics presents a strong argument in favor of the approach developed in this paper. Despite the richness of the theoretical results, the empirical implications are intuitive. Excess returns underreact to news because agents underparameterize their forecast model, and information slowly diffuses through the property that prices partially reveal information. However, excess returns overreact in the sense that large or successive shocks to, say, dividends will cause the economy to switch from the neighborhood of one underparameterized equilibrium to another. This switching between models generates regime switching means and volatilities similar to the Markov chains in Guidolin and Timmermann.

Our paper fits into a broader literature on misspecification and asset pricing. We employ the same simple asset pricing model presented in Brock and Hommes (1998). They also consider agents' real-time choice of trading strategy, though their approach is more in line with the trading strategies of Hong and Stein (1999). In Guidolin and Timmermann (2005b), traders form high order expectations using a Bayesian learning mechanism. Timmermann (1994, 1996) also assumes that agents form expectations via a statistical forecasting model. Cagetti, Hansen, Sargent, and Williams (2002) study robust decision making where agents are concerned with (potential) model misspecification. Anderson, Ghysels, and Juergens (2005) derive a Consumption CAPM where agents are structurally heterogeneous in their expectations.

This paper proceeds as follows. Section 2 presents the model. Section 3 presents theoretical results. Section 4 discusses the empirical implications while Section 5 presents the calibrated version of the model. Section 6 concludes.

2 Asset Pricing Model with Restricted Perceptions

We follow Grossman and Stiglitz (1980), Brock and Hommes (1998), Hong, Scheinkman, and Xiong (2005), among many others, in assuming that agents optimize with respect to the mean-variance efficient frontier. We make this assumption for two analytical reasons: first, so that demands remain bounded; second, so that demand is linear. One can justify the assumption based on a log approximation to exponential utility with Gaussian returns (Cochrane (2001)). The assumption (at least of linear demand) is standard in the literature and is made in, for example, Hong, Stein, and Yu (2005).

The objective of this paper is to provide a framework in which limited cognitive abilities and restricted perceptions impinge on equilibrium asset prices. Our approach to bounded rationality is to impose that when agents deviate from full rationality, they do so in a statistically optimal manner. We make our points in a simple model in which asset prices are driven by expected capital gains and by exogenous processes for dividends and asset share supply. We argue that our main results would extend to a model with more explicit microfoundations.[4]

We introduce share supply into the model to capture float. Recent papers by Cochrane (2005), Lamont and Thaler (2003), Hong, Scheinkman, and Xiong (2005) show that float can have an effect on price in the presence of short sales constraints. The idea is that with incomplete markets the demand curve for a risky asset might be downward sloping. We interpret our highly stylized model as a proxy for a more complicated setting where there exist many investors, all constrained by computational and cognitive limitations as well as facing short sale constraints. Because the full information rational expectations outcome is precluded by both of these constraints, the issue facing investors is what "technical trading" strategy to follow. The novelty of our approach is that we can pin down the boundedly rational trading strategy in an equilibrium. One important implication of this equilibrium is that prices are partially revealing. We then show via simulations that our approach has important implications for the time-series of asset prices and excess returns.

The household's problem at time t is:

$$\max_{z} EW_{t+1} - \frac{a}{2} EVarW_{t+1}$$

subject to

$$W_{t+1} = RW_t + (p_{t+1} + y_{t+1} - Rp_t)z_t$$

where z_t is the holdings of the risky asset, p is its price, y are dividends, and $R > 1$ is the nominal risk-free rate of return. We assume that dividends follow a stationary AR(1) process, which has a deviations from mean-form

$$y_t = \rho y_{t-1} + \varepsilon_t$$

where ε_t is mean-zero with variance σ_ε^2. In equilibrium, the demand of shares must equal supply. The usual assumption is that the supply of shares is constant and normalized to one. We, however, assume a stationary AR(1) process representing the supply of shares:

$$z_{st} = \phi z_{st-1} + \nu_t$$

The stochastic disturbance ν_t is mean-zero, with variance σ_ν^2, and is possibly correlated with ε_t, i.e. we allow for $\sigma_{\nu\varepsilon} \neq 0$. We assume, without loss of generality, that

[4] As Cochrane (2001) points out one can derive the reduced-form of this model from one in which the household consumes only in the last-period. Successive generations would give the dynamics implied below.

the exogenous processes are mean-zero. Ideally, the risky asset would be a composite index of stocks and y_t, z_{st} would be high dimensional VARs. We abstract from such a specification for analytical convenience. We also interpret the share process z_{st} as a proxy for asset float. Asset float is the change in the supply of shares usually after a lock-up period following an initial public offering. We view asset float and the supply of shares more generally so that it also includes stock repurchases. Assuming that the supply of shares follows an AR(1) is an obvious analytic device. Implicitly we are assuming short sales constraints and that future increases or decreases in supply are known only to the extent that they are forecastable from the AR(1) law of motion. This paper is a first step at incorporating equilibrium underparameterization and learning into an asset pricing model and leaves more realistic strategic and hedging considerations for future research. In the calibrated version of the model, below, we estimate an AR(1) for share supply using U.S. data.

There are two types of agents, each omitting some relevant information from their forecasting model when they solve the above problem. One type omits the role of supply in affecting price while the other omits the dividends process. We make this assumption to bring some realism to the asset pricing model. Because of computing constraints agents are forced to underparameterize their model. This is the same motivation of Hong, Stein, and Yu (2005) in the case where agents omit a portion of the dividend process from their forecasting model. Heterogeneous expectations also arise in Hong, Scheinkman, and Xiong (2005). A novelty to our approach is that we endogenize the distribution of agents across these underparameterized models. Our approach can be viewed as an extension of these other papers in the direction of parameter and trading strategy uncertainty. Although we assume that dividends are a univariate stochastic process, one could easily extend dividends and share supply to bivariate VAR processes along the lines of Branch and Evans (2006a). The main innovation to our approach is that we pin down *both* the forecasting model parameters *and* the distribution of agents across models as an equilibrium object. We then can use real-time learning to study the dynamics and to speculate on the model's ability to address some of these financial market puzzles.

Each agent type j solves

$$\max_{z_j} RW_t + E_t^j \left(p_{t+1} + y_{t+1} - Rp_t\right) z_{jt} - \frac{a}{2}\sigma^2 E_t^j z_{jt}^2$$

where $\sigma^2 = Var_t(p_{t+1} + y_{t+1} - Rp_t)$ is the subjective conditional variance of the rate of return, which for simplicity is assumed constant over time and uniform across agent types.[5] The first-order condition leads to the demand for type j of,

$$z_{jt} = \frac{1}{a\sigma^2} E_t^j \left(p_{t+1} + y_{t+1} - Rp_t\right)$$

[5]This assumption follows Brock and Hommes (1998). The case of heterogeneous and time-varying σ^2 was considered in Gaunersdorfer (2001). Allowing for heterogeneous and time-varying σ^2 is a topic of our current research.

The responsiveness of demands z_{jt} to expected rates of return depends on $a\sigma^2$, which it will be convenient for us to call "perceived risk." Note that perceived risk is a product of the subjective conditional variance and the degree of risk aversion.

Financial market equilibrium requires that price adjusts to ensure market clearing. Let n denote the fraction of agents with expectations E_t^1. In equilibrium,

$$nz_{1t} + (1-n)z_{2t} = z_{st}$$

which leads to the equilibrium process for stock prices,

$$p_t = \beta n E_t^1 p_{t+1} + \beta(1-n)E_t^2 p_{t+1} + \beta\rho y_t - \beta a\sigma^2 z_{st} \tag{1}$$

where for convenience we write $\beta = R^{-1}$. To derive (1) we have assumed that $E_t^1 y_{t+1} = E_t^2 y_{t+1} = \rho y_t$. This is a natural assumption. We envision underparameterization because computing and degrees of freedom constraints prevent agents from regressing price on all available information. Agents know the univariate processes for dividends and supply, but we assume that it is prohibitively costly to incorporate both elements into their forecasting model for *price*.

It might appear contradictory that agents know the processes for dividends and supply, yet they do not use all known information when forecasting stock price. In this simple setting this is, of course, unrealistic. But, if one thinks of all of the factors that might be influencing dividends, share supply, and price the total number of factors with non-trivial predictive power would exceed computational and degree of freedom constraints. If dividends and share supply were actually high order vector autoregressive processes, possibly correlated, then forecasting future dividends and supplies are curtailed by the number of parameters of the model. For example, an n-variable VAR(p) has $n^2 \times p$ coefficients to estimate, plus the parameters of the autocovariance matrix. At the monthly frequency, the degrees of freedom would quickly evaporate.

Hong, Stein, and Yu (2005) and Barberis, Shleifer, and Vishny (1998) also assume underparameterized forecast models. These authors, however, motivate the assumption by appealing to psychology research that suggests people forecast using simple paradigms or reference models. One could also extend their motivations to our approach. Our primary motivation for underparameterization, though, is to model agents as econometricians. Given computational and degree of freedom limitations, VAR practitioners specify parsimonious forecasting models. In complex environments such as the stock market, we would expect similar behavior on the part of traders. Our theoretical interest, though, is to impose some modeling discipline on these deviations from full information: in our framework, within the context of their forecasting models, agents are unable to detect their misspecification. Remarkably, the theoretical and empirical implications of this approach are rich.

Agents forecast by projecting a perceived law of motion (PLM) for price. The set of PLM's, given the underparameterization restriction, are:

$$\begin{aligned} PLM_1 : p_t &= b^1 y_t + \eta_t \\ PLM_2 : p_t &= b^2 z_{st} + \eta_t \end{aligned}$$

where η_t is a perceived exogenous white noise shock. This implies expectations of the form,

$$\begin{aligned} E_t^1 p_{t+1} &= b^1 \rho y_t \\ E_t^2 p_{t+1} &= b^2 \phi z_{st} \end{aligned}$$

Plugging these expectations into (1) leads to the following actual law of motion (ALM) for price,

$$\begin{aligned} p_t &= \beta n b^1 \rho y_t + \beta(1-n) b^2 \phi z_{st} + \beta \rho y_t - a\beta\sigma^2 z_{st}, \text{ or} \\ p_t &= \xi_1(n) y_t + \xi_2(n) z_{st} \end{aligned} \tag{2}$$

where

$$\begin{aligned} \xi_1(n) &= \beta\left(nb^1 + 1\right)\rho \\ \xi_2(n) &= \beta\left((1-n)b^2\phi - a\sigma^2\right) \end{aligned}$$

In the sequel, we will suppress the dependence of ξ_j on n. In a rational expectations equilibrium (REE),

$$\begin{aligned} \xi_1 &= \frac{\beta\rho}{1-\beta\rho} \\ \xi_2 &= -\frac{\beta a\sigma^2}{1-\beta\phi} \end{aligned}$$

Although agents in the model are assumed to have underparameterized forecasting models (restricted perceptions), we require that they forecast in a statistically optimal manner. We require that the forecast model parameters are optimal linear projections. That is, the belief parameters $b^j, j = 1, 2$ satisfy the following least-squares orthogonality conditions,

$$Ey_t\left(\xi_1 y_t + \xi_2 z_{st} - b^1 y_t\right) = 0 \tag{3}$$

$$Ez_{st}\left(\xi_1 y_t + \xi_2 z_{st} - b^2 z_{st}\right) = 0 \tag{4}$$

or,

$$\begin{aligned} b^1 &= \xi_1 + \xi_2 r \\ b^2 &= \xi_2 + \xi_1 \tilde{r} \end{aligned}$$

where $r = Ey_t z_{st}/Ey_t^2, \tilde{r} = Ey_t z_{st}/Ez_{st}^2$. Orthogonality conditions like (3) appear frequently in the macroeconomics literature. For example, Sargent (1999), Cho, Williams, and Sargent (2003) define a self-confirming equilibrium with respect to a very similar condition. Evans and Honkapohja (2001) show that under adaptive learning an underparameterized forecasting model may converge to a set of parameters that satisfy an orthogonality condition like (3). Many other applications that employ (3) are discussed in Branch (2006). The key feature of orthogonality conditions like (3),(4), are that within the context of their forecasting model, agents are unable to detect their misspecification.[6]

Given exogenous processes y_t, z_{st}, and ξ_j, $j = 1, 2$, and given the proportion n of agents using forecast model $j = 1$, a Restricted Perceptions Equilibrium (RPE) is then defined as a stochastic process $\{p_t\}$ of the form (2), where the coefficients satisfy

$$\begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} = \begin{bmatrix} 1 - \beta\rho n & -\beta\rho n r \\ -\beta\phi(1-n)\tilde{r} & 1 - \beta(1-n)\phi \end{bmatrix}^{-1} \begin{bmatrix} \beta\rho \\ -a\sigma^2\beta \end{bmatrix}$$

We have our first result.[7]

Proposition 1 *There exists a unique RPE for every* $0 \leq n \leq 1$.

Although agents in the model are underparameterizing their forecasting models, each agent's forecast does reflect the influence of that part of the omitted variable that is correlated with the variables used in their forecast. This property arises because of the orthogonality condition, which is satisfied in equilibrium: b^j depends on the two reduced-form parameters ξ_1, ξ_2 and also on the regression coefficient $(r, \tilde{r})$. In addition, asset prices aggregate and reflect all available information – in this sense asset prices are partially revealing.

It is important to note that the model is self-referential: b^j, hence ξ_j are not free parameters but are equilibrium objects. For similar reasons, we do not want to treat n as a free parameter and now proceed to make it endogenous. In consequence, although agents use misspecified forecast models, there are still important cross-equation restrictions imposed on the dynamics that are analogous to the restrictions obtained under fully rational expectations.

In order to pin down n, we need a metric for evaluating forecast success. In Brock and Hommes (1998) this metric is trading profit from the most recent period. However, they mention that one might expect distinct results if a risk-adjusted fitness measure is adopted instead. In order to stay in line with the assumption that agents

[6]Of course, if they step out of their model and run specification tests they could detect the misspecification.

[7]All proofs are in the Appendix.

are mean-variance maximizers, we also assume that agents adjust their trading profits for variance when deciding on forecast success. Thus, we assume that each agent ranks the two forecasting models according to,

$$U^j = E\pi_t^j - \frac{a}{2}\sigma^2 E z_{jt}^2$$

where $\pi_t^j = (p_{t+1} + y_{t+1} - Rp_t)\, z_{jt}$ and E is the (unconditional) expectations operator.

The Appendix computes π_t^j and Ez_{jt}^2 for $j = 1, 2$. Predictor selection depends on the difference in fitness measures. Define $F(n) : [0, 1] \to \mathbb{R}$ as $F(n) = U^1 - U^2 = (E\pi_t^1 - E\pi_t^2) + (a\sigma^2/2)(Ez_{2t}^2 - Ez_{1t}^2)$. Then we can write this expression as

$$F(n) = \frac{1}{a\sigma^2}\left(B_y E y_t^2 - B_{yz} E y_t z_{st} + B_z E z_{st}^2\right)$$

where B_y, B_z and B_{yz} are given by

$$\begin{aligned} B_y &= \frac{\rho^2}{2}\left(\xi_1^2 - r^2\xi_2^2\right) \\ B_z &= \frac{\phi^2}{2}\left(\tilde{r}\xi_1^2 - \xi_2^2\right) \\ B_{yz} &= \phi\rho\left(r\xi_2^2 - \tilde{r}\xi_1^2\right). \end{aligned}$$

Note that B_y, B_z and B_{yz} are functions of n because ξ_1 and ξ_2 depend on n.

As in our earlier papers, we follow Brock and Hommes (1997) in assuming a multinomial logit (MNL) approach to predictor selection. The MNL approach has a venerable history in discrete decision making. In this setting, agents are selecting their forecasting models from a discrete choice set and so the MNL map is natural in this setting:

$$n = \frac{\exp(\alpha U^1)}{\exp(\alpha U^1) + \exp(\alpha U^2)}$$

which can be written,

$$n = \frac{1}{2}\left[\tanh\left\{\alpha F(n)\right\} + 1\right] \equiv T_\alpha(n) \tag{5}$$

In particular, $T : [0, 1] \to [0, 1]$ is a continuous and well-defined function provided that an RPE exists.

Definition. *A Misspecification Equilibrium n^* is a fixed point of the map T: $n^* = T(n^*)$.*

By Brouwer's theorem, a Misspecification Equilibrium (ME) exists in this model. The T-mapping is indexed by the parameter α which is typically called the 'intensity of choice' parameter. Since the MNL map derives from a random utility setting, finite values of α parameterize deviations from full utility maximization. The 'neoclassical' case is $\alpha \to \infty$. Our interest is mainly in equilibria where all agents choose only the best performing statistical model and so we will focus on the $\alpha \to \infty$ case.

3 Analytic Results

It is useful to re-write the function $F(n)$ as,

$$\frac{F(n)}{Ey_t^2} = \frac{1}{a\sigma^2}\left(B_y - B_{yz}r + B_z Q\right)$$

where $Q = Ez_{st}^2/Ey_t^2$. The number and nature of Misspecification Equilibria depend on the properties of $F(n)$. Furthermore, one can calculate

$$\begin{aligned} F(0) &= \beta^2\left[\rho^2 C_2 + \frac{\left(a\sigma^2 - \tilde{r}\beta\rho\phi\right)^2}{(1-\beta\phi)^2}C_1\right] \\ F(1) &= \frac{\beta^2}{(1-\beta\rho)^2}\left[\rho^2(1-a\sigma^2\beta r)^2 C_2 + (a\sigma^2)^2(1-\beta\rho)^2 C_1\right] \end{aligned}$$

where $C_1 = -(1/2)Q\phi^2 + r^2\rho(-(1/2)\rho + \phi)$, $C_2 = (1/2)\rho^2 - r\tilde{r}\rho\phi + (1/2)\tilde{r}^2\phi^2 Q$.

These are complicated expressions and general results are not available. However, using the argument in Branch and Evans (2006b), the following result can be used to characterize possible equilibria:[8]

Proposition 2 *Let $N_\alpha^* = \{n^* | n^* = T_\alpha(n^*)\}$ denote the set of Misspecification Equilibria. In the case of large α, N^* has one of the following properties:*

1. *If $F(0) < 0$ and $F(1) < 0$ (Condition P0) then $n^* = 0 \in N^*$.*
2. *If $F(0) > 0$ and $F(1) > 0$ (Condition P1) then $n^* = 1 \in N^*$.*
3. *If $F(0) < 0$ and $F(1) > 0$ (Condition PM) then $n^* \in \{0, \hat{n}, 1\} \subseteq N^*$, where $\hat{n} \in (0,1)$ is such that $F(\hat{n}) = 0$.*
4. *If $F(0) > 0$ and $F(1) < 0$ (Condition P) then $n^* = \hat{n} \in N^*$, where $\hat{n} \in (0,1)$ is such that $F(\hat{n}) = 0$.*

Because we do not know whether F is monotonic, we cannot rule out the existence of additional equilibria besides those listed. When Condition P0 or Condition P1 holds then either $n^* = 0$ or $n^* = 1$ is a Misspecification Equilibrium. If Condition PM holds then both $n^* = 0$ and $n^* = 1$ are Misspecification Equilibria. This is the case of multiple equilibria that will receive further attention below. Condition P implies that there exists an interior Misspecification Equilibrium with heterogeneous

[8] We remark that for perfectly correlated disturbances the system is degenerate in the sense that any $0 \leq n \leq 1$ is an equilibrium. This follows since $F(n) \to 0$ as $r\tilde{r} \to 1$.

expectations. In Branch and Evans (2006a) we said that when Condition P holds the model exhibits Intrinsic Heterogeneity. As indicated below, this case does not appear to arise in the current paper for empirically realistic cases with two exogenous shocks. Notice that under Condition PM there must also be an interior equilibrium $\hat{n}$ for large α. However, because $F(n)$ is a continuous function, Condition PM implies that this equilibrium satisfies $T'(F(\hat{n})) > 1$ and hence is unstable. Conversely, under Condition P, for large α there is an $\hat{n}$ at which $F(n)$ crosses through zero from above and, as we showed in our earlier paper, this equilibrium is locally stable.

Proposition 2 does not state under which circumstances these conditions will arise. In fact, it does not even state whether all of the cases are possible. The signs of $F(0), F(1)$ depend in a complicated way on $\phi, \rho, a\sigma^2, \sigma_\varepsilon^2, \sigma_\nu^2, E\varepsilon\nu$. The conditions in Proposition 2 essentially place bounds on the relative variances Q. Analytic results are available for special limiting cases given below. Section 3.2 then presents numerical examples.

Corollary 3 *Conditions P0, P1, PM and P can each be satisfied for appropriate choices of structural parameters.*

We remark, however, that Condition P only appears to arise if either $\rho < 0$ or $\phi < 0$, or both, which seems unrealistic empirically. The other cases do not require negative serial correlation of the exogenous variables. Numerical examples are given below.

Additional analytical results are available for certain limiting cases of interest. In particular, we have:

Corollary 4 *Assume $\rho, \phi > 0$. For $|r|, |\tilde{r}|$ sufficiently small we have:*

(i) Condition P0 holds if $a\sigma^2\sqrt{Q} < \rho^2(1-\beta\phi)/\phi$;

(ii) Condition P1 holds if $a\sigma^2\sqrt{Q} > \rho^2/(\phi(1-\beta\rho))$;

(iii) Condition PM holds if $\rho^2(1-\beta\phi)/\phi < a\sigma^2\sqrt{Q} < \rho^2/(\phi(1-\beta\rho))$.

This corollary shows the importance of risk aversion and the relative variance of supply shocks. For a given Q, values of perceived risk $a\sigma^2$ that are neither too high nor too low lead to multiple equilibria even in the case of low contemporaneous correlation between the exogenous shocks.

3.1 Some intuition

There are two exogenous processes driving asset prices: dividends and the supply of shares. Both stochastic processes though have two effects in (1): the direct effect and an indirect effect acting through expectations.[9] The number and nature of equilibria depend on the balancing of these two effects. Notice that p_t depends positively on expectations. Thus, whether these direct effects are positively or negatively projected onto the asset price depends on the equilibrium belief parameters, which in turn depend on the equilibrium proportion of agents adopting the dividend forecasting model.

The feedback effects are:

$$\begin{aligned} E_t^1 p_{t+1} &= (\xi_1 + \xi_2 r)\rho y_t \\ E_t^2 p_{t+1} &= (\xi_2 + \xi_1 \tilde{r})\phi z_{st} \end{aligned}$$

Notice in the expressions for ξ_1, ξ_2 in the special case above of $r, \tilde{r} \to 0$ that $a\sigma^2$ directly influences the size of ξ_2 and that ξ_2 is negative (because z_{st} has a negative direct effect):

$$\begin{aligned} \xi_1 &= \frac{\beta\rho}{1-\beta\rho n} \\ \xi_2 &= \frac{-a\sigma^2\beta}{1-\beta(1-n)\phi} \end{aligned}$$

In this case where the shocks are uncorrelated, $r = \tilde{r} = 0$, beliefs reinforce the direct effect of dividends and supply of shares. Multiple equilibria arise naturally in this case for a range of perceived risk. The condition on $a\sigma^2$ required for multiple equilibria puts bounds on the importance of the direct effect of z_{st} relative to dividends. If $a\sigma^2\sqrt{Q}$ is large then the share supply forecast model always dominates, while when $a\sigma^2\sqrt{Q}$ is sufficiently low the dividend model is necessarily superior. For intermediate values of perceived risk, either model can emerge as an equilibrium.

3.2 Numerical Examples

In this subsection we turn to numerical examples to illustrate our theoretical results. In each case we plot the T-map, $F(n), \xi_1(n)$, and $\xi_2(n)$. We are interested in large α, so we set $\alpha = 10000$. Above we presented analytic results, for the special case of weakly correlated exogenous processes, and provided some more general intuition. We here choose particular parameter values to illustrate the rich theoretical properties of the model.

[9] Strictly speaking, the timing of the model is that dividends are paid at time $t+1$, so that it is expected dividends that matter for price. Since agents have common beliefs on dividends, these have a common effect effect on stock prices, as specified in (1).

3.2.1 Multiple Equilibria

We adopt the parameter values $\rho = .6, \phi = .4, \beta = .95, \sigma_\nu = 1.7, \sigma_\varepsilon = 1, \sigma_{\nu\varepsilon} = .25, a\sigma^2 = 1$. Figure 1 plots (clockwise starting from the northwest frontier): the T-map, the reduced-form RPE parameters ξ_1 and ξ_2 respectively and the risk-adjusted profit difference function $F(n)$. Each frontier plots these values against n. A Misspecification Equilibrium occurs when the T-map crosses the 45-degree line.

Notice first that $F(n)$ is monotonically increasing with $F(0) < 0, F(1) > 0$. As a result, the northwest panel demonstrates that there are multiple equilibria, in particular at $n = 0$, $n = 1$. This result is in line with our earlier intuition of the effect of positive feedback in self-referential models. In this case, there exist three equilibria. Notice, though, that the interior equilibrium occurs at $\hat{n}$ where $F(\hat{n}) = 0$. This equilibrium is unstable in the sense that $T'(\hat{n}) > 1$ and so under a real-time learning and predictor selection dynamic, as considered below, we would not observe the interior equilibrium as an outcome.

INSERT FIGURE 1 HERE

In the right-most panels of the figures $\xi_1(n), \xi_2(n)$ are plotted. These panels illustrate the intuition that it is the trade-off between the positive and negative feedback of the two exogenous processes that makes multiple equilibria possible. It is worth noting that although $\xi_1(n)$ is increasing in the figure it is not always so even in the case of multiple ME. This makes it harder to draw general conclusions.

The propositions and these numerical examples suggest that $a\sigma^2$ plays a significant role in the nature of the equilibria.[10] To study this further Figure 2 plots the comparative static of $a\sigma^2$ on the value of n^* for the parameterization used to generate Figure 1. In particular, Figure 2 is a bifurcation diagram with $a\sigma^2$ as the bifurcating parameter. To generate the figure we consider all values of $a\sigma^2$ in the interval $[0.5, 1.5]$ and plot all corresponding fixed points to T.[11]

INSERT FIGURE 2 HERE

Figure 2 plots the bifurcation diagram. For low values of $a\sigma^2$ there is a unique equilibrium at $n = 1$, for medium $a\sigma^2$ there are multiple equilibria, and for large

[10] We remind the reader σ^2 is the perceived one-step ahead variance of the excess rate of return. The actual average value of this quantity can be made to match this perception by tuning up or down the variance of the intrinsic shocks. Since there is no consensus for the value of a, we look at various values of $a\sigma^2$ assuming agents estimate σ^2 correctly.

[11] In Brock and Hommes (1997) the 'intensity of choice' parameter α was treated as a bifurcation parameter. In this paper, we are primarily interested in $\alpha \to \infty$ to concentrate on equilibria where all agents only choose the best performing models. Thus, in this stochastic setting the degree of risk-aversion is a more interesting and relevant bifurcation parameter.

enough $a\sigma^2$ there is a unique equilibrium at $n = 0$ thereafter. These results are in line with Corollary 4. A similar diagram, of course, exists for Q, the relative variance of dividends.

3.2.2 Comparison to Rational Expectations

Above, it was shown that a restricted perceptions equilibrium is distinct from a rational expectations equilibrium. One obvious question is how badly the underparameterized agents are faring relative to if they had full information. To address this issue this subsection conducts the following experiment: suppose there exists a zero-mass hyper-rational agent who knows the actual law of motion for price, including the distribution of agents across models, and this agent could trade on that information. How much better would the hyper-agent fare compared to the underparameterized forecasting models? If the additional (risk-adjusted) profits for the hyper-rational agent are relatively small, then we would not expect trading profits to be so high as to justify overcoming the higher computational costs associated with using all information.

Figure 3 reports the results of this experiment. In Figure 3, the risk-adjusted profits U_1, U_2 and U_r (the risk-adjusted profits of the hyper-rational agent), are plotted for various values of $a\sigma^2$ using the same parameterization used to generate Figure 1. Each panel corresponds to a different value of $a\sigma^2$. The top panel is for a value of $a\sigma^2 = 0.3$ that corresponds to $n = 1$ as the unique equilibrium. The middle panel fixes $a\sigma^2 = 0.83$ so that there exists multiple equilibria, as in Figure 1. Finally, the bottom panel ($a\sigma^2 = 1.40$) reports risk-adjusted profits in the event $n = 0$ is the unique equilibrium. As mentioned above U_r are the risk-adjusted profits for a trader who formulates their asset holdings using the actual law of motion (2) as their forecasting model for next period's price.

INSERT FIGURE 3 HERE

Figure 3 demonstrates that a hyper-rational agent will always do better than underparameterized agents. This is expected since such an agent makes use of all information and there are no feedback effects from their trading since they have zero mass. However, the difference between the hyper-rational and the two underparameterized models is small, and negligible for some values of $a\sigma^2$. The top panel is for the case of a small $a\sigma^2$, so that there is a unique equilibrium at $n = 1$. At the equilibrium, the $n = 1$ model and hyper-rational return almost identical returns. In the middle panel, for moderate values of $a\sigma^2$ implying the existence of multiple equilibria, the difference is slightly more pronounced. Notice also that the difference is greatest at the unstable interior equilibrium. The bottom panel (the case when $n = 0$

is the unique equilibrium) the difference is negligible between the supply model and rational profits.

From this graph one can conclude that the profit difference between hyper-rational and underparameterized agents is relatively small. Provided complex forecasting models are costly, it is plausible to assume agents would use the best fitting univariate model. We return briefly to this issue later in the context of real-time dynamics.

4 Empirical Implications

As a means of highlighting the model's empirical implications we focus on two dynamic properties of asset markets that have received significant attention in the finance literature: (i) under and overreaction to fundamentals, and (ii) regime switching of means and volatilities in long-term excess returns. In the next Section, we calibrate the model and demonstrate its ability to match these empirical features. The current Section aims to illustrate the channels through which the model is capable of matching the empirical regularities.

4.1 Over/underreaction

The under/overreaction puzzle has been stated in many ways, e.g. as deriving from momentum traders as in Hong and Stein (1999) or from multi-factor models as in Fama and French (1996). These effects have been demonstrated both for price and excess returns. In this section, we illustrate over/underreaction to economic news via excess returns impulse response functions. We compare the impulse responses of two distinct Misspecification Equilibria: $n = 0, 1$. The $n = 0$ equilibrium corresponds to traders underparameterizing by omitting dividends from their price regression, while in the $n = 1$ equilibrium agents instead omit supply shares. We present two scenarios: a dividend shock and a supply shock scenario. We then compare each equilibrium's dynamic response to the REE in which an unanticipated shock has an initial effect on excess returns but then returns quickly converge to their steady-state values. Our notion of under or overreaction is relative to the rational expectations benchmark. In particular, under/overreaction will be reflected in impulse responses as a more gradual response to news innovations.

Intuitively, one would expect that in a $n = 0$ ME, the market would initially underreact to dividends innovations relative to supply shocks in short horizons as agents do not fully incorporate dividend news into their price forecasts, and vice-versa for the $n = 1$ ME. At longer horizons, though, as the information diffuses into their expectations via the orthogonality condition, they overreact *vis a vis* rational

expectations.

Figure 4 confirms this intuition. To make the intuition sharp, Figure 4 uses the same parameterization as was used to generate Figure 1 except that it sets $\sigma_{\nu\varepsilon} = 0$. We then draw from the exogenous processes for dividends and supply shares and plot out the dynamic response of excess returns for ME and the REE. The top panel plots the impulse response for a (positive) dividend shock and the bottom for a (negative) supply shock.[12] Both shocks produce a positive innovation to price. In the top panel, the solid line corresponds to $n = 0$ and in the bottom panel it corresponds to $n = 1$. In response to a dividend shock, the $n = 0$ initially underreacts. Over subsequent periods, however, it overreacts as it slowly mean-reverts. Similar results appear for the supply shock.

INSERT FIGURE 4 HERE

4.2 Persistence and Volatility Clustering

In addition to the specific issue of under/overreaction we also address persistence and excess volatility in net returns. There is a large and recent literature that shows excess volatility and conditional heteroskedasticity in stock returns. Additionally, a recent study by Guidolin and Timmermann (2005a) demonstrates that there is regime switching means in excess returns. To illustrate the type of dynamics our model exhibits we turn to a real-time learning and dynamic predictor selection version of the model. We assume that agents update their risk-adjusted fitness and forecasting model parameters using a recursive updating algorithm with constant gain.

In the real-time learning and dynamic predictor selection version of the model agents do not have fixed beliefs. Beliefs are generated using least-squares in real-time. Time varying parameter estimates make it possible that a sequence of shocks could move the economy from one equilibrium to another (in the case of multiple ME). For this reason, agents will want to remain guarded against the possibility of a regime change and choose their forecasting strategy in real time as well.

Price is now given by the law of motion,

$$p_t = \xi_1(b^1_{t-1}, n_{t-1})y_t + \xi_2(b^2_{t-1}, n_{t-1})z_{st}$$

The timing of the model is that at the end of each period agents update their beliefs of b^1, b^2, their risk-adjusted expected profits, and their model choice n. At time t then price depends on the real-time learning and dynamic predictor selection from the end

[12] We omit the $n = 1$ case for the dividend shock and the $n = 0$ case for the supply shock because with uncorrelated disturbances these ME will coincide with the REE impulse response.

of period $t-1$. We make this timing assumption to avoid the simultaneity between prices and beliefs.[13] Using recursive least-squares (RLS), the belief parameters are calculated as

$$b_t^j = b_{t-1}^j + \lambda_t R_{j,t}^{-1} x_{j,t-1} \left(p_t - b_{t-1}^j x_{j,t-1}\right), \ j = 1, 2.$$

where

$$R_{j,t} = R_{j,t-1} + \lambda_t \left(x_{j,t-1}^2 - R_{j,t-1}\right), \ j = 1, 2,$$

is the estimated state covariance matrix, and $x_{jt} \in \{y_t, z_{st}\}$.

The term λ_t is typically referred to as the gain sequence. Two cases are assumed in the literature: a decreasing gain, $\lambda_t = t^{-1}$ so that $\lambda_t \to 0$; and a constant gain, $\lambda_t = \lambda \in (0, 1)$. With a decreasing gain, convergence to the restricted perceptions values of b^1, b^2 is possible. Our interest, though, is in demonstrating the model's implications for its asymptotic dynamics, which will be the central interest in the calibrated version of the model. Thus, we focus on the constant gain case where agents respond to past forecast errors with a time-invariant weight λ.

In order to choose their predictors, agents also estimate in real-time the (risk adjusted) expected profits:

$$\hat{E}U_t^j = \hat{E}\pi_t^j - \frac{a}{2}\sigma^2 \hat{E}z_{jt}^2$$

where

$$\begin{aligned}
\hat{E}\pi_t^j &= \hat{E}\pi_{t-1}^j + \kappa((1/a\sigma)\left(p_t + y_t - (1/\beta)p_{t-1}\right)\left((E_{t-1}^j p_t + \rho y_{t-1} - (1/\beta)p_{t-1}) - \hat{E}\pi_{t-1}^j\right) \\
\hat{E}z_{jt}^2 &= \hat{E}z_{jt-1}^2 + \kappa\left(\left((1/a\sigma)^2(E_{t-1}^j p_t + \rho y_{t-1} - (1/\beta)p_{t-1}\right)^2 - \hat{E}z_{jt-1}^2\right)
\end{aligned}$$

We will also assume a constant value for κ. Using this recursive estimate of expected trading profits, the law of motion for predictor proportions now follows,

$$n_t = \frac{1}{2}\left(\tanh\left[\frac{\alpha}{2}\left(\hat{E}U_t^1 - \hat{E}U_t^2\right)\right] + 1\right)$$

We turn to simulations of the real-time version of the model to illustrate the sense in which the model generates persistence and volatility clustering in excess returns. We assume the same parameter values as used to generate Figure 1. We choose $\lambda = .05, \kappa = .12$. A larger value for κ than λ implies that agents are more concerned with the possibility of regime change in equilibrium trading strategies than belief parameters. We draw initial values for $n, b^j, R^j, j = 1, 2$ randomly and then simulate the model for a transient period of length 10,000 assuming a decreasing gain for λ, κ. The assumption of a decreasing gain during the transient period ensures that at the

[13] See Evans and Honkapohja (2001) and Brock and Hommes (1997) for further discussion of these issues.

beginning of the simulation period the model will be near their equilibrium values. We then simulate the model for 1000 periods. Figure 5 plots the results from a typical simulation.

In Figure 5 the solid heavy line represents the simulated values for excess returns. The figure also superimposes the value for n_t, which alternates between the $n = 0$ and $n = 1$ equilibrium values. Two interesting features arise. First, the $n = 1$ equilibrium has a higher mean excess return than the $n = 0$ equilibrium. Second, the dividend only model has higher volatility than the share supply model. This leads to, in real-time, the economy switching between high return/high volatility periods and low return/low volatility periods. The switches between these two regimes occur frequently and persistently. The intuition for why the economy may switch from one equilibrium to another revolves around the interaction between the exogenous shocks and the gain parameters λ, κ. A particularly large shock, mediated through beliefs via λ, κ, may induce agents to switch forecasting models – thus, jumping the economy from one basin of attraction to another. Because λ, κ are at constant values there are repeated realizations of shocks sufficiently large to switch the economy between equilibria. The persistence in a particular shock, and the frequency with which these regime switches occur, are governed by a complicated interaction between the gain parameters λ, κ, the intensity of choice parameter α, and the stochastic shocks y_t, z_{st}.

Notice also that the nature of the over/underreaction will vary over time as well. For example, if one were only looking at underreaction to dividend news, then in the $n = 1$ periods you would not identify underreaction effects. This insight of real-time learning and dynamic predictor selection leading to interesting and complicated long-run dynamics in excess returns is the motivation for the next Section which turns to a calibrated version of the real-time model.

INSERT FIGURE 5 HERE

One might wonder whether the switching between equilibria evident in Figure 5 might present an exploitable trading opportunity for agents who incorporate into their forecasting model both dividends and share supply. In Section 3.2.2 we showed that a hyper-rational agent would receive (somewhat) higher (risk-adjusted) trading profits. But that result only applies in a misspecification equilibrium. It is no longer obvious how an agent with a bivariate model will fare in the real-time learning and dynamic predictor selection framework. Indeed, an agent with a bivariate forecasting model, and constant parameters, would not earn higher (risk-adjusted) profits than an agent who switched between univariate forecasting models. To demonstrate this point we conducted the following experiment: after a 10,000 period simulation, along the lines of Figure 5, we regressed price on dividends and share supply; using these constant parameter values, we calculated the demand function for a zero-mass agent who generates forecasts from this bivariate model; we then simulated the model for

10,000 periods and computed the realized excess returns for the bivariate model and the univariate model with switching. We found that the univariate model with switching outperformed the bivariate model. This result was also found for a real-time version in which the parameters of the bivariate model were updated using a constant gain.

The reason for this last finding is clear: with a constant parameter bivariate model and a true process generated by learning and predictor selection (e.g. Figure 5), the agent would not take advantage of those periods with high excess returns. Whether, and under what conditions, a bivariate model will fare better than the univariate models is a delicate point: for an agent to generate higher profits than the univariate models in an environment subject to switching between multiple equilibria would require a complicated econometric model able to capture the drift and volatility induced by the switching. If there is a cost to complexity, it would be reasonable for agents to stick with the univariate models, remaining alert to potential regime change.

5 Asset Return Dynamics in a Calibrated Model

Long-run Excess Return properties hold a special interest for financial economists. It has long been known that long-run excess returns exhibit higher volatility than implied by standard asset pricing models with rational expectations. Moreover, recent research into the dot-com burst, for instance Ofek and Richardson (2003) and Cochrane (2005), study the role of short-sale constraints, sudden supply bursts, liquidity and volume effects, for excessive long-run returns and then a collapse of the same average returns for internet stocks. What motivates these papers is that there was a large increase in the supply of shares at the beginning of 2000 (as IPO lock-ups expired) that were not previously priced into the market via short-sales. This increase in supply then looked like an unexpected increase in supply that led to a precipitous market crash as the market became aware of this supply effect. The story that Ofek and Richardson tell is that the market is populated by optimistic and pessimistic traders: optimistic agents trade on the spot market and pessimistic traders sell short. Short sale constraints (these are not legal constraints but economic ones, as claimed by the authors) then limit the proportion of pessimistic traders. Hence, the dynamics.

The dot-com burst was an example of how not accounting for the way in which share supply variation impacts prices can lead to a collapse in average stock returns. This example may open up the door for further modeling of supply share dynamics. The examples presented in Section 4 demonstrate that modeling supply shares and bounded rationality may have implications that address important empirical issues. The regime switching in Figure 5 raises the possibility that, in simulations, we could

see under/overreaction to both dividend and supply innovations. In particular, the interaction of real-time learning and dynamic predictor selection, underparameterization, and dividend-share dynamics can lead to over and underreaction, persistence, and volatility clustering in excess returns. This section seeks to demonstrate that a calibrated version of our model can reproduce many of the empirical findings.

5.1 Calibration

In this Section we assess to what degree the simple model presented here can account for some empirical regularities in excess return dynamics. In order to make a meaningful comparison we need to choose parameter values for the model. This subsection discusses our choice of parameter values.

For the purposes of the model at hand, the most crucial parameters for calibration are the autoregressive parameters and covariances for the dividend and share supply processes. Data on U.S. dividends are widely available. Data on share supply are more limited. For share supply we adopt the series constructed by Baker and Wurgler (2000).[14] Baker and Wurgler (2000) calculate total new annual (nominal) equity issues in the U.S.. Ideally, one would have a data series on all new issues, repurchases, bankruptcies, etc. Such data is not readily available and so the Baker-Wurgler data is the most comprehensive accounting of the U.S. time series of share supply. We calibrate the dividend process from data on corporate profits (after tax) and net dividends from the Economic Report of the President.[15] The data are reported in nominal terms and we adjust them to 1995 dollars using the consumer price index obtained from the Economic Report of the President.

Both dividends (or corporate profits) and share supply exhibit a trend. We detrend the data and estimate an AR(1) for the resulting series. We then calculate the associated AR(1) parameter and standard deviation implied by this regression's residuals. These are then used as the calibrated values for $\rho, \phi, \sigma_\varepsilon, \sigma_\nu, \sigma_{\nu\varepsilon}$. Table 1 reports the results.

INSERT TABLE 1 HERE

The remaining parameters are $\beta, \alpha, \lambda, \kappa, a\sigma^2$. Following a large literature, we set $\beta = .9957$, based on the one-month risk free rate. λ is calibrated at .01, the value reported in a VAR forecasting exercise in Branch and Evans (2006c). We choose a value of $\kappa = .5$, in accordance with Branch (2004), in estimating a discrete

[14]Data obtained from Wurgler's website: http://pages.stern.nyu.edu/~ jwurgler.

[15]Specifically, we look at corporate profits with inventory evaluation and capital adjustments. The data are obtained from Table B90.

choice forecasting model of the Michigan survey of consumers. We fixed $\sigma^2 = 15$ in accordance with the variance of excess returns observed in monthly data (Campbell and Ammer (1993)). Picking the value for a is difficult. We choose a value that is empirically not implausible and that leads to the kind of dynamics described above. The parameter a can be thought of as the coefficient of absolute risk aversion. Most experimental studies tend to favor CRRA over CARA, though Holt and Laury (2002) report values in the range of $0.1-0.2$.[16] In our model, given all of the other parameters, a controls the basin of attraction between equilibria and so will have implications for the frequency of switching and the size of shocks that will induce switching. We set $a = .05$, which implies that $a\sigma^2 = 0.75$. Smaller values of $a\sigma^2$ tend to increase the proportion of time spent near the $n = 0$ equilibrium and larger values of $a\sigma^2$ to increase the proportion of time spent near the $n = 0$ equilibrium. Finally, we fix $\alpha = 2$, in line with the value considered large but finite in Brock and Hommes (1997). Figure 6 plots the T-map for the calibrated values, and it is clear that $\alpha = 2$ mimics the large α case considered in the theoretical results above. In fact, for the calibrated parameter values all values of α produce a similar T-map and so our numerical results are robust to the calibrated 'intensity of choice'. Notice that there exist multiple implying dynamics similar to Figure 5.

INSERT FIGURE 6 HERE

5.2 Results

There is a large empirical literature on under/overreaction. Many of the earliest papers framed the issue according to its time series properties: a positive autocorrelation over short horizons implies a slow diffusion of information and a negative autocorrelation over longer horizons suggests mean reversion. Thus, underreaction would arise as a positive autocorrelation and overreaction as negative autocorrelation. Cutler, Poterba, and Summers (1991) show under and overreaction in time-series data for various countries. DeBondt and Thaler (1985) provide evidence of overreaction in the sense that stocks that receive good news for many periods will have lower average excess returns than stocks that receive successive rounds of bad news. Barberis, Shleifer, and Vishny (1998) and Hong and Stein (1999) summarize many other studies that find under/overreaction. Barberis, Shleifer, and Vishny argue that the most persuasive evidence is cross-sectional. Hong and Stein (1999) in their simulations construct broad cross-sections of excess returns data.

In this Section we generate time-series data on excess returns in the calibrated

[16]Holt and Laury (2002) emphasize that results are not in favor of CARA. We have employed a CARA specification anyway, because this leads to a convenient and illuminatiung closed form solution.

version of the model in order to illustrate the time-series properties for the U.S. economy documented by Cutler, Poterba, and Summers (1991). Our methodology is to take the real-time learning and dynamic predictor selection version of the model, as developed in Section 4, parameterize the model according to Table 1, and generate estimates of the autocorrelation function. We then compare our simulated autocorrelation function with the estimated autocorrelation function reported in Cutler, Poterba, and Summers (1991). To generate these estimates we simulate the model for a transient period of length 5000, we then store as data the next 5000 periods. For this model run, we then estimate the autocorrelation function. We repeat this 5000 times and report the mean estimates of the autocorrelation function.[17]

Table 2 presents the results for three different treatments: the real-time learning and dynamic predictor selection model (RTL), the $n = 0$ Misspecification Equilibrium model, the $n = 1$ Misspecification Equilibrium model.[18] The table reports the first-order autocorrelation and then average autocorrelations across periods. Two results standout. First, the RTL version produces positive autocorrelations at short horizons and negative autocorrelations at long horizons. The $n = 1$ version produces negative autocorrelations at all time-horizons. The $n = 0$ formulation leads to negative autocorrelations at short and long horizons and positive autocorrelations in between. Second, for the RTL model, the short-horizon autocorrelations are closer in magnitude to Cutler, Poterba, and Summers (1991). Cutler, et al. find in the U.S. data a first order autocorrelation of .106 and .021 in subsequent periods. At longer horizons they report negative autocorrelations ranging from -.017 to -.006. The RTL version captures these broad qualitative properties. Quantitatively, the RTL model underestimates the degree of over-reaction. There is one caveat to the interpretation of these findings. We have calibrated our model based on annual data and the Cutler, et al. data is based on monthly data. Thus, we would not expect our model to exactly conform to the data unless the calibrated values are identical for monthly rates and for the monthly rates converted from annual frequencies. Thus, this exercise is limited by the availability of only-annual data on share supply.

INSERT TABLE 2 HERE

Nonetheless it is striking how well the RTL model fits the data. The intuition for these findings is clear. As Figure 4 demonstrates, the underparameterization of a particular shock induces an underreaction at first and then an overreaction as the information about the shock diffuses through the economy. Fixing all agents at a particular trading strategy will miss some of this underreaction as there are two

[17]Numerical explorations suggest that 5000 simulations of 10000 periods each produced stable results, suggesting that the model has converged to a unique invariant distribution.

[18]We found that allowing for parameter learning with fixed predictor proportions did not change the qualitative findings.

kinds of shocks in this economy. The RTL model fits the autocorrelation pattern better because it includes underreaction to both supply and share shocks. Moreover, because switching between equilibria has a bearing on the size of the underreaction, the RTL model is better able to capture the magnitudes of underreaction in the data.

There is also a large literature on excess volatility and volatility clustering in returns. For example, Turner, Startz, and Nelson (1989) find evidence for regime switching conditional heteroskedasticity in stock market returns. Bollerslev, Chou, and Kroner (1992) find ARCH effects in stock returns. The discussion of under/overreaction above is a subset of a much broader literature on long-run predictability of stock returns. Recently, one way this predictability has arisen is through Markov-switching in mean returns for financial variables (Ang and Bekaert (2002)). There is also a literature that makes the link, at the individual stock level, between idiosyncratic volatility and average returns (Merton (1987)). Guidolin and Timmermann (2005a) provide evidence for aggregate U.K. stock data that suggests that average returns and volatility follow a three-state Markov switching process.

Section 4 showed that our model implies that excess returns may follow a regime switching ARCH model, thereby exhibiting both persistence and volatility clustering. We now study this issue with the calibrated model. As above, we repeatedly run simulations consisting of transient periods and periods for which we record the time-series. We then identify the data in each simulation according to its "regime," i.e. whether $n = 0$ or $n = 1$. Then within each regime we calculate the average excess return $\bar{R}_j, j = 0, 1$ and the variance $\sigma_j^2, j = 0, 1$. To get a sense of the relative magnitudes we then calculate $\frac{\bar{R}_1 - \bar{R}_0}{\bar{R}_1}$ and $\frac{\sigma_1^2 - \sigma_0^2}{\sigma_1^2}$. Table 3 reports the mean value of these calculations across all simulations, and compares them to Guidolin and Timmermann.

Table 3 shows that the model yields volatility clustering and persistence in returns. The $n = 1$ state has higher average excess returns than the $n = 0$ state and higher volatility. Because the model switches between states these patterns are persistent across time. However, unlike in Guidolin and Timmermann, the switches are not governed by a Markov chain but occur as unanticipated shocks push the stock price from one basin of attraction to another.

INSERT TABLE 3 HERE

Table 3 also shows that the calibrated model delivers estimates that are similar to the data of Guidolin and Timmermann (2005a). The table reports the relative average returns and variances across the "bull" and "normal" states, calculated from the data in Guidolin and Timmermann (2005a). The bull state has a higher return and lower variance than in the simulated data, but the magnitudes are close. As with the under/overreaction results there are reasons to be cautious in interpreting these findings as evidence. First, Guidolin and Timmermann estimate their econometric

model for U.K. data, while we have calibrated our model to monthly (converted from annual) U.S. data. Second, Guidolin and Timmermann estimate a 3-state Markov chain. Our model predicts two states. In the U.K. data the third state ("bear") is associated with negative excess returns. Our model does not deliver negative average excess returns and so we compare our two-state model with their bull and normal states. We should, therefore, not expect exact accordance of results, but once again the match is striking.

6 Conclusion

This paper has developed a theory of underparameterization and learning in a simple asset pricing model. Asset price is driven by expectations of future price and exogenous processes for dividends and the supply of asset shares, where the latter is viewed as a proxy for asset float. Agents forecast price by projecting it onto either dividends or share supply. Although agents are forced to underparameterize, we assume that they attempt to do so in an optimal way, through our twin assumptions that the forecast models impose the relevant orthogonality conditions and that agents choose only models that maximize, or almost maximize, risk-adjusted expected trading profits. In our framework, model parameters and the distribution of agents across forecasting models are jointly determined in equilibrium. The approach advocated in this paper can be seen as a generalization of Hong, Stein, and Yu (2005) and Barberis, Shleifer, and Vishny (1998) to a framework in which parameters and models are determined endogenously in equilibrium.

We demonstrate that underparameterization and misspecification equilibria can arise in this simple asset pricing model. Depending on the complicated interaction between the exogenous processes and the degree of risk-aversion of agents, multiple Misspecification Equilibria can arise as an equilibrium outcome. Adding real-time learning and dynamic predictor selection generates regime-switching dynamics in excess returns.

When the model is calibrated to U.S. stock data we find that the model is capable of capturing many of the salient empirical features of excess returns dynamics such as under/overreaction, persistence, and volatility clustering. Because of the richness of the theoretical results, and the broad empirical implications for excess returns, the approach in this paper seems to provide a suitable balance between rational expectations and fully behavioral approaches.

A Appendix

Detailed Computations for Section 2: It is straightforward to compute that

$$\begin{aligned}
E\pi_t^1 &= \frac{1}{a\sigma^2}\left[\{(1+\xi_1)\rho-\beta^{-1}\xi_1\}\{(1+\xi_1+\xi_2 r)\rho-\beta^{-1}\xi_1\}Ey_t^2\right.\\
&+ \beta^{-2}\xi_2\{\xi_1(\phi\beta-2)(\rho\beta-1)+\rho\beta(-2+\phi\beta+r\xi_2(\phi\beta-1))\}Ey_t z_{st}\\
&- \left.\beta^{-1}\xi_2^2(\phi-\beta^{-1})Ez_{st}^2\right]\\
E\pi_t^2 &= \frac{1}{a\sigma^2}\left[\{(\rho-\beta^{-1}\xi_1)^2+\xi_1\rho(\rho-\beta^{-1}\xi_1)\}Ey_t^2\right.\\
&+ \beta^{-2}\left[2\rho\xi_2\beta(-1+\phi\beta)+\phi\tilde{r}\xi_1^2\beta(-1+\rho\beta)+\xi_1(\phi\rho\tilde{r}\beta^2+\xi_2(-1+\phi\beta)(-2+\rho\beta)\right]Ey_t z_{st}\\
&+ \left.\{\xi_2^2(\phi-\beta^{-1})^2+\xi_2(\phi-\beta^{-1})\xi_1\tilde{r}\phi\}Ez_{st}^2\right]\\
Ez_{1t}^2 &= \left(\frac{1}{a\sigma^2}\right)^2\left\{\left[(1+\xi_1+\xi_2 r)\rho-\beta^{-1}\xi_1\right]^2 Ey_t^2\right.\\
&- \left.2\beta^{-1}\xi_2\left[(1+\xi_1+\xi_2 r)\rho-\beta^{-1}\xi_1\right]Ey_t z_{st}+\beta^{-2}\xi_2^2 Ez_{st}^2\right\}\\
Ez_{2t}^2 &= \left(\frac{1}{a\sigma^2}\right)^2\left\{(\rho-\beta^{-1}\xi_1)^2 Ey_t^2+2(\rho-\beta^{-1}\xi_1)\left[(\xi_2+\xi_1\tilde{r})\phi-\beta^{-1}\xi_2\right]Ey_t z_{st}\right.\\
&+ \left.\left[(\xi_2+\xi_1\tilde{r})\phi-\beta^{-1}\xi_2\right]^2 Ez_{st}^2\right\}
\end{aligned}$$

Proof of Proposition 1. A unique RPE exists if and only if

$$\begin{bmatrix} 1-\beta\rho n & -\beta\rho n r \\ -\beta\phi(1-n)\tilde{r} & 1-\beta(1-n)\phi \end{bmatrix}^{-1}$$

exists. This will be true provided that the characteristic equation

$$(1-\beta\rho n)(1-\beta(1-n)\phi)-\beta^2\rho\phi r\tilde{r}n(1-n)\neq 0$$

This is equivalent to,

$$1-\beta(1-n)\phi-\beta\rho n+\beta^2\rho\phi n(1-n)(1-r\tilde{r})\neq 0$$

The second term on the l.h.s. is positive, so it suffices to show that $1-\beta(1-n)\phi-\beta\rho n\neq 0$. Because $\beta<1$ and it is assumed that $|\rho|,|\phi|<1$, the result follows. ■

Proof of Corollary 4. As $r,\tilde{r}\to 0$ we have

$$\begin{aligned}
F(0) &\to \beta^2\left((1/2)\rho^4-(1/2)\frac{(a\sigma^2)^2}{(1-\beta\phi)^2}\phi^2 Q\right)\\
F(1) &\to \frac{\beta^2}{(1-\beta\rho)^2}\left((1/2)\rho^4-(1/2)\phi^2 Q(a\sigma^2)^2(1-\beta\rho)^2\right)
\end{aligned}$$

Then straightforward algebra leads to the conditions in Corollary 4. ■

References

[1] Ang, A., and G. Bekaert, 2002, "International Asset Allocation with Regime Switches," *Review of Financial Studies*, 15, 1137-1187.

[2] Anderson, Evan, Eric Ghysels, and Jennifer L. Juergens, 2005, "Do Heterogeneous Beliefs Matter for Asset Pricing?" *Review of Financial Studies*, 18, 3, 875-924.

[3] Baker, Malcolm, and Jeffrey Wurgler, 2000, "The Equity Share in New Issues and Aggregate Stock Returns," *Journal of Finance*, October.

[4] Barberis, Nicholas, Andrei Shleifer, and Robert Vishny, 1998, "A Model of Investor Sentiment," *Journal of Financial Economics*.

[5] Bekaert, Geert, and Robert J. Hodrick, 1992, "Characterizing Predictable Components in Excess Returns on Equity and Foreign Exchange Markets," *Journal of Finance*, 47, 2, 467-509.

[6] Bernard, Victor L., 1992, "Stock price reactions to earnings announcements," in R. Thaler, ed.: *Advances in Behavioral Finance*, Russell Sage Foundation, New York.

[7] Bollerslev, T., Chou, R., and K. Kroner, 1992, "ARCH Modeling in Finance: a Review of the Theory and Empirical Evidence," *Journal of Econometrics*, 52, 5-59.

[8] Branch, William A., 2004, "The Theory of Rationally Heterogeneous Expectations: Evidence from Survey Data on Inflation Expectations," *Economic Journal*, 114, 592-621.

[9] Branch, William A., 2006, "Restricted Perceptions Equilibria and Learning in Macroeconomics," in D. Colander, ed.: *Post Walrasian Macroeconomics: Beyond the Dynamic Stochastic General Equilibrium Model*, Cambridge University Press, New York.

[10] Branch, William A., and George W. Evans, 2006a, "Intrinsic Heterogeneity in Expectation Formation," *Journal of Economic Theory*, 127, 264-295..

[11] Branch, William A., and George W. Evans, 2006b, "Model Uncertainty and Endogenous Volatility," mimeo.

[12] Branch, William A., and George W. Evans, 2006c, "A Simple Recursive Forecasting Model," *Economics Letters*, 91, 158-166.

[13] Brock, W.A., and Cars H. Hommes, 1997, "A Rational Route to Randomness," *Econometrica*, 65, 1059-1160.

[14] Brock, W.A., and Cars H. Hommes, 1998, "Heterogeneous Beliefs and Routes to Chaos in a Simple Asset Pricing Model", *Journal of Economic Dynamics and Control*, 22, 1235-74.

[15] Brock, W.A. and Cars H. Hommes, "Heterogeneous Beliefs and Routes to Complex Dynamics in Asset Pricing Models with Price Contingent Contracts," mimeo.

[16] Cagetti, Marco, Lars Peter Hansen, Thomas J. Sargent, and Noah Williams, 2002, "Robustness and Pricing with Uncertain Growth," *Review of Financial Studies*, 15, 363-404.

[17] Campbell, John Y., and John Ammer, 1993, "What Moves the Stock and Bond Markets? A Variance Decomposition for Long-Term Asset Returns," *Journal of Finance*, 48, 1, 3-37.

[18] Cho, In Koo, Noah Williams, and Thomas J. Sargent, 2003, "Escaping Nash Inflation," *Review of Economic Studies*, 69, 1, 1-40.

[19] Cochrane, John H., 2001, *Asset Pricing*, Princeton University Press, Princeton, NJ.

[20] Cochrane, John H., 2005, "Asset Pricing Program Review: Liquidity, Trading and Asset Prices," mimeo.

[21] Cutler, David M., James M. Poterba, and Lawrence H. Summers, 1991, "Speculative Dynamics," *Review of Economic Studies,* 58, 529-546.

[22] Debondt, W., and R. Thaler, 1985, "Does the Stock Market Overreact?" *Journal of Finance*, 40, 793-808.

[23] Evans, George W., and Seppo Honkapohja, 2001, *Learning and Expectations in Macroeconomics*, Princeton University Press, Princeton, NJ.

[24] Fama, Eugene F., and Kenneth R. French, 1996, "Multifactor Explanations of Asset-Pricing Anomalies," *Journal of Finance*, 47, 426-465.

[25] Gaunersdorfer, Andrea, 2001, "Endogenous Fluctuations in a Simple Asset Pricing Model with Heterogeneous Agents," *Journal of Economic Dynamics and Control*, 24, 799-831.

[26] Grossman, Sanford J., and Joseph E. Stiglitz, 1980, "On the Impossibility of Informationally Efficient Markets," *American Economic Review*, 70, 383-408.

[27] Guidolin, Massimo, and Allan Timmermann, 2005a, "Economic Implications of Bull and Bear Regimes in UK Stock and Bond Returns," *Economic Journal*, 115, 111-143.

[28] Guidolin, Massimo, and Allan Timmermann, 2005b, "Properties of Equilibrium Asset Prices under Alternative Learning Schemes," forthcoming *Journal of Economic Dynamics and Control*.

[29] Holt, Charles, and Susan Laury, 2002, "Risk Aversion and Incentive Effects," *American Economic Review*, 92, 5, 1644-1655.

[30] Hong, Harrison, and Jeremy C. Stein, 1999, "A Unified Theory of Underreaction, Momentum Trading, and Overreaction in Asset Markets," *Journal of Finance*, 54,6, 2143-2184.

[31] Hong, Harrison, Jeremy C. Stein. and Jialin Yu, 2005, "Simple Forecasts and Paradigm Shifts," forthcoming *Journal of Finance*.

[32] Hong, Harrison, Jose Scheinkman, and Wei Xiong, 2005, "Asset Float and Speculative Bubbles," forthcoming *Journal of Finance*.

[33] Lamont, Owen, and Richard Thaler, 2003, "Can the Market Add and Subtract? Mispricing in Tech Stock Carve-outs," *Journal of Political Economy*, 111, 227-268.

[34] Lansing, Kevin J., 2006, "Lock-in of Extrapolative Expectations in an Asset Pricing Model," *Macroeconomic Dynamics*, 10, 317-348.

[35] Lettau, Martin, and Sydney C. Ludvigson, 2005, "Expected Returns and Expected Dividend Growth," *Journal of Financial Economics*, 76, 583-626.

[36] Manski, C.F., and D. McFadden, 1981, *Structural Analysis of Discrete Data with Econometric Applications*, MIT Press, Cambridge, MA.

[37] Marcet, Albert, and Thomas J. Sargent, 1989, "Convergence of Least-Squares Learning Mechanisms in Self-Referential Linear Stochastic Models," *Journal of Economic Theory*, 48, 337-368.

[38] Merton, Robert C., 1987, "A Simple Model of Capital Market Equilibrium with Incomplete Information," *Journal of Finance*, 42.

[39] Ofek, Eli, and Matthew Richardson, 2003, "Dotcom Mania: The Rise and Fall of Internet Stock Prices," *Journal of Finance*, 58, 1113-1137.

[40] Sargent, Thomas J., 1999, *The Conquest of American Inflation*, Princeton University Press.

[41] Timmermann, Allan, 1994, "Can Agents Learn to Form Rational Expectations? Some Results on Convergence and Stability of Learning in the UK Stock Market," *Economic Journal*, 104, 777-798.

[42] Timmermann, Allan, 1996, "Excess Volatility and Predictability of Stock Returns in Autoregressive Dividend Models with Learning," *Review of Economic Studies,* 523-557.

[43] Turner, C., Startz, C., and C. Nelson, 1989, "A Markov Model of Heteroskedasticity, Risk, and Learning in the Stock Market," *Journal of Financial Economics*, 25, 3-22.

[44] Williams, Noah, 2004, "Escape Dynamics in Learning Models," mimeo., Princeton University.

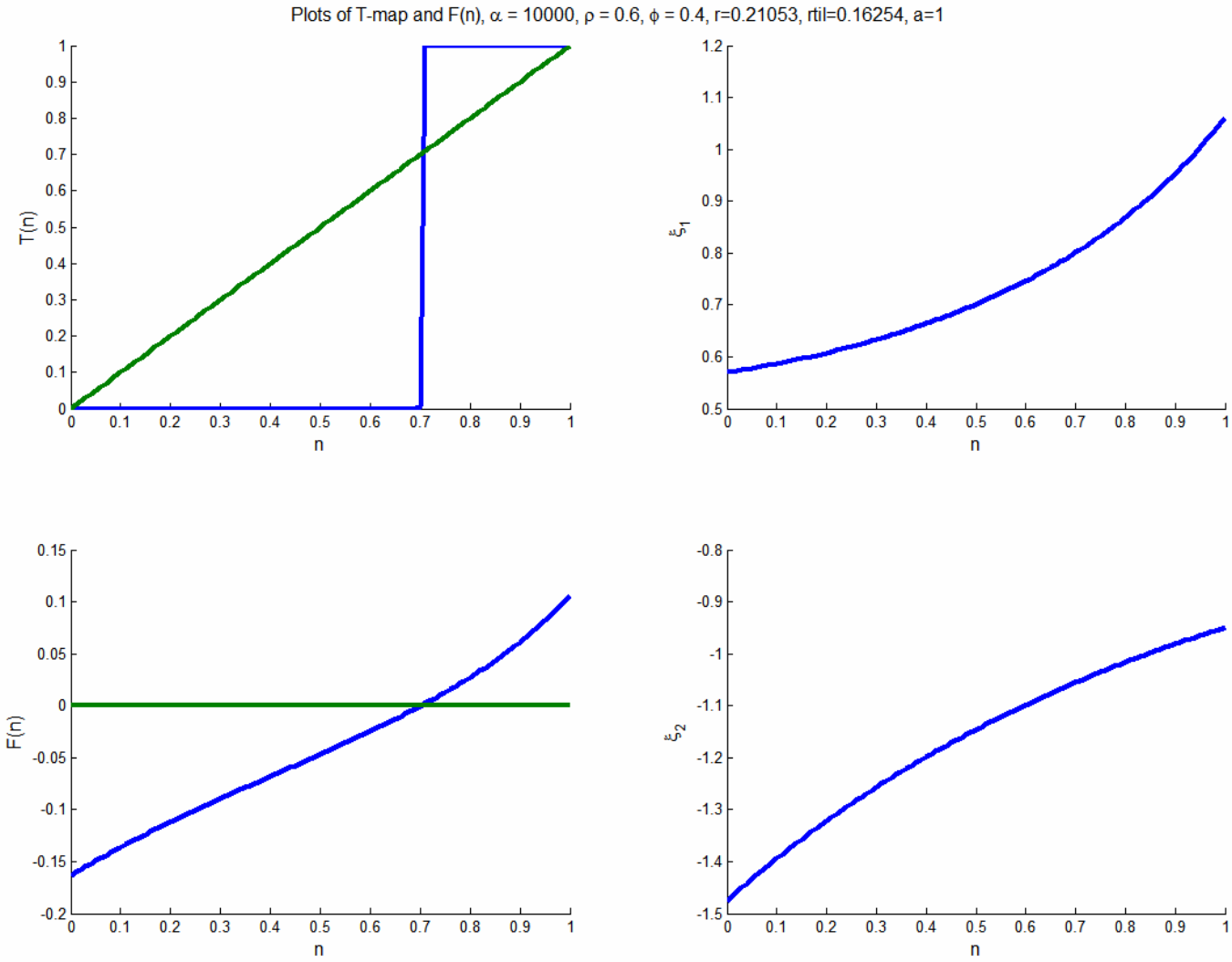

Figure 1. Multiple equilibria.

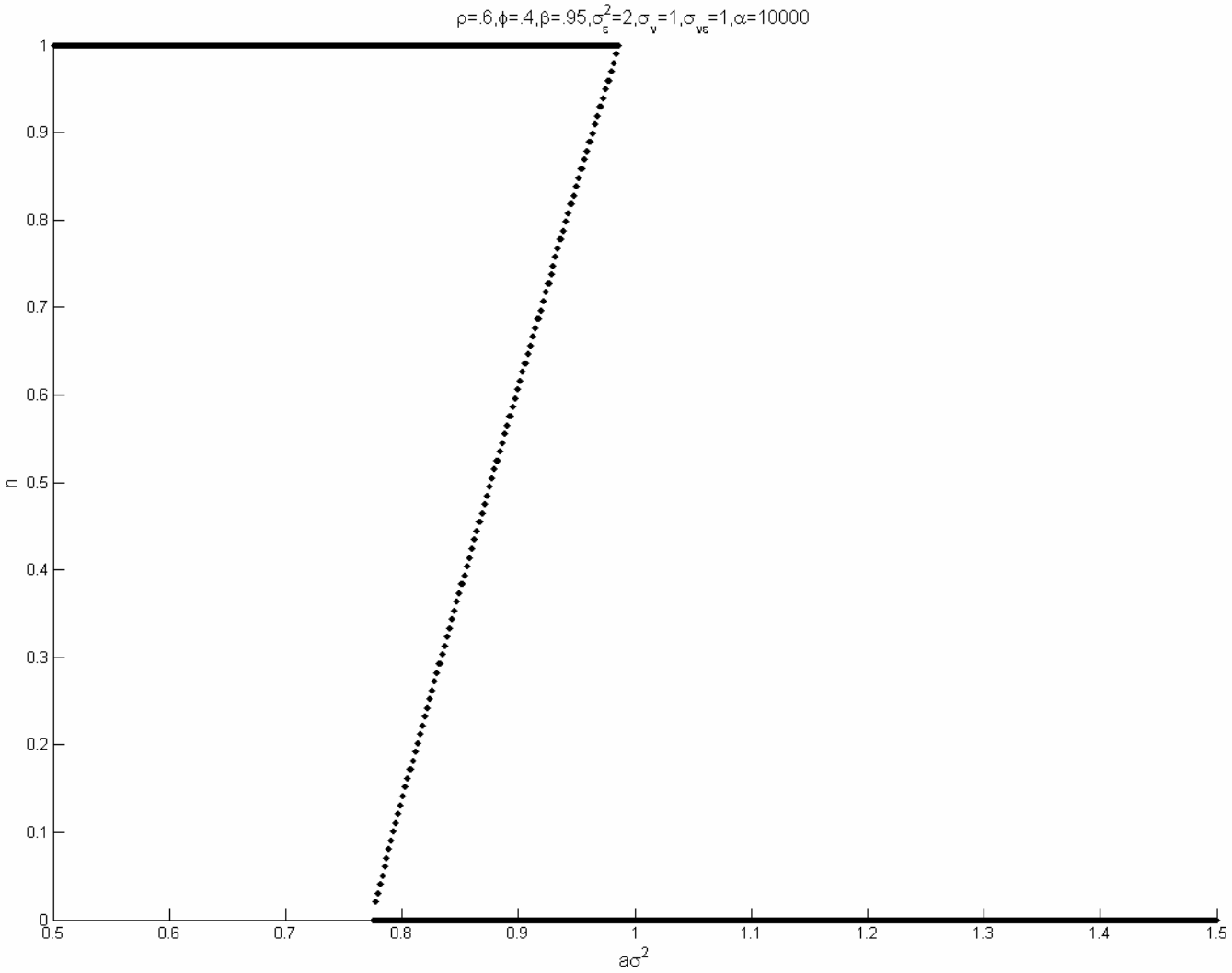

Figure 2. Bifurcation diagram.

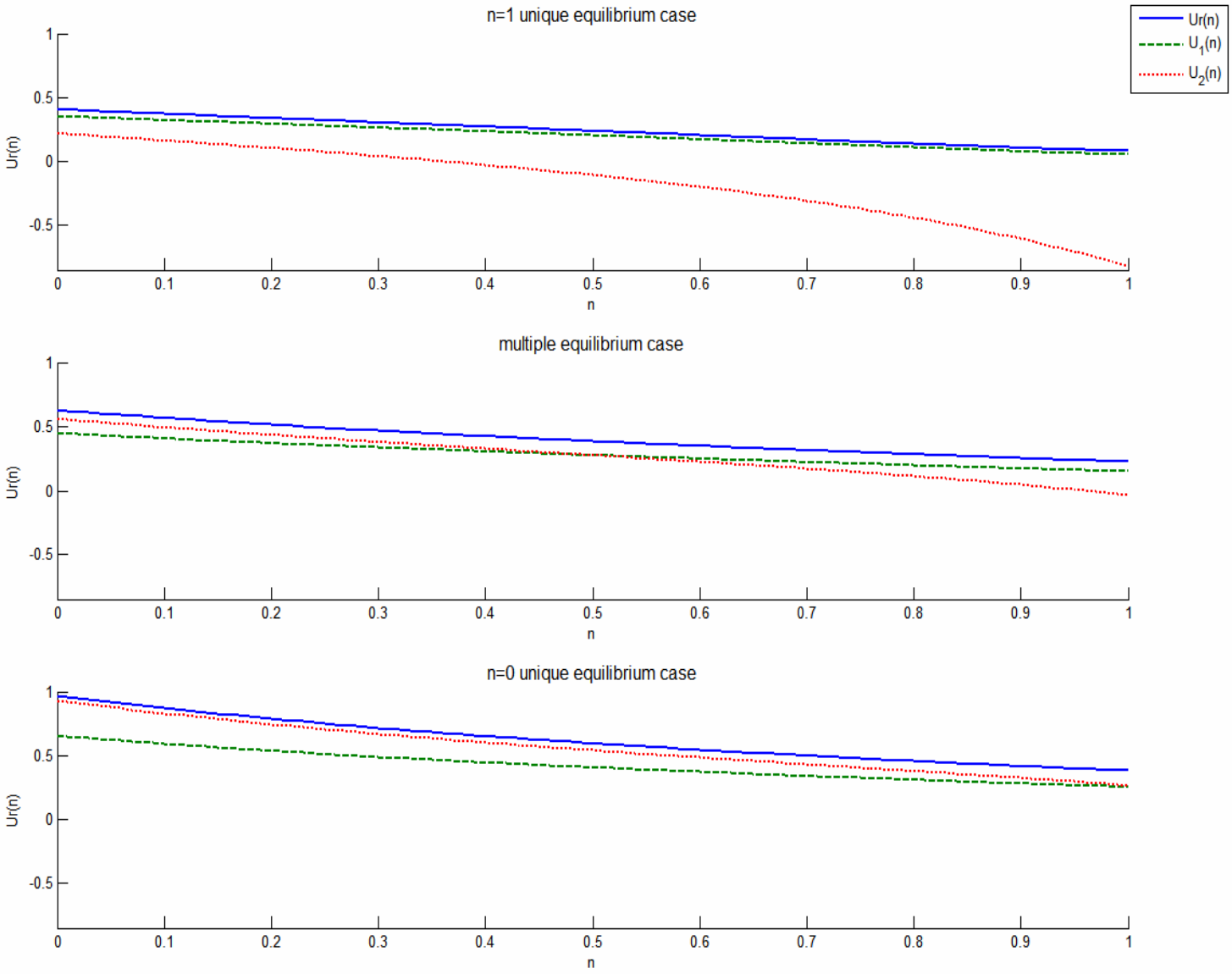

Figure 3. Comparison of (risk-adjusted) trading profits between underparameterized models and hyper-rational model for various degrees of risk-aversion. The top panel is for the case *a=0.3*, corresponding to *n=1* as the unique equilibrium. The middle panel sets *a=0.83* which leads to multiple equilibria. The bottom panel is the case of *a=1.4* and *n=0* as the unique equilibrium.

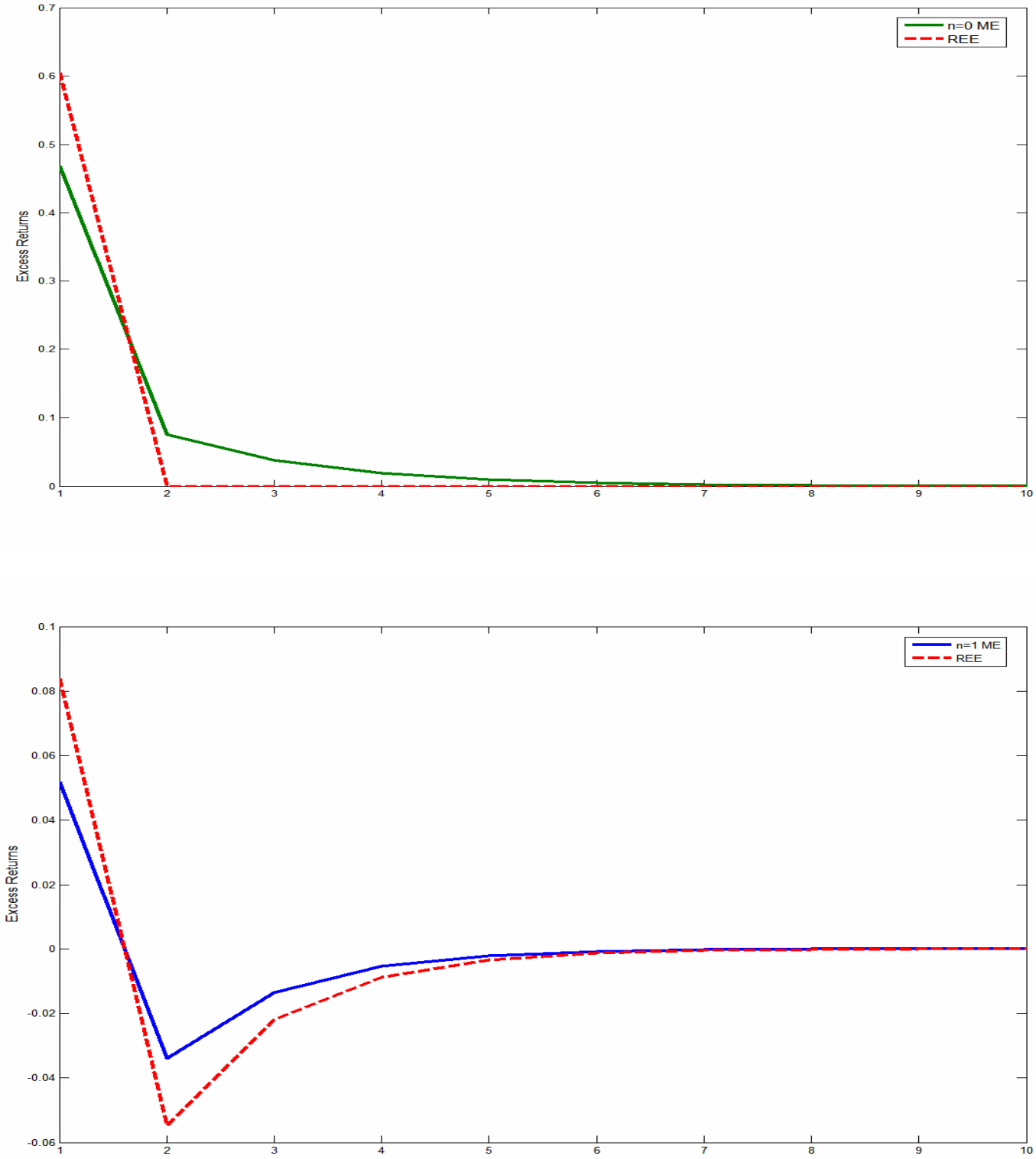

Figure 4. Impulse responses for excess returns. Top panel plots the impulse responses, for the underparameterized Misspecification Equilibrium and the REE, in the event of a positive dividend shock. The bottom panel plots the same in response to a negative share supply shock. The *n=0* model exhibits under reaction at short horizons and over reaction at longer horizons in response to a dividend shock. The *n=1* model exhibits under reaction and over reaction in response to a supply shock.

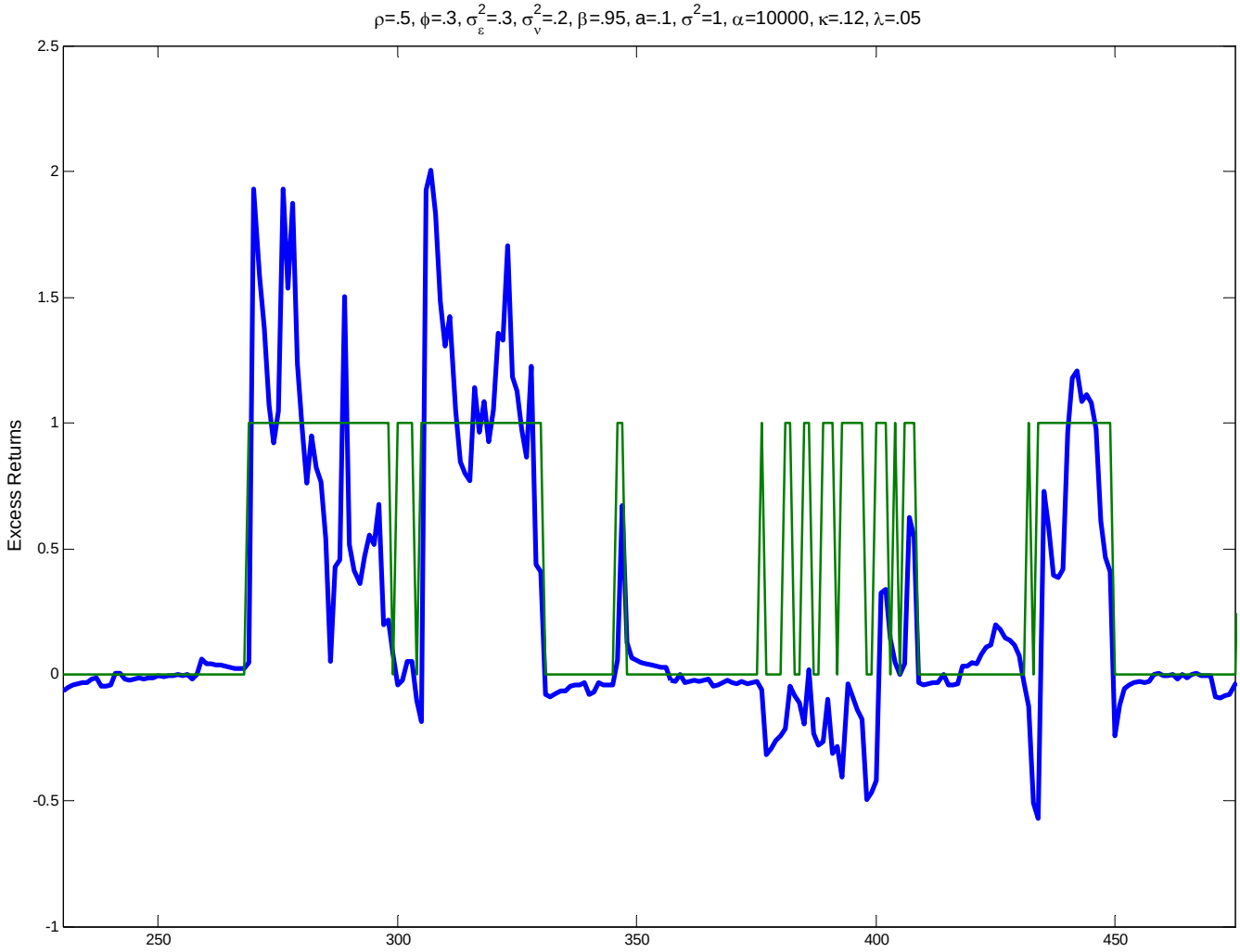

Figure 5. Simulated Excess returns with predictor proportions superimposed.

Parameter	Calibration
ϕ	0.8061
ρ	0.9766
σ_ε	0.2163
σ_ν	1.3178
$\sigma_{\nu\varepsilon}$	0
β	.9957
σ^2	15
a	.05
λ	.01
κ	.50
α	2

Table 1. Calibrated parameter values. Share data comes from Baker and Wurgler (2000). Dividend and C.P.I. data are from the 2005 Economic Report of the President, Table B90.

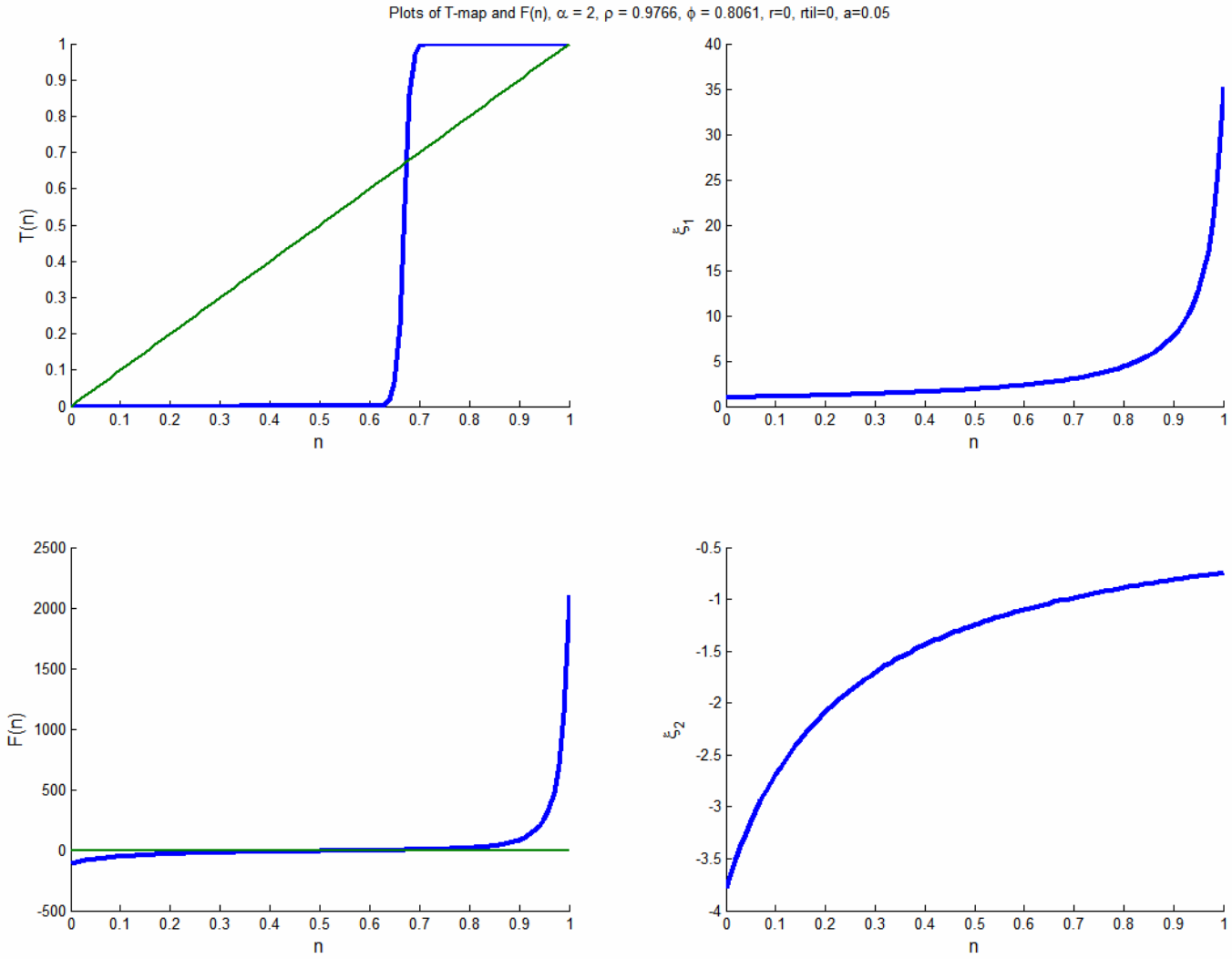

Figure 6. Multiple equilibria in the calibrated model

	Autocorrelations for excess returns									
	Autocorrelations – excess returns relative to risk-free rate – periods averaged in autocorrelations									
Model	1	1-12	13-24	25-36	37-48	48-60	61-72	73-84	85-96	97-5000
RTL	.0985	.0894	.0699	.0457	.0276	.0127	.0026	-.0042	-.0078	-.0083
$n = 1$ M.E.	-.0045	-.0026	-.0017	-.0017	-.00023	-.0019	-.0016	-.0019	-.0011	-.0008
$n = 0$ M.E.	-.0713	-.0168	.0114	.0090	.0052	.0033	.0004	-.0002	-.0016	-.0016

Table 2. Autocorrelations generated from simulated data. *RTL* refers to the model specification with real-time learning and dynamic predictor selection. $n = j, j = 0, 1$ refers to the Misspecification Equilibrium with n and the belief parameters held fixed at their equilibrium values. Autocorrelations were calculated by simulating the model for 5000 time periods (after a transient period of length 5000) and computing the sample autocovariance function. The reported values are the mean autocorrelations across 5000 simulations.

Persistence in Returns and Volatility Clustering		
	$\frac{\bar{R}_1-\bar{R}_0}{\bar{R}_1}$	$\frac{\sigma_1^2-\sigma_0^2}{\sigma_1^2}$
simulated data	.6043	.2762
bull-normal U.K. data	.4768	.4252

Table 3. Regime switching returns and volatility. The model is simulated and the mean and variance for each regime, i.e. $n = 0$ or $n = 1$, are calculated. The table reports the mean values across 1000 simulations. The bull-normal U.K. data are calculated from Table 3 of Guidolin and Timmermann (2005a), who estimate a 3-state model with ARCH effects. For comparison purposes, Table 2 only includes the relative means and variances for the "normal" and "bull' states.