Computational Thinking and Math Maturity: Improving Math Education in K-8 Schools

The title of this book has been changed. The original title was *Improving Math Education in Elementary Schools: A Short Book for Teachers.*

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This document can be accessed at: http://darkwing.uoregon.edu/~moursund/Books/ElMath/ElMath.html.

Date of this version: 6/3/06. It incorporates significant changes from the 1/1/05 version.

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Preface

> The saddest aspect of life right now is that science gathers knowledge faster than society gathers wisdom. (Isaac Asimov, 1988)

> An educated mind is, as it were, composed of all the minds of preceding ages. (Bernard Le Bovier Fontenelle, mathematical historian, 1657-1757)

This book is motivated by the problem that our K-8 school math education system is not as successful as many people would like it to be, and it is not as successful as it could be. It is designed as supplementary material for use in a Math Methods course for preservice K-8 teachers. However, it can also be used by inservice K-8 teachers and for students enrolled in Math for Elementary and Middle School teachers' courses.

Many people and organizations have put forth ideas on how to improve our math education system. However, in spite of decades of well-meaning reform effort, national assessments in mathematics at the precollege level in the United States do not indicate significant progress. Rather, scores on these national assessments have essentially flat lined during the past 40 years.

The results of the past 40 years of attempts to improve math education suggest that doing more of the same is not likely to improve the situation. We can continue to argue about whether back to basics or a stronger focus on new math is the better approach. From time to time, both such approaches have produced small pockets of excellence. In general, however, our overall math education system is struggling to achieve even modest gains.

This book draws upon and explores four Big Ideas that, taken together, have the potential to significantly improve out math education. The Big Ideas are:

1. Thinking of learning math as a process of both learning math content and a process of gaining in math maturity. Our current math education system is does a poor job of building math maturity.
2. Thinking of a student's math cognitive development in terms of the roles of both nature and nurture. Research in cognitive acceleration in mathematics and other disciplines indicates we can do much better in fostering math cognitive development.
3. Understanding the power of computer systems and computational thinking as an aid to representing and solving math problems and as an aid to effectively using math in all other disciplines.
4. Placing increased emphasis on learning to learn math, making effective use of use computer-based aids to learning, and information retrieval.

Math Maturity

Math maturity is a relatively commonly used term, especially in higher education. In higher education, the dominant components of math maturity are "proof" and the logical, critical, creative reasoning and thinking involved in understanding and doing proofs. The focus is on mathematical thinking, on being able to read and write math, and on being able to learn math using a wide range of resources such as print materials, courses, colloquium talks, and so on.

Many of the same ideas are applicable to defining math maturity at the precollege level. However, the cognitive development work of Piaget and others provides another quite useful approach. Piaget's four-stage cognitive development scale is useful in tracking and facilitating cognitive development through the levels: sensory motor, preoperational, concrete operations, and formal operations. Piaget and many more recent researcher have recognized that one can look at the formal operation end of this scale both in general, and also in specific disciplines. Thus, we can explore the math education curriculum in terms of how well it helps students gain in math cognitive development.

The past 20 years have brought quite rapid progress in cognitive neural science and other aspects of brain science. Researchers have gained considerable insight into how the brain functions in math learning and math problem solving. This research is beginning to contribute to the design of more effective aids to learning math and to increasing math maturity.

Nature and Nurture

People are born with a certain "amount" of innate mathematical ability. In dealing with quantity, for example, this innate ability roughly corresponds to dealing with 1, 2, 3, and many. Howard Gardner has identified logical/mathematical as one of the eight multiple intelligences that in his theory of intelligence.

However, most of what we call mathematics has been invented by people. It is part of the accumulated knowledge of the human race, and it is passed on from generation to generation by informal and formal education. Children who grow up in a hunter-gather society do not learn the types of math that we expect children to learn in our information age society.

In recent years, use of brain imaging equipment and brain modeling using computers have been added to earlier tools used to study cognitive development. Research in cognitive development and cognitive acceleration suggests that our informal and formal educational system could be doing much better.

Computational Thinking

Many people now divide the discipline of mathematics into three major sub disciplines: pure math, applied math, and computational math. The term *computational* denotes the study and use of computer modeling and simulation. The table in figure P.1 contains data from Google searches on the three sub disciplines.

Search Expression	**Google Hits 5/10/06**
"applied math" OR "applied mathematics"	50,200,000
"pure math" OR "pure mathematics"	5,450,000
"computational math" OR "computational mathematics"	3,090,000

Figure P.1 Google searches on some math sub disciplines

Of the three sub disciplines listed, the most recent to emerge is computational. The addition of computational as a subdivision of math and various other disciplines has occurred because of the steadily increasing role of computers as an integral component of the content of many different disciplines. For example, in 1998 one of the two winners of the Nobel Prize in Chemistry received the prize for his previous 15 years of work in computational chemistry. He had developed computer models of chemical processes that significantly advanced the discipline of chemistry.

Similarly, physics is now divided into the three components: theoretical, experimental, and computational. Here is a brief quote from page 2 of the April 22, 2006 issue of Science News:

> When black holes collide, they cause surrounding space-time to wiggle, generating a torrent of radiation known as gravitational waves. That's what Einstein's general theory of relativity predicts, but **computer models [modelers]** have struggled for more than 30 years to reproduce those waves. Because of the relativity theory's mathematical complexity and the extreme gravity of black holes, modelers haven't succeeded in getting black holes to crash.
>
> Now, two teams independently reported that they have successfully **simulated** the merger of two black holes and the event's production of gradational waves. [Bold added for emphasis.]

As you can see, computational means far more than just doing arithmetic calculations. Indeed, it has emerged as a way of thinking.

An excellent, brief introduction to *computational thinking* is provided in Jeannette Wing (2006). She is the Head of the Computer Science Department at Carnegie Mellon University. Quoting from her article:

> Computational thinking builds on the power and limits of computing processes, whether they are executed by a human or by a machine. Computational methods and models give us the courage to solve problems and design systems that no one of us would be capable of tackling alone. Computational thinking confronts the riddle of machine intelligence: What can humans do better than computers, and What can computers do better than humans? Most fundamentally it addresses the question: What is computable? Today, we know only parts of the answer to such questions.
>
> Computational thinking is a fundamental skill for everybody, not just for computer scientists. To reading, writing, and arithmetic, we should add computational thinking to every child's analytical ability.

Learning to Learn Math

All teachers recognize that to be effective, they need to know the content they are teaching and they need to know how to teach the content. Much teaching knowledge and skill cuts across the school disciplines. However, there is considerable discipline-specific pedagogical knowledge and skill for each discipline. To be an effective teacher of math, one needs to know math and one needs to have significant math pedagogical knowledge and skill.

A somewhat similar idea holds for learning math. The human brain is naturally curious and has an innate ability to learn. A child is born with a modest amount of math capability, such as being able to distinguish among quantities such as one, two, and three. As a person's brain grows and matures, one's innate mathematical ability grows.

However, mathematical development depends heavily upon the informal and formal math learning environments that are available to the learner. In addition, math development is highly dependent on learning to learn math—in making progress toward being a more effective and efficient learner of math.

A good example of what is entailed by this is inherent to the idea of reading across the curriculum. We know that there is a difference between general reading skills and discipline-specific reading skills. We also know that students first learn to read and eventually can read to learn. From a math education point of view, students need to learn to read math. Progress in this endeavor is important to learning math by reading. Our current math education system is weak in helping students learn to read math and they learn math through reading.

Nowadays, most students have relatively easy access to the world's largest library—the Web. Thus, as they learn to read math and to learn math by reading, they can take advantage of the math components of this huge and steadily growing library. Because math is an important component of many disciplines, learning to read math is an important part of learning to read across the curriculum

Computers have also brought us computer-assisted learning (CAL). In recent years, some of the best CAL falls into the category highly interactive intelligent computer-assisted learning (HIICAL). Such materials are a powerful aid to learning. Research on HIICAL in math suggest that some of the available materials are considerably more effective aids to student learning than are the traditional aids.

Contents of this Book

The 10 chapters of this book weave together various approaches to the four Big Ideas discussed above. Each chapter includes a set of activities for preservice and inservice teachers who are studying this book, and a set of activities useful in working with K-8 students. The latter activities are quite general, and they certainly do not constitute a curriculum that can be picked up and implemented at the K-8 level. Rather, they suggest some ideas to explore with young students and to try out in the K-8 curriculum.

Final Comment

As with most of my current writing efforts, this book is a "work in progress." It is regularly being added to and revised. Your input and suggestions are welcome.

The change in title from the previous edition represents my growing insights into the problems faced by our math education system.

Dave Moursund

June 2006

Chapter 1

Introduction

> ...we discovered that education is not something which the teacher does, but that it is a natural process which develops spontaneously in the human being. It is not acquired by listening to words, but in virtue of experiences in which the child acts on his environment. The teacher's task is not to talk, but to prepare and arrange a series of motives for cultural activity in a special environment made for the child. (Dr. Maria Montessori)

> Technology is a gift of God. After the gift of life it is perhaps the greatest of God's gifts. It is the mother of civilizations, of arts and of sciences. (Freeman Dyson)

This book is about the craft and science of teaching and learning math at the elementary and middle school levels. The goal of this book is to help improve math education. This chapter introduces the whole book and covers some needed background materials.

The Preface contains four unifying Big Ideas. If you skipped over the Preface, I recommend that you read it now. Think about your current level of expertise as a learner and as a teacher in each of the Big Idea areas. Do this again as you finish reading each chapter of this book.

Reading and Math

At the current time, reading and math are the two most emphasized components of the K-8 curriculum. Although reading and math are taught as two separate and distinct subjects, it is clear that they are related. Numbers and many "math" words and expressions are part of our everyday speaking and reading vocabulary. Indeed, many people think of math as a language and talk about gaining fluency in the use of this language, or gaining fluency in mathematics.

Robert Logan (2004) provides an excellent discussion of the development of six languages. Quoting from Chapter 1 of his book:

> In this book we will develop the hypothesis that speech, writing, mathematics, science, computing and the Internet form an evolutionary chain of languages. **Two of the languages mathematics and writing we shall see emerged at exactly the same point in time around 3100 BC** followed approximately 1000 years later by science. Within my life time two languages have appeared in rapid succession, computing and the Internet, the fifth and sixth language. I hope to demonstrate that computing and the Internet (which includes the World Wide Web) will play a role as important as that of any of the four languages that preceded them many years ago. [Bold added for emphasis.]

Throughout our country, there is a top down movement to establish high standards for student achievement in reading and math, and to improve our educational system so that these high standards are met. In recent years, education has become a political issue, and many politicians want to be considered as leaders of educational reform.

In educational circles, both reading instruction and math instruction tend to evoke considerable controversy. In essence, the issues are what the standards should be—what students should learn—along with how students should be taught, and how this learning should be assessed. In reading, there is considerable agreement about the goal of having students achieve an adequate level of reading fluency (speed, accuracy, comprehension) by the end of the third grade so that they can begin to make effective use of reading as an aid to learning throughout the curriculum. The controversy tends to lie in teaching methods, such as phonics versus whole language, and in the content of the materials that students read. Controversy also lies in who or what to blame because a large number of students do not achieve the reading level fluency goals.

In math, both the content and the pedagogy issues remain unresolved. However, there is considerable agreement that the results being produced by our current math education system, whether the approach is "back to basics" or "new-new math," are not nearly as successful as many people would like. Michael Battista provides an excellent summary of the situation in a 1999 article.

> For most students, school mathematics is an endless sequence of memorizing and forgetting facts and procedures that make little sense to them. Though the same topics are taught and retaught year after year, the students do not learn them. Numerous scientific studies have shown that traditional methods of teaching mathematics not only are ineffective but also seriously stunt the growth of students' mathematical reasoning and problem-solving skills. Traditional methods ignore recommendations by professional organizations in mathematics education, and they ignore modern scientific research on how children learn mathematics (Battista, 1999).

Think about the quote from Michael Battista. Is it a good description of your personal math learning experiences? Does the description fit some of the children and adults that you know? Many math education leaders agree that Battista is correct. There is much less agreement about how to make progress in solving this educational problem.

Computer Science and Mathematics

The disciplines of mathematics and computer science are closely related. Many of the current college and university Departments of Computer Science were formed by groups of faculty who split off from Mathematics Departments. Even now, it is not uncommon to find mathematics and computer science combined in a single department.

At the current time, research and instruction in mathematics can be divided into three main categories: Pure Mathematics, Applied Mathematics, and Computational Mathematics. The term *Computational* has come to be an important descriptor in many other disciplines. For example, it is now common for a person to be a Computational Biologist, Computational Chemists, or Computational Physicist. A excellent, brief introduction to computational thinking is provided in Jeannette Wing (2006), who is chair of the Computer Science Department at Carnegie Mellon University. Quoting from her article:

> Computational thinking builds on the power and limits of computing processes, whether they are executed by a human or by a machine. Computational methods and models give us the courage to solve problems and design systems that no one of us would be capable of tackling alone. Computational thinking confronts the riddle of machine intelligence: What can humans do better than computers, and What can computers do better than humans? Most fundamentally it addresses the question: What is computable? Today, we know only parts of the answer to such questions.

> Computational thinking is a fundamental skill for everybody, not just for computer scientists. To reading, writing, and arithmetic, we should add computational thinking to every child's analytical ability. Just as the printing press facilitated the spread of the three Rs, what is appropriately incestuous about the vision is that computing and computers facilitate the spread of computational thinking.

Computational thinking has always been a part of math and math education. However, computers add a new dimension to computational thinking. This broadened view of computational thinking adds a new challenge to math teachers at all grade levels.

From time to time throughout this book, there are brief comments about possible roles of computers in math education. Computers are both an aid to instruction and part of the content in mathematics. Chapter 9 takes a deeper look at roles of computers in K-8 math education.

Math Expertise: Content and Maturity

You have a level of math expertise that you have developed over years of informal and formal study and use of math. Likely, you know some people who have greater math expertise than you, and you know some people who have less math expertise than you. You may have an opinion about yourself, such as "I am good at math." or "I am not very good at math." As a preservice or inservice K-8 teacher, you need to be concerned about whether your level of math expertise is sufficient to help your future students make satisfactory progress in building their own math expertise.

Math expertise can be divided into two major components: math content and math maturity. Much of the math coursework you have taken focused on math content—for example, learning many different arithmetic, algebraic, and geometric procedures and how to use these procedures to solve a wide range of math problems.

Math maturity focuses on areas such as understanding, solving math problems you have not previously encountered, theorem proving, precise mathematical communication, mathematical logic and reasoning, knowing how to learn math, problem posing, transfer of learning (being able to use one's math knowledge and make math connections over a wide range of disciplines and in novel settings), and interest—including intrinsic motivation—in math.

The idea that a math problem may have no solutions, one solution, or more than one solution is part of math maturity. The idea that a solution or a solution process may be more or less clever, beautiful, or elegant is also part of math maturity. Math maturity is an idea that is not specific to any particular content area in math. To a large extent, math maturity does not depend on knowing some specific part of the content of math. A person may have a high level of math content knowledge and a low level of math maturity, or vice versa. Figure 1.1 provides an example of two hypothetical students: Student A (S-A) and Student B (S-B)

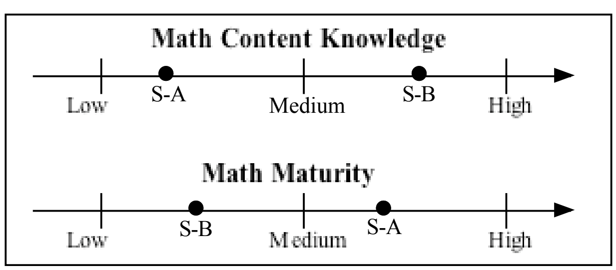

Figure 1.1 Separate expertise scales for math content and math maturity.

A Good Math Teacher

A good teacher of math has an appropriate level of expertise both in the discipline of math and in the discipline of teaching. Lee Shulman coined the phrase *content pedagogical knowledge* in order to emphasize the importance of a teachers having specific pedagogical knowledge and skills within the disciplines that they teach (Shulman, 1987). Lee Shulman is President of the Carnegie Foundation for the Improvement of Education (Carnegie, n.d.) Figure 1.2 expands on Shulman's work to emphasize *discipline pedagogical knowledge* as one of the keys to good teaching in any discipline.

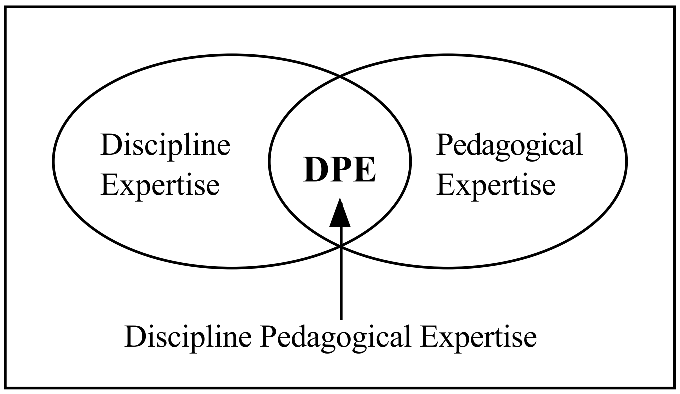

Figure 1.2 Discipline Pedagogical Expertise.

In summary, to be an effective teacher of math, you need both math content knowledge and math maturity. In addition, you need to know how to teach math—that is, you need math pedagogical knowledge designed to help your students learn math content and gain in their math maturity. Research by Liping Ma (1999) and others suggests that the majority of K-8 teachers in the United States are relatively weak in math pedagogical knowledge. My research into math maturity suggests that this is also an area of relative weakness for many K-8 math teachers.

It is evident that not all people agree with the statements in the previous paragraph. Many states have alternative routes to teacher certification that are based mainly on content knowledge. Many people seem to feel that content knowledge is the "be all, end all" to the qualifications needed to be a good teacher. A summary of current research on this issue is available in Emerick et al. (2004). In brief summary, this research summary argues that content knowledge does not suffice—that a focus just on content knowledge of teachers is doing students a great disservice.

K-8 teachers tend to teach math in the way that they were taught. That is, much of what you know about being a teacher of K-8 mathematics you learned while you were in K-8 school. This creates a cycle in which the next generation of students is taught in much the same manner as the previous generation. This cycle can and must be broken if the quality of math education that our students receive is to be significantly improved. You, personally, can make a significant difference for your students. The ideas presented in this book will help you.

Read & Write Across the Curriculum

The Sumerians—who lived in the area that is now Iraq—developed writing about 5,200 years ago (Acosta, n.d.). This soon led to the development of schools and formal schooling to teach reading, writing, and arithmetic. While schools have made considerable progress over the years, there is still a considerable similarity between schools 5,000 years ago and schools today.

You are undoubtedly familiar with the curriculum ideas of "reading across the curriculum" and "writing across the curriculum." Reading and writing are important components of each discipline, and we want students to learn to read and write within each discipline they study. Marilyn Burns is well known for her many math education books (Burns, 1995). The following quotation is from Burns (2004), an article that contains a number of examples of having children write during their K-8 math instruction.

> … and for my first 20 years as a middle school and elementary school teacher, writing played no role in my math teaching.
>
> Today, my view has changed completely. I can no longer imagine teaching math without making writing an integral aspect of student learning.

Later in the article, Marilyn Burns explains some of the roles of writing in math instruction:

> Writing in math class supports learning because it requires students to organize, clarify, and reflect on their ideas—all useful processes for making sense of mathematics. In addition, when students write, their papers provide a window into their understandings, their misconceptions, and their feelings about the content there're learning.

Marilyn Burns then goes on to describe general categories of writing in math, including keeping journals, solving math problems, explaining math ideas, and writing about learning processes. She argues that such writing is an important component of a modern math education. In essence, she is providing her insights into math maturity.

Learning to read and then reading to learn is a widely accepted idea in education. Some people argue that these two topics should be more thoroughly integrated, so that from the very beginnings of learning to read, there is a focus on using one's reading skills to learn. Others point to how great a challenge it to learn to read, and support the somewhat traditional two-phase process whereby students concentrate on learning to read during K-2 or K-3 education, and then begin making a transition into a reading to learn mode.

Students who are beginning to learn to read in kindergarten or the first grade already have a large speaking and listening vocabulary. To a large extent, learning to read is a process of learning to decode in a manner that ties in to one's oral knowledge and skills. It takes a substantial effort for most students to learn to decode and then attach meaning and understanding to what they are reading.

Learning to Read Math

Contrast this general effort of learning to read with learning to read in a specific discipline such as math. A typical kindergartner or first grader knows very little math. Suppose that as a child is learning to read, we also want the child to learn to read across the various disciplines emphasized in school. From the very beginning, we then run into the issues of a child's content knowledge within a discipline and the child's oral fluency in the discipline.

In essence, if we want students to be learning to read math while in the first grade, then they will need to be learning to read the math that they are learning in the first grade. This is sharply different than the child learning reading in general, with the reading mainly focusing on areas where the child already has oral fluency and knowledge.

Our math education system has adopted a compromise position. From kindergarten on, students are exposed to math symbols. As they learn the alphabet and punctuation marks, they learn the digits and some math symbols such as + and -. As they learn to recognize and spell certain words, they learn to recognize and write some two digit and three digit numbers.

Almost immediately, however, the parallel between learning to read in their oral language and learning to read math breaks down. In learning to read one's oral language, the meaning and understanding have already been learned. The learning to read is a process of learning to decode. In learning to read math, the meaning and understanding have not been already learned. Learning to decode math symbols, when one does not know the meaning of the math the symbols represent, is fraught with difficulty.

Perhaps for this reason, math in the early grades tends to be taught using oral methods. Students learn math symbols and some math vocabulary, but they do not learn to read math for understanding. This gap between a student's math reading skills and reading to learn math skills tends to persist throughout K-12 education and perhaps through a couple of years of college math.

The Blog discussion (Reading and Math, 2006) captures the essence of the situation. The discussion focuses on difficulties that students have in reading the math questions on state tests. Some states allow the teacher to read a question to the students, and others strictly forbid this. Many teachers view state and national math tests as mainly being reading tests, and thus not being a fair test of students who have reading difficulties and/or come from a different cultural and socioeconomic background than what is assumed in the test questions. Research indicates that students who are in the bottom third of readers tend to do poorly in math.

There is quite a bit of literature on reading across the curriculum. A 5/19/06 Google search of the quoted phrase *"reading across the curriculum"* produced 75,900 hits. Many of these documents mention a variety of subject areas that need special emphasis, such as reading in math and in science.

However, surprisingly few of these document move beyond the level of saying this should occur, to the level of providing appropriate reading material, talking about teacher education, and providing research evidence on the effectiveness of teaching students to read math and science.

Moreover, in math education, quite a bit of the focus on students learning to read math is on students learning to read "word problems." This focus differs considerably from a focus on learning to read math well enough so that one can read math texts and other math materials in order to learn math. For many students, learning to read math tends to mean learning to browse the part of a math book that comes just before a set of assigned exercises, in order to find examples that seem to be the same as the assigned exercises. From the student point of view, the goal is not reading math in order to learn math. Rather, the goal is to complete the required exercises.

Math, a Human Endeavor

From a historical point of view, writing has facilitated a steady accumulation of human knowledge, including math. Here is a statement that mathematicians often quote:

> God created the integers; all the rest is the work of man. (Leopold Kronecker, 1923-1891)

The quotation from Kronecker captures the idea that math is a steadily growing discipline. The invention of writing has made possible more than 5,000 years of the development and accumulation of mathematical knowledge that can be shared with others. You know lots of things about math that people did not know 5,000 years ago. For example, you know about fractions, the number zero, the decimal point, and decimal notation. You also make routine use of applications that are strongly based on math. For example, you tell time using a digital or analog watch. You use money. You understand the concept of distance and you know how to make use of instruments such as a ruler to measure distance.

Math has become so important and routinely used in our that children begin to learn math well before they enter kindergarten, and math is a required part of the school curriculum well into high school. Most colleges require students to take some math.

> Mathematics is one of humanity's great achievements. By enhancing the capabilities of the human mind, mathematics has facilitated the development of science, technology, engineering, business, and government. (Kilpatrick, Swafford, and Findell, 2002)

The idea that people created math and that math is a human endeavor are thoroughly embedded in many books about math. Indeed, Harold Jacobs wrote a secondary school math book titled *Mathematics, a Human Endeavor* that has been widely used (Jacobs, 1994). There is a tremendous amount of materials about the history of math available on the Web (History Topics Index, n.d.).

Piaget and Vygotsky

Jean Piaget (1896-1980) and Lev Vygotsky (1896-1934) both made major contributions to developmental psychology. At an initial glance, one might say that Piaget focused more strongly on the nature aspects of human development, why Vygotsky focused more on the nurture aspects. However, they both understood nature and nurture, and a careful reading of their works suggests that their thinking was not too far apart (Cole and Wertsch, n.d.).

Preservice and inservice teachers have all studied cognitive development. Most likely, you have studied Piaget's 4-stage cognitive development model; sensory motor, preoperational, concrete operations, and formal operations (abstract thinking). The general theory posits that people move through these stages at different rates that depend on interaction of hereditary and environmental factor—that is, nature and nurture.

Vygotsky is well known for his work on social constructivism and for what he calls the Zone of Proximal Development. Quoting Vygotsky, this is "the distance between the actual development level as determined by independent problem solving and the level of potential development as determined through problem solving under adult guidance or in collaboration with more capable peers." Vygotsky argues that instruction is most effective when it at the level of the Zone of Proximal Development.

In Piaget's four-state model of cognitive development, the formal operations stage is open ended. Moreover, there is no fine dividing line between people who function at a formal operations level from those who do not. In addition, a person may be further along in their development of formal operations in one discipline than in another. Thus, it is useful to study a student's movement along a Piagetian-type developmental scale in a specific discipline such as mathematics. Piaget, himself, had a considerable interest in mathematical development.

A person's brain/mind stores patterns. It is helpful to think of such patterns as models or representations of data, information, knowledge, and procedures. Learning is a process of revising models (perhaps even to the extent of discontinuing use of a model), and building new models.

Piaget describes this learning process in terms assimilation and accommodation. Assimilation involves the incorporation of new events into one's current brain/mind models. Accommodation involves changing one's existing brain/mind models, building new, better models.

For example, a young child's mental model of the number line might consist only of a representation of a small set of counting numbers such as 1, 2, 3, and 4. This mental model changes to assimilate larger sets of objects that can be counted.

However, eventually the child encounters number line concepts that require a significant change in the model. Zero and negative integers require an accommodation. A more major change in the model is needed to accommodate fractions. A still more major change may be necessary to accommodate rational numbers that are neither counting numbers nor fractions, and then to accommodate irrational numbers.

The number line is very complex and abstract relative to a young child's innate ability to deal with one, two, three and many. Informal and formal education make possible a child's growing understanding of (mental model of) the number line. As recently as the past century, anthropologists have discovered small tribes of hunter-gatherer people whose language and mental models incorporate provisions for dealing with just a small number of integers, and "many." Children growing up in such societies learn the number line model of their parents and others in the tribe.

Contrast the "1, 2, 3, many" model with the math models we now expect children to learn. Recently I read about a runner setting a new world record in the 100-meter dash. His time was 1/100 of a second faster than the old record. Such high precision measurements of length and time are not relevant in a hunter-gatherer society. How about paper and pencil algorithms to do long division of multi digit decimal numbers? How about solving quadratic equations and systems of linear equations? How about figuring miles per gallon, or costs per mile for car transportation? How about doing a state and federal income tax return, budgeting, dealing with credit cards and interest rates, and saving for one's retirement? How about calculators, computers, GPS systems, text messaging on a cell phone, the speed, capacity, and reliability of multimedia storage devices, and so on? Wow! Overwhelming!

The point is, we live in a society that is mathematically very complex. It takes many years of good quality informal and formal math education for an average person to gain the math knowledge and skills to learn to deal effectively with this complexity.

Assimilation or Accommodation to a Calculator

I want to give one more example to conclude this section. Much of your initial informal and formal math education focused on learning to count, gain an initial understanding of the number line, and learning to do simple calculations. Over time, your understanding of the number line grew and you learned more about calculations that could be performed on numbers. Your mental models of number line and of calculation changed through assimilation and accommodation.

At some stage, calculators entered the scene. Perhaps you were told that calculators could be used to check the answers obtained from doing paper and pencil calculation or mental calculations. Probably your mental models for the number line and for calculation easily assimilated the simple, 4-function calculator.

Unfortunately, the 8-digit calculator's number line is not the same as the number line that you had been learning about in school. Divide 1 by 3, multiply the result by 3, and the result is not 1.

Aha, you say. That is just an example of a rounding error, and is certainly easy to spot and to explain. (Note that one can memorize the term *rounding error* without really understanding what it means.) The fraction 1/3 corresponds to an infinite repeating decimal, and the calculator only handles a limited number of decimal places.

However, here is a more challenging example. Do you think that the average of two positive numbers can be smaller than either of the numbers? Well, here is an example from the number line of an 8-digit calculator:

$(5.000006 + 5.0000008)/2 = 5.000005$

The point is, a calculator's number line is not the same as the number line that you take for granted. Right now, how is your mental model of the number line dealing with the idea that the model does not fit with a calculator's number line? I wonder what you are now thinking about a computer's number line?

The chances are that the calculator you were learning to use had a square root key. The square root key, along with the rapid growth in availability of computers, led to an accommodation by second year high school algebra courses. Calculating square roots by hand (using an algorithm that is vaguely like long division) was dropped from the curriculum. Indeed, scientific calculators and graphing calculators have led to significant changes in the secondary school math curriculum. In some sense, the math education system assimilated and accommodated the electronic digital calculator.

Quite likely, the inexpensive calculator that you routinely use memory keys with labels such as M+, M-, MR, and MC. The memory features of a calculator are much like those of a computer. Do you know how to make effective use of the memory keys? Can you teach their use to students? Can you explain computer memory in terms of calculator memory? If your answers to some or all of these questions is no, then you have encountered a situation in which the math education you have received failed to fully accommodate to calculators.

One of My Pet Peeves

This book explores a variety of ways of making progress on the problem of improving math education. Improving math education is a very difficult task. There are many possible approaches, and certainly there is not uniform agreement as to what constitutes an improvement in math education. That is, the problems of math education are very complex, real-world problems.

An earlier part of this chapter mentioned, "The idea that a math problem may have no solutions, one solution, or more than one solution is part of math maturity." Did you "blip" over the idea that a math problem may have no solution, one solution, or more than one solution? Can you give examples of each of these situations?

Problem solving is one of the key elements of mathematics, and all teachers of math teach problem solving. K-8 teachers often stress that a major goal in math is for students to get “the” right answer to a problem. Students grow up with the idea that each math problem has exactly one and only one right answer. Students often carry this incorrect idea over to other disciplines, so that the term *problem* tends to mean to them a situation in which there is exactly one right answer.

I am always peeved and become agitated when I hear a math teacher talking about getting “the” right answer. That is because the teachers are teaching an incorrect (a false) idea. Before reading the examples given below, see if you can explain why this is a wrong idea.

Read the math problem examples given below. In the future, I hope you will no longer talk about getting “the” right answer in math.

1. Find two integers that are greater than 1 and less than 10. [There are lots of correct answers.]
2. Find two odd integers that add up to an even integer. [There are lots of correct answers.]
3. Find two even integers that add up to an odd integer. [There are no such integers.]
4. Find an integer that lies between 0 and 1. [There is no such integer.]
5. Find a fraction that lies between 0 and 1. [There are lots of correct answers.]

This book explores a number of educational problems related to math education. Such problems tend to have the characteristic that “one size does not fit all.” It is quite likely that some of these problems have no solution and that some have many quite acceptable solutions.

Final Remarks

Math education is a complex and challenging field. It takes a considerable period of informal and formal education for a math teacher to gain needed levels of expertise in math content, math maturity, general pedagogy, math pedagogy, and the many other things that go into being a good teacher. A commitment to becoming a good teacher of math is a commitment to a lifetime of learning. It is a rewarding career!

K-8 School Applications

Each chapter contains a few ideas for classroom applications at the K-8 level. These are meant to be suggestive, and are by no means comprehensive. Each teacher will need to build their own pieces of curriculum, instruction, and assessment to appropriately implement the ideas. An underlying goal in the use of such classroom applications is for you, the preservice or inservice teacher, to learn more about how the minds of your students work.

1.1 Ask your students, “What is math?” Younger students can provide oral answers, while older students can both talk about and write on this topic. Look for responses that seem to focus on math content and other responses that seem to focus on math maturity. Use responses to carry on class discussion designed to broaden student insights into the discipline of math. If your students mention the idea of getting “the” right answer, use that as a teachable moment to increase their mathematical maturity. If all of your student responses focus on doing arithmetic calculations, you may begin to suspect that there previous education has been weak in areas leading to increased math maturity.

1.2 Ask your students, "How do you learn math?" Use responses to help students gain insights into the fact that there are a variety of ways to learn math, and that different students may learn math in different ways. As a variation on this question, explore student insights into how one knows that they have learned a math topic (or, indeed, a topic in any discipline) well enough. You might get an answer such as, "When I get a good grade on the test." If that answer comes up, you have a teachable moment. I am assuming that you agree that there is much more to learning and understanding than getting high scores on tests.

Activities for Self-Assessment, Assignments, and Group Discussions

The last section of each chapter contains activities that can be used in teaching from and/or learning from this book.

1.1 The four Big Ideas given in the Preface—that serve to unify the components of this book—are: A) Math content versus math maturity; B) Math developmental theory; C) Computational math and computational thinking; and D) Learning to learn math. Explain what each of these ideas means to you, and what (if anything) this chapter has contributed to your current level of understanding of the topics. For example, what does computational thinking mean to you right now? What, if any, accommodation (in the sense of Piaget) did you need to make to your mental model of calculation to accommodate or assimilate the ideas of a calculator number line? This is a good chance to practice metacognition.

1.2 Think about your own K-8 math education experiences and what you have observed in visits to K-8 schools since then. What seems to be working well and what does not seem to be working well? Be as specific as possible. Add your insights into why the things that are working well are working well, and why the things that are not working well are not working well. That is, practice your causality thinking.

1.3 Think about your knowledge and experience in the areas of reading math (reading in the content areas) and writing math (writing in the content areas). How is such reading and writing the same as and different from just plain reading and writing? From this thinking, develop some ideas you can implement as you help your students gain increased expertise in reading and writing math.

1.4 This book is specifically designed for preservice teachers who are currently taking a math pedagogy course. Prior to this, such students have had years and years of instruction in mathematics. When I think about this, I conclude that most of what they know about math pedagogy will have come from what they happened to pick up through their years and years of math coursework. Share your thinking about this situation. What might you be learning in your teacher education program of study that will help to break the model of teachers teaching math in the way that they were taught?

Chapter 2

Academic Disciplines

> An individual understands a concept, skill, theory, or domain of knowledge to the extent that he or she can apply it appropriately in a new situation. (Howard Gardner, The Disciplined Mind: What All Students Should Understand, Simon & Schuster, 1999.)

> ... pedagogy is what our species does best. We are teachers, and we want to teach while sitting around the campfire rather than being continually present during our offspring's trial-and-error experiences. (Michael S. Gazzaniga, 1998, p 8)

Elementary school teachers are responsible for teaching a wide range of disciplines such as art, language arts, math, music, science, and social science. Some middle school teachers specialize in just one or two disciplines, while others cover a broader range of disciplines.

The disciplines that K-8 students study in school are both broad and deep. The initial instruction that students receive is designed to provide an initial level of knowledge and skill, and to lay foundations for future learning. It is important that careful thought be given to laying appropriate foundations.

What is a Discipline?

Although the focus of this book is on the discipline called mathematics, let’s begin this chapter by taking a more general approach. What is a discipline, and how does one distinguish between disciplines?

Each discipline can be defined by its unique combination of:

- The types of problems, tasks, and activities it addresses.
- Its tools, methodologies, and types of evidence and arguments used in solving problems, accomplishing tasks, and recording and sharing accumulated results.
- Its accumulated accomplishments such as results, achievements, products, performances, scope, power, uses, impact on the societies of the world, and so on.
- Its history, culture, unifying principles and standards of rigor, language (including notation and special vocabulary), and methods of teaching, learning, and assessment.
- Its particular sense of beauty and wonder. A mathematician’s idea of a “beautiful proof” is quite a bit different than an artist’s idea of a beautiful painting or a musician’s idea of a beautiful piece of music.

When you read this *What is a Discipline* list, did you just “bleep” over the details, or did you pause at each bulleted item and reflect on its meaning to you and to our educational system? Did you select a discipline that you know well and check on your insights into how each of the listed items fits or fails to fit your knowledge of the discipline? Did you think about what items you think should be added to the list, and what might be deleted?

Reading the *What is a Discipline* list may give you some insight into reading across the curriculum. Each discipline has its own vocabulary and special symbols. The accumulated knowledge and results in a discipline make it difficult for a novice to read with understanding within the discipline. Some disciplines are more challenging than others, and math is one of the more challenging disciplines when it comes to reading.

I constructed this five-item *What is a Discipline* list over a period of several years, tried it in various presentations and courses, and revised it several times. It is “dense,” in the sense that it contains a lot of ideas packed into a small number of words. To me, it has the meaning that I want to convey. However, it is up to you to read it carefully, think about what it means to you, and eventually to construct your own meaning and understanding.

This is similar to what one does when reading math. Reading math for understanding and for construction of personal meaning is usually a slow and arduous process.

Here are a few examples of questions that you might ask yourself about some of the idea in the *What is a Discipline* list. What is the same and what is different between a math problem and a problem in art, language arts, health, music, science, or social science? What is the same and what is different when using math to solve a math problem versus using math to help solve a problem in art, language arts, etc.? You can think of these questions as a small part of an assessment of your current level of math maturity. If you answer these questions with ease, this is an indication that you have made good progress in developing your math maturity.

Research in brain science is beginning to give us important insights about one of the things that are the same across various disciplines. The brain learns by storing patterns. Mathematics is often described as the study of patterns. However, that statement is applicable to every discipline. Of course, one can define math to be the discipline in which one studies math patterns. However, that isn’t a very useful definition. (Compare that definition to defining music as the discipline in which one studies musical patterns of sound.)

Learning can be thought of as a process of storing patterns in one’s brain, and developing skill in retrieving and making use of these stored patterns. When a person encounters a problem situation, his or her brain attempts to match the perceived pattern of the problem situation with one or more stored patterns. If an appropriately similar stored pattern is recognized, this becomes a starting point for dealing with the problem situation. Thus, learning and problem solving in all disciplines have to do with developing and storing patterns in one’s brain, in pattern matching or pattern recognition, and in making use of stored patterns (Goldberg, 2005).

Many of the patterns that you have stored in your brain are quite specific to a specific domain or a specific situation. For example, in your formal study of algebra you have undoubtedly encountered quadratic functions and equations. You probably memorized the quadratic formula and developed skill in its use. You have a “chunk” of quadratic functions and equations information stored as patterns in your brain. (Chapter 8 contains a section on chunking.) This chunk, or collection of patterns, is your brains model or representation for what you know about the topic. Such chunks are interconnected with many other different chunks or models. Thus, the term “quadraphonic sound” might well lead you to think briefly about quadratic equations or perhaps quadruplet births.

Now, can you give me examples of use of quadratic functions and equations in your everyday life and in disciplines outside of math? Can you explain why these topics are so important that they are a standard part of second year high school algebra courses? What are some examples of equations that are not quadratic equations? Does learning how to solve quadratic equations help one to learn to solve other kids of equations? What do you think about the idea of high school students learning to make use of graphing and equation-solving calculators that can automatically graph quadratic and many other functions and can automatically solve a wide range of equations?

These are math content, math content-pedagogy, and math maturity types of questions. As a teacher of math, you should be interested in posing and answering similar types questions for all of the math content that you teach. Your answers should help shape the curriculum, instruction, and assessment you use with your students.

The Web—A Steadily Growing, Global Library

You know how to read, and you have had experience in reading in many different disciplines. That does not automatically mean that you are skilled in reading in each discipline, or that you are skilled in reading to learn within a specific discipline. The *What is a Discipline* list is full of relatively complex words and ideas. It is easy to read such a list and gain almost no understanding of the information it is attempting to convey. Reading for deep understanding and learning is a lot different than reading for entertainment. Students need to learn to read in a reflective manner that leads to learning and understanding. They need specific help and practice in learning to read in different disciplines.

Reflect on your learning experiences in learning to read math. When you were taking math courses in high school or college, how did you learn math? Did you learn mainly by listening to the lecture/demonstrations from the teacher? Did you learn a substantial amount by reading the math book? At the current time, are you confident in your ability to learn math by reading math books and articles?

Chapter 3 of this book discusses problem solving. One of the most important ideas in problem solving is building on the previous work of others. To do this, one must learn to access the previous work of others, and understand it at a level useful to aid in solving problems and accomplishing tasks.

Once reading and writing had been invented, information began to be accumulated in libraries. For thousands of years, only a very small percentage of the population learned to read, and few people could afford to own a book. It is only in the past couple of hundred years that literacy and public libraries have become widespread.

Electronic digital computers have made possible the Internet and the interactive electronic global library that is called the World Wide Web. The Web is now larger than any physical library and is continuing to grow quite rapidly.

The size and availability of the Web provides considerable added importance to students learning to read across the curriculum. Education has many goals. One of the goals is for students to develop the knowledge and skills to effectively deal with a broad range of problems that they will face as adults, at home, at work, and at play. Thus, we want students to learn to read well enough in each discipline that they study so they can make effective use of this reading to learn the discipline and help solve problems they encounter within the discipline. Part of a modern education is learning to make effective use of the Web (global library).

It is helpful to think of the Web (global library) in terms of a physical library that contains books, journals, magazines, maps, and other physical documents. Certainly, the Web "contains" huge numbers of such documents. However, the Web is far more than a static library. Many of the Web pages that one can access are not merely passive storage sites. They are interactive aids to learning and to solving problems. Think of it this way. A handheld calculator is an interactive, active aid to solving certain calculation problems. Give it a calculation to perform, and it can automatically carry out the calculation. Similarly, many Web sites can automatically solve quite complex problems, drawing upon the power of sophisticated computers to do the computations.

The Web is many other things. For example, the Web includes a very large number of business sites that sell goods and services. The Web provides ways for people to communicate, collaborate, and to share their pictures, writings, and music. The Internet and the Web are contributing significantly to the development of what Marshall McLuhan called the *global village* (McLuhan, n.d.).

To conclude this section, think about math aspects of the Internet and the Web. How do they affect what constitutes a good math education for life in today's and tomorrow's societies? What can you be doing as a teacher of math to better prepare your students to deal with the math aspects of their lives as they become adults?

Assimilation and Accommodation in a Discipline

What happens to a discipline when something new such as computers and then the Internet and then the Web come along? I find it helpful to think in terms of Piaget's ideas of assimilation and accommodation in human development.

Because ICT provides such a wide range of tools, essentially every academic discipline has been faced by issues of assimilation and accommodation. When computers began to come into widespread use, one of the first things to happen in higher education was the decision to create or not to create a new department focusing on computer and information science. During the 1960s and 1970s, such departments were created in Business, Engineering, and Liberal Arts. These new departments can be thought of as accommodations on the part of the colleges and universities.

Eventually many non-computer departments began to offer computer-related coursework. Initially a very few courses were offered, and ICT was integrated into some existing courses. This can be thought of as an assimilation approach. Thus, for example, an English Department might offer sections of Freshman English Composition in computer labs. Students would learn to compose using a computer, and their inclass composition quizzes would require use of the computer. A Library School might offer a course on information retrieval from both hard copy and electronic sources.

Other parts of higher education found that assimilation did not suffice. The Art Department at the University of Oregon provides a good example. In addition to the traditional art programs of study they offered, they eventually established a very large program in Digital Art. The Music School at the University of Oregon has a large Digital Music program of study.

What happened in Mathematics Departments is quite interesting. Many such departments began to offer courses in computer programming and then courses in computer science. Often this led to the establishment of a Department of Computer Science through a transfer of courses and faculty to the new department. In other cases, computer science remained in the Mathematics Department. In either case, graphing calculators have been assimilated into the math curriculum. However, most coursework in math departments has been little changed by computers.

The same statements hold true for precollege math. The National Council of Teachers of Mathematics has been recommending use of calculators through the precollege curriculum since 1980, and calculators are part of the NCTM Principles and Standards. Math assessment at state and national levels typically now allow use of calculators. At the elementary school level, however, many teachers do not routinely integrate use of calculators into the curriculum. They maintain a traditional curriculum, while them make some modest efforts to assimilate calculators. They do not make significant changes in the curriculum that are required for effective use of calculators and computers. For the most part, computers have had little impact on the precollege math curriculum.

Your Knowledge of Medicine vs. Your Knowledge of Math

The *What is a Discipline* list given earlier in this chapter includes the idea of accumulated accomplishments within a discipline. Just for the fun of it, think about your knowledge of the accumulated accomplishments in medicine. You know quite a bit about a wide range of diseases, germs, bacteria, viruses, a wide range of drugs and vaccines, various types of surgery, and so on. You know some things about DNA, cloning, and genetic engineering. Perhaps you know your blood type, and that there are different blood types. Your accumulated knowledge in medicine is well beyond that of the world's best physicians and medical researchers of a few hundred years ago.

Now, contrast that with your knowledge of the accumulated accomplishments in math. Can you name some of the accumulated accomplishments of math? How does your list compare to your knowledge of medicine? (Remember, Isaac Newton and others developed calculus about 350 years ago, and its mathematical foundations go back a long time before then.)

What might you conclude from this activity? You have learned a lot about the discipline of medicine through your informal efforts and the efforts of our schools. Your informally learned knowledge of medicine may well exceed your school learning in this area. This is because medicine is relevant to your everyday life. Think about what aspects of math are relevant to your everyday life. Think about what aspects of the K-8 school math curriculum are relevant to the everyday lives of K-8 students. What might you and other K-8 teachers do to make math more relevant across the entire curriculum and in the lives of your students?

Reflexive Reading for Constructivist Learning

This book contains a relatively high density of Big Ideas. If you read this book in the same manner and at the same rate as you read a short story or a novel, you will gain very little from it. To gain appreciable benefit from reading this book, you will need to read in a reflective manner, pausing frequently to think about what you already know and how it fits in with what you are reading. You will need to construct meaning that integrates into and adds to your current knowledge and understanding. That is, you will need to practice constructivist learning (Ryder , n.d.) and reflexivity.

In essence, that is what the learning theory called constructivism is all about. Constructivism is a learning theory applicable to learning in each discipline. It is a theory about developing patterns in one's brain, and then building on these patterns. It is important in the teaching and learning of math, as well as all other disciplines. Thus, you might want to spend a little time thinking about your preparation to help your future students learn math (and other disciplines) in a constructivist manner (Math Forum, n.d.).

The activities in this document are intended to encourage you to think, and to think about your thinking. Thinking about your thinking is called metacognition. It is an important component of formal and informal education at all grade levels and in all disciplines. Metacognition about a discipline you are studying is an important aid to increasing your level of maturity in the discipline.

Learning About Learning

One of the goals of this book is to encourage you to think about what you, personally, can do to improve our educational system. Teaching is a very challenging and demanding profession. Good teachers are always learning and growing professionally. You may find it useful to make a copy of the discipline-defining bulleted list so you can refer back to it as you develop lesson plans and as you engage in your everyday activities as a (constructivist) teacher.

Moreover, you should structure your professional career as a teacher to allow significant time for learning. There is a huge amount of research and practitioner knowledge on the craft and science of teaching and learning. Bransford (1999) provides an excellent overview of this field, and the book can be read free from the Website listed in the reference.

On September 30, 2004 the National Science Foundation announced it had committed $36.5 million to fund three major research centers in the area of Learning About Learning (NSF, 2004). Quoting from this announcement:

> How do we learn? This most fundamental ability comes about through the complex interplay of genes, brain-based neural mechanisms, developmental trajectories, and social and physical environments. These processes of learning are just beginning to be understood. A deeper understanding of learning will allow scientists and educators to devise methods for improving how humans learn and develop machines that can perform tasks intelligently and independently.
>
> NSF has launched the new Science of Learning Centers to meet the challenge of learning about learning. Their goal is to make new discoveries about the foundations of learning across a wide range of learning situations—from processes at the cellular level to complex processes engaging different brain areas, to behaviors of individuals, to interactions in the classroom, to learning in informal settings, to learning performed by computer algorithms.

See http://www.nsf.gov/awardsearch/showAward.do?AwardNumber=0354400 for information about an April 1, 2005 award of more than $4 million made under this grant program. Here is a quote from the abstract of the proposal.

> Education changes the brain, and understanding this complex process will be fundamental to creating a science of learning. Based on our understanding of how the brain encodes, stores, and activates knowledge, what are the barriers to learning, and how can ways around those barriers be implemented? **What is the brain-basis of core content areas of learning, including language, math, science, and literacy?** [Bold added for emphasis.]

See http://www.nsf.gov/awardsearch/showAward.do?AwardNumber=0350277 for information about a September 1, 2004 award of about $180,000 made under this grant program. Here is a quote from the abstract of the proposal:

> This catalyst project lays the foundation for a future NSF Science of Learning Center that will **focus on the interrelationships among mathematics, language, and cognition in the learning process.** It will examine how deaf and hard-of-hearing students (K-12 through post-secondary education) learn mathematics as compared to hearing peers. **The operational definition of mathematics for this project is the target learners' abilities for conceptual understanding, procedural knowledge, and problem solving, as well as their powers to reason, make connections, and communicate mathematical knowledge.** [Bold added for emphasis. Notice the math and language connection. The last sentence in the quoted material seems closely aligned with helping students develop mathematical maturity.]

Learning about learning is an important research topic. However, it is also a core component of learning throughout all formal and informal education. One of your jobs as a teacher of mathematics is to help your students gain increasing knowledge and skills about how to learn math—that is, help them to increase this aspect of their math maturity.

Final Remarks

There are literally thousands of disciplines sub disciplines, sub sub disciplines,…, in which various people work to develop a useful or higher level of expertise. It can take many thousands of hours of study and practice to achieve a high level of expertise in a narrow sub discipline or a sub sub discipline. For example, think of the discipline of medicine, the sub discipline of surgery, and the sub sub disciplines of eye surgery or brain surgery.

Our educational system is designed to lay the foundations for students to eventually move into disciplines and sub disciplines of their choice. Learning to read and then reading to learn is stressed in school because reading to learn is very important in learning other disciplines.

Similarly, math is stressed because math plays an important role in many other disciplines. In parallel with reading to learn, students studying math need to learn to learn math. An important component of this is learning to read math both as an aid to reading and understanding problems that contain math, and also as an aid to learning math.

K-8 School Applications

2.1 Carry on a discussion with your students about two or three of the subjects (disciplines, with math being one of the disciplines) they are learning about. Students are to talk about how the subjects are the same and how they are different. They are to talk about how one shows knowledge and skill in each of the subjects. They are to talk about which subject is the most fun and which is the least fun, and why. As you listen to and participate in this conversation, listen for comments about problems and problem solving. If a student talks about problem solving in a non-math discipline, or if no student mentions this idea, use this as a teachable moment to expand on the fact that problem solving is part of every discipline.

2.2 Carry on a discussion with your class about uses they have made of things learned in school. Help them to explore what it means to make use of things they are learning. A use might be just bringing a topic up in a conversation with parents, siblings, or others. Or, it might be to answer a question, help solve a problem, or help accomplish a task. Make sure the discussion includes a focus on uses the students have made of math learned in school. As you listen to their comments about use of math, pay particular attention to whether the applications are based on what they are learning in school, or whether they can be learned and used without going to school. Children throughout the world who grow up in environments that do not include formal schooling still manage to learn a lot of math that they use in an everyday basis.

Activities for Self-Assessment, Assignments, and Group Discussions

2.1 Spend some time thinking about the *What is a Discipline* list from the point of view of your preparation to teach the various subjects you currently teach or are preparing to teach. Select two disciplines—one being math—and do a compare/contrast between the two disciplines. Share some of your insights and feelings from doing this activity.

2.2 Think about the K-8 school math curriculum that was in place when you were in school and/or that you have observed in more recent visits to schools. Discuss some of the aspects of the discipline of mathematics that are in the curriculum, some aspects that you feel should be added to the curriculum, and some aspects that you feel should be deleted from the curriculum. Make sure you give careful thought to how calculators and computers are affecting or could be affecting the curriculum.

2.3 Reexamine the four Big Ideas given in the Preface: A) Math content versus math maturity; B) Math developmental theory; C) Computational math and computational thinking; and D) Learning to learn math. Explain what each of these ideas means to you, and what (if anything) this chapter has contributed to your current level of understanding of the topics.

Chapter 3

Problem Solving

> If I had eight hours to chop down a tree, I'd spend six sharpening my axe. (Abraham Lincoln)
>
> The reason most kids don't like school is not that the work is too hard, but that it is utterly boring. (Seymour Papert)
>
> Judge a man by his questions rather than his answers. (Voltaire)

Problem solving lies at the heart of each discipline. However, the nature of the problems being addressed and the methodologies being used varies considerably from discipline to discipline. This chapter provides a brief introduction to *problem* and *problem solving.*

As you read this chapter, keep in mind that math is both a discipline in its own right and is also a powerful aid to representing and solving problems in many other disciplines. Every discipline deals with problems, and math is often a useful aid to dealing with the problems in disciplines outside of mathematics.

Problem Solving

I use the term problem solving in a very broad sense. For me, *problem solving* includes dealing with:

- Question situations: recognizing, posing, clarifying, and answering questions.
- Problem situations: recognizing, posing, clarifying, and solving problems.
- Task situations: recognizing, posing, clarifying, and accomplishing tasks.
- Decision situation: recognizing, posing, clarifying, and making decisions.
- Using higher-order, critical, creative, and wise thinking to do all of the above. Often the "result" is shared or demonstrated as a product, performance, or presentation.

Here is a definition of the word *problem* that I have found useful in my teaching of preservice and inservice teachers at all grade levels and in a variety of subject areas:

> You (personally) have a problem if the following four conditions are satisfied:
>
> 1. You have a clearly defined given initial situation.
> 2. You have a clearly defined goal (a desired end situation). Some writers talk about having multiple goals in a problem. However, such a multiple goal situation can be broken down into a number of single goal problems.
> 3. You have a clearly defined set of resources that may be applicable in helping you move from the given initial situation to the desired goal situation. These typically include some of your time, knowledge, and skills. Resources might include money, the Web, and the telephone

system. There may be specified limitations on resources, such as rules, regulations, guidelines, and timelines for what you are allowed to do in attempting to solve a particular problem.

4. You have some ownership—you are committed to using some of your own resources, such as your knowledge, skills, time, and energy, to achieve the desired final goal.

The fourth component of this definition is particularly important. Unless a student has ownership—an appropriate combination of intrinsic and extrinsic motivation—the student does not have a problem. Motivation, especially intrinsic motivation, is a huge topic in its own right, and I will not attempt to explore it in detail in this book. A book chapter on motivation is available at (Retrieved 5/25/06) http://education.calumet.purdue.edu/vockell/EdPsyBook/Edpsy5/Edpsy5_intro.htm.) You certainly know that many teachers are not very successful in helping their students to develop intrinsic motivation in their math studies. As students progress through elementary school and into secondary school, the math they study seems to have less and less meaning and intrinsic motivation for many students.

As noted at the start of this chapter, problem solving lies at the core of each discipline. Perhaps you have heard people ask questions such as "Why do I need to study math?" or "Why do I need to study xyz (where xyz is some other discipline that is a required part of the curriculum)?"

While there are many possible answers to such questions, a unifying answer is that by doing so you will be able to solve a variety of problems that you cannot currently solve. You will learn about some of the important accomplishments within the discipline, some of its history, and some of its language. As you learn the language and notation, you will get better in making use of and building on the accumulated knowledge of the discipline. You will learn to precisely represent problems to be solved and tasks to be accomplished so that you can communicate your needs and interests to other people and to Information and Communication Technology (ICT) systems.

ICT provides powerful information retrieval systems (an aid to building on the previous work of others) as well as tools that can solve or greatly aid in solving a wide range of problems. A later chapter of this book is devoted to ICT and math education.

George Polya

George Polya was one of the leading mathematicians of the 20th century, and he wrote extensively about problem solving. His 1945 book, *How to Solve It: A New Aspect of Mathematical Method,* is well known in math education circles (Polya, 1957).

The Goals of Mathematical Education (Polya, 1969) is a talk that he gave to a group of elementary school teachers.

> To understand mathematics means to be able to do mathematics. And what does it mean doing mathematics? In the first place it means to be able to solve mathematical problems. For the higher aims about which I am now talking are some general tactics of problems—to have the right attitude for problems and to be able to attack all kinds of problems, not only very simple problems, which can be solved with the skills of the primary school, but more complicated problems of engineering, physics and so on, which will be further developed in the high school. **But the foundations should be started in the primary school. And so I think an essential point in the primary school is to introduce the children to the tactics of problem solving. Not to solve this or that kind of problem, not to make just long divisions or some such thing, but to develop a general attitude for the solution of problems.** [Bold added for emphasis.]

In this statement, Polya is talking both about problem solving throughout the field of math, and also about use of math in solving problems in other disciplines. He is also talking about "the right attitude and to be able to attack all kinds of problems." This statement is about math maturity, rather than about knowledge of any specific math content.

As the following quotation from the same talk indicates, Polya was particularly concerned with helping students learn to think mathematically when working on problems.

> We wish to develop all the resources of the growing child. And the part that mathematics plays is mostly about thinking. Mathematics is a good school of thinking. But what is thinking? The thinking that you can learn in mathematics is, for instance, to handle abstractions. Mathematics is about numbers. Numbers are an abstraction. When we solve a practical problem, then from this practical problem we must first make an abstract problem. Mathematics applies directly to abstractions. Some mathematics should enable a child at least to handle abstractions, to handle abstract structures.

Notice the emphasis on representing problems in the abstract words and symbols of math. Later in this book I will present some ideas from Piaget and others on cognitive developmental theory. Problem solving and abstraction lie at the Formal Operations end of the Piagetian scale for cognitive development. As we teach math, we are attempting to help students move up this cognitive development scale.

Building on Previous Work

One of the most important ideas in problem solving is to build on the previous work of yourself and others. That is, one way to solve a problem is to retrieve from your own memory either a solution to the problem or a method for solving the problem. Another way is to retrieve this information from another person, from a book, from a machine such as a cash register, or from a calculator or a computer. If you are repeatedly faced by a particular problem or type of problem, it is very useful to memorize one or more solutions to the problem, or a general method for solving the problem in a timely fashion.

Mathematics is a very large discipline because a large number of people have been working throughout recorded history to build and accumulate knowledge in this field. A research mathematician may spend years working on a single problem or a small group of related problems. If the mathematician is successful, then information about solving the problem or group of problems is published and becomes part of the accumulated knowledge of the field.

The human race's accumulated knowledge in mathematics is stored in hundreds of thousands of books, monographs, journals, Web publications, and other forms of publication. Much of this accumulated knowledge is only accessible to those who have studied math at a graduate school level. While it is easy to talk about the importance of building on the accumulated knowledge of oneself and others, it can take many years of hard work to develop the knowledge needed to read and understand the accumulated research knowledge in a discipline.

Moreover, currently most of the accumulated knowledge in a field such as math is not readily available. It is scattered throughout the libraries of the world, and it is written in many different languages. Over time, such difficulties of accessing materials will decrease as the materials are digitized and become accessible through the Web. Progress in the computer translation of languages will also help.

To summarize, one goal in math education needs to be that students learn to access the accumulated math knowledge that is appropriate to their educational level and needs, and to learn to make use of this accumulated knowledge to solve problems and accomplish tasks. That is, students need to learn to read math with understanding. One aspect of this is having students learn to read math well enough so that they can "look up" and read the math that they have studied in their previous years of studying math in school. A somewhat different way to think about this is that when a student is learning a math topic, the student should be learning enough to "relearn" the topic in the future, after a substantial amount of forgetting has occurred.

To Memorize or not to Memorize: That is the Question

Rote memory is useful in problem solving. However, a focus on rote memory tends to be a poor approach to getting better at math problem solving. Indeed, rote memorization without understanding is a very poor approach to getting better at solving novel, challenging problems in math or in any other discipline.

Our math education system has trouble in creating an appropriate balance between memorization and learning with understanding. There are various reasons for this. One is that initial learning of math tends to be rote memorization with relatively little understanding. It is easy to memorize a sequence of sounds such as one, two, three, four, … and repeat them when prompted. For many, it is a significant step to move from this to creating a one to one correspondence that results in naming the number of items in a small set.

As children enter school, they and their math teachers are faced by memorization versus understanding. The line of least resistance and seemingly quickest results tends to be memorization, often accompanied by very little understanding. It is impressive to hear a young child parrot addition and multiplication facts. Often, however, if you delve into such a child's understanding of the numbers, operations, and memorized facts, you find that the learning is quite shallow.

This realization provides one of the major underpinnings of new new math. The NCTM strongly recommends that starting at the earliest grade levels in school, the math curriculum should strive for student understanding rather than for students being able to quickly parrot memorized answers.

At the early grade levels, computers enter into this discussion because drill and practice, along with speed and accuracy drills, are easily administered by computers. Moreover, the software can keep task of errors and provide extra practice on items that the student has missed. Thus, the rote memorization process can be speeded up for many students.

Calculators and computers also enter this discussion. One way to think about a calculator is that it is an information retrieval device. It can do calculations so fast it in almost as though the calculator has memorized the answers and is merely recalling from its rote memory. Extending this idea to computers and the Web, we can think of a computer both as having the ability to quickly "figure out" answers to certain types of problems, and also to quickly "look up" answers from the huge amount of stored information it has available. In that sense, a computer can be thought of as an auxiliary brain, able to do certain types of brain-like tasks and able to store a huge amount of information.

In terms of education, the computer's computational and storage capabilities compete with a human's computational and information storage and retrieval capabilities. A good education provides a balance between the learner and the computer that is appropriate to the needs and capabilities of the learner.

Polya's 6-Step (Heuristic) Strategy

The research literature on problem solving is quite large, and math education includes a number of heuristic strategies for attaching math problems. Examples of heuristic strategies include: draw a picture; break a big problem into smaller pieces; trial and error; develop a somewhat similar but simpler problem; and do library research. Each of these examples is a heuristic—a plan of action that may help, but is not guaranteed to help. This is in contrast with an algorithm, which is guaranteed to solve a particular category of problem or accomplish a particular task in a finite number of steps.

Thinking mathematically and solving math problems are large topics and are important components of any math or math education course. While these two topics are beyond the scope of this short book, all readers should be interested in Polya's (1957) general heuristic strategy for attempting to solve any math problem. I have reworded his strategy so that it is applicable to a wide range of problems in a wide range of disciplines—not just in math. This six-step strategy can be called the Polya Strategy or the Six Step strategy. Note that there is no guarantee that use of the Six Step strategy will lead to success in solving a particular problem. You may lack the knowledge, skills, time, and other resources needed to solve a particular problem, or the problem might not be solvable.

1. Understand the problem. Among other things, this includes working toward having a well-defined (clearly defined) problem. You need an initial understanding of the Givens, Resources, and Goal. This requires knowledge of the domain(s) of the problem, which could well be interdisciplinary. You need to make a personal commitment (Ownership) to solving the problem.

2. Determine a plan of action. This is a thinking activity. What strategies will you apply? What resources will you use, how will you use them, in what order will you use them? Are the resources adequate to the task? On hard problems, it is often difficult to develop a plan of action. Research into this situation suggests that many good problem solvers "sleep on the problem." That is, after working on a problem for quite awhile with little or no success, they put the problem out of mind and do something else for days or even weeks. What may well happen is that a subconscious level the mind continues to work on the problem. Eventually, an "ah-ha" sometimes occurs.

3. Think carefully about possible consequences of carrying out your plan of action. Focus major emphasis on trying to anticipate undesirable outcomes. What new problems will be created? You may decide to stop working on the problem or return to step 1 as a consequence of this thinking.

4. Carry out your plan of action. Do so in a thoughtful manner. This thinking may lead you to the conclusion that you need to return to one of the earlier steps. Note that this reflective thinking leads to increased expertise.

5. Check to see if the desired goal has been achieved by carrying out your plan of action. Then do one of the following:

A. If the problem has been solved, go to step 6.

B. If the problem has not been solved and you are willing to devote more time and energy to it, make use of the knowledge and experience you have gained as you return to step 1 or step 2.

C. Make a decision to stop working on the problem. This might be a temporary or a permanent decision. Keep in mind that the problem you are working on may not be solvable, or it may be beyond your current capabilities and resources.

6. Do a careful analysis of the steps you have carried out and the results you have achieved to see if you have created new, additional problems that need to be addressed. Reflect on what you have learned by solving the problem. Think about how your increased knowledge and skills can be used in other problem-solving situations. (Work to increase your reflective intelligence!)

Many of the steps in this six-step strategy require careful thinking. However, there are a steadily growing number of situations in which much of the work of step 4 can be carried out by a computer. The person who is skilled at using a computer for this purpose may gain a significant advantage in problem solving, as compared to a person who lacks computer knowledge and skill.

Computers and Math Problem Solving

I find the diagram given in figure 3.1 to be particularly useful when I talk about computers and math problem solving at the precollege level. With some effort, this diagram can be modified to fit problem solving in other disciplines.

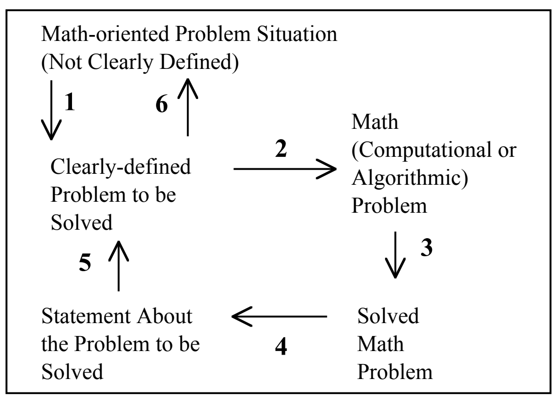

Figure 3.1 Math problem solving.

The six steps illustrated are 1) Problem posing and problem recognition; 2) mathematical modeling; 3) Using a computational or algorithmic procedure to solve a computational or algorithmic math problem; 4) Mathematical "unmodeling"; 5) Thinking about the results to see if the Clearly-defined Problem has been solved; and 6) Thinking about whether the original Problem Situation has been resolved. Steps 5 and 6 also involve thinking about related problems and problem situations that one might want to address or that are created by the process or attempting to solve the original Clearly-defined Problem or resolve the original Problem Situation.

In some sense, all of the steps except (3) involve higher-order knowledge and skills. They require a significant level of math maturity and cognitive activity. Step (3) lends itself to a rote memory approach. It is highly desirable that students develop speed and accuracy in certain types of mathematical operations. However, the human mind is not good at memorizing math procedures and then carrying them out rapidly and accurately with the assistance of pencil and paper. On the other hand, calculators and computers are really good at carrying out math procedures.

Precollege teachers who teach math tend to estimate that about 75% of the math education curriculum focuses on (3). [Note: This is an estimate I have made based upon working with a very large number of teachers. I don't know of any published research that backs up my assertion.] This leaves about 25% of the learning time and effort focusing on the remaining five steps. Appropriate use of calculators and computers as tools, and Computer-Assisted Learning, could easily decrease the time spent on (3) to 50% or less of the total math education time. This would allow a doubling of the time (from 25% to 50%) devoted to instruction and practice on the higher-order knowledge and skill areas.

Correctness of a Solution

Suppose that you were given the task of writing a persuasive paper about some aspect of our national election system. You ask the teacher, "How long does it need to be?" The teacher says that it needs to be sufficiently long to accomplish the task, and that grading will be based on the quality of the paper. The question for you is, how can you tell when you have accomplished the task?

Remember, problem solving is part of every discipline. With the broad definition of *problem* that we are using, your writing task is a writing problem to be solved. It is certainly different than a math problem! Think about doing a compare and contrast with a math assignment. Here are some of my thoughts as I pretend to be a student:

1. If the teacher had just said how long the paper was to be, I would know I was close to done when I had achieved the required length. That is a little bit like an assignment in math where I am supposed to do all of the odd numbered problems at the end of a chapter. I know I am done when I have completed all of the odd numbered problems. But, I may have made mistakes in solving some of the problems. I guess that is a little like having errors in the writing and in the logic of the persuasive arguments.
2. The teacher didn't tell me if I needed to have a bibliography. I suppose I do, because this seems like the type of writing problem that requires research. When I am doing a math assignment, I sometimes need to look back in the book to see how to solve a particular type of problem. Occasionally I can't find an example in the book, perhaps because it is a problem from last year or several years ago. I guess it is easier to do library research in non-math areas.
3. In my writing, I will have a goal of convincing the reader of "something," through my careful logic and using information from the literature. First, I need to get a clear idea of my goal—what I want to convince my reader about. I suppose this is a little bit like solving a math problem. In solving a math problem, I usually have a clear goal, and I carry out a sequence of

steps. Each step is sort of like a piece of an argument, moving me in a logical fashion towards my goal.

4. I know that writing is a process, and that I will be doing "revise, revise, revise" to produce as good a product as possible in the time that I am willing to devote to the writing task. I know that my paper will not be perfect—that with more time, I could make it better. This seems different than solving a math problem. When I solve a math problem and get an answer, I am done. That assumes, of course, that I have some way of telling that I have gotten a correct answer. Of course, my math problem (task) might be to make a proof. That is sort of like making a persuasive argument. But, in math it is possible to make a persuasive argument that is really convincing. I guess that is what a math theorem is all about.

5. Etcetera, etcetera, etcetera.

All of the items in the list can be considered as aspects of math maturity. The 4th point in the list is especially important in math. In some math problem-solving situations, it is possible to check an answer. For example, addition can be checked by subtraction, division by multiplication, and so on. In some math problems, one can check an answer by testing to see if it meets the conditions specified in the problem. For example, suppose I am supposed to find three consecutive positive integers whose sum is a perfect square of an integer. If I find an answer, I can easily check to see if it is correct. If I can't find an answer, I can always try to prove that there is no answer. However, that requires me to develop a carefully constructed chain of logical argument that will be convincing to my readers.

Final Remarks

Remember, problem solving lies at the core of every discipline. When solving school problems, students often come to believe that the goal is to get a correct answer. Actually, the main goal is to get better at solving problems. Once one starts to face real world problems, the there is no answer book or teacher to provide immediate feedback on correctness or usefulness of one's answer or answers.

In math and in all other disciplines, students need to learn to depend upon themselves and the quality of their own work as an aid to checking the quality and usefulness of the results.

K-8 School Applications

3.1 Having a person "think out loud" as they attempt to solve a problem is a standard research tool. (To learn more about facilitating this type of activity, see http://www.stcsig.org/usability/topics/articles/tt-think_outloud_proc.html.) It can also be useful both as an aid to learning and as a vehicle through which a teacher can gain insight into a student's learning and problem-solving difficulties. Select some math problems (as distinguished from math exercises) of a difficulty level appropriate to your students. Train your students in carrying out this thinking out loud activity through use of volunteers who role model it. In this training process, you are role modeling how to interact with the out loud-thinker, and how to provide appropriate feedback. Gradually work toward the situation in which students can work in pairs or small teams, with a student thinking out loud in the team, explaining his or her thinking processes when attempting to solve a problem. The listeners or listeners practice interaction with the talker, gaining skill in listening and providing appropriate feedback.

3.2 This chapter contains a 4-part definition of the term "problem." Since problem and problem solving are key components of each discipline you teach, it seems reasonable that your students should be learning definitions of these terms that are appropriate to their developmental level and the disciplines they are studying. Set yourself a teaching goal of having your students understand meanings for math problem and math problem solving that are appropriate to the level at which you teach. You might begin such a lesson by first asking students to say what they think a math problem is, and what they think math problem solving is. You might then continue by looking at some examples of problems and problem situations that may or may not be math problems, and carrying on a discussion with your students about these examples and non-examples. You might continue by asking your students what it means to solve a math problem. For example, in this discussion you might hear a student say, "Do things to get the right answer." You might use that response to explore situations in which a math problem has no solution, only one solution, or more than one solution. You might raise the question, how can one tell if a proposed answer is right? This is a big and important topic in its own right.

Activities for Self-Assessment, Assignments, and Group Discussions

3.1 People teaching math often try to distinguish between an exercise and a problem. An exercise is practice in applying and carrying out a procedure that the students have recently encountered. A problem is more challenging, requiring higher-order cognition. The diagram in figure 5.1 shows that a number of steps are required in working from a typical math-related problem situation to a solved problem. What are your personal insights into the amount of math education time in K-8 school spent on exercises versus time spent on problems?

3.2 You know that there are 50 states in the United States, that each has a geographical location, Governor, state capital, two Senators, a number of Representatives, and so on. Think about what data for each state is worthwhile for most students to memorize. As you do this, think about the concepts such as geographical location, state capital, government and governmental officials, and so on. If a person learns the concepts, then information about specific details can be retrieved relatively quickly from the Web or other resources. What are your current thoughts on what to memorize and what to "understand" and be able to look up? What would it take to change your current position?

3.2 Think about some "real world" math problems that you have encountered recently. How did you go about solving these problems? For example, which did you solve by quick recall of memorized information, on which did you seek help on, on which did you make use of calculators or computers, and what other approaches did you use?

Chapter 4

What is Mathematics?

> Woodrow Wilson, like most Americans of his time, despised mathematics, complaining that "the natural man inevitably rebels against mathematics, a mild form of torture that could only be learned by painful processes of drill." (Page 52, A Beautiful Mind, Sylvia Nasar)

> The tools of mathematics are abstraction, symbolic representation, and symbolic manipulation. However, being trained in the use of these tools no more means that one thinks mathematically than knowing how to use shop tools makes one a craftsman. (Alan Schoenfeld, 1992).

It is not easy to give a useful and simple answer to the question: What is mathematics? Many mathematicians and math educators have attempted to answer this question. This chapter provides some answers from a math content point of view, while the next chapter provides an answer from a math maturity point of view. In both chapters, the goal is to help increase your math pedagogical knowledge in a manner that will help you be a better teacher of mathematics.

What is Math?

Many people have addressed the question, “What is mathematics?” See, for example, (Lewis, n.d.) and the many publications of the National Council of Teachers of Mathematics. Here are two good examples of answers to the question, “What is mathematics?”

> Mathematics is an inherently social activity, in which a community of trained practitioners (mathematical scientists) engages in the science of patterns—systematic attempts, based on observation, study, and experimentation, to determine the nature or principles of regularities in systems … The tools of mathematics are abstraction, symbolic representation, and symbolic manipulation. However, being trained in the use of these tools no more means that one thinks mathematically than knowing how to use shop tools makes one a craftsman. Learning to think mathematically means (a) developing a mathematical point of view—valuing the processes of mathematization and abstraction and having the predilection to apply them, and (b) developing competence with the tools of the trade, and using those tools in the service of the goal of understanding structure—mathematical sense-making (Schoenfeld, 1992).

Notice the emphasis on thinking mathematically. One gains increased expertise in math by both learning more math and by getting better at thinking and problem solving using one’s knowledge of math.

> Mathematics is built on a foundation which includes axiomatics, intuitionism, formalism, logic, application, and principles. Proof is pivotal to mathematics as reasoning whether it be applied, computational, statistical, or theoretical mathematics. The many branches of mathematics are not

> mutually exclusive. Oft times applied projects raise questions that form the basis for theory and result in a need for proof. Other times theory develops and later applications are formed or discovered for the theory. Hence, mathematical education should be centered on encouraging students to think for themselves: to conjecture, to analyze, to argue, to critique, to prove or disprove, and to know when an argument is valid or invalid. **Perhaps the unique component of mathematics which sets it apart from other disciplines in the academy is proof—the demand for succinct argument that from a logical foundation for the veracity of a claim** (Padraig & McLoughlin, 2002). [Bold added for emphasis.]

Notice the emphasis on proof or disproof.

Proof

The word *proof* comes up in most attempts to define mathematics. Of course, the idea of proof or proving something is not restricted just to mathematics. A trial lawyer attempts to prove his or her case. A person attempts to prove that another person is wrong in a particular situation. Researchers in science attempt to prove scientific theories.

Each discipline has its own ideas and standards about what constitutes a proof. Math proofs are designed to answer, once and for all, the correctness or incorrectness of a "mathematical" assertion. Suppose, for example, that I am exploring the sum of three consecutive integers. I see that 6 + 7 + 8 = 21, and 11 + 12 + 13 = 26. After looking at a lot of examples, I conjecture that if the first of the three consecutive integers is odd, then the sum is an even integer; if the first integer is even, then the sum is an odd integer. Looking at lots of example, and not finding any counter examples, may increase my confidence that my conjectures are correct. However, my failure to find a counter example does not constitute a proof. Think about definitions of odd and even integers. See if you can construct a convincing proof that my conjectures are correct.

Then think about whether K-8 students, once they have encountered definitions of odd and even integers, might be able to develop convincing proofs. If the conjecture given above is too difficult for students at a particular age, how about considering the simpler conjecture that the sum of two consecutive integers is odd. A young child attacking this task might make use of small cubes, physically lining up rows of cubes to represent integers, and then arguing from the patterns that result.

Finally, be aware that there are lots of simple proof-type situations that can be constructed for use in the K-8 school setting. To give one more example, suppose that students have learned the mathematical word *mean.* You might then have them compute the mean of various sets of three consecutive integers, looking for a pattern. Quite likely some of the students will note that the answers they obtain are always the middle one of the three consecutive integers. Can they construct a convincing argument that this is always the case? What if one wants to find the mean of five consecutive integers?

When you present young students with such problems, you want to think carefully about what they may be learning. The examples given above might lead some students to think that the mean of a set of consecutive integers is an integer. With a little encouragement, some of your students might conjecture and then attempt to prove that "The mean of an odd number of consecutive integers is an integer, and the mean of an even number of consecutive integers is not an integer."

Fluency and Proficiency

The terms fluency and proficiency are often used in talking about goals and expertise in mathematics. The following definition of math proficiency is quoted from Kilpatrick et al. (2001), a report written for the National Academy of Sciences.

> Mathematical proficiency, as we see it, has five components, or *strands:*
>
> - *conceptual understanding*—comprehension of mathematical concepts, operations, and relations
> - *procedural fluency*—skill in carrying out procedures flexibly, accurately, efficiently, and appropriately
> - *strategic competence*—ability to formulate, represent, and solve mathematical problems
> - *adaptive reasoning*—capacity for logical thought, reflection, explanation, and justification
> - *productive disposition*—habitual inclination to see mathematics as sensible, useful, and worthwhile, coupled with a belief in diligence and one's own efficacy.

Warning! The mathematical proficiency bulleted list reflects many hundreds of hours of thinking by some of the world's leading math educators. Did you read it in a reflective manner? Did you work to construct your own meaning? What aspects of the presented ideas will you remember five minutes from now, a day from now, or a year from now?

For the most part, answers to the "what is math" question do not depend on specific areas of math content. The question and answers are part of math maturity. As you think about the mathematical proficiency bulleted list, you are working to increase an aspect of your math maturity that is very important to being a good teacher of math.

As you construct and/or make use of a math lesson plan, you can think about how it fits in with and contributes to increasing your students' mathematical proficiency. For example, compare having students work on a drill and practice page of arithmetic computations, versus students solving word problems, versus students creating word problems, versus students reading a science book and identifying math usage in science.

Weaknesses in Our Math Education System

Mathematics is a huge discipline. It has both great breadth and great depth. Depth, in this case, refers to a vertical structuring, in which "higher" levels of math build on "lower" levels of math. The scope and sequence of the precollege math curriculum is designed to help students gain both breadth and depth of knowledge and understanding.

Out math education system faces a number of challenges. One is the issue of breadth versus depth. Out current system is sometimes criticized as being "A mile wide and an inch deep." This is a criticism that students are not making enough progress to the "higher" levels (sometimes called "deeper" levels) of math.

Another challenge is to decide specifically what to teach and the level of proficiency that is required to move on to the next unit or course.

Many people argue that our math education system is not as good as it could be. They argue that students are not acquiring a sufficient level of math proficiency. Deborah Ball was the chair of a group of people studying the development of proficiency in math. Their report noted:

> Developing proficiency in mathematics is important for all students. However, when considered in light of current standards, or compared with performance in other countries, evidence on student achievement in mathematics makes clear the need for substantial improvement. U.S. students do not, as a group, achieve high levels of mathematical proficiency. The nation must seek to narrow the achievement gaps between white students and students of color, between middle-class students and students living in poverty; gaps that have persisted over the past decade (Ball, 2002).

Over the years, there have been several important international studies that help us understand math education in the United States versus math education in other countries.

> If you look at state and national assessments of math and science competence among our country's elementary and secondary schools today, you'll discover small pockets of excellence amid a broad swath of mediocrity. In fact, only a minority of U.S. students are meeting math and science proficiency benchmarks.
>
> International assessments from the Trends in International Mathematics and Science Study (TIMSS) show U.S. students are at or below the international average and significantly behind their peers in Japan and Canada. TIMSS compared our most advanced students with those from 15 other nations, and the brightest U.S. students scored dead last against international competitors in advanced math and physics assessments (Ruetters, 2002).

Math Education Reform

There are two obvious ways to go in considering math education reform. One can propose changes in the content scope and sequence, and one can propose changes in teaching methodologies. Of course, these two major issues can be merged, and that is what has tended to happen. Thus, at the current time there is a back to basics group and a new-new math group. The first group tends to emphasize both a back to basics" content and also a back to basics teaching methodology. The second group tends to emphasis “new” content and teaching methodology .

The Mathematically Correct (n.d.) Website presents arguments supporting back to basics and against the ideas of the new-new math education reformers. Quoting from their Website:

> Mathematics achievement in America is far below what we would like it to be. Recent "*reform*" efforts only aggravate the problem. As a result, our children have less and less exposure to rigorous, content-rich mathematics.
>
> The advocates of the new, fuzzy math have practiced their rhetoric well. They speak of higher-order thinking, conceptual understanding and solving problems, but they neglect the systematic mastery of the fundamental building blocks necessary for success in any of these areas. Their focus is on things like calculators, blocks, guesswork, and group activities and they shun things like algorithms and repeated practice. The new programs are shy on fundamentals and they also lack the mathematical depth and rigor that promotes greater achievement.

The Mathematically Sane (n.d.) Website presents arguments against the Mathematical Correct group and arguments supporting the new-new math.

The Standards produced by the National Council of Teachers of Mathematics (NCTM, n.d.) represent the sense of direction of new-new math reform. The NCTM Standards are divided into five content standards and five process standards. None of the ten standards say anything about computation in their titles. The ten NCTM Standards contain 33 goals. Exactly one of the 33 goals talks about computation—the traditional focus of much of the elementary school math curriculum! This particular goal statement is the Numbers and Operations standard, and it says, “compute fluently and make reasonable estimates.”

However, the NCTM emphasizes the importance of procedures and procedural thinking. For example, is a quote from (NCTM, n.d.):

> Learning the "basics" is important; however, **students who memorize facts or procedures without understanding** often are not sure when or how to use what they know. In contrast, **conceptual understanding** enables students to deal with novel problems and settings. They can solve problems that they have not encountered before. [Bold added for emphasis.]

Like all educators, the math education community is struggling with computers. Here is material quoted from the NCTM Principles and Standards (NCTM, n.d.):

> Technology is essential in teaching and learning mathematics; it influences the mathematics that is taught and enhances students' learning.
>
> Calculators and computers are reshaping the mathematical landscape, and school mathematics should reflect those changes. **Students can learn more mathematics more deeply with the appropriate and responsible use of technology.** They can make and test conjectures. They can work at higher levels of generalization or abstraction. In the mathematics classrooms envisioned in Principles and Standards, every student has access to technology to facilitate his or her mathematics learning. [Bold added for emphasis.]

Notice how this statement seems to separate computer technology from the content of mathematics. It seems to miss the point that computer science and mathematics strongly overlap. It does not reflect the importance of computational thinking or procedural thinking from a combined computer science and mathematics point of view.

Research-based Reform

There seem to be no end to suggestions on how to improve our math education system. There is a huge amount of literature containing suggestions. However, the research supporting these suggestions is not as strong as one might like (Gersten, 2002). Moreover, many of these suggestions involve significant changes, and our math education system seems to have considerable resistance to such changes.

Suggestions for improvement and/or approaches to improvement can be divided into major categories such as:

1. Develop and implement a better curriculum. The National Science Foundation and other funding agencies have funded a number of such endeavors, and these have led to new research-based curriculum and supportive materials that have come into widespread use. This approach has the advantage that once the curriculum and materials have been developed, they can be mass produced and widely distributed.
2. Require students to take more math courses. Require students to pass state tests in math in order to graduate from high school. Right now, the latter approach is gaining in popularity. If students do not meet high standards in math, don’t award them a high school diploma. This is a top down approach, with a major threat overhanging students that fail to achieve the standards that are being set.
3. Require math teachers to be better prepared. Every child should have “good” math teachers. While such an approach has general appeal, it fails to take into consideration the difficulties of substantially improving the

math education preparedness of the huge number of teachers who teach math.

4. Require that math instruction be given more minutes of school time per day and per school year. One approach to doing this is by requiring that the school day be longer and/or that the school year be longer.
5. Related to (1), develop Computer-Assisted Learning or HIICAL math materials and make them widely available. Such materials can incorporate a number of important research-based aspects of the theory and practice of teaching and learning math. For example, instruction can be individualized so that it is appropriate to the math developmental level and the content/maturity levels of the learner. Moreover, if students are learning math in a HIICAL environment, then they have computers available to do and use the math they are learning. They also have computers available for information retrieval and for review of and further instruction in topics that they have studied in the past. Thus, this approach can facilitate the thorough integration of ICT into the content and assessment of math education.
6. Educate students to gradually take an increasing level of responsibility for their own learning. As students mature, the personal responsibility level is increased. This approach requires providing students with good opportunities to learn and suitable feedback mechanism. Some of the ideas of this approach are built into the computer games that so many students like to play. Such games are often very challenging. They provide a combination of intrinsic and extrinsic motivation that leads to learners focusing a tremendous amount of learning effort.

Since new approaches are always in the process of being developed and implemented, it is difficult to predict what the future results will be. However, the past 40 years of such activities have not brought us significant improvement.

Researchers talk about the fidelity of implementation of research-based changes in education. It turns out that it is very difficult to achieve a high level of fidelity—implementation of a quality equal to what was achieved during the research. In most situations, high quality implementation requires high quality teachers, school administrators, and others who are committed to making significant changes. The needed lengthy and ongoing staff development and support is seldom available.

The items (5) and (6) on the list are gradually receiving more attention from researchers and funding agencies. As noted in Gersten (2002), math education research points to the value of providing good feedback to students as to how well they are doing. It also supports providing positive reinforcement for good performance. My personal opinion is that the approaches described in (5) and (6) are most likely to lead to significant improvements in math education.

Final Remarks

Perhaps the most important thing to understand about math education reform is that it is complex and controversial. Progress in brain science and in the field of computers and information science contribute to this complexity. Lots of people feel that our math education system needs to be changed. Different stakeholder groups have widely varying opinions on the types of changes that will produce an increased level of mathematical proficiency in our students.

The complexity of needed changes, along with the difficulties of achieving widespread high fidelity implementation, suggest to me that ICT will play a major role.

K-8 School Applications

4.1 Pick a simple math exercise that is appropriate to the math level of your students. For example, the exercise might we, "What is two plus three?" at a first grade level. After you students agree on an answer, carry on a discussion using questions such as: A) How do you know that this is a correct answer? B) Is there more than one correct answer? and C) How would you go about changing the mind of someone who thinks that this is a wrong answer?

4.2 Take a careful look at a math unit that you have taught or are preparing to teach. Think about what you want your students to gain in *conceptual understanding, procedural fluency, strategic competence, adaptive reasoning, and productive disposition.* Analyze the math unit from the point of view of how it contributes in these five different areas.

4.3 Ask your students if they can think of a math problem that has more than one right answer. The goal is to lead the class to find examples that are appropriate to their current level of understanding of math. You may need to provide a first or second example before the class is able to generate additional example. First graders can deal with, "Find two counting numbers that add up to six. Somewhat older students can deal with, "Find two counting numbers that multiply together to give 12." and "Using the unit squares, make a rectangular pattern whose area is 24."

Activities for Self-Assessment, Assignments, and Group Discussions

4.1 As a K-8 teacher, you will likely encounter both the back to basics approach to math education and the new-new math approach to math education. Think about how easy it is to fall back into the mode of teaching the way you were taught (thus, revert to a focus on computation and the other basics), versus learning and teaching a new-new math curriculum. Spend some time making a list of topics and ideas that you feel are new-new math, and a list of topics and ideas that you feel are stressed by the back to basics movement.

4.2 Review the "what is math" quotations given in this chapter. Think about which (if any) of the ideas in these quotations can be integrated into K-8 school math. Think about this from the point of view of, "The way the twig is bent is the way the tree will grow." Argue for and against the idea that K-8 school math should place much less emphasis on paper and pencil computation and much more emphasis on topics that lay a different type of foundation for students as they continue to study math in middle school, high school, and beyond.

4.3 Name one big and important idea from this chapter that you are apt to remember and make use of as a teacher of math. What is it about this idea that resonates with you and is likely to stay with you?

Chapter 5

Mathematical Maturity

Be the change you want to see in the world. (Mahatma. Gandhi)

The future depends on what we do in the present. (Mahatma Gandhi)

One of your goals as a teacher is to help your students increase their levels of expertise within the various disciplines you teach. To be an effective math teacher, for example, you need an appropriate balance of math content knowledge and math maturity as you help your students to gain both increasing math content knowledge and skills, and increasing math maturity. You also need both general pedagogical expertise and math-specific pedagogical expertise.

Cognitive Maturity

Piaget's research and cognitive development scale gives us a "broad strokes" picture of how nature and nurture combine as a child grows older in an environment containing informal and formal education. Over the years, many people have added to Piaget's work and have helped to develop instruments that are more finely calibrated than Piaget's 4-level scale.

An example is provided by the Columbia Mental Maturity Scale (CMMS, n.d.) published by The Psychological Corporation. Quoting from their Website:

> **Description:** The Columbia Mental Maturity Scale (CMMS) is an individually administered instrument designed to assess the general reasoning ability of children between the ages of 3 years, 6 months to 9 years, 11 months. The CMMS consists of 92 pictorial and figural, classification items arranged in a series of eight overlapping levels. Each of the eight levels contains between 51 and 65 items that are appropriate for a specific chronological age.

Cognitive maturity includes components such as judgment, associating cause and effect, rational behavior and decision making, and abstract thinking. Quoting from Healthy Futures (2005):

> The adolescent years are the period of time during which a person grows from puberty to cognitive maturity. This period extends well past the teen years. In fact, most college students are still adolescents. The purpose of this paper is to discuss the data proving that—of physical, mental, and cognitive maturity—**it is cognitive maturity that develops last, usually not reaching completion until the mid-twenties.** [Bold added for emphasis.]

It is appropriate to consider both a person's overall level of cognitive maturity, and also the person's cognitive developmental level in various disciplines or in various Multiple Intelligence areas. Thus, a student might have a relatively low or a relatively high level of mathematical; maturity compared with other students of approximately the same age. Piaget was particularly interested in math cognitive development.

Figure 5.1 provides results for Google searches on different types of maturity. Notice music, math, and art hold down positions 3-5 in this list. Compare these results with the bottom five on the list. While it is reasonably common to talk about science maturity or business maturity, it is uncommon to write about maturity is specific disciplines in science such as biology, chemistry, or physics.

Google Search Expression	Hits on 5/8/06
"literary maturity"	2,790000
"cognitive maturity" OR "mental maturity"	86,000
"music maturity" OR "musical maturity"	71,000
"math maturity" OR "mathematical maturity" OR "mathematics maturity"	61,000
"art maturity" OR "artistic maturity"	52,000
"business maturity"	26,000
"cognitive maturity"	24,000
"science maturity" OR "scientific maturity"	12,000
"writing maturity"	431
"reading maturity"	282
"chemistry maturity"	33
"accounting maturity"	31
"biology maturity"	27
"physics maturity"	24
"economics maturity"	19

Figure 5.1 Google searches on various types of maturity.

Figure 5.2 combines figure P1 with the math row of the above table.

Google Search Expression	Hits 5/10/06
"applied math" OR "applied mathematics"	50,200,000
"pure math" OR "pure mathematics"	5,450,000
"computational math" OR "computational mathematics"	3,090,000
"math maturity" OR "mathematical maturity" OR "mathematics maturity"	61,000
"math maturity" OR "mathematical maturity" OR "mathematics maturity" AND "elementary school" OR "middle school" OR "secondary school" OR "high school"	14,100
"math maturity" OR "mathematical maturity" OR "mathematics maturity" AND "applied math" OR "applied mathematics"	901
"math maturity" OR "mathematical maturity" OR "mathematics maturity" AND "pure math" OR "pure mathematics"	575
"math maturity" OR "mathematical maturity" OR "mathematics maturity" AND "computational math" OR "computational mathematics"	101

Figure 5.2 Google searches of math maturity within various math sub disciplines.

Notice that about a quarter of the math maturity hits are at the precollege level. Often the term math maturity is used in these documents without being defined. Occasionally, the use gives some hint of a definition. For example, here is a quote from a Website describing the Courant Institute's festival of math and science for high school students Saturday 25 March 2006 at New York University.

> We have developed a color grading system in an attempt to **indicate the pace and mathematical maturity required of each class.** Green means that everybody in the class should be able to follow. **No mathematical maturity is required beyond that acquired in high school.** Black means the class will move at a fast pace, and **students should have a high level of mathematical maturity.** Experience with abstract mathematics beyond the high school level (eg math camps, college courses, competition preparation, etc) is highly recommended. Blue and purple are somewhere in between. Note that this system is just an approximation, and the actual level of each class depends on the teacher. [Bold added for emphasis.]

Expertise in Math Content and Math Maturity

The main focus in this chapter is on math maturity. However, there is no fine dividing line between math content and math maturity. This is illustrated using a Venn diagram in figure 5.3.

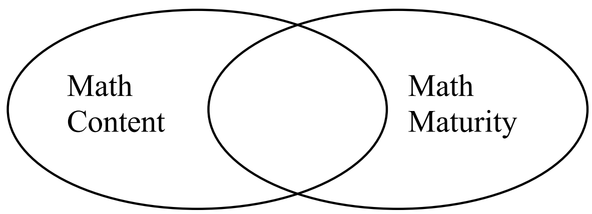

Figure 5.3. Math content and maturity overlap.

A student's level of expertise may well differ in math content and math maturity. This is illustrated for Student A and Student B (S-A and S-B) in figure 5.4.

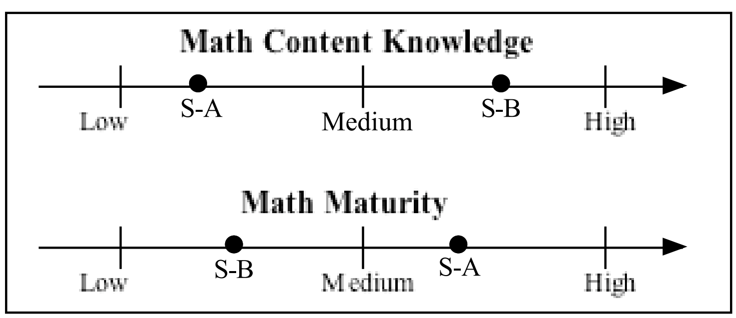

Figure 5.4 Two students (S-A and S-B) on Content and Maturity scales.

A person may be at substantially different levels on these two scales. An appropriate balance between the two scales for one person may not be appropriate for another person, since it depends on interests, abilities, goals, and so on. My personal opinion is that our math education system places much more emphasis on math content than on math maturity. Likely, this is because it is easier to teach content and assess learning of content than it is to teach and assess for increasing maturity. This is a challenging area of math education research.

You may wonder what research-oriented math educators do. One answer is that they formulate hard math education research questions that have not been previously answered, and they attempt to answer them. Consider the challenge of doing research in the area of balance between a student's math content knowledge and math maturity. Do we have a good definition of math content knowledge and good measures of a student's math content knowledge? Do we have a good definition of math maturity and good measures of a student's math maturity? What might we mean by saying that for a particular student, the student's math content knowledge is appropriately in balance with the student's math maturity? What types of instructional interventions do we have available that lead to relatively precisely measurable increases in math content knowledge or in math maturity? As you can see, this is a complex and challenging area of research.

What I find particularly interesting is that ordinary, everyday math teachers are expected to take appropriate classroom action in this content vs. maturity area, even though the needed research has not been done. In that sense, each teacher is doing *action research* to determine what works best for them and their students (Action Research. n.d.).

The same type of analysis can be applied to students learning arithmetic procedurals and students learning other math procedures. What is procedural thinking and what is computational thinking? How do we make useful measurements of a student's progress in procedural or computation thinking from a math education or computer education point of view? How does the continued rapid improvement in computer systems affect our math education goals?

Math Content

There is considerable agreement about the scope and sequence of PK-12 math education content in the US. At the elementary school level, for example, a modest number of textbook series capture most of the market. This also holds true at the secondary school level and on into higher education. Clearly, one measure of a person's progress toward increasing math content expertise is the level of coursework that has been completed, the grades received in these courses, and the quality and rigor of the coursework.

Math can be learned through other ways than just taking courses. Moreover, there is a large amount of math that is not included in the commonly available coursework. Although a modest number of textbook producers tend to dominate the market, there are many other materials available. Moreover, many teachers do not rigorously follow the textbooks adopted by their school districts. However:

> Shadow studies that track teachers' activities have shown that between 80 and 90 percent of classroom and homework assignments are textbook-driven. Which suggests that the Big Four textbook publishers—McGraw-Hill, Harcourt, Houghton Mifflin, and Pearson have swallowed other publishing companies and made them imprints—have established a *de facto* national curriculum. (Jones, 2000)

Finally, (long pause, drum roll), we need to remember that many students forget most of the math content they have "covered" in their math courses. Through introspection, you can decide to what extent you have forgotten much of what was covered in some of your high school math courses, such as the geometry course that you likely had.

Teachers of math tend to be driven by the need to "cover" the curriculum, to "get through" the book and the planned lessons. They do this even though they know that students will forget most of what is covered. As I reflect about this situation, I tend to feel embarrassed about much of the teaching that I have done over the years.

Components of Math Maturity

The term math maturity is widely used by mathematicians and math educators. For example, a middle school teacher may say, “I don’t think Pat has the necessary math maturity to take an algebra course right now.” It is clear that the teacher is not talking about Pat’s math content knowledge. Probably Pat has completed the prerequisite coursework. Perhaps Pat is weak in math reasoning and thinking, tends to learn math by rote memorization, has little interest in math, and shows little persistence in working on challenging math problems.

Perhaps the dominant component in the literature of math maturity is “proof” and the logical, critical, creative reasoning and thinking involved in understanding and doing proofs. The following list contains this and some additional components of math maturity. An increasing level of math maturity is demonstrated by:

1. An increasing capacity in the logical, critical, creative reasoning and thinking involved in understanding and doing proofs.
2. An increasing capacity to move beyond rote memorization in recognizing, posing, representing, and solving math problems. This includes transfer of learning of one’s math knowledge and skills to problems in many different disciplines.
3. An increasing capability to communicate effectively in the language and ideas of mathematics. This includes:
   A. Mathematical speaking and listening fluency.
   B. Mathematical reading and writing fluency.
   C. Thinking and reasoning in the language and images of mathematics.
4. An increasing capacity to learn mathematics—to build upon one’s current mathematical knowledge and to take increasing personal responsibility for this learning.
5. Improvements in other factors affecting math maturity such as attitude, interest, intrinsic motivation, focused attention, perseverance, having math-oriented habits of mind, and acceptance of and fitting into the “culture” of the discipline of mathematics.

Discussion of Some Math Maturity Ideas

This section contains brief discussions of a variety of ideas related to math maturity. Notice that the list in the previous section did not name any specific math content. The focus was on things that things that people do in the discipline of mathematics, such as solve problems, make proofs, communicate, and learn. In essence, these ideas about maturity apply to any discipline.

Logical/mathematical Proof

One of the most fundamental ideas in math is that of proof. The very precise language of mathematics makes it possible to make very precise (mathematical) statements and then to give arguments as to whether the statements are true, false, or undecidable. Such arguments (proofs) can be communicated to other people who can attempt to verify their correctness.

Many students first encounter the idea of a formal proof when they take a high school geometry course. Math department faculty in higher education argue about how proof-oriented a freshman calculus course should be. Part of the argument is whether the course has enough focus on this aspect of math maturity so that students will have the needed math maturity for the next high level of math courses.

Less formal proofs are often built into the math curriculum starting at the lowest grade levels. For example, perhaps a student provides a very quick response to the question, what is 15 + 17? When asked to explain, the student says: "I know that 16 + 16 is 32. I can see that 15 + 17 is just the same as 16 + 16. You just add one on to the 15 and take one off the 17. This argument may well be convincing to the teacher and other students. As a follow-up to this, perhaps a student will ask, "How do you know that 16 + 16 = 32? The student might respond with a singsong response of 2 + 2 = 4, 4 + 4 = 8, 8 + 8 = 16, 16 + 16 = 32. The student "knows" that 16 + 16 = 32 because this is a memorized fact with memory recall aided by music.

Another student might respond with a correct answer and note that," I know that 15 and 15 are 30. The answer must be 2 more than this, because 17 is 2 more than 15." Again, this builds on a memorized or easily reconstructed fact, and some understanding of the number line.

Think about the understanding of the number line that the student displayed in modifying the problem in to an equivalent, but easier problem. Was this an explicitly taught part of the curriculum, was it merely illustrated a few times in different situations, or did the student discover this on his or her own? Increasing math maturity is shown by generalizing from example or by making up one's own methods.

In new-new math, informal proofs, explaining how one knows an answer is correct, solving a problem in more than one way, and so on have become an important part of the curriculum. That is, the "proof" component of math maturity is gradually being embedded into the math curriculum at the lowest levels and tested at the state and national levels.

Problem Solving and Proof

There is no easy dividing line between making a proof and solving a problem. If the problem solving includes "show your work and reasoning, and check your answer," then in essence the student is being asked to prove that the sequence of steps used to solve the problem is a correct sequence, and that each step was done correctly.

Much the same type of analysis holds for computer programming. A computer program is like a proof. Testing a computer program is a process of giving arguments (often based on test cases) that the program is correct. Computer scientists have also made progress in developing ways to prove that a program is correct, somewhat in the same manner that one proves a theorem.

Chapter 3 this book gave a formal definition of *problem*. At an informal, intuitive level, a problem is a challenge, a situation in which a solution or a solution process is not immediately evident. This type of definition means that something may be a problem for one person and not a problem for another person. It also means that through study and practice, something that began as a problem is can be reduced to being an exercise and is no longer a problem.

Word problems (story problems) are used throughout the K-8 math curriculum. There are a variety of reasons for this. One is to have students increase their capabilities to read and understand mathematical problem situations, formulate a well-defined math problem, solve the math problem, and then see if this work resolves the original problem situation. One sign of increasing math maturity is increasing capability to carry out such math activity. Notice how success in this endeavor requires being able to read with understanding in both the math content area and in the content area of the problem.

Many teachers and students often work to defeat the purpose of word problems. They think of the goal in a word problem as being to get an answer (or, "the" answer), rather than to learn. Thus, the students memorize things such as, "in a word problem, *of* means *times*."

Thought and Language

A 5/8/06 Google search of the term *"mathematical thinking"* produced more than 600,000 hits. As with any discipline, mathematics has its own particular ways of thinking and communicating. Formal math education is designed to help students get better at thinking mathematically. An excellent introduction to this topic is given in Math Forum (n.d.; The teachers role.). Quoting from the article:

> Within the mathematics education community there is strong interest in the use of discourse for teaching and learning mathematics (NCTM, 1991; Atkins, 1999; Schifter, 1996). The teacher's role is described in broad terms as facilitative, to include listening carefully to students, framing appropriate questions, and mediating competing perspectives. Students are expected to develop problem-solving skills: defining problems, formulating conjectures, and discussing the validity of solutions. Stigler and Hiebert (1998) report similar roles for teachers and students in mathematics classrooms in Japan, where mathematical discourse is an integral part of instruction.

Gary Marcus (2004, p. 124) indicates that thought and language are only loosely connected. Many mathematicians and other people clearly develop and make use of mental representations (mental images, mental pictures) that are not words. For example, Albert Einstein, when describing his discovery of special relativity said:

> Words and sentences, whether written or spoken, do not seem to play any part in my thought processes. The psychological entities that serve as building blocks for my thoughts are certain signs or images, more or less clear, that I can reproduce and recombine at will. (Marcus, 2004, p219.)

However, precise mathematical communication with oneself and with others is an important component of increasing math maturity. The process of communicating with oneself (thinking) is not necessarily the same as the process of communicating with another person. Examples of this are easily found as one works with students who are Talented and Gifted in math. They will often arrive at correct solutions to problems without being able to explain in words how they solved the problem.

One of the teaching techniques that math teachers employ is to have their students keep math journals and write in them on a regular basis. The goal is to get students to think about the math they are studying and to practice written communication about this thinking and math.

An earlier part of this book introduced the idea of *computational thinking*. This idea is important in the discipline of computer science, and it is also important in the overlap between computer science and each other discipline. Since math and computer science have a large overlap, computational thinking is an important part of math. Computational mathematics is already established as an important component of mathematics. Computational thinking is not only an important aspect of computational mathematics, it is also important in both pure and applied mathematics. Thus, from a math education point of view, computational thinking needs to be integrated into mathematical thinking, and it is an important component of math maturity.

Understanding Understanding

The subsections given above include ideas such as understanding a proof, reading with understanding, communicating with understanding, understanding the meaning of an answer produced by a problem-solving process, and so on. Perhaps the term *understanding* is being overused?

At a faculty meeting a few years ago, I listened to faculty in my College of Education arguing about whether the term *understanding* was an appropriate one to use in specifying goals and objectives in a lesson plan. The faculty were somewhat split on this issue, but the majority felt that it was not appropriate to write objectives such as, “Students will understand …” Those against using the term *understand* argued that objectives need to be measurable, and that it was difficult or impossible to say what is meant by and to measure understanding. They argued that objectives need to be much more specific.

It is clear that one can learn some things with little or no conscious understanding. For example, when I was a child, I had trouble learning to tie my shoes. I received instruction (over and over again) on how to tie my shoes. Eventually, after much trial and error, I learned to tie the type of bowknot that is usually used in shoe tying.

I did not learn why a bowknot is used, rather than some other type of knot. I did not learn that there are many different kinds of knots and that they have a variety of uses and characteristics. I did not learn anything about friction. I did not learn that a bowknot is useful in tying bow ties and in decorating packages. I learned little about symmetry, although I am sure that I was told that the loops should be about the same size.

Now, consider a simple math example. Suppose I ask you, “What is three plus five?” Probably you say “eight” in a stimulus-response manner, with little or no conscious thought. Suppose I then ask you to explain why three plus five is eight. Aha! Now I am probing your understanding of the number system you are using and the meaning of the word *plus*. I am looking for understanding that can be applied to doing computations where you have not memorized an answer. I am looking for understanding that allows you to detect errors in your memory as well as errors in using memorized processes.

Learning to Math, and Mathing to Learn

Reading math obviously requires having some knowledge of the math content being read. However, there are some aspects of reading math that are independent of any particular math knowledge and that endure even as one forgets much of the math one has learned. They are a blend of math content and math maturity. An analogy might help. When you are doing a paper and pencil calculation, the smallest error can lead to an incorrect result. When reading math, the smallest misunderstanding can lead to an overall lack of understanding of what one is reading. Math writing tends to be very precise and concise. Terms such as line, line segment, variable, function, equation, and quadrilateral all have precise meanings.

I am a relatively slow reader. When I am reading a novel, I read 30 to 40 pages an hour. I tend to read newspaper and magazine articles at approximately this same speed. However, when reading a math textbook or research article, I may end up reading only 1/10 to 1/5 as fast. Indeed, I may puzzle over a single paragraph for an hour or more. Much depends on how familiar the material is.

It is easy to think about the ideas of learning to read and reading to learn. It is much harder to understand what might be meant by learning to math and then mathing to learn.

Learning to math includes learning to read, write, speak, and listen (in math) with understanding. However, there is much more than just such math communication skills. Learning to math includes learning problem posing, problem representation, problem solving, theorem posing, and theorem proving. Learning math includes learning to do math.

Mathing to learn includes mathing to learn math. However, math is part of the content of many other disciplines. Mathing to learn includes mathing to learn and do the math inherent to non-math disciplines. The area of probability and statistics provides an excellent example. Many disciplines make use of probability and statistics. Without an appropriate level of probability and statistics, much of the literature in many non-math disciplines cannot be read with understanding. Similar statements hold for graphing and graphical representations of data.

Problem Posing and Question Asking

Posing math problems and asking math questions constitute one of the most important topics in the math maturity list, and this topic is often overlooked in the math curriculum. In January 2004 the NCTM issued the following call for papers for an October 2005 Focus issue of Teaching Children Mathematics:

> The Editorial Panel of *Teaching Children Mathematics (TCM)* is seeking manuscripts that discuss or exemplify the role of problem posing and problem solving in the pre-K–6 mathematics classroom. The importance of this focus topic is reflected in NCTM's *Principles and Standards for School Mathematics*, which calls for teachers to regularly ask students to pose and solve interesting problems based on a wide variety of situations. By highlighting problem posing and problem solving, the Editorial Panel aims to provide teachers and teacher educators with resources to assist in their efforts to integrate problem posing and problem solving in the pre-K–6 mathematics classroom. Although problem posing and problem solving go hand in hand, manuscripts that specifically address problem posing are welcome. Accessed 4/26/06: http://my.nctm.org/eresources/article_summary.asp?URI=TCM2004-01-253a&from=B.

Of course, posing problems and asking questions are an essential component of every discipline.

> In any discipline, it is essential to help students understand our ignorance. They should come to appreciate the range of questions that remain open and, most importantly, the fact that countless interesting questions have yet to be thought of. Such an understanding is an invitation to join in the discussion. **When teachers present mathematics as a predetermined set of facts to be transmitted, the implicit message is that students are separate from those who created the mathematic** (Problem Posing, n.d.). [Bold added for emphasis.]

Problem posing is a key component of becoming an independent learner. One poses problems and questions of personal interest, and then one seeks answers, driven by intrinsic motivation.

> Once you have learned how to ask relevant and appropriate questions, you have learned how to learn and no one can keep you from learning whatever you want or need to know. Neil Postman and Charles Weingartner. *Teaching as a Subversive Activity*

Learning to pose and/or recognize math problems and math questions in "real world" and school settings contributes to understanding of math and transfer of learning of one's math knowledge. There is a substantial amount of literature on math problem posing that can be accessed from the Web. Using the search engine Google to search *mathematical OR math "problem posing"* produced over 40,000 hits on 5/27/06. The literature indicates that math problem posing has been extensively studied, can be used at all grade levels and in college, can be an important component in a Math Methods course, and is a challenge to teachers.

As an example of this challenge, many discussions about what is mathematics include a statement about finding patterns. To expand on this a bit, think about the mathematics involved in finding a possible pattern, describing the pattern, testing if the description seems to be accurate, conjecturing that the description is accurate, and proving that the description is correct. Perhaps I am a student at an early level in grade school. I have just learned about odd and even integers. In playing around, I add some pairs of odd integers and see a pattern that each time the sum is an even integer. I conjecture that the sum of two odd integers is always an even integer. But, I may lack the wherewithal to create a convincing proof of this.

Now, think about my teacher. Does my teacher have an appropriate math pedagogical knowledge, math content knowledge, and math maturity to facilitate my exploration of this topic, to provide feedback on the correctness or incorrectness of steps I am taking to "prove" my conjecture, or to actually construct a proof that will be convincing to students in my classroom?

Much of the literature on math problem posing focuses on students developing word problems that are suggested by a particular environment or by a particular math calculation. Liping Ma (1999), for example, based part of her doctoral research on asking elementary school teachers in the US and China to create a word problem that is solved by the calculation (1 ¾) ÷ (1/2). See if you can do this calculation and if you think of a "real world" problem in which it is desirable to carry out this calculation. This type of problem-posing activity can be used with any computational procedure students are studying.

Final Remarks

Think about uses that you make of math in a typical day. Your list might include telling time, estimating or measuring distances, counting a variety of things (such as calories or carbohydrates), doing exact or approximate arithmetic calculations, spending and keeping track of money, using your mental map of a town in order to drive from one locations to another, telling a friend how to drive to where you live, and so on. Here are a couple of interesting ways to think about your list:

1. Which of the uses on your list were learned in school, and how did you become skilled at transferring this school learning to settings in your everyday life?
2. Which of these uses did you learn outside of school (perhaps from other people, by discovery, by reading), and what does this tell you about your ability to learn math-types of things outside of formal schooling?
3. How is your list similar to and different from the lists your colleagues would likely create, and how do such differences get taken into consideration in the math curriculum, instruction, and assessment in the K-8 school?

The heart of math maturity and math content is being able to use your math knowledge and skills to deal with the types of math-related problems and tasks that you encounter. If your life and career depend heavily on "school math," you will build a working knowledge of this school math and it will become a part of your everyday life. You will develop the math-related knowledge, skills, and habits of mind that are important to you in this type of everyday life.

On the other hand, if much of the math that you studied in school has little use in your everyday life, then you will likely forget most of that content. Your math-type knowledge, skills, and habits of mind will grow in the areas and types of uses that are useful to you in your everyday life.

K-8 School Applications

5.1 Once a week, at the beginning of the math instruction period, ask your students to give examples of any use they have made (outside of math periods) of the math studied in the past week. Younger students can do this orally, in a whole class discussion. Older students might write about this in their math journals.

5.2 This is for students near the end of the first grade, and older. Ask your students, "Which are you better at—reading, or math? Explain why you gave the answer you did." After this discussion has gone on for a period of time, ask your students to talk about their thoughts and feelings concerning word problems in math. Look for insights that you feel represent increasing understanding and maturity.

Activities for Self-Assessment, Assignments, and Group Discussions

5.1 Think about the mathematics instruction you received before you started college and while in college. Focus specifically on those aspects of your math education that seemed to be designed to increase your math maturity. Name some of these activities and analyze their effectiveness. For example, have you received specific instruction on how to read math, how to learn math, and how to retrieve math information from reference books and the Web?

5.2 Two of the Big Ideas in math are *variable* and *function.* What do these two words (concepts) mean to you? What sort of mental model, picture, or idea pops into your conscious working memory when you think about the term *variable* or the term *function*? In what sense is each a part of your math content knowledge and in what sense is each a part of your math maturity? It might help you in your thinking if you make a list of times or situations in your everyday life where you make use of these two concepts in a math-related manner. For example, you might say, "I've got so many balls in the air, I don't know what is most apt to happen." Roughly speaking, this is a statement about dealing with a lot of variables and how they relate to each other.

5.3 Make up some questions that you feel are appropriate to use with students at a particular grade level, and that are designed to help assess the math maturity of such students. Try your instrument with some students (probably in a one-on-one setting) and report on the results.

Chapter 6

Intelligence

Intelligence is quickness in seeing things as they are. (George Santayana)

The real problem is not whether machines think but whether people do. (B.F. Skinner, *Contingencies of Reinforcement*, 1969)

Historically, the study of the human brain (one of a person's organs) and the study of the human mind (think of the mind as a product of the brain) have been distinct disciplines. Computer-oriented people tend to think of the brain as hardware (they call it wetware) and the mind as software.

The study of the mind is currently part of the field of psychology, while the study of the brain is part of the discipline of neuroscience. However, in recent years, the mind and brain disciplines have begun to merge, in a discipline called cognitive neuroscience.

> The Cognitive Neuroscience Society (CNS) is committed to the development of mind and brain research aimed at investigating the psychological, computational, and neuroscientific bases of cognition. The term cognitive neuroscience has now been with us for almost three decades, and identifies an interdisciplinary approach to understanding the nature of thought. Retrieved 5/27/06: http://www.cogneurosociety.org/content/welcome.

Jacques Hadamard (1865-1963) was a prolific and well-respected research mathematician and teacher. In one of his books, he explored the working of the mathematical mind (Hadamard, 1945). In the first chapter, while talking about the difficulty of this task, he notes:

> That difficulty is not only an intrinsic one, but one which, in an increasing number of instances, hampers the progress of our knowledge: I mean the fact that the subjects involves two disciplines, psychology and mathematics, and would require, in order to be treated adequately, that one be both a psychologist and a mathematician.

If Hadamard were alive today, he would likely be impressed by the progress that is occurring in brain and mind science, and in applications of computers to the teaching, learning, and doing math. However, he would likely argue that we still have a long way to go before we have a thorough understanding of the psychology of invention in the mathematical field. This is, indeed, a challenging area of research and development.

What is Intelligence?

Intelligence is the ability to learn and to take actions that make use of one's learning. Clearly, intelligence is not limited just to humans (NSF Press Release, 10/27/04). However, the ability of an ordinary person to learn a natural language such as English demonstrates a very high level of intelligence on the intelligence scale of all life on earth. Indeed, although students in our regular K-8 classrooms vary in intelligence, all are highly intelligent on the scale of all intelligent creatures on earth.

For many years, psychologists studying the human brain/mind have tried to measure its capabilities. Quite a bit of this work has focused on defining intelligence and measuring a person's intelligence.

> The concept that intelligence could be or should be tested began with a nineteenth-century British scientist, Sir Francis Galton. Galton was known as a dabbler in many different fields, including biology and early forms of psychology. After the shake-up from the 1859 publishing of Charles Darwin's "The Origin of Species," Galton spent the majority of his time trying to discover the relationship between heredity and human ability (History of I.Q., n.d.).

Howard Gardner (1993), David Perkins (1995), and Robert Sternberg (1988) are researchers who have written widely sold books about intelligence. Of these three, Howard Gardner is probably the best known by PK-12 educators, because his theory of Multiple Intelligences has proven quite popular in PK-12 education (Mckenzie). However, there are many researchers who have contributed to the extensive and continually growing collection of research papers on the intelligence (Yekovich 1994). The following definition of human intelligence is a composite from various authors, especially Gardner, Perkins, and Sternberg. Intelligence is a combination of the abilities to:

1. Learn. This includes all kinds of informal and formal learning via any combination of experience, education, and training.
2. Pose problems. This includes recognizing problem situations and transforming them into more clearly defined problems.
3. Solve problems. This includes solving problems, accomplishing tasks, and fashioning products.

Ways to measure intelligence were first developed more than 120 years ago, and this continues to be an active field of research and development. A very simplified summary of the current situation consists of:

1. There are a variety of IQ tests that produce one number or a small collection of numbers as measures of a person's intelligence. **Most of these tests place a high emphasis on the linguistic and mathematical/logical aspects of intelligence.** Increases in math content knowledge and in math maturity tend to contribute to scoring higher on IQ tests.
2. The "one number" approach (the general intelligence, or "g" factor) was developed by Charles Spearman in 1904, and it still has considerable prominence.
3. Many people have proposed and discussed the idea of multiple intelligences. In the past two decades, the work of Howard Gardner has helped to publicize this idea. Logical/mathematical, spatial, and linguistic are three of the eight Multiple Intelligences identified by Gardner, and they all relate to learning and using mathematics
4. Significant decreases in the intelligence of children result from starvation, lack of needed vitamins and minerals, and exposure to various poisons such as lead and mercury (Nutrition, n.d.). Significant differences also result from other aspects of a child's home environment, such as education

> level of the adults in the environment and socioeconomic status (ASCD, 2004).

Fluid and Crystallized Intelligence

While Howard Gardner and Robert Sternberg have garnered a lot of publicity during the past couple of decades for their work on intelligence, many really important ideas have been developed by other people. One of these is the idea that "g" can be divided into two major components: fluid intelligence (biologically-based) (gF) and crystallized intelligence (acquired knowledge base) (gC).

> The theory of fluid and crystallized intelligence … proposes that primary abilities are structured into two principal dimensions, namely, fluid (***Gf***) and crystallized (***Gc***) intelligence. The first common factor, ***Gf***, represents a measurable outcome of the influence of biological factors on intellectual development (i.e., heredity, injury to the central nervous system), whereas the second common factor, ***Gc***, is considered the main manifestation of influence from education, experience, and acculturation. ***Gf-Gc*** theory disputes the notion of a unitary structure, or general intelligence, as well as, especially in the origins of the theory, the idea of a structure comprising many restricted, slightly different abilities (McArdle , et al., 2002).

In casual conversations about intelligence and IQ, people tend to forget about the meaning of the "Q" in IQ. The human brain grows considerably during a person's childhood, with full maturity being reached in the early to mid 20s for most people. Both gF and gC increase during this time. Recent research suggests that gF then begins a slow decline. However, with appropriate education and cognitive experiences, gC continues to grow well into a person's 50s (McArdle et al.; 2002).

Rate of Learning Math

Research in multiple intelligences indicates that a person may have varying levels of intelligence in different areas. Moreover, a person may learn faster and better in one area than in another. As a personal example, my logical/mathematical intelligence is far above my linguistic intelligence and my spatial intelligence. I learn math much faster and better than I learn languages. My current level of performance on spatial tasks might best be described as in the feeble to modest range.

IQ tends to be a good measure of how fast and how well a person can learn. The following quoted material provide information about the rate of learning of slow versus fast learners (Gottfredson, 1998):

> High-ability students also master material at many times the rate of their low-ability peers. Many investigations have helped quantify this discrepancy. For example, a 1969 study done for the U.S. Army by the Human Resources Research Office found that enlistees in the bottom fifth of the ability distribution required two to six times as many teaching trials and prompts as did their higher-ability peers to attain minimal proficiency in rifle assembly, monitoring signals, combat plotting and other basic military tasks. **Similarly, in school settings the ratio of learning rates between "fast" and "slow" students is typically five to one.** [Bold added for emphasis.]
>
> …
>
> Half a century of military and civilian research has converged to draw a portrait of occupational opportunity along the IQ continuum. **Individuals in the top 5 percent of the adult IQ distribution (above IQ 125) can essentially train themselves, and few occupations are beyond their reach mentally.** … Serious problems in training low-IQ military recruits during World War II led Congress to ban enlistment from the lowest 10 percent (below 80) of the population, and no civilian occupation in modern economies routinely recruits its workers from that range. Current

> military enlistment standards exclude any individual whose IQ is below about 85. [Bold added for emphasis.]

Notice the bold parts of the above quote. The literature seems to suggest that students with IQs below 75 will learn two to three times or more slowly than average students, while students with IQs above 125 will learn two to three times as fast as average students. The higher IQ students are able to train themselves—to learn on their own. Obviously, there are exceptions to these general findings.

There has been some research on math learning. For example, here is a research-based statement about the rate and quality of math learning for students with mild disabilities (Cawley et al., 2001):

> The background literature of special education has long shown that students with mild disabilities (a) demonstrate levels of achievement approximating 1 year of academic growth for every 2 or 3 years they are in school (Cawley & Miller, 1989); (b) exit school achieving approximately 5th- to 6th-grade levels (Warner, Alley, Schumaker, Deshler, & Clark, 1980); and (c) demonstrate that on tests of minimum competency at the secondary level, their performance is lower for mathematics than it is for other areas (Grise, 1980). Warner et al. showed that students with learning disabilities attained only one-grade equivalent level in mathematics from Grade 7 through Grade 12.
>
> The data presented by Grise show that on a test of *minimum competency* for students in the 11th grade, 48% of students with learning disabilities passed the language/reading component, but only 16% of the students passed the mathematics component. Data from the State of New York (Mills, 2000) show that on performance on state administered mathematics assessments, 61% of 3rd-grade students with disabilities and 58% of Grade 6 students with disabilities in low socioeconomic districts met criterion whereas 90% of Grade 3 students with disabilities and 83% of Grade 6 students with disabilities in upper socioeconomic districts met criterion.

This information about math education provides strong evidence of the importance of social economic status (and its related aspects of home environment) in math education. It indicates that the rate and quality of math learning is substantially less for students with learning disabilities.

I have not found good data on upper limits on rate of learning. Here are quotes from two world-class mathematicians, talking about their high school days:

> I was involved with the math team. There was a fairly substantial amount of activity. In the beginning of high school I spent a lot of time going through old English algebra texts, and that was fantastic training for the math team. … **I remember learning all of what is now considered to be freshman [college] calculus in about two weeks or so, just sitting down and doing it** (Bloom, 1985, page 309). [Bold added for emphasis.]

> Then I started auditing math courses at the university, and suddenly, my life changed. I mean, here I was, sixteen years old, and I was taking graduate courses in mathematics and doing well, if not better, than anyone in the class (Bloom, 1985, page 313).

The vertical structure of a mathematics curriculum tends to highlight the ranges of math learning described above. At the bottom end of the scale are students who learn math relatively slowly and who are severely limited in how far they can progress in math. At the top end of the scale are students who learn math very rapidly and can progress to very high levels of mathematical knowledge and understanding.

Street Smarts and Folk Math

Robert Sternberg is well known for his triarchic model of intelligence. Very roughly speaking, he divides intelligence into the three parts: creativity, street smarts, and school smarts. Here is a somewhat different way of explaining his theory. Think of creativity as being gF, while street smarts and school smarts are two broad categories in which one develops gC. If a person is raised in a preliterate hunter-gather community living in a jungle, the person will develop a high level of “hunter-gather living in a jungle” street smarts. Since the person will not be exposed to reading, writing, and books, the person will not develop an appreciable level of school smarts.

The following is quoted from Cianciolo and Sternberg (2004, p20).

> **School’s eye views of intelligence**
>
> Shirley Brice Heath (Heath, 1983), an ethnographer, studied mismatches between notions of intelligence held in the home and those held in the school environment, and observed the effects of these mismatches on the development of language in children. In three communities, Heath discovered that as home socialization practices diverged from those valued by school environments, performance in school suffered. For example, in one community, verbal interaction typically involved highly fanciful storytelling and clever put-downs. Students from this community experienced difficulty in school, where fanciful stories were perceived as lies, and putdowns were not a valued part of the school’s social environment. In another community, parents modeled their verbal exchanges after modes of knowledge transmission in the church, which discouraged dialogue and fantasy. Students from this community excelled in verbatim recall, but experienced great difficulty when novel storytelling was required.

Research suggests similar findings in math. Quoting from Sternberg (2002) in which he argues that there is more to intelligence than just IQ:

> For example, Carraher, Carraher, and Schliemann (1985) studied a group of children that is especially relevant for assessing intelligence as adaptation to the environment. The group was of Brazilian street children. Brazilian street children are under great contextual pressure to form a successful street business. If they do not, they risk death at the hands of so-called "death squads," which may murder children who, unable to earn money, resort to robbing stores (or who are suspected of resorting to robbing stores). The researchers found that the same children who are able to do the mathematics needed to run their street business are often little able or unable to do school mathematics. In fact, the more abstract and removed from real-world contexts the problems are in their form of presentation, the worse the children do on the problems. These results suggest that differences in context can have a powerful effect on performance.
>
> Such differences are not limited to Brazilian street children. Lave (1988) showed that Berkeley housewives who successfully could do the mathematics needed for comparison shopping in the supermarket were unable to do the same mathematics when they were placed in a classroom and given isomorphic problems presented in an abstract form. In other words, their problem was not at the level of mental processes but at the level of applying the processes in specific environmental contexts.

Gene Maier (n.d.) was one of the founders of the Math Learning Center, and he served as its President for many years. One of his areas of interest is “folk math” versus school math. He notes that many people (including cabinet makers, carpenters, mill wrights, street urchins throughout the world, and lots of other people with little or no formal education) make routine use of math to help solve the types of problems they encounter on the job and in their day-to-day lives. By and large, they make use of folk math (their math-oriented street smarts) rather than school math.

Of course, many other people have thought about the ideas underlying street smarts and school smarts. For example, Jerome Bruner has had a significant impact on our educational system. Quoting from Bruner (n.d.):

> It is surely the case that schooling is only one small part of how a culture inducts the young into its canonical ways. Indeed, schooling may even be at odds with a culture's other ways of inducting the young into the requirements of communal living.... What has become increasingly clear... is that education is not *just* about conventional school matters like curriculum or standards or testing. What we resolve to do in school only makes sense when considered in the broader context of what the society intends to accomplish through its educational investment in the young. How one conceives of education, we have finally come to recognize, is a function of how one conceives of culture and its aims, professed and otherwise. (Jerome S. Bruner 1996: ix-x)

The street smarts versus school smarts analysis helps to explain why children raised in poverty (low socioeconomic environments) tend to be a year behind average in school smarts by the time they begin school. Their early childhood learning focuses on gaining street smarts knowledge and skills that help them survive and prosper in a poverty environment. Here is a brief summary of recent research in this area (ASCD, 2004):

> In general, as socioeconomic status increased, the degree of environmental influence on measured IQ scores decreased. For the most impoverished families, almost 60 percent of the variability in scores was explained by environmental differences, whereas the percentage of variation in scores attributable to genetic difference was essentially zero. For the high-SES grouping, almost 90 percent of the variance in scores was explained by genetic differences.
>
> The effect of environment on the IQ of young children can be significant, particularly for children living in poverty. As the influence of poverty decreases, the importance of environmental conditions as a limiting factor on intelligence also decreases. By addressing the environmental issues created by poverty, it may be possible to weaken the link between low socioeconomic status and poor student performance on IQ (and other) tests.

It is interesting to carry this line of thought a little further. Some children grow up in an environment that is school smarts mathematically "rich." I am an example of such a person, since both my father and mother were on the faculty in the Department of Mathematics at the University of Oregon. I grew up in a culture that placed high value on knowing and using math. This environment helped to "grow" my math oriented gF and gC.

Final Remarks

My conclusion is that one of the reasons for the relatively poor success of our formal math education system is that the math environment many of our children grow up in before they start school and the math environment they encounter both at home and in school during the early years of their formal education is not particularly "rich" in its support of school mathematical development. This idea illustrated in the following quote from an American Association for the Advancement of Science report (New, 1998). The article by Rebecca New is one of many related articles available at AAAS (1998).

> Teacher attitudes and knowledge may also account for much of the inequitable treatment of preschool mathematics, science, and technology. The field of early childhood education has struggled for much of the second half of this century to establish a reputation of professionalism. However, the knowledge base deemed essential for teachers' scientific and professional status derives almost exclusively from the child study movement and the field of developmental psychology. Few states require early childhood educators to have formal professional knowledge in the content areas as a condition of certification. Consequently, the experiences in science, mathematics, and technology that many early childhood educators bring with them to the classroom are limited by their personal histories as learners in those domains.

I also conclude that many people grow up rather weak in their folk math development, because they are not raised and taught in environments that are explicitly designed to foster cognitive growth of street smarts mathematics (folk math). Many students find that much of the school math they learn is not particularly to their outside-of-school interests and needs.

K-8 School Applications

6.1 Quite a bit of a young student's attitude toward math comes from math-related attitudes in the home environment. As you work with individual students in K-8 schools, it is helpful to have insights into the home math environment and attitudes that your students have grown up in. You can garner some of this information by engaging your students in whole class discussions about the interests in and attitudes towards math that they encounter at home. You might ask, for example, if there is someone in their home situation who particularly likes math, or someone who thinks that math is really hard, or that boys are better at math than girls (or, vice versa).

6.2 Many people find that math is fun. Indeed, most children find that math is fun while they are at the primary level, but many then find it to be less fun is they move into the upper K-8 grades. As a teacher, you need to learn what aspects of math are fun (hence, perhaps intrinsically motivating) to your students. You can do this by observing and talking to your students as you try out a wide range of different math activities that other teachers have found to be fun. Many fine examples, along with videos of teachers using the ideas and materials, can be found at the PBS Teacher Source (n.d.). There are a large number of other Websites that contain free math materials for use in the K-8 school, and many of these are "fun" oriented. For example, see Elementary School Math Center (n.d.).

Activities for Self-Assessment, Assignments, and Group Discussions

6.1 What are your personal thoughts on nature versus nurture as determiners of intelligence? What personal knowledge and experience do you have that supports your position? How does your position fit into the way you plan to work with young students?

6.2 Think about the math that you routinely use in your day-to-day life. Give examples of the folk math aspects that you see in this use of math. Give some ideas about what schools might do to increase the folk math knowledge and skills of students.

6.3 What are your personal attitudes towards math and the learning of math, and what seems to have led to these attitudes?

Chapter 7

Math Cognitive Development

> The most dangerous experiment we can conduct with our children is to keep schooling the same at a time when every other aspect of our society is dramatically changing. (Chris Dede, written statement to the PCAST panel, 1997)
>
> It is not the strongest of the species that survive, nor the most intelligent, but the one most responsive to change. (Charles Darwin)

This chapter is about cognitive development of the mind. Many of the ideas discussed in this chapter are based on the work of Jean Piaget and people who have built on his work. This chapter focuses specifically on math cognitive development.

Mind

The word *mind* has a number of different definitions. Quoting from Encarta® World English Dictionary © 1999 Microsoft Corporation:

1. the center of consciousness that generates thoughts, feelings, ideas, and perceptions and stores knowledge and memories
2. the capacity to think, understand, and reason (often used in combination)

Most definitions of mind include the term consciousness, which is a very complex idea. Many people consider the neurobiology of consciousness to be the last major unsolved problem in biology.

> Since school activities focus principally on conscious learning and behavior, the biology of consciousness will thus help to formulate credible 21st century theories of teaching and learning. But since consciousness is also integral to religious belief and cultural behavior, its relationship to educational theory will certainly be controversial. Educational leaders will obviously have to understand consciousness in order to deal intelligently with the complex issues it will raise. (Sylwester, 2004).

Piaget's Cognitive Developmental Scale

You are probably familiar with the four-stage Piagetian Developmental Scale shown in figure 7.1 (Huitt and Hummel, 1998).

Approximate Age	Stage	Major Developments
Level 1. Birth to 2 years	Sensorimotor	Infants use sensory and motor capabilities to explore and gain understanding of their environments.
Level 2. 2 to 7 years	Preoperational	Children begin to use symbols. They respond to objects and events according to how they appear to be.

Level 3. 7 to 11 years	Concrete operations	Children begin to think logically. This stage is characterized by 7 types of conservation: number, length, liquid, mass, weight, area, volume. Increasing intelligence is demonstrated through logical and systematic manipulation of symbols related to concrete objects. Operational thinking—mental actions that are reversible—develops.
Level 4. 11 years and beyond	formal operations	Thought begins to be systematic and abstract. In this stage, intelligence is demonstrated through the logical use of symbols related to abstract concepts, problem solving, thinking logically about abstract propositions, testing hypotheses, and gaining and using higher-order knowledge and skills.

Figure 7.1 Piaget's Stages of Cognitive Development

Piaget's stages of cognitive development are not specific to any particular discipline. However, a math-oriented reader of figure 7.1 might decide that Concrete Operations and Formal Operations seem to be somewhat math oriented. Piaget was particularly interested in math aspects of cognitive development. You may want to reread the material quoted from George Polya given in Chapter 3. Even at its most elementary levels, school math tends to be rather abstract.

Cognitive development is dependent on both nature and nurture. Roughly speaking, a child's progress though the first two Piagetian Developmental stages is more strongly dependent on nature, while progress in the latter two stages is more strongly dependent on nurture. However, nature versus nurture is not that simple. Marcus (2004) argues that nature and nurture are so thoroughly intertwined that is hopeless to attempt to separate them. Moreover, his arguments provide strong support for the value of high quality informal and formal education.

Although the Piagetian scale has only four labeled levels, it is a continuous scale. It is a common mistake to think of a person either being at Formal Operations or not being at Formal Operations. It is much more accurate to think of a person making progress in moving through a stage and gradually moving into the early part of the next stage. The rate of movement strongly depends on formal and informal education and the environment in which one operates. Moreover, a person may be well into Formal Operations in a one discipline such as history, and not yet have reached the beginnings of Formal Operations in another discipline such as math.

There are a variety of instruments used to measure cognitive development, and with such an instrument one can define a specific score as being the minimum score to be labeled "Formal Operations." When that is done, researchers find that only about 35% of children in industrialized societies have achieved Formal Operations by the time they finish high school (Chiappe and MacDonald , n.d.).

The following quoted materials provide additional information about the attainment of formal operations.

> However, data from similar cross-sectional studies of adolescents do not support the assertion that all individuals will automatically move to the next cognitive stage as they biologically mature. Data from adult populations provides essentially the same result: Between 30 to 35% of adults attain the cognitive development stage of formal operations (Kuhn, Langer, Kohlberg & Haan, 1977). For formal operations, it appears that maturation establishes the basis, but a special environment is required for most adolescents and adults to attain this stage (Huitt & Hummel, 2003).

> Many studies suggest our [college] students' ability to reason with abstractions is strikingly limited, that a majority are not yet "formal operational" (Gardiner, 1998).

These findings suggest that we need to take a careful look at the cognitive expectations in courses in all disciplines and at all grade levels. For example, the study of causality and the generating and testing of hypotheses are key ideas in the discipline of history and in the sciences. A ninth grade history or science course is apt to have a significant emphasis on these ideas and reasoning. However, these ideas and reasoning are part of Formal Operations. Unless they are presented and explored in a careful and appropriate Concrete Operations manner, they will be well over the heads of most of the ninth graders. This difficulty grows as one attempts to teach such ideas to still less cognitively developmentally mature students.

Geometry Cognitive Development Scale

The same sort of analysis is applicable to our math curriculum. About 50 years ago, the Dutch educators Dina and Pierre van Hiele focused some of their research efforts on defining a Piagetian-type developmental scale for Geometry (van Hiele, n.d.). Their five-level scale is shown in figure 7.2. (Notice that the van Hieles, being mathematicians, labeled their first stage Level 0. This is a common practice that mathematicians use when labeling the terms of a sequence.)

Stage	Description
Level 0 (Visualization)	Students recognize figures as total entities (triangles, squares), but do not recognize properties of these figures (right angles in a square).
Level 1 (Analysis)	Students analyze component parts of the figures (opposite angles of parallelograms are congruent), but interrelationships between figures and properties cannot be explained.
Level 2 (Informal Deduction)	Students can establish interrelationships of properties within figures (in a quadrilateral, opposite sides being parallel necessitates opposite angles being congruent) and among figures (a square is a rectangle because it has all the properties of a rectangle). Informal proofs can be followed but students do not see how the logical order could be altered nor do they see how to construct a proof starting from different or unfamiliar premises.
Level 3 (Deduction) Roughly speaking, this corresponds to Formal Operations on the Piagetian Scale.	At this level the significance of deduction as a way of establishing geometric theory within an axiom system is understood. The interrelationship and role of undefined terms, axioms, definitions, theorems, and formal proof is seen. The possibility of developing a proof in more than one way is seen.
Level 4 (Rigor)	Students at this level can compare different axiom systems (non-Euclidean geometry can be studied). Geometry is seen in the abstract with a high degree of rigor, even without concrete examples.

Figure 7.2 Van Hiele five-level developmental scale for geometry.

The van Hieles' scale is mainly a school math (as distinguished from folk math) scale. The van Hieles' work suggested that the typical high school geometry course was being taught at a developmental level considerably above that of the typical students taking such courses. Think carefully about your math experiences as you took algebra and geometry courses in high school. Did some of this coursework seem over your head ("I just don't get it."), forcing you into memorize, regurgitate, and forget mode? The same general question applies to students studying math at all grade levels. When students "just don't seem to get it," the chances are good that the content and the way it is being presented are at an inappropriate cognitive developmental level for the students.

It is evident that moving up the van Hiele geometry cognitive developmental scale requires learning quite a bit of school-math geometry. For most students, this means that progress in moving up this scale is highly dependent on their teachers and the math curriculum. The NCTM Standards list geometry as one of the major content strands, and indicate that geometry is an important part of the elementary school math curriculum (NCTM, n.d.). Thus, elementary school teachers have the opportunity to make a major contribution to helping their students increase their geometry-oriented cognitive development.

Math Cognitive Development Scale

Figure 7.3 represents my current thinking on a six-level Piagetian-type scale for school mathematics (as distinguished from folk math). It is an amalgamation and extension of ideas of Piaget and the van Hieles. The first three levels are particularly relevant to K-8 students.

Stage Name	Math Developments
Level 1. Piagetian and Math sensorimotor.	Infants use sensory and motor capabilities to explore and gain increasing understanding of their environments. Research on very young infants suggests some innate ability to deal with small quantities such as 1, 2, and 3. As infants gain crawling or walking mobility, they can display innate spatial sense. For example, they can move to a target along a path requiring moving around obstacles, and can find their way back to a parent after having taken a turn into a room where they can no longer see the parent.
Level 2. Piagetian and Math preoperational.	During the preoperational stage, children begin to use symbols, such as speech. They respond to objects and events according to how they appear to be. The children are making rapid progress in receptive and generative oral language. They accommodate to the language environments (including math as a language) they spend a lot of time in, so can easily become bilingual or trilingual in such environments. During the preoperational stage, children learn some folk math and begin to develop an understanding of number line. They learn number words and to name the number of objects in a collection and how to count them, with the answer being the last number used in this counting process. A majority of children discover or learn "counting on" and counting on from the larger quantity as a way to speed up counting of two or more sets of objects. Children gain increasing proficiency (speed, correctness, and understanding) in such counting activities. In terms of nature and nurture in mathematical development, both are of considerable importance during the preoperational stage.
Level 3. Piagetian and Math concrete operations.	During the concrete operations stage, children begin to think logically. In this stage, which is characterized by 7 types of conservation: number, length, liquid, mass, weight, area, volume, intelligence is demonstrated through logical and systematic manipulation of symbols related to concrete objects. Operational thinking develops (mental actions that are reversible). While concrete objects are an important aspect of learning during this stage, children also begin to learn from words, language, and pictures/video, learning about objects that are not concretely available to them. For the average child, the time span of concrete operations is approximately the time span of elementary school (grades 1-5 or 1-6). During this time, learning math is somewhat linked to having previously developed some knowledge of math words (such as counting numbers) and concepts. However, the level of abstraction in the written and oral math language quickly surpasses a student's previous math experience. That is, math learning tends to proceed in an environment in which the new content materials and ideas are not strongly rooted in verbal, concrete, mental images and understanding of somewhat similar ideas that have already been acquired. There is a substantial difference between developing general ideas and understanding of conservation of number, length, liquid, mass, weight, area, and volume, and learning the mathematics that corresponds to this. These tend to be relatively deep and abstract topics,

	although they can be taught in very concrete manners.
Level 4. Piagetian and Math formal operations. Van Hiele level 2: informal deduction.	Thought begins to be systematic and abstract. In this stage, intelligence is demonstrated through the logical use of symbols related to abstract concepts, problem solving, and gaining and using higher-order knowledge and skills. Math maturity supports the understanding of and proficiency in math at the level of a high school math curriculum. Beginnings of understanding of math-type arguments and proof. Piagetian and Math formal operations includes being able to recognize math aspects of problem situations in both math and non-math disciplines, convert these aspects into math problems (math modeling), and solve the resulting math problems if they are within the range of the math that one has studied. Such transfer of learning and broad application of learning is a core aspect of Level 4.
Level 5. Abstract mathematical operations. Van Hiele level 3: deduction.	Mathematical content proficiency and maturity at the level of contemporary math texts used at the junior and senior undergraduate level in strong math degree programs. Good ability to learn math through some combination of reading required texts and other math literature, listening to lectures, participating in class discussions, studying on your own, studying in groups, and so on. Solve relatively high level math problems posed by others (such as in the text books and course assignments). Pose and solve problems at the level of one's math reading skills and knowledge. Follow the logic and arguments in mathematical proofs. Fill in details of proofs when steps are left out in textbooks and other representations of such proofs.
Level 6. Mathematician. Van Hiele level 4: rigor.	A very high level of mathematical proficiency and maturity. This includes speed, accuracy, and understanding in reading the research literature, writing research literature, and in oral communication (speak, listen) of research-level mathematics. Pose and solve original math problems at the level of contemporary research frontiers. Function as an independent learner in math.

Figure 7.3. Six-stage mathematical cognitive developmental scale.

You, and each of the students you teach, are at some place on this six-level continuous scale. As you teach math, think carefully about what you are doing that will help your students move up this scale. As you study math, think carefully about how this helps you move up the scale.

Of course, this is easier said than done. Suppose a young child is presented with the addition task 3 + 5. As an adult, you have probably memorized that 3 + 5 = 8, and so can respond quickly to this computational problem. However, think about the following list of ways that this task might be completed mentally.

1. Count-all: count 3 objects, "1, 2, 3," then 5 objects, "1, 2, 3, 4, 5" then count all the objects, "1, 2, 3, 4, 5, 6, 7, 8" to get 8 objects. This approach might be based on visualizing 3 objects (such as small cubes) and 5 objects, and then mentally counting from this visualization.

2. **Count-on from first number**: count-on 5 after 3; "4, 5, 6, 7, 8." This seems to require more understanding of how numbers work than does the first approach. It seems to require some sort of mental visualization of the 5 as 5 objects, so that one can start counting after the counting number 3, moving up one in the counting sequence for each of the 5 objects.

3. **Count-on from larger**: turn the problem round and count-on 3 after 4 as "5, 6, 7. 8." This requires an understanding that 3 + 5 is the same as 5 + 3.

4. **Derived fact**: "3+5 is 2 more than 3 + 3, so it is 8." "3 + 5 is the same as 4 + 4, so it is 8." "3 + 5 is 2 less than 5 + 5, so it is 8." These are relatively

sophisticated approaches, using both a reformulation of the problem and memorized facts.

5. Known (memorized) fact: "3+5 is 8."

Many children will discover or be taught the first approach by the time they enter kindergarten. Some will discover or be taught the next two approaches during kindergarten. Some will eventually discover the strategy in the fourth approach, but many will not discover this on their own. If the formal school instruction focuses strongly on memorization, students may well not learn the strategy and become proficient in its use.

This example illustrates a major issue in math education. Suppose there is a clearly defined problem or closely related category of problems that we really want students to be able to solve. We can have them memorize (learn by rote) answers or quickly applied algorithms for arriving at answers. Or, we can take a longer route of teaching and learning for understanding. In some cases, rote memorization is "the" right approach. However, learning without understanding is quite fragile and provides a very weak framework for further learning. Each math teacher and each math learner is faced by the difficulty of achieving an appropriate balance between the two approached.

Probability and Math Cognitive Development

The van Hieles examined the secondary school geometry course from the point of view of student cognitive development in geometry. They concluded that there is a significant mismatch between student cognitive development and the typical proof-oriented course being taught at the time they developed their scale.

Other researchers have examined other parts of the math curriculum from a cognitive development point of view. A number of math education researchers have explored the issue of cognitive development and learning probability. A good example of such work is provided in Soen (1997). Quoting from that article:

> Piaget and Inhelder (1975) were the first researchers to study the development of the idea of chance in children. According to them, the concept of probability as a formal set of ideas develops only during the formal operational stage, which occurs about twelve years of age. By that age, children can reason probabilistically about a variety of randomizing devices.
>
> …
>
> Garfield and Ahlgren (1988) contend that before the teaching of probability, students must have an understanding of ratio and proportion. Students must be able to function at the formal operational level. They must have the necessary skills in dealing with abstractions.

The research relating the learning of probability and a student's level of cognitive development suggests that learning for understanding requires students to be at a formal operations level. Remember, even though age 11 or 12 is a biological time for beginning to move into formal operations, only about a third of students have achieved formal operations by the time they finish high school. Thus, research in this area tells us that K-8 students are not ready to develop a formal understanding of probability.

At the current time, the K-8 school math curriculum includes a focus on "intuitive" probability. From the point of view of math educators familiar with ideas such as those quoted above, the goal is to have students gain an intuitive (not a formal math-oriented) understanding of some "simple" probability concepts. The realization is that students are simply not developmentally ready for a formal treatment of the topic.

Unfortunately, many K-8 teachers do not have a good intuitive understanding of probability and have themselves not achieved formal operations in math. Thus, the teaching situation is often best described as "the blind leading the blind" and the results are not very good. The following quoted example falls into the category of intuitive probability (Brainchild, n.d.) and helps to illustrate my point.

Math, 3rd Grade

> Jackie's dad baked 36 chocolate chip cookies and 24 peanut butter cookies on Monday. On Tuesday, he baked 12 cherry chip, and 15 mixed nut cookies. Jackie reaches into the cookie jar and pulls out a cookie. Which kind of cookie is she least likely to pull out?
>
> A. mixed nuts
>
> B. cherry chip
>
> C. peanut butter
>
> D. chocolate chip

I believe that this problem was made up by a person with a poor intuitive understanding of probability and the real world. Consider, for example, what you know about cookie jars. One typically puts cookies into the jar through an opening in the top, and draws them out in the same manner. Thus, the situation tends to be one of "last in, first out."

The problem statement does not indicate that the cookies were placed in a cookie jar, nor whether this was done at the end of each day of baking. Cookies placed in a cookie jar do not (magically) arrange themselves in a random order. Thus, a correct answer to this problem depends on whether the Monday's cookies were put into the cookie jar before the Tuesday's cookies. It also depends on whether Tuesday's cookies were put into the jar in the order that they are mentioned in the word problem—thus, the cherry chip cookies going in first and the mixed nut cookies going in last. If the mixed nut cookies go in last, I think that the probability of reaching into the jar and drawing out a mixed nut cookie is 100%.

The problem was written to be somewhat politically correct—the father baking the cookies. But, I am concerned about the small number of cookies baked on Tuesday. That does not fit with my understanding and experience in baking cookies. All in all, I find that this example problem is poorly conceived.

It is easy to talk about an "intuitive understanding," but it is more difficult to state clearly what this might mean. Herbert A. Simon (1916-2001) was a Nobel Prize (in Economics) winning researcher and scholar who made many significant contributions in the areas of problem solving, computers, cognitive psychology, and economics. Many years ago he gave a talk at the University of Oregon, during the celebration of the addition of a new, major building in the Business School complex. In this talk he said, "Intuition is frozen analysis."

Mathematicians often talk about mathematical intuition.

> The meanings I wish, rather, to emphasize here, are those falling under the heading of *geometrical and physical intuition*. I would question whether there is such a thing as innate, "raw", untutored intuition of these or indeed of any kind. In any case, it is clear that our intuitions can be cultivated through training and practice. These may accord with tacit knowledge gained through experience,

> but, equally, one may gain intuitions that help one maneuver through subject matter that is initially highly nonintuitive. Moreover, intuitive knowledge or understanding is not simply separated from that obtained by more or less systematic reasoning—the two frequently go hand in hand, and neither is dispensable in practice (Feferman, 1998).

Notice that both Simon and Feferman make it clear that the intuition arises from (is based on) careful analysis of lots of examples or cases of the situation in which one eventually has a good intuition. There is quite a bit of literature on math intuition. . For example a Google search 5/29/06 of the term *"math intuition" OR "mathematical intuition"* produced over 32,000 hits.

The literature on mathematical intuition makes me suspicious of the idea that K-8 students can develop a good intuitive understanding of a topic such as probability on their own or through the teachings of a teacher who lacks such intuition. Part of my conclusion is rooted in the fact that people had been gambling for thousands of years (presumably some of the people developing a good intuition of the odds in various situations), but that the development of a solid mathematical footing for probability was a major mathematical achievement beginning about 400 years ago. Quoting Apostol (1969):

> A gambler's dispute in 1654 led to the creation of a mathematical theory of probability by two famous French mathematicians, Blaise Pascal and Pierre de Fermat. Antoine Gombaud, Chevalier de Méré, a French nobleman with an interest in gaming and gambling questions, called Pascal's attention to an apparent contradiction concerning a popular dice game. The game consisted in throwing a pair of dice 24 times; the problem was to decide whether or not to bet even money on the occurrence of at least one "double six" during the 24 throws. A seemingly well-established gambling rule led de Méré to believe that betting on a double six in 24 throws would be profitable, but his own calculations indicated just the opposite.

The second of the two quoted paragraphs at the beginning of this section on probability (the statement by Garfield and Ahlgren) points out another problem in elementary school math. Even at the middle school level, ratio and proportion tend to confound students. They can memorize procedures, but most gain relatively little understanding of what they are doing. Attempts to provide an "intuitive understanding" type of treatment of these topics at still lower grade levels tend to be relatively unsuccessful.

In summary, research on math and cognitive development suggest that attempts to teach these topics at the elementary school and middle school will be fraught with significant difficulties. My conversations with a very large number of teachers suggest that this research result is correct.

Math Manipulatives

My analysis of Piagetian cognitive development and mathematical cognitive development is that much of the math curriculum students encounter at the precollege level is not being taught in a manner consistent with our understanding of cognitive development. It is being taught at a level of abstraction that is too far above the developmental levels of students.

As previously mentioned, this situation tends to force the majority of students into memorize and regurgitate mode, where they develop only a modest understanding of what they are doing. Such mathematical knowledge is fragile and tends to disappear over time. It provides a very weak foundation for a student's future studying of math.

There is general agreement in the math education leadership that math should be taught in a manner that builds understanding, and that a successful math education program can and does help students to achieve understanding. Much of the current reform movement in math focuses on students gaining a higher level of understanding of the content being covered. One approach that is showing good signs of success is to make extensive use of math manipulatives. Math manipulatives fit in well with—help to bridge the gap—of students being at a concrete operations level, and gradually moving toward formal operations.

The ready availability of computers in schools has facilitated the development of computer-based manipulatives (virtual manipulatives), and these are now commonly used in school. Douglas Clements (1999) has written an excellent analysis of physical manipulatives versus virtual manipulatives. Many useful virtual manipulative materials are available free at the Website Virtual Manipulatives (n.d.).

I believe that our math education system is thinking way too small as it considers the use of physical and virtual manipulatives. Yes, indeed, such manipulative are useful in developing an understanding of important mathematical concepts. Yes, indeed, such manipulatives are quite useful in moving students from the preoperational level to the concrete operations level. But in addition, physical and virtual manipulative lie at the very core of problem solving in many different disciplines, and are key to computational mathematics.

For example, consider a businessperson developing a spreadsheet model of a certain aspect of a business. The development of such a model is an example of doing computational mathematics. The resulting spreadsheet model can be thought of as a virtual manipulative. Using this spreadsheet in posing and answering "What if?" types of questions is doing a type of computational thinking.

Consider researchers developing an appropriate shape for an airplane or a car. They develop physical models that they test in wind tunnels. Nowadays, they develop computer models that they use as they pose and answer "What if?" types of questions. Physical and virtual manipulative are routine tools of these researchers. They are aids to computational thinking.

Think about architects. In the past, they developed physical models as well as blueprints and other drawings. Now, they develop computer models. They have long recognized the value of various types of physical and virtual models (physical and virtual manipulatives) in representing and solving the problems they face.

In 1998, one of the Chemists who received a Nobel Prize did so on the basis of his work on developing computer models of molecules in chemical reactions. Over the previous 15-20 years he had developed virtual manipulatives that proved to be powerful aids to understanding and attacking certain types of problems in chemistry.

I could continue to extend the list, but perhaps the message is becoming clear. Computer models—virtual manipulatives developed through the use of computers and computational math—are now commonly used to help represent and solve problems in many different disciplines. Math educators should take this into consideration as they make use of manipulative to help students learn math. At the same time their students are learning to use manipulative as an aid to learning math, they could be learning about use of manipulative to help represent and solve problems in many other disciplines.

FOSS, Example from Science Education

Educators in each discipline are aware of the work of Piaget and other research in cognitive development. Thus, curriculum developers in each discipline pay attention to how their materials align with the cognitive development of the students who will use the materials.

The Full Option Science System (FOSS) is based on the teaching and learning philosophy of the Lawrence Hall of Science, University of California, Berkley (n.d.). The Lawrence Hall of Science is one of the world's more successful and well-known hands-on museums of science. Quoting from Foss (n.d.):

> The FOSS program is correlated to human cognitive development. The activities are matched to the way students think at different times in their lives. The research that guides the FOSS developers indicates that humans proceed systematically through predictable, describable years, and that students learn science best from direct experiences in which they describe, sort, and organize observations about objects and organisms. Upper elementary students construct more advanced concepts by classifying, testing, experimenting, and determining cause and effect relationships among objects, organisms, and systems.
>
> FOSS investigations are carefully crafted to guarantee that the cognitive demands placed on students are appropriate for their cognitive abilities.

The FOSS curriculum is based upon a combination of research in science education and years of practical experience in working with young learners. Here are a few of the key ideas quoted from the FOSS Website:

- learning moves from experience to abstractions. FOSS modules begin with hands-on investigations, then move students toward abstract ideas related to those investigations using simulations, models and readings.
- a child's ability to reason changes over time. FOSS designs investigations to enhance their reasoning abilities.
- fewer topics experienced in depth enhance learning better than many topics briefly visited. FOSS provides long-term (8-10 weeks) topical modules for each grade level, and the modules build upon each other within and across each strand, progressively moving students toward the grand ideas of science. The grand ideas of science are never learned in one lesson or in one class year.

It is interesting to compare and contrast these ideas and the FOSS approach to education with the various math curricula that are widely used in this country. The first bulleted item notes the challenge of abstraction, and emphasizes the need to move from the concrete to the abstract. The second bulleted item emphasizes working over time to enhance the growing reasoning ability of learners. The third bulleted item addresses the issue of breadth versus depth, indicating that the developers of FOSS favor depth over breadth. Math educators would do well to carefully investigate the FOSS work

Final Remarks

Five thousand years of cumulative progress in the discipline of mathematics have led to a broad and deep discipline. While math has many uses in many different disciplines, math tends to have a high level of abstraction. Thus, our math education system is caught between the need to help K-8 students develop practical, down to earth understanding and use of math, and the abstractions that underlie "higher" math such as the algebra and geometry that students will face in secondary school.

This has led to the teaching of a number of rather abstract and difficult math topics at lower and lower grade levels. My personal opinion is that a significant fraction of students regularly encounter math instruction and learning requirements that are quite a bit above their mathematical cognitive developmental levels. For such students, their main recourses are memorization without much understanding, or just giving up.

K-8 School Applications

7.1 You have an understanding of the number line. Probably your understanding is quite a bit different than that of most K-8 students. For example, you have insight into the existence of irrational numbers such as the positive square root of 2, and you can probably make a mark on a number line close to where this number lies. Your broader and deeper understanding developed over the years as you studied math and as your brain continued to develop. Think about the understanding of the number line you expect your average student to have at the beginning of the school year and then at the end of the school year for a specific grade level that interests you. Carry on a whole class discussion with your students to gain insight into their current understanding of the number line. Examine the math content you are teaching and how it relates to increasing student understanding of the number line. When appropriate, engage your students in a discussion about how the math content topic fits in with and expands their understanding of the number line.

7.2 Watch your students as they do paper and pencil arithmetic and as they make use of math manipulative to explore various math topics. Likely, you will see some students who are better at (more comfortable with) one of these activities as compared to the other. There can be transfer of learning in either direction—from manipulative to abstract symbols, or vice versa. If you see an example of this happening, point it out to the whole class and use the situation to help your students to learn to find and make use of such connections.

7.3 Talk with your students about models, such as toy cars, model airplanes, toy figurines of people and animals, and so on. Move the focus toward the similarities and differences between models and the "real thing." Then focus the conversation on mathematical models. Math modeling lies at the very heart of use of math to help represent and solve problems. Work to learn the current level of your students' understanding of this idea, and then to help them expand their understanding.

Activities for Self-Assessment, Assignments, and Group Discussions

7.1 The chances are good that you are at the Formal Operations level on the four-level Piagetian Cognitive Developmental Scale. What evidence can you provide that you are at this level? Think about where you fall on the six-stage mathematical cognitive developmental scale. What evidence can you provide that supports your conclusion? Share your insights into your mathematical self that result from this activity.

7.1 Can a teacher be an effective teacher of K-8 school mathematics if the teacher is not at Level 4 (Piagetian and Math formal operations) on the six-stage mathematical cognitive developmental scale? Present arguments on each side of this issue, as well as suggestions for a K-8 math teacher who is not at this math cognitive developmental level.

7.3 Explore and share your insights into how math manipulative fit into helping students learn math while at various states in their mathematical development. What do you know about uses of and the effects of using physical manipulatives versus virtual manipulatives (that is, computerized manipulatives)?

7.4 Develop a lesson plan in which students use math manipulatives to help learn some math ideas and, at the same time, increase their understanding of math modeling.

Chapter 8

Cognitive Neuroscience

> The illiterate of the 21st century will not be the one who can not read and write, but the one who can not learn, unlearn, and relearn. (Alvin Toffler)
>
> Civilization advances by extending the number of important operations which we can perform without thinking of them. (Alfred North Whitehead)

Cognitive neuroscience is a relatively new discipline, combining aspects of brain science and mind science that specifically focus on cognition. Research in cognitive neuroscience is helping to advance the field of education. Quite a bit of the research that is being done focuses on topics such as attention, math, and reading. Since reading is an important aspect of math, it is not surprising that students with reading difficulties often have difficulties in math. A number of articles about math disabilities are available at (Retrieved 5/28/06) http://www.ldonline.org/indepth/math. See in particular Geary (1999).

Cognitive Neuroscience in Education

Cognitive neuroscience research using brain imaging is beginning to make significant contributions to our understanding of learning and using math, although this type of research is still in its infancy. For example, by 1999 brain imaging showed different parts of the brain being used in exact calculations than being used in estimations or approximate calculations (Dehaene et al. 1999). This provides scientific evidence to support the idea that teaching students to do exact calculations and teaching students to estimate are distinct topics, and that transfer of learning between these two topics may be a challenge to students and their teachers.

> Cognitive neuroscience has emerged in the last decade as an intensely active and influential discipline, forged from interactions among the cognitive sciences, neurology, neuroimaging (including physics and statistics), physiology, neuroscience, psychiatry, and other fields.
>
> …
>
> The cross-disciplinary integration and exploitation of new techniques in cognitive neuroscience has generated a rapid growth in significant scientific advances (NSF, 2002).

As an example of cognitive neuroscience progress, chapter 6 mentioned that research on gF suggests that this component of g increases into early adulthood. A recently published longitudinal brain imaging study reports results that seem to be consistent with this gF result (Gogtay et al., 2004).

Robert Sylwester is a well-known educator and authority on how better understanding of the brain can shed light on education practices that directly impact the classroom. He writes a monthly column that is available on the Web (Sylwester, 2004). Quoting from his October 2004 article:

> John Dewey, Jean Piaget, and B. F. Skinner helped shape 20th century educational policy and practice by connecting teaching and learning to emerging cultural and scientific developments. Recent dramatic advances in the cognitive neurosciences and computer technology suggest that a similar set of creative educational theorists will soon emerge to help schools connect teaching and learning to 21st century biology and technology.

In brief summary, the discipline of brain and mind science has progressed to a level in which it can and is making significant contributions to teaching and learning.

Dyscalculia

Brain imaging has identified regions of the brain associated with different types of dyscalculia, a difficulty in learning certain aspects of math (Stanescu-Cosson et al., 2000; Pearson, 2003).

> Over the past several decades important advances have been made in the understanding of the genetic, neural, and cognitive deficits that underlie reading disability (RD), and in the ability to identify and remediate this form of learning disability (LD). Research on learning disabilities in mathematics (MD) has also progressed over the past ten years, but more slowly than the study of RD. One of the difficulties in studying children with MD is the complexity of the field of mathematics. In theory, MD could result from difficulties in the skills that comprise one or many of the domains of mathematics, such as arithmetic, algebra, or geometry. Moreover, each of these domains is very complex, in that each has many subdomains and a learning disability can result from difficulties in understanding or learning basic skills in one or several of these subdomains .
>
> As an example, to master arithmetic, children must understand numbers (e.g., the quantity that each number represents), counting (there are many basic principles of counting that children must come to understand), and the conceptual (e.g., understanding the Base-10 number system) and procedural (e.g., borrowing from one column to the next, as in 43-9) features involved in solving simple and complex arithmetic problems. A learning disability in math can result from difficulties in learning any one, or any combination, of these more basic skills. To complicate matters further, it is possible, and in fact it appears to be the case, that different children with MD have different patterns of strengths and weakness when it comes to understanding and learning these basic skills (Geary,1999).

Perhaps 5-7 percent of students have some form of dyscalculia. Early identification of dyscalculia can make a significant contribution to helping students deal with this learning disability. Symptoms of dyscalculia include (Dyscalculia, n.d.):

- Difficulty with numbers;
- Poor understanding of the signs +, -, / and x, or may confuse these mathematical symbols;
- Difficulty with addition, subtraction, multiplication and division or may find it difficult to understand the words "plus," "add," "add-together";
- May reverse or transpose numbers for example 63 for 36, or 785 for 875;
- Difficulty with times tables;
- Poor mental arithmetic skills;
- Difficulty telling the time and following directions.

Our current educational system is not good at early identification of students with dyscalculia. That is unfortunate, because early identification and a strong intervention can help a student overcome or more effectively real with this disability. The possible parallel with dyslexia (a serious reading disability) is interesting. Research in this area now strongly suggests that a strong, early intervention can lead to "rewiring" of the brain in a manner that contributes significantly to a person being able to become a fluent reader.

To do a precise diagnosis that a student has dyscalculia requires considerable knowledge and skill. However, an K-8 teacher or a parent can easily study the bulleted list given above and do a preliminary screening of students who seem to be having considerable difficulty in learning math. In addition, students identified as dyslexic should also be carefully screened for dyscalculia. (Remember the emphasis earlier in this book that math is a language.)

Dyslexia

Dyslexia is a type of reading disability. Quoting from Dyslexia (n.d.):

> Dr. Sally Shaywitz, a researcher at the Yale University of Medicine showed in 1998 that areas in the back of the brain that are usually activated when readers sounded out words are significantly *less activated* in dyslexics' brains. Moreover, areas in the front of the brain displayed more activity in dyslexics' brains than in the brains of normal readers. More recently, researchers at the University of Washington have shown that dyslexics' brains work up to five times harder than non-dyslexic brains. Girard Sagmiller, in his website called What is Dyslexia?, describes dyslexia "like running a 100-meter race. In your lane you have hurdles, but no one else does. You feel that it's unfair but you try running like the other competitors anyway."

Dyslexia has been studied much more than dyscalculia, and it may be about three times as prevalent as dyscalculia.

> It appears that many … children with RD [reading disability] also have difficulties learning basic arithmetic. In particular, children and adults with RD often have difficulties retrieving basic arithmetic facts from long-term memory. The issue is whether the co-occurrence of RD and difficulties in remembering arithmetic facts are due to a common underlying memory problem. The answer to this question is by no means resolved. Nonetheless, some evidence suggests that the same basic memory deficit that results in common features of RD, such as difficulties making letter-sound correspondences and retrieving words from memory, is also responsible for the fact-retrieval problems of many children with MD [mathematics disability]. If future research confirms this relationship, then a core memory problem that is independent of IQ, motivation and other factors, may underlie RD and at least one form of MD. (Cleary, 1999)

Attention

Attention is a large and important component of the discipline of cognitive neuroscience. A human's five senses bring in an overwhelming amount of data. The brain, at a conscious and subconscious level, pays attention to some of this data; however, it filters out and ignores most of this data.

This presents a challenge both to teachers and to students. In a schoolroom class, a student's brain is processing input from five senses, and it is thinking about lots of other things. For example, it may be sensing that he or she is hungry, bored, will have a lot of fun later in the day, would rather be listening to some good music, is worried about a recent interaction with a friend, and so on. A teacher needs to teach in a manner that catches and holds student attention, and the student needs to learn to focus his or her attention on a learning task.

Michael Posner is a world-class expert in attention. The following is from Posner & Fan (2004).

> "Everyone knows what attention is. It is the taking possession of the mind in clear and vivid form of one out of what seem several simultaneous objects or trains of thought." (James, 1890).

> However, this subjective definition does not provide hints that might lead to an understanding of attentional development or pathologies. The theme of our paper is that it is now possible to view attention much more concretely as an organ system.
>
> …
>
> We believe that viewing attention as an organ system aids in answering many perplexing issues raised in cognitive psychology, psychiatry and neurology. … **We can view attention as involving specialized networks to carry out functions such as achieving and maintaining the alert state, orienting to sensory events and controlling thoughts and feelings.** [Bold added for emphasis.]

The study of attention as an organ is now being facilitated by brain imaging technology. Researchers are beginning to understand which parts of the brain are active when a person is paying attention, or focusing attention in a particular manner. This is contributing to increased understanding of Attention Deficit Disorder (ADD) and other attention pathologies.

As a potential or current teacher at the K-8 level, you know that there are many different things that attract student attention away from the topics being addressed in class. One of the reasons that this happens is that some of the school topics are, from a student's point of view, "just plain boring." In math education, for example, this helps to explain why (very roughly speaking) most children find math class time reasonably interesting up through about the 3rd or 4th grade. During those first few years of formal schooling, math tends to contain many new, interesting, empowering, and attention grabbing ideas. After about the 3rd to 4th grade, an increasing number of students find that math class is not particularly interesting and does not hold their attention.

Chapter 9 of this book focuses specifically on computers in education. It includes a brief introduction to games in education. For more on games in education, see Moursund (2006, Games Website). One aspect of computer games is that they are attention grabbing and attention holding. At the current time, research indicates that K-8 children are spending more time playing computer games than they are watching television (Science of Mental Health, 2004). Both television and computer games can be viewed as major competitors for a student's attention! Repetitious paper and pencil drill and practice of computational algorithms does not compete well with such media.

Genetics

The past decade has seen a very high rate of progress in genetics and in decoding the human genome. We now have theory and instrumentation that helps us gain increased understanding of the human brain. We have steadily increasing knowledge of the human genome, noninvasive tools for brain imaging, and tools and skills for manipulation of individual genes. This progress has raised the nature versus nurture discussion to an entirely new level. We are gaining increased understanding of nature, and we now have the ability to change nature.

> In the coming decades, we will all collectively as a society need to decide what we think about biotechnology and what applications we are and are not willing to allow. The debates we have now, about cloning and stem cell research, pale in comparison to debates we are likely to encounter as the technology for manipulating genes advances. We are already at the point where it is possible to screen embryos for the predisposition to certain life-threatening illnesses; as we unravel more and more of the genome, we will be able to detect more and more disorders (or predispositions to disorders) well in advance of birth. Ultimately, if we so choose, we may be able to directly manipulate embryonic genomes—add a gene here, delete a gene there. The genes of a child might eventually be more a matter of choice than of chance (Marcus, 2004, p174).

We are all aware of the issue of athletes taking drugs to enhance the development and performance of their physical bodies. Perhaps you drink beverages that contain caffeine, and you know that caffeine enhances brain alertness and performance. In the coming years, there will be a steadily increasing number of drugs that can enhance brain development and performance. Thus, as a teacher, you can look forward to having to deal with issues of students who have been genetically enhanced and/or enhanced by a variety of drugs.

Chunks and Chunking

Here are three different types of human memory:

- Sensory memory stores data from one's senses, and for only a short time. For example, visual sensory memory stores an image for less than a second, and auditory sensory memory stores aural information for less than four seconds.
- Working memory (short term memory) can store and actively process a small number of chunks. It retains these chunks for less than 20 seconds.
- Long-term memory has large capacity and stores information for a long time.

Research on working memory indicates that for most people the size of this memory is about 7 ± 2 chunks (Miller, 1956). This means, for example, that a typical person can read or hear a seven-digit telephone number and remember it long enough to key into a telephone keypad. The word *chunk* is very important. When I was a child, my home phone number was the first two letters of the word diamond, followed by five digits. Thus, to remember the number (which I still do, to this day) I needed to remember only six chunks. But, I had to be able to decipher the first chunk, the word "diamond." That is, it was a combination of rote memorization and some level of understanding that allowed me to make use of this memory aid.

Long-term memory has a very large capacity, but this does not work like computer memory. Input to computer memory can be very rapid (for example, the equivalent of an entire book in a second), and a computer can store such data letter perfect for a long period. The human brain can memorize large amount of music, poetry, or other text. However, for most people this is a long and slow process for most people. By dint of hard and sustained effort, an ordinary person can memorize nearly letter perfect the equivalent of one or two books. At the current time, the Web contains the equivalent of tens of millions of books.

On the other hand, the human brain is very good at learning meaningful chunks of information. Think about some of your personal chunks such as constructivism, multiplication, democracy, transfer of learning, and Mozart. Undoubtedly these chunks have different meanings to me than they do for you. As an example, for me, the chunk "multiplication" covers multiplication of positive and negative integers, fractions, decimal fractions, irrational numbers, complex numbers, functions (such as trigonometric and polynomial), matrices, and so on. My breadth and depth of meaning and understanding were developed through years of undergraduate and graduate work in mathematics.

It is useful to think of a chunk as a label or representation (perhaps a word, phrase, visual image, sound, smell, taste, or touch) and a collection of pointers. A chunk has two important characteristics:

1. It can be used by short-term memory in a conscious, thinking, problem-solving process.

2. It can be used to retrieve more detailed information from long-term memory.

Our education system can be substantially improved by taking advantage of our steadily increasing understanding of how the mind/brain learns and then uses its learning in problem solving. Chunking information to be learned and used is a powerful aid to learning and problem solving. However, even if two people receive the same education about a topic, and use the same label for a chunk that they form on that topic, their chunks will be quite different. This is a key idea in constructivism.

Brain Versus Computer

In the early days of computers, people often referred to such machines as *electronic brains*. Even now, more than 50 years later, many people still use this term. Certainly a human brain and a computer have some characteristics in common. However:

- Computers are very good at carrying out tasks in a mechanical, "non-thinking" manner. They are millions of times as fast as humans in tasks such as doing arithmetic calculations or searching through millions of pages of text to find occurrences of a certain set of words. Moreover, they can do such tasks without making any errors.
- Human brains are very good at doing the thinking and orchestrating the processes required in many different very complex tasks such as carrying on a conversation with a person, reading for understanding, posing problems, and solving complex problems. Humans have minds and consciousness. A human's brain/mind capability for "meaningful understanding" is far beyond the capabilities of the most advanced computers we currently have.

There are many things that computers can do much better than human brains, and vice versa. Our educational system can be significantly improved by building on the relative strengths of brains and computers, and decreasing the emphasis on attempting to "train" students to compete with computers. We need to increase the focus on students learning to solve problems using the strengths of their brains and the strengths of Information and Communication Technology.

One of the key issues in studying human brains and computer-as-brains is the human brain-computer brain interface. If we go back to the time of the first computers, the interface was mainly via an electric typewriter device, punch paper tape, and punch cards. Later came display screens, touch screens, and the mouse. Now we also have voice input and output. We also have wireless connectivity and cell phones.

At the current time, some research projects are working on implanting ICT systems into people's brains. Cochlear implants and retinal implants can be considered as part of this overall endeavor, and cochlear implants are now relatively common. Brain implants have been used to help deal with epilepsy. Research is being done on creating a direct connection between a person's brain and a computer located outside the brain. For example, a volunteer in this research program is able to play a simple computer game involving movements of the curser by "thinking" up, down, right, and left.

The point I am making here is that in the past, and continuing into the future, there has been substantial research on improving human-computer interfaces. As a computer user, you likely make routine use of a mouse, video display screen, and a keyboard. In the future, we will see significant progress in building still more direct brain-computer interfaces. Improvements in the interface will have a significant impact on education. In essence, such improvements contribute to the idea of the computer as an auxiliary brain, or as a brain enhancement. The way I view it is that I and my human brain/mind, along with my computer system, can accomplish a wide range of tasks and solve a wide range of problems that I, all by myself, cannot do.

Augmentation to Brain/Mind

Reading and writing provide an augmentation to short (working) term and long-term memory for personal use and that can be shared with others. Data and information can be stored and retrieved with great fidelity. As Confucius noted about 2,500 years ago, "The strongest memory is not as strong as the weakest ink."

Writing onto paper provides a passive storage of data and information. The "using" of such data and information is done by a human's brain/mind.

Computers add a new dimension to the storage and retrieval of data and information. Computers can process (carry out operations on) data and information. Thus, one can think of a computer as a more powerful augmentation to brain/mind than is provided by static storage on paper or other hardcopy medium. The power, capability, and value of this type of augmentation continue to grow rapidly. Certainly, this is one of the most important challenges in education at the current time. For the most part, our formal educational system has yet to understand the idea of ICT as an augmentation to the mind/brain.

In thinking about chunks and learning, I see two approaches. In the first approach, a clear framework is provided. Think of the framework as scaffolding for a chunk along with a label for the chunk. One learns the framework and then fits new knowledge and experiences into the framework. In the second approach, one creates their own framework. This is less efficient initially, but perhaps more productive over the long run in the task of helping students learn to learn and to take increasing responsibility for their own learning.

To illustrate, suppose I want to know a modest amount about something that others have carefully studied. Since part of a discipline is how to teach and learn it, I decide to take advantage of this accumulated knowledge. I have the discipline taught to me by an expert teacher.

Next, suppose that I want to extend my new knowledge to my personal world and to situations not covered in the standard curriculum. In this situation, I hope that I have learned to learn on my own. I hope that I have the creativity and skill to discover, invent, find, and so on, and fit my new learning into the old framework. I hope that I can restructure the old framework so that it better fits the new and my needs.

There is one more important piece to this. Suppose that the area that I want to study is one in which computers provides powerful aids to solving its problems. Then I want my chunk to include a link to the capabilities and limitations of computers as an aid to solving the problems. I want to have the knowledge and skills to make use of this computer augmentation to my brain. The next chapter focuses on computers in education.

Final Remarks

Chapters 6-8 are interrelated. Taken together, they provide some insight into logical/mathematical intelligence and into difficulties in learning math. Research into children with learning disabilities provides clear evidence of the rate of learning math and upper limits faced by certain students. For me, this raises the question of: What about students who are not classified as learning disabled, but have similar learning difficulties, but as a somewhat lesser level?

Math educators have some insights into such difficulties. For example, some secondary schools offer a two-year sequence that covers first-year high school algebra. It has long been recognized that the theorem proof aspects of high school geometry are beyond the math maturity of many students who attempt such a course.

In summary, research into cognitive development is helping to make clear that math is often taught assuming a math cognitive developmental level that is well above what many of the students have achieved. It appears to me that this difficulty starts quite early in the math curriculum. Because of the vertical structure of the math curriculum, for many students the problem grows from year to year. In some sense, the math curriculum and math requirements in school seem designed to lead the majority of students down a path in which they do poorly and eventually develop an attitude of "I can't do math."

K-8 School Applications

8.1 Brainstorming is a very useful strategy in thinking about a problem situation. In some sense, one's brainstorming around a particular idea is like putting together some of the topics that are chunked (in one's brain) with the idea. Present your class with a problem or task related to the math they are studying. Do a whole class brainstorm to illustrate the process of brainstorming. When done, facilitate discussion about how the results relate to the problem or task at hand, and see if the class collectively can solve the problem or accomplish the task. This activity can be used a number of times over a school year. Among other things, it helps to expand an existing chunk that some or perhaps all of the students have.

8.2 Play a short-term memory game with your students. For example, hold up a picture of a geometric figure for the class to view for a few seconds. Then, each class member is to draw the figure from memory. Alternatively, invite a student to come to your desk and view a geometric figure. The student must then go to another student and tell the student how to draw the geometric figure. These activities can be more challenging by adding letters to designate various vertices or edges of the figure, so that these also must be held in short term memory. The activity tends to be easy for students if the figure is one that they know a word for, such as square, rectangle, or triangle. Then the figure is just one chunk. Chunking helps in dealing with the burden of labeled vertices and edges if they are labeled systematically, such as a rectangle with vertices A, B, C, D working counterclockwise.

Activities for Self-Assessment, Assignments, and Group Discussions

8.1 Select a math topic that you feel it is important for a K-8 student to learn. Think about this from the point of view of being a "chunk" that the student will construct in his or her brain/mind. What is a good name for this chunk? What is a good mental image or picture for this chunk? How do you expect this chunk to grow in breadth and depth over time? What are some aspects of this chunk that you expect will serve the student over a lifetime?

8.2 Think carefully about your rate and ease of learning math versus your rate and ease of learning some other discipline that is in the K-8 curriculum. How have you and the school curriculum accommodated these rates of learning during your many years of being a student?

Chapter 9

Computer Science Education and Math Education

Chapter 1 contained a short section noting the strong historical and actual overlap between the two disciplines Mathematics and Computer Science. Chapter 9 explores this topic in more detail.

Our K-8 math education system is reasonably consistent throughout the country. The Standards developed by the National Council of Teachers of Mathematics help to guide individual states, school districts, and schools as they develop and implement their math curriculum content, instructional processes, and assessment. In additional, there is a reasonable level of uniformity in the standards that teacher education programs require in the math preparation of elementary school teachers and middle school math teachers.

The International Society for Technology in Education has developed National Educational Technology Standards for PK-12 students, and has made significant contributions to improving the computer technology education of preservice and inservice teachers. At the current time, however, the computer technology preparation of most PK-12 teachers is relatively weak. On average, there is substantial room for improvement in the computer technology education of precollege students.

Computer Science

The Venn diagram of figure 9.1 is designed to illustrate that there is a substantial overlap between the two disciplines Computer Science and Mathematics.

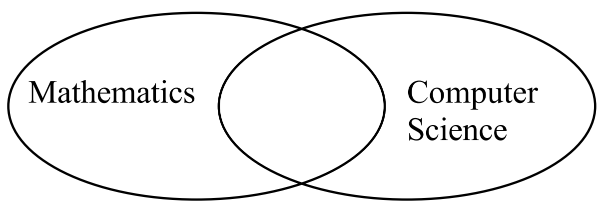

Figure 9.1. Computer Science and Mathematics have a considerable overlap.

This history of mathematics details many efforts to develop aids to computation. The counting board dates back about 2,500 years. Various forms of the abacus (a counting board with the markers strung on rods) date back as much as 2,000 years and are still in use today. The first mechanical calculators were built more than 450 years ago. Since then, there has been steady progress in the development of aids to computation.

Initially, the focus was on developing aids to quickly and accurately carry out math algorithms for addition, subtraction, multiplication, and division. The advent of the electronic digital computer substantially broadened this orientation. Computers can work with words and graphics as well as with numbers. Computers can carry out heuristic procedures as well as algorithmic procedures.

> **Computer science** is the study of the theoretical foundations of information and computation and their implementation and application in computer systems. Many diverse fields exist within the broader discipline of computer science; some emphasize the computation of specific results (such as computer graphics), while others (such as computational complexity theory) relate to properties of computational problems. Still others focus on the challenges in implementing computations. For example, programming language theory studies approaches to describing a computation, while computer programming applies specific programming languages to craft a solution to some concrete computational problem. (Retrieved 4/30/06: http://en.wikipedia.org/wiki/Computer_science.)

Chapter 1 contains a quotation Jeannette Wind, head of the Computer Science Department at Carnegie Mellon University. This department is one of the leading CS departments in the world, and is especially known for its work in Artificial Intelligence.

> Computational thinking builds on the power and limits of computing processes, whether they are executed by a human or by a machine. Computational methods and models give us the courage to solve problems and design systems that no one of us would be capable of tackling alone. Computational thinking confronts the riddle of machine intelligence: What can humans do better than computers, and What can computers do better than humans? Most fundamentally it addresses the question: What is computable? Today, we know only parts of the answer to such questions. (Wing, 2006)

Computational thinking can be thought of as an extension or broadening of the idea of procedural thinking mentioned in the chapter 4. In summary, computational thinking deals with the representation and solving of problems using human, machine, and other forms of intelligence and aids to problem solving. Algorithmic thinking deals with the development and use of algorithms that can be proven to solve a specific type of problem or accomplish a specific type of task. Procedural thinking deals with the development and use of procedures that are designed to solve a specific type of problem or accomplish a specific type of task—but that are not proven to always be successful.

Disciplines such as math and computer science are so large and complex that they are broken into components. Thus, a mathematics researcher might specialize in algebra, analysis, statistics, topology, and so on. The mathematician might specialize in the pure mathematics, applied mathematics, or computational mathematics aspects of his or her areas of specialization.

Computer science is divided into sub disciplines such as analysis of algorithms, artificial intelligence, computer engineering, databases, networks, and so on. All can be studied from a theoretical or an applied point of view.

Computer science differs significantly from mathematics in the visibility and ease of use of some of its results. Thus, even quite young children search the Web, use email, play computer games that involve very sophisticated computer graphics, and so on. Mathematics plays a key underlying role in all of these applications, but the computer and the people working in the computer field tend to get the credit. Many of the computer applications packages of software are relatively easy to learn to use at a "modest"—but useful—level.

People working in applied mathematics focus developing the theory and practice of solving math problems that tend to be of immediate use people who make use of math to help solve their problems. Generally speaking, the people using the math tend to need a fairly strong background in mathematics in order to appropriately and successfully make use of the applied mathematics.

Contrast this with applied computer science. Computer scientists and others working in applications of computer science are often able to develop the theory and practice to a level where the solution of a particular category of problems becomes highly or fully automated. Thus, we see millions of people who have never had a computer science course making relatively sophisticated uses of computers.

Many computer scientists make a distinction between Computer Science and ICT. They think of CS in much the same way that mathematicians think about Mathematics—as a deep, very rigorous, and demanding discipline of study. They tend to be quick to point out that learning to make use of the widely used computer tools tends to require little or no knowledge or understanding about the underlying computer science.

In higher education, the distinction between CS and ICT—which, of course, has no fine dividing line—has led to the development of ICT courses both in the Computer Science Department and in many other departments. Each department develops their ICT courses to best fit the needs of their discipline. Thus, an ICT course offered by a Computer Science Department is apt to contain more of the underlying CS theory than a course on the same topic taught outside of CS.

CS and ICT in K-8 Education

The mathematic education of a child begins at a very young age. Children are exposed to counting, some arithmetic ideas, and some geometry ideas from very early on. Some of the games they play involve counting, dice, and spinners. They gradually learn about money and time. More formal instruction in arithmetic and other math begins at the Kindergarten level and usually continues through two or three years of required math in grades 9-12 of high school. Most college and universities require some math in bachelor's degree programs.

Contrast this with our current approach to computer education. Today's children grow up with toys that contain computers and that are able to seemingly act on their own volition or in an interactive manner. Many children develop considerable skill in using a mouse and some initial skills in keyboarding well before them begin Kindergarten. Thus, in some sense the computer education of many children is more advanced than their math education before they begin school.

Beginning with the beginning of formal schooling, however, math education and computer education take substantially different approaches. Math education is driven by detailed scope and sequence, state and national standards, books and other materials tied to the standards, and state and national assessment. Even at the first grade level, a child's math teacher is apt to have had at least 11 years of formal math instruction in grades 1-12 and a half-year or year of formal math instruction in college, along with a Math Methods course.

In contrast to this, the typical K-8 teacher's preparation in computer science comes from some combination of formal and self-instruction in ICT. While it is hard to quantify this level of preparation, perhaps on average the formal instruction is equivalent to one or one and a half years of four credit courses in higher education. Typically, this includes instruction in computer methods in a Computer Methods course and/or integrated into other courses in the Teacher Education program. The informal and self-instruction may well include many years of using a word processor, email, the Web, computer games, text messaging on a cell phone, digital still and video cameras, digital music storage and playback devices, and so on.

I find it interesting to compare the idea of folk math (street smarts math) and folk ICT (street smarts ICT). In both cases, people gain the knowledge and skills that fit their needs. Much of the folk math and folk ICT learning occurs in a highly intrinsically motivated, situated learning environment. The learner tends to make immediate use of the learning in dealing with outside of school problems.

ICT and CS Education of K-8 Students

For the majority of K-8 students, formal instruction in computer technology comes from some combination of a computer technology specialist and from regular classroom teachers. The computer technology specialist may be a computer lab coordinator who has some teaching duties. A few students are enrolled in computer camps, after school computer programs, and other outside of school formal computer experiences.

The informal computer technology education of a young person is often highly dependent on the computer knowledge and skills of parents, older siblings, grandparents, and other people in the immediate family. Thus, for example, my older daughter had a relatively high level of skill in computer programming by the time she finished the sixth grade, and well before she received any instruction in this area from her school teachers.

In K-8 schools that do have a computer specialist, the student to computer specialist ratio is usually very large. Moreover, the computer specialist has many non-teaching responsibilities such as providing computer support and staff development for teachers, maintaining the computer systems, acquiring software and hardware, scheduling labs, and so on. Thus, the average K-8 student receives only a modest amount of formal instruction from a computer specialist, even in situations where the school has one or more such personnel. Moreover, this instruction tends to be skills oriented—how to use a computer system in an ICT environment. There tends to be little emphasis on representing and solving problems in a computer environment or on developing computational thinking.

This means that much of the responsibility for computer education of K-8 students falls on the shoulders of regular classroom teachers. What they need to be doing can be divided into four categories:

1. Providing students with the opportunity (or requiring) regular use of the ICT that they have learned. This is best accomplished by having computers in the teacher's classroom. An alternative is to make use of laptops that are regularly made available to students in their classroom. Taking students to a computer lab occasionally is a relatively poor approach.
2. Adding to the ICT tool knowledge of their students. Every teacher should have the computer knowledge and skills to teach their students how to use a new piece of software that is relevant to the curriculum
3. Helping their students to learn uses of ICT to represent and help solve problems within the various disciplines the teacher teaches. Remember, problem solving is part of every discipline, and computers are a general-purpose aid to problem solving that cuts across all disciplines. Procedural and computational thinking need to be integrated into the everyday curriculum. A school's computer specialist (if the school is fortunate enough to have one) cannot be expected to teach details of roles of computers in art, music, reading, writing, science, math, social science,

health, and other disciplines that make up part of the everyday content in a student's curriculum

4. Helping students gain some underlying and/or introductory knowledge of computer science. Any calculator or computer use in the regular classroom provides an opportunity for the teacher to help students learn some computer science. For example, the number system on a calculator is limited by the number of digits of accuracy the calculator uses. This is different than the number system that students are learning in "traditional" math.

Computer Science Cognitive Development Scale

The original 4-level Piagetian Cognitive Developmental Scale is still a useful tool to educators. Piaget himself recognized some of its shortcomings, and the need to consider development in specific discipline areas. He was particularly interested in cognitive development in mathematics.

Neo-Piagetians direct a strong focus on students increasing their level of Piagetian Development in various disciplines. All teachers have a responsibility for helping their students increase their general levels of cognitive development. A math teacher has a responsibility of helping students move up a Math Cognitive Developmental Scale, such as was illustrated in chapter 7. All teachers have a responsibility of helping students move up a Computer Science Cognitive Developmental Scale. This section presents some initial ideas about such a scale. The model is based on the diagram given in figure 9.2.

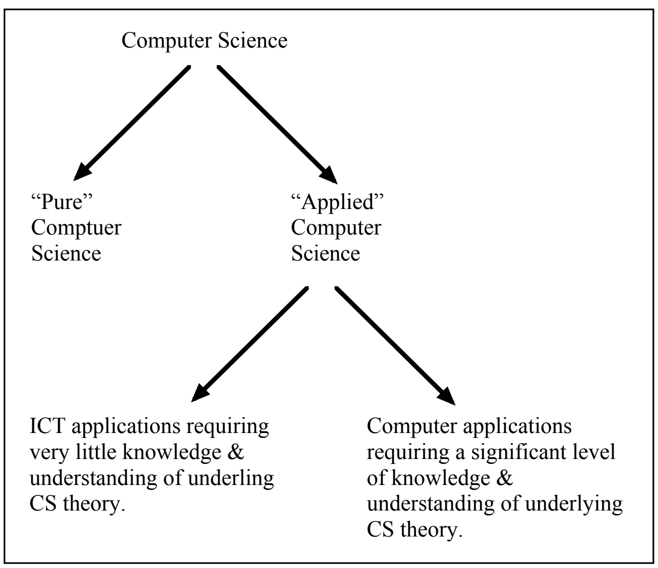

Figure 9.2 A simplified model of Pure and Applied CS.

This diagram is of somewhat limited value in dealing with the complexities of CS. For example, there is a strong overlap between pure and applied CS. Both draw heavily upon mathematics and the underlying theories of CS. Figure 9.3 considers the situation from a computers in education point of view.

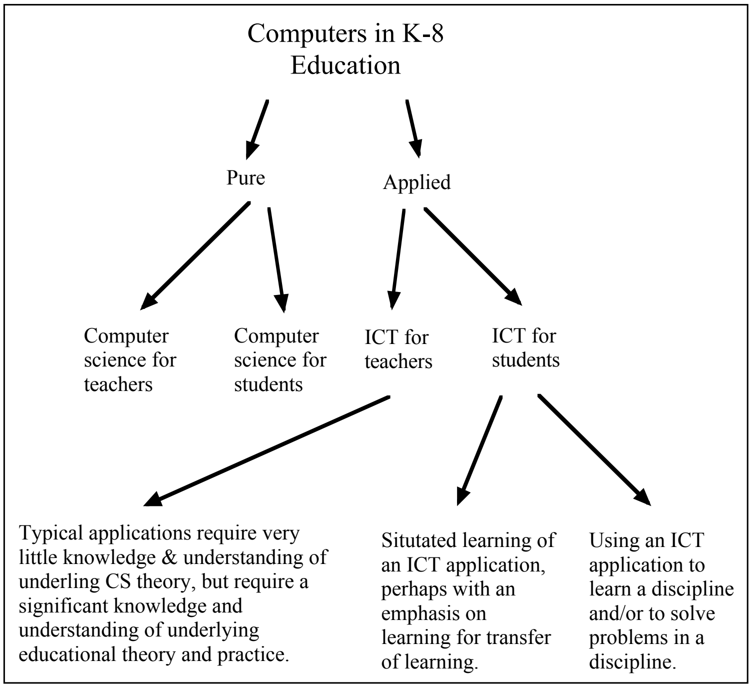

Figure 9.3. Computers in education.

Computer Science is a large, vibrant, and rapidly growing field. The International Society for Technology in Education (ISTE) has developed national educational technology standards for students, teachers, and school administrators. These are ICT applications-oriented (see bottom left corner of figure 9.2 and right side of figure 9.3) and have been widely adopted and serve to provide a good sense of direction for the ICT preparation of teachers and their students (ISTE NETS, n.d.).

Each discipline and sub discipline can be divided into content and maturity. This book contains a heavy emphasis on helping students increase their math maturity. Similar ideas apply to computer science.

The discipline of ICT can be divided into ICT Content and ICT Maturity, much like I have done for math earlier in this document. ISTE NETS for Students (ISTE NETS-S) provides recommendations for ICT content in the PreK-12 curriculum. ISTE NETS for Teachers provides the recommendation that precollege teachers should meet the ISTE NETS-S and should have a substantial amount of knowledge and skill in educational uses of computers.

Following the same line of reasoning that led to the math cognitive development scale given earlier in this document, I have been working on an computer science cognitive development scale. My current version (very rough draft) is in figure 9.4. I have not done the needed empirical research to help support ideas in this scale. Rather, the scale is my current attempt to make use of Piagetian-related research and to apply it to my explorations of CS.

Stage in CS Cognitive Development	**Age and/or Education Levels**	**Brief Discussion**
Stage 1. Piagetian	Age birth to 2 years. Informal education	Infants use sensory and motor capabilities to explore and gain increasing understanding of their environments.

Sensorimotor.	provided by parents, and other caregivers.	ICT has brought us a wide range of sound and music-producing, talking, moving, walking, interactive, and developmentally appropriate toys for children in Stage 1. These contribute both to general progress in sensory motor development and to becoming acquainted with an ICT environment.
Stage 2. ICT Preoperational.	Age 2 to 7 years. Includes both informal education and increasingly formal ICT education in preschool, kindergarten, and first grade.	During the Piagetian Preoperational stage, children begin to use symbols, such as speech. They respond to objects and events according to how they appear to be. They accommodate to the language environments they spend a lot of time in. ICT provides a type of symbols and symbol sets that are different from the speech, gestures, and other symbol sets that have traditionally been available. TV and interactive ICT-based games and edutainment are a significant environmental component of many children during Stage 2. During this stage, children can develop considerable speed and accuracy in using a mouse, touch pad, and touch screen to interact and problem solve in a 3-dimensional multimedia environment displayed on a 2-dimensional screen. The work of Seymour Papert and others demonstrates that children at the upper end of this developmental level can learn rudiments of programming in Logo and other graphic-oriented programming languages.
Stage 3. ICT Concrete Operations.	Age 7 to 11 years. Includes informal education and steadily increasing importance of formal education in grades 2-5.	During the Piagetian Concrete Operations stage, children begin to think logically. In this stage intelligence is demonstrated through logical and systematic manipulation of symbols related to concrete objects. Operational thinking (mental actions that are reversible) develops. ISTE has established NETS-Student that includes a statement of what students should be able to do by the end of the fifth grade. During the ICT Concrete Operations stage children: • Learn to use a variety of software tools such as those listed in the 5th grade ISTE NETS-Student, and begin to understand some of the capabilities and limitations of these tools. (They do logical and systematic manipulation of symbols in a computer environment.) • Learn to apply these software tools at a Piagetian Concrete Operations level as an aid to solving a wide range of general curriculum-appropriate problems and tasks. • With appropriate instruction and opportunity, can gain considerable skill in programming in languages such as BASIC, Logo, and Squeak, as well in graphical manipulation environments, such as working with still and video digital photography.
Stage 4. CS Formal Operations.	Age 11 and beyond. This is an open ended developmental stage, continuing well into adulthood. Requires ICT knowledge, skills, speed, and understanding of topics in ISTE NETS for students finishing the 12th grade. Requires knowledge and understanding of CS equivalent to An Advanced Placement high school CS course	During the Piagetian Formal Operations stage, thought begins to be systematic and abstract. In this stage, intelligence is demonstrated through the logical use of symbols related to abstract concepts. Formal Operations in ICT includes functioning at a Piagetian Formal Operations level in specific activities such as: 1. Communicate accurately, fluently, and with good understanding using the vocabulary, notation, and content of ISTE NETS-S for the 12th grade. 2. Given a piece of software and a computer, install and run the software, learn to use the software, explain and demonstrate some of the uses of the software, save a document you have created, and later return to make further use of your saved document. 3. Problem solve at the level of detecting and debugging hardware and software problems that occur in routine use of ICT hardware and software. 4. Convert (represent, model, pose) real world problems from non-ICT disciplines into ICT problems, and then solve these problems. 5. Routinely and comfortably use ICT in the other disciplines you have

	or a college freshman CS year sequence for potential CS majors.	studied, at a level consistent with and supportive of your cognitive developmental level in these disciplines. In addition, this developmental level requires proficiency in representing and solving problems in a computer programming environment comparable to what is achieved by students in a rigorous college freshman computer science course for potential majors in the field.
Stage 5. Abstract CS operations.	As with math, there are CS developmental levels well above what are typically achieved by well educated high school graduates. This stage corresponds to solid undergraduate work in CS	CS content proficiency and maturity at the level of contemporary CS texts used at the senior undergraduate level in strong programs, or first year graduate level in less strong programs. Good ability to learn CS through some combination of reading required texts and other CS literature, listening to lectures, participating in class discussions, studying on your own, studying in groups, and so on. Solve relatively high level CS problems posed by others (such as in the text books and course assignments). Pose and solve problems at the level of one's CS reading skills and knowledge. Follow the logic and arguments in CS theory. Fill in details of CS proofs and underlying theory when steps are left out in textbooks, lectures, and so on.
State 6. Computer Scientist.	This stage corresponds to success in high level graduate work, post doctoral work, and a continuing lifelong career in CS. People at this stage have a high level of CS expertise, and they may well have invested 10,000 to 20,000 hours or more of effort to attain this level.	A very high level breadth and depth of CS proficiency and maturity. Speed, accuracy, and understanding in reading the research literature, writing research literature, and in oral communication (speak, listen) of research-level CS. Pose and solve original CS problems at the level of contemporary research frontiers. Do research that advances the field. Design complex computer hardware and software systems, and participate in their implementation.

Figure 9.4. Computer science cognitive development and expertise scale.

A key aspect of this CS Cognitive Developmental Scale is Stage 4, the rigorous formal operations level of learning and performance required in a solid college-level CS course. Some student take an equivalent course given as a one year or two years sequence in high school, and a few students reach this level at a still younger age. Roughly speaking, CS Stage 4 requires a person to be at the formal operations level on the traditional Piagetian scale, and to have achieved a comparable developmental level in CS.

Procedures and Procedural Thinking

From a computer programmer point of view, a computer program is a procedure—a step-by-step set of directions—that can be carried out by a computer. Programmers develop procedures to solve or help solve problems. In doing this, they make use of procedures (both algorithms and heuristics) written by others and by themselves. Such building on the previous work of oneself and others is a standard approach used by computer programmers.

For an example, consider a team of programmers developing a word processor. It is a relatively simple task to write a program that accepts input from a keyboard, displays the text on a screen, stores it in computer memory, and outputs it when commanded to do so. The task gets a little more difficult when the text contains a variety of fonts and character sizes, bold and italic, and so on. Still, the challenge is not too big.

It is sure nice to have an outliner built into a word processor, as well as the ability to insert pictures and other graphics. The programmer's challenge is growing.

Next, consider the idea that a person using a word processor to write a paper may want the software to help in various ways, such as spell checking, grammar checking, formatting for final publication, perhaps generating a table of contents and/or an index, and so on. It is helpful to have provisions for creating tables and lists, alphabetizing a list, or sorting a list into numerical order. The challenge to the programmer continues to grow.

However, a pattern is beginning to emerge. There is the basic word processor. Then, there are a lot of different features (in essence, separate procedures) that can be added to it. Thus, a number of programmers can work on the overall task because it can readily be broken down into a collection of smaller, more manageable tasks.

Moreover, some of the tasks have been done lots of times by other programmers. Sorting a list alphabetically or numerically is a common programming task in a first term computer-programming course. Thus, the team of professional programmers working to develop a word processor will make extensive use of these and other procedures that are stored in a library of computer procedures.

Other components may be really challenging. Consider a grammar checker. This task is an area of research, and it is very challenging. The grammar checker in a word processor such as Microsoft Word in gradually getting better due to the efforts of researchers from a variety of different disciplines, including the work of people in artificial intelligence.

In summary, writing a computer program is a particular type of problem solving task. It involves procedural thinking and developing procedures to solve or help solve a problem. The programmer is developing a tool that may be used by problem solvers in many different disciplines, or that may be quite narrow and scope and used only by a few narrow specialists in a particular narrow part of some discipline. Typically, the programming tasks faced by programmers are complex, and a program is typically a large and relatively complex set of instructions.

Math-Related ICT Topics

Listed below are some math-related ICT topics. The intent of this list is to provide you with a hint of the breadth and depth of this discipline. More detail on a number of these topics is available in my math Website (Moursund, n.d., Math website). If you are interested in still broader aspects of ICT in education, a number of appropriate and free materials are available at Moursund (n.d., Free Materials).

1. The discipline of mathematics is now commonly divided into three major components: pure math, applied math, and computational math. Computational math includes developing and making effective use of math models and simulation. Some of the key ideas in math modeling are inherent to math manipulatives—both physical and virtual manipulatives. Graphic arts software provides useful tools in modeling and simulation, and thus is a valuable resource in teaching computational math. Spreadsheet software provides an excellent environment for teaching math modeling.

2. Computer algebra systems (CAS) provide very powerful tools to carry out a wide range of mathematical procedures.

   It is now common for students taking high school math courses to learn to use graphing, equation-solving calculators that have some built-in rudiments of CAS. Ideas such as function, equation, and graphing are very important ideas in math. As you work with K-8 students, you are laying the foundations for their future learning of these topics. Among other things, this means you are helping students to develop chunks in their mind/brain that can grow to include these topics. For more information see Moursund (n.d., Computational Math).

3. Computer-assisted learning (CAL) is gradually improving. We now have Highly Interactive Intelligent Computer-Assisted Learning (HIICAL) systems that are quite good. The meaning of "quite good" can be debated. Research in this area tends to compare test scores of students taught by conventional instructional methods versus test scores of students taught by HIICAL. There is now a significant amount of such software that, on average, leads to better test scores than does conventional instruction (Moursund, 2002).

   HIICAL software can be developed that integrates the power of computer-assisted instruction with the power of CAS systems. That is, we are gradually seeing a merger of powerful computer tools and powerful aids to learning and using the tools. Such software has the potential to lead to major changes in math education. The goal might become to educate students so that they function well mathematically in a world in which such systems are readily available.

4. The terms (concepts) of *variable* and *function* are essential in both math and computer science. In a calculator or in a computer (when doing computer programming) a variable is a memory location that has a name. Different values can be placed in this memory location. Why do you think some calculators are called four-function calculators? It is because they can carry out the four functions that we call addition, subtraction, multiplication, and division. A scientific calculator may have a hundred or more built-in functions. The point is, calculators and computers provide a concrete way to think about variable and function.

5. It is helpful to think about math *training* versus math *education.* Most of what an animal trainer does falls into the category of training, as contrasted with education. Education has a focus on understanding; training has a focus on rote performance.

   Our educational system consists of a mixture of training and education, and it is not easy to draw a clear distinction between the two. Research in computer-assisted learning suggests that this approach to teaching and learning is currently more effective in training than it is in education. Suppose, for example, that we want students to memorize the single digit multiplication facts and to be able to retrieve these facts with great speed and accuracy. This can be considered as a training task, and CAL is quite effective in this teaching/learning situation. Even the simplest of HIICAL designed for such training is able to individualize instruction, detect student weaknesses and address these weaknesses, and assess student speed and accuracy. From those points of view, such a CAL system is definitely more effective than a teacher working with a whole class. As we look to the future of math education, we will see HIICAL becoming a common component.

Math manipulatives can be used in both training and education modes. However, the current focus on using math manipulatives is in education for understanding and problem solving, rather than on training. For a list of resources on virtual (that is, computer-based) manipulatives for use in math education see Virtual Manipulatives (n.d.).

6. Artificial intelligence (AI) is a branch of the discipline of Computer and Information Science. It focuses on developing hardware and software systems that solve problems and accomplish tasks that—if accomplished by humans—would be considered a display of intelligence. As I look toward the future, I see a steady increase in situations where people and AI systems work together to solve problems and accomplish tasks.

   What is artificial intelligence? It is often difficult to construct a definition of a discipline that is satisfying to all of its practitioners. AI research encompasses a spectrum of related topics. Broadly, AI is the computer-based exploration of methods for solving challenging tasks that have traditionally depended on people for solution. Such tasks include complex logical inference, diagnosis, visual recognition, comprehension of natural language, game playing, explanation, and planning (Horvitz, 1990).

   AI is of steadily growing importance in education (Moursund, 2005, 2006). K-8 students already have a mind/brain chunk in this area, based on the robots and computers they see on television, their electronic toys, and so on. One of your jobs as a teacher is to shape this chunk so that it is more accurate and so that it can better accommodate future learning. For example, a handheld calculator has some intelligence. Think about how this intelligence is similar to and different from human intelligence in math.

7. Distance education is a rapidly growing field. If we use a rather broad definition of distance education, then it is already in common use in K-8 schools. When a student uses the Web to retrieve information, this is a form of distance education. When a student uses a help feature in a software package, this is a form of distance education. Much of the CAL that students use is accessed through a computer that is remotely located; thus, much of current CAL is a type of distance education.

   Imagine the situation in which HIICAL that covers the entire math curriculum is routinely available to students at home, at school, and wherever else they have access to the Internet. Such a system would also provide access to CAS, large numbers of math resource books, and other aids to learning and using math. While the progress in this direction seems relatively slow, I believe that this situation will be a standard part of many educational systems within the next two decades.

8. In light of goals for students learning math content and gaining in math maturity, how authentic is math assessment? Outside of school testing situations, people who need to make appreciable use of math tend to make use of calculators, computers, and many specialized devices (such as a global positioning system, computerized laser measuring and surveying systems) as aids to math problem solving. This suggests that authentic assessment in math should be moving in the direction of open book, open notes, open calculator, open computer, and similar forms of assessment. Some progress in this direction has occurred in the use of calculators, but little progress is occurring other aspects of authentic math assessment. See Moursund (n.d., Project-Based learning).

Final Remarks

Our math education system has done some assimilation and accommodation for computer science and ICT. However, to a large extent it has ignored computer science and ICT. There are many reasons for this. One of the most challenging is that it takes a large amount of time and effort for a preservice or inservice teacher to develop a level of expertise in math that meet contemporary standards. While CS (including ICT) is not nearly as broad and deep a discipline as math, it still takes a large amount of time and effort to learn this discipline at a level that meets reasonable standards.

People who become K-8 math teachers gain much of their math content knowledge and math maturity through their PK-12 education, where they take math year after year. No such system exists in CS and ICT. Thus, the CS and ICT preparation of most preservice and inservice teachers is very weak relative to their math preparation. It is not adequate for them to do a good job of teaching the CS and ICT components of a modern math curriculum. Even after completing a good teacher education program of study, most beginning teachers of K-8 math need substantial inservice education in CS and ICT to adequately handle the computer aspects of a modern math education.

K-8 School Applications

9.1 Many math educators and others feel it is important for students to develop high accuracy and speed on number facts and simple arithmetic calculators. Thus, they make use of timed tests along with a lot of drill and practice. (Note that many other math educators think that this is not an appropriate way to teach math!). Locate a suitable piece of math education software that provides timed test and/or drill and practice. Have your students use it. Then hold a whole class discussion about what they like and what they don't like about use of the software. Make sure the discussion includes a focus on what students are actually learning and how well their increased knowledge and skill transfers to other settings.

9.2 Provide your students with calculators that have a M+ (that is, a memory that can be added to) key. Carry on a whole class discussion about what is the same and what is different between this calculator and a computer. Make sure that the discussion eventually includes calculator and/or computer memory. Both a calculator and a computer have memory and a central processing unit (CPU). A CPU on a simple calculator can carry out a very limited number of operations such as add, subtract, multiply, and divide. The CPU on a computer may well be able to carry out a hundred or more different operations. Both a calculator and a computer can automatically follow a step-by-step set of instructions.

Activities for Self-Assessment, Assignments, and Group Discussions

9.1 Many leaders in the field of ICT in education argue that the development of writing, the mass printing and distribution of printed materials made possible by Gutenberg's movable type printing press, and the development of computers are the three most important developments in the history of education. Compare and contrast current and potential roles of ICT in education relative to the contributions made by writing and the printing press.

9.2 Make a list of things that you can do much better than ICT systems, things that ICT systems can do much better than you, and things that you and ICT systems working together can do much better than either can do alone. Analyze your list from the point of view of our current K-8 school and teacher education systems.

9.3 Summarize and analyze your thoughts on having most math tests be open book, open calculator, and open computer.

Chapter 10

Conclusions and Final Thoughts

> Give a man a fish and you feed him for a day. Teach a man to fish and you feed him for a lifetime. (Chinese proverb)
>
> I hear and I forget. I see and I remember. I do and I understand. (Confucius)

Math education is a large, complex, and challenging discipline. The formal teaching of math began at the time of the first formal teaching of reading and writing, a little more than 5,000 years ago. During the past 5,000 years, the collected mathematical knowledge of the human race has grown immensely. A number of ideas that challenged the mathematical geniuses of their time have trickled down into the precollege school math curriculum—indeed, even into K-8 schools.

As the agriculture age has given way to the industrial age and now the information age, the math-related demands placed on people have grown. In information age societies such as the United States, there are now much higher math education expectations than there were in the industrial age or the agricultural age. As our society continues to raise its math education expectations, it is not achieving the math learning gains that it would like.

Because math knowledge and skills are so important in our information age society, you can expect to see continued efforts to "reform" our math education system. This book supports the idea that with appropriate informal and formal teaching and support, students (on average) can gain greater math content knowledge and greater math maturity than they are currently obtaining. However, such math education goals leave us with many challenging issues. Here are a few examples:

1. A substantial fraction of parents and K-8 teachers have not achieved Math Formal Operations. Their levels of school-math maturity and school-math content knowledge are low. Thus, many children growing up in our society tend gain their first 13 to 14 years (birth through grade eight) of informal and formal math education in what I would call relatively poor math education environments. If we want to significantly improve our math education system, we will have to make significant progress toward addressing this problem.

   This means, of course, that significant progress will take decades. As we gradually improve the math education of preservice and inservice K-8 teachers, we will see progress in improving K-8 school math education. As we gradually improve K-12 math education, this will eventually lead to parents who will provide a better math education environment for their children. It will also lead to preservice teachers entering teacher education programs with a better preparation in math and in math pedagogy.

2. Our current math education curriculum is often described as being "a mile wide and an inch deep" (Ruetters, 2002). I have some trouble understanding what this means, as I don't use linear measure when I am taking about the breadth and depth of a curriculum. However, what I think it means is that many people are concerned about how our curriculum has expanded in breadth, covering more and more topics in a shallower and shallower manner. The curriculum lacks the depth needed for students to gain understanding and a number of other aspects of increasing math maturity. Our curriculum is not well designed in terms of helping students learn to make connections and to transfer their math knowledge and skills to areas outside of the formal math curriculum.

3. ICT brings new dimensions to both school math and folk math. We have yet to appropriately understand and implement a math education system that adequately takes into consideration the capabilities of ICT as aids to teaching, learning, and using math.

   For example, consider computer tools that are routinely used by graphic artists. They are based on a very large amount of mathematics. However, very few graphic artists feel the need to have studied this underlying mathematics, and few people who teach graphic arts use of computers have appreciable insights into the underlying mathematics. The issue here is somewhat similar to the issue of children using calculators rather than paper and pencil algorithms, or researchers using statistical packages of computer programs without having mastered the underlying mathematics.

   However, the issue is also quite different. The goal of a graphic artist is to solve a graphic artist problem or complete a graphic artist task. The graphic artist has graphic arts knowledge and skills that can provide feedback on progress toward solving the problem or accomplishing the task.

   This example identifies a major hole in the overall math curriculum. We are not very successful in helping students understand math at a level where they can detect their own errors. People who routinely use math are able to detect their errors because they have knowledge (intuition, deep insights) into the problems that they are addressing. Even though our math curriculum makes considerable use of word problems that provide some context for the problem to be solved, it is rare that a student has a sufficient grasp of the problem setting and meaning to be able to detect errors in math thinking and in carrying out needed math procedures.

4. There are a variety of math topics that require a student to be at or near math Formal Operations in order to gain a significant understanding of the topic. Examples include probability, ratio and proportion, and algebra. Roughly speaking, if many of the students you are teaching "just don't seem to get it" for certain topics, then there is a good chance that they are not developmentally ready for the topic.

I think what has happened in the school math curriculum is that it has developed a severe imbalance between the immediate success and long-term success. For a specific category of problems, immediate success is achieved by memorizing (without understanding) how to solve the specific category of problems. Such learning is fragile and brittle, does not transfer well to new situations, and tends to be quickly forgotten.

Long-term success requires learning for understanding and developing a significant level of math maturity. To do this, without increasing the amount of time devoted to math instruction, requires decreasing the breadth of the math content covered, and devoting much more time to learning for understanding and increased math maturity.

5. At the current time, the order of many topics in the math curriculum is dependent on the math that curriculum developers expect students can learn at various grade levels. However, calculators and computers add a twist to this. For example, third graders can learn to read and make use of various types of graphs, including a pie chart. However, creating such graphs tends to require use of math that students do not learn until the 4th or 5th grade. Since a computer can create such graphs, this raised the possibility of students creating and using these graphs at an earlier grade level than they currently do. James Fey, a math education professor at the University of Maryland, developed a number of examples of inverted curriculum in math more than 20 years ago.

6. One of the most important ideas in math education is learning to build upon and make effective use of the accumulated knowledge in the discipline of math. An important requirement in this endeavor is that students learn to read (with understanding) math at the levels they have studied. ICT is a powerful aid to learning, a powerful aid to information retrieval, and a powerful aid to carrying out many of the types of procedures that are important in solving math problems. Our current math education system is not doing well in helping students learn to read math and to make effective use of ICT.

In brief summary, our math education system can be a lot better. However, this will require significant improvement in teachers, in appropriate use of ICT, and in our understanding of the human brain and learning processes.

Final Remarks

The Preface contains four Big Ideas that help to unify this book. They are:

1. **Math content and math maturity**. Learning math is a process of both learning math content and a process of gaining in math maturity.

2. **Nature, nurture, and increasing math cognitive development.** Think of a student's math cognitive development in terms of the roles of both nature and nurture. Research in cognitive acceleration in mathematics and other disciplines indicates we can do much better in fostering math cognitive development.

3. **Computers and computational thinking.** Understanding the power of computer systems and computational thinking as an aid to modeling (representing) and solving math problems, and as an aid to effectively using math in all other disciplines.
4. **Learning to learn math.** Placing increased emphasis on learning to learn math, learning to read math with understanding, and making effective use of use computer-based aids to learning, and information retrieval.

Every student taking math courses at the K-8 level should gradually come to understand these Big Ideas and to take increased personal responsibility for achieving them. However, in K-8 education, teachers must play a major role in helping their students make progress in these four areas.

These four areas are not topics to be placed someplace in the math curriculum, taught as individual units, and then forgotten. Rather, they are topics to be thoroughly integrated throughout the math curriculum at all grade levels. As a teacher of math, before you start to teach a new unit, think carefully about what aspects of these four Big Ideas are included. If necessary, do some lesson plan revision, so that these ideas are a routine component of each of your math units.

K-8 School Applications

10.1 Present the following quotation to your students:

> Give a man a fish and you feed him for a day. Teach a man to fish and you feed him for a lifetime. (Chinese proverb)

Have your students talk about or write about what this means in terms of learning math. For older students, have them do this for math and another core discipline in the curriculum, and then do a compare and contrast between the two disciplines.

10.2 If you teach math at a grade level where students first learn the word *variable* or where variable is an important part of the curriculum, facilitate a whole class discussion on what the term means. Look for ideas both in math and in other disciplines, and then for commonalities among these various examples and ideas.

Activities for Self-Assessment, Assignments, and Group Discussions

10.1 Make up a definition of math maturity that you feel is relevant to the students you teach. Based on your definition, determine a way of assessing your student's growth in math maturity throughout a school year.

10.2 Computer graphics software facilitates students accomplishing various tasks that they cannot do without the software. How does this relate to James Fey's idea of inverted curriculum? Give some examples of possible uses of graphics software in a math education curriculum.

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I have made this letter longer than usual, only because I have not had the time to make it shorter. (Blaise Pascal, almost 400 years ago.)

Fortune favors the prepared mind. (Louis Pasteur)

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