# Pseudo Symmetric Multifuntors: Coherence and Examples

by

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DISSERTATION ABSTRACT

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Donald Yau introduced pseudo symmetric **Cat**-multifunctors and proved that Mandell's inverse $K$-theory multifunctor is stably equivalent to a pseudo symmetric one. We prove a coherence result for pseudo symmetric **Cat**-multifunctors in the form of a 2-adjunction. As a consequence, we obtain that pseudo symmetric **Cat**-multifunctors preserve $E_n$-algebras parameterized by $\Sigma$-free **Cat**-operads at the cost of changing the parameterizing **Cat**-operad $\mathcal{O}$ by $\mathcal{O} \times E\Sigma_*$, where $E\Sigma_*$ is the categorical Barrat-Eccles operad. Since Mandell's inverse $K$-theory is pseudo symmetric we derive that $E_n$-algebras parameterized by free $E_n$ **Cat**-operads in the symmetric monoidal category of $\Gamma$-categories can be realized, up to stable equivalence, as the $K$-theory of some $E_n$-algebra in the multicategory of permutative categories. This result can be regarded as a multiplicative version of a theorem by Thomason that says that any connective spectrum can be realized as the $K$-theory of a suitable symmetric monoidal category up to stable equivalence. Our coherence theorem also allows for a simple description of a 2-category defined by Yau which has **Cat**-multicategories as 0-cells and pseudo symmetric **Cat**-multifunctors as 1-cells. We also provide new examples of pseudo symmetric **Cat**-multifunctors by proving that the free algebra functor of a symmetric, pseudo commutative, strong 2-monad, as defined by Hyland and Power, can be seen as a pseudo symmetric **Cat**-multifunctor. This result can be interpreted as a coherence result for symmetric, pseudo commutative, strong 2-monads and it implies a coherence result for pseudo commutative, strong 2-monads conjectured by Hyland and Power.

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Para Mamá, Papá y Naty

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# CHAPTER 1

# INTRODUCTION

Algebraic $K$-theory can be seen as a technique to build spectra from algebraic data. Segal's infinite loop space machine [Seg74] constructs spectra from symmetric monoidal categories. Permutative categories, i.e., symmetric monoidal categories that are strictly associative and unital, provide another input for $K$-theory, which a construction of May [May74] turns into spectra. These two $K$-theory constructions are equivalent [MT78]. We are interested in multiplicative structures in spectra, so it is natural to ask what kind of conditions are necessary to impose on a permutative category so that its $K$-theory is, for example, an $E_\infty$-ring spectrum. Although this question was first tackled by May [May77] who defined bipermutative categories, it wasn't until later that an answer was provided independently by May [May09], and Elmendorf and Mandell [EM06]. They proved that the $K$-theory of a bipermutative category is an $E_\infty$-ring. Elmendorf and Mandell also describe the categorical input that gives rise to a range of rings, modules and algebras in spectra after applying May's $K$-theory construction.

In their apporach, Elmendorf and Mandell use two main tools. On the one hand, the homotopy theory of spectra was made more transparent with the introduction of the modern symmetric monoidal model categories of spectra. These model categories allow for the treatment of various multiplicative structures in spectra and thus, are a natural target for Elmendorf and Mandell's $K$-theory construction. On the other hand, the domain of their construction, **Perm**, is not a symmetric monoidal category, although it is in a 2-categorical sense [GJO22]. The usual definitions of ring, algebra and module that use monoidal structures are thus not available. To overcome this obstruction, Elmendorf and Mandell introduced multicategories in homotopy theory. Multicategories are a generalization of categories that allow for maps with multiple inputs, even in the absence of a symmetric monoidal structure. In a sense, they allow the handling of multilinear maps in the absence of tensor products. They are a generalization of symmetric monoidal categories and operads at the same time, with operads being multicategories with one object. One can define multiplicative structures in a given multicategory, with multifunctors between multicategories preserving these structures. Elmendorf and Mandell define $K$-theory as a multifunctor from the multicategory **Perm** to symmetric spectra. Multifunctoriality implies that $K$-theory preserves multiplicative structures.

Thomason proved that every connective spectrum can be realized as the $K$-theory of some permutative category [Tho95]. Mandell [Man10] made Thomason's construction functorial by providing a stable homotopy inverse functor $\mathcal{P}$ to $K$-theory. The functor $\mathcal{P}$ takes as input $\Gamma$-categories (modelling connective spectra [Tho80; Cis99; BF78]) and its target is **Perm**. Elmendorf [Elm21], and independently Johnson and Yau [JY22] extended $\mathcal{P}$ to a **Cat**-enriched multifunctor, but one that is not symmetric: it is not compatible with the permutation of elements in the domains of multicategory mapping spaces. To account for this Yau [Yau24b] introduced pseudo symmetric multifunctors, where there is compatibility only up to coherent natural isomorphisms, and he proved that Mandell's inverse $K$-theory multifunctor $\mathcal{P}$ is pseudo symmetric in this sense.

A natural question to ask about pseudo symmetric multifunctors is whether they preserve multiplicative structure and if so, in what sense. We answer this question by proving a coherence result for pseudo symmetric multifunctors. If $F : \mathcal{M} \to \mathcal{N}$ is a pseudo symmetric multifunctor between multicategories enriched in **Cat**, we prove that the natural isomorphisms attesting the pseudo symmetry of $F$ assemble together to give a symmetric multifunctor $\phi(F) \colon \mathcal{M} \times E\Sigma_* \to \mathcal{N}$ satisfying a universal property, where $E\Sigma_*$ is the categorical Barratt-Eccles operad defined in Example 2.1.5. We can also think about our result as a rigidification result. We can rigidify $F$ and turn it into a symmetric multifunctor $\phi(F)$, at the cost of changing its domain. This is the main result of Chapter 2.

**Theorem 1.0.1.** (Theorem 2.2.3) Let $\mathcal{M}$ be a **Cat**-enriched multicategory. There is a pseudo symmetric multifunctor $\eta_{\mathcal{M}} \colon \mathcal{M} \to \mathcal{M} \times E\Sigma_*$ such that for every **Cat**-enriched multicategory $\mathcal{N}$ and every pseudo symmetric multifunctor $F \colon \mathcal{M} \to \mathcal{N}$, there exists a unique symmetric **Cat**-enriched multifunctor $\phi(F) \colon \mathcal{M} \times E\Sigma_* \to \mathcal{N}$ such that the following diagram commutes:

$$\begin{array}{ccccc} & & \mathcal{M} \times E\Sigma_* & & \\ & \overset{\eta_{\mathcal{M}}}{\nearrow} & & \overset{\phi(F)}{\searrow} & \\ \mathcal{M} & & \xrightarrow[F]{\hspace{6em}} & & \mathcal{N}. \end{array}$$

That is, $F = \phi(F) \circ \eta_{\mathcal{M}}$ as pseudo symmetric multifunctors.

Thus, if $\mathcal{O}$ is an operad in **Cat**, pseudo symmetric algebras in a **Cat**-enriched multicategory $\mathcal{M}$ over $\mathcal{O}$ (i.e., pseudo symmetric multifunctors $\mathcal{O} \to \mathcal{M}$) can be rigidified to get symmetric algebras in $\mathcal{M}$ over $\mathcal{O} \times E\Sigma_*$. The following result, which appears as Corollary 2.3.7, holds since multiplying by $E\Sigma_*$ sends the commutative operad $\{*\}$ to the $E_\infty$ operad $E\Sigma_*$ and sends $\Sigma$-free $E_n$-operads in **Cat**, like the ones defined in [Ber96] and [BFSV03], to $E_n$-operads.

**Corollary 1.0.2.** (Corollary 2.3.7) Let $F\colon \mathcal{M} \to \mathcal{N}$ be a **Cat**-enriched pseudo symmetric multifunctor. Then,

1. $F$ sends commutative monoids to $E_\infty$-algebras.
2. $F$ sends $E_n$-algebras over $\Sigma$-free $E_n$ **Cat**-operads to $E_n$-algebras for $n = 1, 2, \ldots, \infty$.

In this sense, $F$ preserves symmetric $E_n$-algebras parameterized by $\Sigma$-free $E_n$-operads at the cost of changing the parameterizing operad. This corollary extends our understanding of the behavior of inverse $K$-theory since it implies that the inverse $K$-theory pseudo symmetric multifunctor $\mathcal{P}$ from [Yau24b] sends commutative monoids to $E_\infty$-algebras and sends $E_n$-algebras ($n = 1, 2, \ldots$) parameterized by $\Sigma$-free operads to $E_n$-algebras. Since $\mathcal{P}$ provides a stable inverse to $K$-theory, and $K$-theory is a symmetric multifunctor, this implies that every symmetric $E_n$-algebra parameterized by a $\Sigma$-free **Cat**-operad in $\Gamma$-categories is stably equivalent to the $K$-theory of a symmetric $E_n$-algebra in permutative categories for $n = 1, 2, \ldots, \infty$. Our result can thus be seen as a multiplicative version of Thomason's theorem [Tho95]. This also shows how Theorem 1.0.1 can be used to grasp the behavior of pseudo symmetric multifunctors on structures parameterized by symmetric operads in general.

This rigidification structure can be extended in a 2-categorical sense. In [Yau24b] Yau defines the 2-category **Cat-Multicat** having **Cat**-enriched multicategories as 0-cells, symmetric multifunctors as 1-cells and multinatural transformations as 2-cells. He also defines the 2-category $\mathbf{Cat\text{-}Multicat^{ps}}$ with 0-cells **Cat**-enriched multicategories, 1-cells pseudo symmetric multifunctors, and 2-cells pseudo symmetric **Cat**-multinatural transformations. Every symmetric **Cat**-enriched multifunctor (respectively, multinatural transformation) is canonically a pseudo symmetric multifunctor (respectively multinatural transformation), so there is a 2-functorial inclusion $j\colon \mathbf{Cat\text{-}Multicat} \to \mathbf{Cat\text{-}Multicat^{ps}}$. Taking into account these 2-categorical structures, we can extend our previous result by providing a left 2-adjoint $\psi$ to $j$, which, at the 0-cell level, sends a multicategory $\mathcal{M}$ to $\psi(\mathcal{M}) = \mathcal{M} \times E\Sigma_*$.

**Theorem 1.0.3.** (Corollary 2.2.5 and Theorem 2.2.7) The 2-categorical inclusion $j\colon \mathbf{Cat}\text{ -}\mathbf{Multicat} \to \mathbf{Cat}\text{-}\mathbf{Multicat}^{\mathbf{ps}}$ admits a left 2-adjoint

$$\psi\colon \mathbf{Cat} - \mathbf{Multicat}^{\mathbf{ps}} \to \mathbf{Cat} - \mathbf{Multicat}$$

with $\psi(\mathcal{M}) = \mathcal{M} \times E\Sigma_*$ for $\mathcal{M}$ a **Cat**-multicategory. In particular, for **Cat**-multicategories $\mathcal{M}$ and $\mathcal{N}$ we have an isomorphism of categories

$$\mathbf{Cat}\text{-}\mathbf{Multicat}^{\mathbf{ps}}(\mathcal{M},\mathcal{N}) \cong \mathbf{Cat}\text{-}\mathbf{Multicat}(\mathcal{M} \times E\Sigma_*, \mathcal{N}).$$

An important consequence of this theorem is that we can give a very simple and compact description of the 2-category $\mathbf{Cat}\text{-}\mathbf{Multicat}^{\mathbf{ps}}$ solely in terms of symmetric **Cat**-multifunctors and **Cat**-mutinatural transformations, which we do in Definition 2.2.8.

The question about the existence of a multiplicative equivariant $K$-theory machine taking some algebraic input to $G$-spectra and preserving multiplicative structures has received some attention in recent years. On the one hand Barwick, Glasman and Shah [BGS20] and Kong, May, and Zou [KMZ24] among others provide examples of multiplicative structures in $G$-spectra built from various kinds of inputs, but don't provide a systematic approach. On the other hand the works of Guillou, May, Merling and Osorno [GMMO23], and Yau [Yau24a] introduce equivariant multiplicative $K$-theory machines using multicategories. The $K$-theory machine of [GMMO23] is a non-symmetric multifunctor from the multicategory of algebras and pseudo morphisms over a pseudo commutative operad enriched in $G$-**Cat** to orthogonal $G$-spectra. It is conjectured to be a pseudo symmetric functor. Yau's $G$-equivariant $K$-theory machine is a symmetric multifunctor with domain the multicategory of pseudo algebras over a pseudo commutative operad in $G$-**Cat** and produces orthogonal spectra. In both constructions pseudo commutative operads play an important role.

Pseudo commutative operads were defined by Corner and Gurski [CG23]. These are operads whose associated monads are pseudo commutative. Now, commutative monads were introduced by Anders Kock in [Koc70] and are designed to capture the concept of a monoidal 2-monad. Let's make this a little more precise. Suppose that $\mathcal{K}$ is a 2-category with finite products, and consider $\mathcal{K}$ endowed with the symmetric monoidal structure induced by products. Monoidal 2-monads $T : \mathcal{K} \to \mathcal{K}$ are strong in the sense that there is a 2-natural transfomation with components $t_2 : A \times TB \to T(A \times B)$ for $A, B$ objects of $\mathcal{K}$. Of course, since $\mathcal{K}$ is symmetric monoidal, there is also a 2-natural transformation with components $t_1 : TA \times B \to T(A \times B)$. Strong 2-monads can be regarded as monoidal 2-functors in two different ways, with the binary components being given by the two 1-cells that form the boundary of the following diagram:

$$\begin{array}{ccccc} TA \times TB & \xrightarrow{t_1} & T(A \times TB) & \xrightarrow{Tt_2} & T^2(A \times B) \\ {\scriptstyle t_2}\downarrow & & & & \downarrow{\scriptstyle \mu} \\ T(TA \times B) & \xrightarrow[Tt_1]{} & T^2(A \times B) & \xrightarrow[\mu]{} & T(A \times B). \end{array}$$

A commutative 2-monad is one where the previous diagram commutes for any $A, B$ objects of $\mathcal{K}$. It is a theorem of Kock [Koc70] that a strong 2-monad $T$ is commutative if and only if $T$ is a monoidal 2-monad. There are a lot of examples of 2-monads $T : \mathbf{Cat} \to \mathbf{Cat}$, e.g., the 2-monad for symmetric strict monoidal categories, that fail to be commutative but that are so up to coherent isomorphisms. Such monads are called pseudo commutative and they were introduce by Hyland and Power [HP02].

From the point of view of multiplicative equivariant algebraic $K$-theory, the most important feature of pseudo commutative 2-monads (and hence pseudo commutative operads) is that they allow for the definition of a multicategory of algebras. Blackwell, Kelly and Power define and study a 2-category of algebras $T\text{-}Alg$ for $T : \mathcal{K} \to \mathcal{K}$ a 2-monad [BKP02]. Hyland and Power [HP02] extend $T\text{-}Alg$ to a non symmetric $\mathbf{Cat}$-multicategory when $T$ is pseudo commutative. If $T$ satisfies another technical condition of being symmetric, then the $\mathbf{Cat}$-multicategory $T\text{-}Alg$ is symmetric. When $\mathcal{K} = \mathbf{Cat}$ and $T$ is accesible, the $\mathbf{Cat}$-multicategory structure in $T\text{-}Alg$ arises from a monoidal bicategorical structure on $T\text{-}Alg$ [Bou02]. The main theorem in Chapter 3 is the following.

**Theorem 1.0.4.** (Theorem 3.3.18) Let $T : \mathcal{K} \to \mathcal{K}$ be a strong, pseudo commutative, symmetric 2-monad. Then, the free algebra multifunctor $T : \mathcal{K} \to T\text{-}Alg$ is pseudo symmetric.

This implies that the free algebra functor for pseudo commutative operads, like those used in [GMMO23] and [Yau24a] is pseudo symmetric. In Remark 3.3.14, we explain how this result implies a coherence result for non symmetric, strong pseudo commutative 2-monads as was conjectured, but not stated clearly or proved in [HP02].

The results in this thesis, together with the author's current work on the definition and coherence of pseudo symmetric **Cat**-multicategories could be used to prove that the multiplicative equivariant $K$-theory machine of [GMMO23] preserves multiplicative structures. This can also be useful in proving that this machine is equivalent to the one constructed by Yau [Yau24a].

In Chapter 2, we prove the coherence theorem for pseudo symmetric **Cat**-multifunctors and extract some 2-categorical consequences as well as some applications to $\mathcal{K}$-theory. In Chapter 3 we prove a coherence for symmetric pseudo commutative 2-monads, and we show how this coherence can be interpreted as the pseudo symmetry of the free algebra multifunctor associated with the 2-monad.

CHAPTER 2

# COHERENCE FOR PSEUDO SYMMETRIC MULTIFUNCTORS

In this chapter we prove a coherence theorem for pseudo symmetric multifunctors as defined by Yau [Yau24a]. In Section 2.1 we introduce multicategories, multifunctors and their enriched versions in **Cat** as well as pseudo symmetric **Cat**-enriched multifunctors [Yau24b]. In Section 2.2 we prove the coherence theorem and extract some 2-categorical consequences. By work of Donald Yau [Yau24b] Mandell's inverse $K$-theory is stably equivalent to a pseudo symmetric **Cat**-enriched multifunctor. We use Section 2.3 to develop the $K$-theoretical consequences of applying our coherence theorem to Mandell's inverse $K$-theory multifunctor.

## 2.1 Symmetric and pseudo symmetric Multifunctors

We begin by reviewing the definition of multicategory enriched in a symmetric monoidal category. In the following definition $(C, 1, \oplus, \lambda, \rho, \xi)$ is a symmetric monoidal category with $\oplus \colon C \times C \to C$ the monoidal product, 1 the monoidal unit, $\lambda$ the left unit isomorphism, $\rho$ the right unit isomorphism and $\xi$ the symmetry. In this paper we will consider only categories enriched over **Cat** with the monoidal structure given by products, but we use a general monoidal category in the definition to make explicit the fact that this definition doesn't make use of the 2-categorical structure of **Cat**.

**Remark 2.1.1.** We will also use the following notation: if $\sigma \in \Sigma_n$ and $\tau_i \in \Sigma_{k_i}$ for $1 \leq i \leq n$, $\sigma\langle \tau_1, \dots, \tau_n \rangle \in \Sigma_{k_1 + \cdots + k_n}$ is the permutation that permutes $n$ blocks of lengths $k_1, \dots, k_n$ according to $\sigma$ and each block of length $k_i$ according to $\tau_i$.

**Definition 2.1.2.** If $C$ is a symmetric monoidal category, a *$C$-multicategory* $(\mathcal{M}, \gamma, 1)$ consists of the following data.

- A class of objects $\mathrm{Ob}(\mathcal{M})$.
- For every $n \geq 0$, $\langle a \rangle = \langle a_i \rangle_{i=1}^n \in \mathrm{Ob}(\mathcal{M})^n$ and $b \in \mathrm{Ob}(\mathcal{M})$, an object in $C$ denoted by
$$\mathcal{M}(\langle a \rangle; b) = \mathcal{M}(a_1, \dots, a_n; b).$$
We will write $\langle a \rangle$ instead of $\langle a_i \rangle_{i=1}^n$ when $n$ is clear from the context or irrelevant. [In the case $C = \mathbf{Cat}$, an object $f$ of $\mathcal{M}(\langle a \rangle; b)$ will be called an $n$-ary

1-cell with input $\langle a\rangle$ and output $b$ and will be denoted as $f\colon \langle a\rangle \to b$. Similarly, we will call $\alpha\colon f \to g$ in $\mathcal{M}(\langle a\rangle; b)(f,g)$ an $n$-ary 2-cell.]

- For each $n \geq 0$, $\langle a\rangle \in \mathrm{Ob}(\mathcal{M})^n$, $b \in \mathrm{Ob}(\mathcal{M})$, and $\sigma \in \Sigma_n$, a $C$-isomorphism

$$\mathcal{M}(\langle a\rangle; b) \xrightarrow[\cong]{\sigma} \mathcal{M}(\langle a\rangle\sigma; b)$$

called the right $\sigma$ action or the symmetric group action. Here

$$\langle a\rangle\sigma = \langle a_1, \ldots, a_n\rangle\sigma = \langle a_{\sigma(1)}, \ldots, a_{\sigma(n)}\rangle.$$

[In the case $C = \mathbf{Cat}$ we write $f\sigma$ for the image of an $n$-ary 1-cell $f\colon \langle a\rangle \to b$ in $\mathcal{M}$ and similarly for 2-cells.]

- For each object $a \in \mathrm{Ob}(\mathcal{M})$, a morphism

$$1 \xrightarrow{1_a} \mathcal{M}(a; a)$$

called the $a$-unit. In the case $C = \mathbf{Cat}$ we notice that if $a \in \mathrm{Ob}(\mathcal{M})$, $1_a\colon a \to a$ is a 1-ary 1-cell while if $f\colon \langle a\rangle \to b$ is an $n$-ary 1-cell, then $1_f\colon f \to f$ is an $n$-ary 2-cell in $\mathcal{M}(\langle a\rangle; b)(f, f)$ so our notation is unambiguous.

- For every $c \in \mathrm{Ob}(\mathcal{M})$, $n \geq 0$, $\langle b\rangle = \langle b_j\rangle_{j=1}^n \in \mathrm{Ob}(M)^n$, $k_j \geq 0$ for $1 \leq j \leq n$, and $\langle a_j\rangle = \langle a_{j,i}\rangle_{i=1}^{k_j} \in \mathrm{Ob}(\mathcal{M})^{k_j}$ for $1 \leq j \leq n$, a morphism in $C$,

$$\mathcal{M}(\langle b\rangle; c) \otimes \bigotimes_{j=1}^{n} \mathcal{M}(\langle a_j\rangle; b_j) \xrightarrow{\gamma} \mathcal{M}(\langle a\rangle; c),$$

where we adopt the convention that $\langle a\rangle \in \mathrm{Ob}(\mathcal{M})^k$, where $k = \sum_{i=1}^n k_j$, denotes the concatenation of the varying $a_j$'s for $j = 1, \ldots, n$. We write this as

$$\langle a\rangle = \langle a_1, \ldots, a_n\rangle = \langle\langle a_j\rangle\rangle_{j=1}^n = \langle a_{1,1}, \ldots, a_{1,k_1}, a_{2,1}, \ldots, a_{n-1,k_{n-1}} a_{n,1}, \ldots, a_{n,k_n}\rangle.$$

The previous data are required to satisfy the following axioms.

- **Symmetric group action**: For every $n \geq 0$, $\langle a\rangle \in \mathrm{Ob}(\mathcal{M})$, $b \in \mathrm{Ob}(\mathcal{M})$, and $\sigma, \tau$ in $\Sigma_n$ the following diagram commutes in $C$ :

$$
\begin{array}{ccc}
\mathcal{M}(\langle a\rangle; b) & \xrightarrow{\sigma} & \mathcal{M}(\langle a\rangle\sigma; b) \\
 & \searrow^{\sigma\tau} & \downarrow{\scriptstyle \tau} \\
 & & \mathcal{M}(\langle a\rangle\sigma\tau; b).
\end{array}
$$

We also require the identity permutation $\mathrm{id}_n \in \Sigma_n$ to act as the identity morphism on $\mathcal{M}(\langle a\rangle; b)$.

- **Associativity:** For every $d \in \mathrm{Ob}(\mathcal{M})$, $n \geq 1$, $\langle c\rangle = \langle c_j\rangle_{j=1}^n \in \mathrm{Ob}(\mathcal{M})^n$, $k_j \geq 0$ for $1 \leq j \leq n$ with $k_j \geq 1$ for at least one $j$, $\langle b_j\rangle = \langle b_{j,i}\rangle_{i=1}^{k_j} \in \mathrm{Ob}(\mathcal{M})^{k_j}$ for $1 \leq j \leq n$, $l_{i,j} \geq 0$ for $1 \leq j \leq n$ and $1 \leq i \leq k_j$, and $\langle a_{j,i}\rangle = \langle a_{j,i,p}\rangle_{p=1}^{l_{i,j}} \in \mathrm{Ob}(\mathcal{M})^{l_{i,j}}$ for $1 \leq j \leq n$ and $1 \leq i \leq k_j$, the following *associativity diagram* commutes in $C$:

$$
\begin{array}{ccc}
\mathcal{M}(\langle c\rangle; d) \otimes \left(\bigotimes_{j=1}^{n} \mathcal{M}(\langle b_j\rangle; c_j)\right) \otimes \bigotimes_{j=1}^{n} \left(\bigotimes_{i=1}^{k_j} \mathcal{M}(\langle a_{j,i}\rangle; b_{j,i})\right) & \xrightarrow{\gamma\otimes 1} & \mathcal{M}(\langle b\rangle; c) \otimes \bigotimes_{j=1}^{n} \left(\bigotimes_{i=1}^{k_j} \mathcal{M}(\langle a_{j,i}\rangle); b_{j,i}\right) \\
\downarrow{\scriptstyle \cong} & & \\
\mathcal{M}(\langle c\rangle; d) \otimes \bigotimes_{j=1}^{n} \left(\mathcal{M}(\langle b_j\rangle; c_j) \otimes \bigotimes_{i=1}^{k_j} \mathcal{M}(\langle a_{j,i}\rangle; b_{j,i})\right) & & \downarrow{\scriptstyle \gamma} \\
\downarrow{\scriptstyle 1\otimes\bigotimes_{j=1}^{n}\gamma} & & \\
\mathcal{M}(\langle c\rangle; d) \otimes \bigotimes_{j=1}^{n} \mathcal{M}(\langle a_j\rangle; c_j) & \xrightarrow{\gamma} & \mathcal{M}(\langle a\rangle; b).
\end{array}
\tag{2.1.1}
$$

- **Unity:** Suppose $b \in \mathrm{Ob}(\mathcal{M})$ and $\langle a\rangle = \langle a_j\rangle_{j=1}^n \in \mathrm{Ob}(\mathcal{M})$, then the following *right unity diagram* commutes in $C$ :

$$
\begin{array}{ccc}
\mathcal{M}(\langle a\rangle; b) \otimes \bigotimes_{j=1}^{n} 1 & & \\
\downarrow{\scriptstyle \mathrm{id}\otimes\bigotimes_{j=1}^{n} 1_{a_j}} & \searrow^{\cong} & \\
\mathcal{M}(\langle a\rangle; b) \otimes \bigotimes_{j=1}^{n} \mathcal{M}(a_j; a_j) & \xrightarrow{\gamma} & \mathcal{M}(\langle a\rangle; b).
\end{array}
$$

With $b, \langle a \rangle$ as before, we also demand that the following *left unity diagram* commutes in $C$.

$$\begin{array}{ccc} 1 \otimes M(\langle a \rangle; b) & & \\ {\scriptstyle 1_b \otimes \mathrm{id}} \downarrow & \overset{\lambda}{\searrow} & \\ \mathcal{M}(b; b) \otimes \mathcal{M}(\langle a \rangle; b) & \xrightarrow[\gamma]{} & \mathcal{M}(\langle a \rangle; b). \end{array}$$

- **Top equivariance:** For every $c \in \mathrm{Ob}(\mathcal{M})$, $n \geq 1$, $\langle b \rangle = \langle b_j \rangle_{j=1}^{n} \in \mathrm{Ob}(\mathcal{M})^n$, $k_j \geq 0$ for $1 \leq j \leq n$, $\langle a_j \rangle = \langle a_{j,i} \rangle_{i=1}^{k_j} \in \mathrm{Ob}(\mathcal{M})^{k_j}$ for $1 \leq j \leq n$, and $\sigma \in \Sigma_n$, the following diagram commutes:

$$\begin{array}{ccc} \mathcal{M}(\langle b \rangle; c) \otimes \bigotimes_{j=1}^{n} \mathcal{M}(\langle a_j \rangle; b_j) & \xrightarrow{\sigma \otimes \sigma^{-1}} & \mathcal{M}(\langle b \rangle \sigma; c) \otimes \bigotimes_{j=1}^{n} \mathcal{M}(\langle a_{\sigma(j)} \rangle; b_{\sigma(j)}) \\ {\scriptstyle \gamma} \downarrow & & \downarrow {\scriptstyle \gamma} \\ \mathcal{M}(\langle a_1 \rangle, \ldots, \langle a_n \rangle; c) & \xrightarrow[\sigma \left\langle \mathrm{id}_{k_{\sigma(1)}}, \ldots, \mathrm{id}_{k_{\sigma(n)}} \right\rangle]{} & \mathcal{M}(\langle a_{\sigma(1)} \rangle, \ldots, \langle a_{\sigma(n)} \rangle; c). \end{array} \tag{2.1.2}$$

Here $\sigma^{-1}$ is the unique isomorphism in $C$, given by the coherence theorem for symmetric monoidal categories, that permutes the factors $\mathcal{M}(\langle a_j \rangle, b_j)$ according to $\sigma^{-1}$.

- **Bottom equivariance:** For $\langle a_j \rangle, \langle b \rangle$ and $c$ as in Top equivariance (2.1.2), the following diagram commutes:

$$\begin{array}{ccc} \mathcal{M}(\langle b \rangle; c) \otimes \bigotimes_{j=1}^{n} \mathcal{M}(\langle a_j \rangle; b_j) & \xrightarrow{\mathrm{id} \otimes \bigotimes_{j=1}^{n} \tau_j} & \mathcal{M}(\langle b \rangle, c) \otimes \bigotimes_{j=1}^{n} \mathcal{M}(\langle a_j \rangle \tau_j; b_j) \\ {\scriptstyle \gamma} \downarrow & & \downarrow {\scriptstyle \gamma} \\ \mathcal{M}(\langle a_1 \rangle, \ldots, \langle a_n \rangle; c) & \xrightarrow[\mathrm{id}_n \left\langle \tau_1, \ldots, \tau_n \right\rangle]{} & \mathcal{M}(\langle a_1 \rangle \tau_1, \ldots, \langle a_n \rangle \tau_n; c). \end{array} \tag{2.1.3}$$

This concludes the definition of a $C$-multicategory.

**Remark 2.1.3.** A $C$-operad is a $C$-multicategory with one object. If $\mathcal{O}$ is a $C$-operad, its $n$-ary operations will be denoted by $\mathcal{O}_n \in \mathrm{Ob}(C)$. A non symmetric $C$-multicategory ($C$-operad) is defined in the same way as a $C$-multicategory ($C$-operad) excluding the data of the symmetric group action as well as the symmetric group, top and bottom equivariance coherence axioms. We will only be concerned with symmetric multicategories and operads. $C$-multicategories are often referred to as colored operads, with the objects of the $C$-multicategory being referred to as colors and $C$-operads having just one color.

**Example 2.1.4.** As examples of **Set**-operads, where **Set** has the monoidal structure induced by products in **Set**, we have the commutative operad $\mathrm{Comm} = \{*\}$ with $\mathrm{Comm}_n = \{*\}$. Another example is the associative operad $\mathrm{Ass} = \Sigma_*$ with $\mathrm{Ass}_n = \Sigma_n$, with the right action of the symmetric product given by right multiplication and $\gamma$ defined in the following way. If $n \geq 1$ and $k_1, \ldots, k_n$ natural numbers with $k = \Sigma_{i=1}^n k_i$, we define $\gamma\colon \Sigma_n \times (\prod_{i=1}^n \Sigma_{k_i}) \to \Sigma_k$ given for $\sigma \in \Sigma_n$ and $\langle \tau_1, \ldots, \tau_n \rangle \in \prod_{i=1}^n \Sigma_{k_i}$ by

$$\gamma(\sigma, \langle \rho_i \rangle_{i=1}^n) = \sigma \langle \rho_i \rangle_{i=1}^n = \sigma \langle \rho_1, \ldots, \rho_n \rangle,$$

as in Remark 2.1.1. When $n$ is clear from the context we will write $\sigma\langle \rho_i \rangle = \sigma \langle \rho_i \rangle_{i=1}^n$.

**Example 2.1.5.** We will consider **Cat**-multicategories where the monoidal structure in **Cat** is given by products. One source of examples is the forgetful functor $\mathrm{Ob}\colon \mathbf{Cat} \to \mathbf{Set}$ which forgets the morphism structure and remembers only the object set. Its right adjoint $E\colon \mathbf{Set} \to \mathbf{Cat}$ is the functor that takes a set $A$ to $EA$, the category with objects $\mathrm{Ob}(EA) = A$, and with a unique isomorphism between each pair of objects. $E$ sends a morphism $f\colon A \to B$ of sets to the functor $Ef\colon EA \to EB$, the only functor such that $f = \mathrm{Ob}(Ef)$. $E$ preserves products, and thus, if $\mathcal{O}$ is a **Set**-operad, $E\mathcal{O}$ is a **Cat**-operad. Similarly, if $\mathcal{M}$ is a **Set**-multicategory, $E\mathcal{M}$ is a **Cat**-multicategory with the same collection of objects as $\mathcal{M}$.
We will call $E\mathrm{Comm} = \{*\}$ the commutative **Cat**-operad. The Barratt-Eccles operad is the **Cat**-operad $E\Sigma_* = EAss$.

**Example 2.1.6.** Another source of examples for multicategories are symmetric monoidal categories, and thus also permutative categories. Each symmetric monoidal category $C$ has an associated **Set**-multicategory $End(C)$, whose objects agree with the objects of $C$ and such that for $\langle a \rangle \in \mathrm{Ob}(C)^n$ and $b \in \mathrm{Ob}(C)$,

$$End(C)(\langle a \rangle; b) = C(a_1 \otimes \cdots \otimes a_n, b).$$

Here we take $a_1 \otimes \cdots \otimes a_n$ with the leftmost parenthesization. Any fixed parenthesization would work. An empty string of objects is interpreted as the monoidal unit $1 \in \mathrm{Ob}(C)$.

Next, we define 1-cells between $C$-multicategories that preserve the action of the symmetric group. These are called symmetric $C$-multifunctors.

**Definition 2.1.7.** A symmetric $C$-multifunctor $F\colon \mathcal{M} \to \mathcal{N}$ between $C$-multicategories $\mathcal{M}$ and $\mathcal{N}$ consists of the following data.

- An object assignment $F\colon \mathrm{Ob}(\mathcal{M}) \to \mathrm{Ob}(\mathcal{N})$.
- For each $n \geq 0$, $\langle a \rangle \in \mathrm{Ob}(\mathcal{M})^n$ and $b \in \mathrm{Ob}(\mathcal{M})$ a $C$ morphism

$$\mathcal{M}(\langle a \rangle; b) \xrightarrow{F} \mathcal{N}(\langle Fa \rangle; Fb).$$

These data are required to preserve units, composition, and the action of the symmetric group.

- **Units:** For each object $a \in \mathrm{Ob}(\mathcal{M})$, $F(1_a) = 1_{Fa}$, i.e., the following diagram commutes in $C$ :

$$\begin{array}{ccccc} & & \mathcal{M}(a,a) & & \\ & \nearrow^{1_a} & & \searrow^{F} & \\ 1 & & \xrightarrow[1_{Fa}]{\qquad\qquad} & & \mathcal{N}(Fa,Fa). \end{array}$$

- **Composition:** For every $c \in \mathrm{Ob}(\mathcal{M})$, $n \geq 0$, $\langle b \rangle = \langle b_j \rangle_{j=1}^n \in \mathrm{Ob}(M)^n$, $k_j \geq 0$ for $1 \leq j \leq n$, and $\langle a_j \rangle = \langle a_{j,i} \rangle_{i=1}^{k_j} \in \mathrm{Ob}(\mathcal{M})^{k_j}$ for $1 \leq j \leq n$ and $1 \leq i \leq n$, the following diagram commutes in $C$ :

$$\begin{array}{ccc} \mathcal{M}(\langle b \rangle; c) \otimes \bigotimes_{j=1}^{n} \mathcal{M}(\langle a_j \rangle; b_j) & \xrightarrow{F \otimes \bigotimes_{j=1}^{n} F} & \mathcal{N}(\langle Fb \rangle; Fc) \otimes \bigotimes_{j=1}^{n} \mathcal{N}(\langle Fa_j \rangle; Fb_j) \\ \gamma \big\downarrow & & \big\downarrow \gamma \\ \mathcal{M}(\langle a \rangle; c) & \xrightarrow[F]{\qquad\qquad} & \mathcal{N}(\langle Fa \rangle; Fc). \end{array} \tag{2.1.4}$$

- **Symmetric Group Action:** For each $\langle a \rangle \in \mathrm{Ob}(\mathcal{M})^n$ and $b \in \mathrm{Ob}(\mathcal{M})$ the following diagram commutes in $C$ :

$$\begin{array}{ccc} \mathcal{M}(\langle a\rangle; b) & \xrightarrow{F} & \mathcal{N}(\langle Fa\rangle; Fb) \\ \cong\downarrow\sigma & & \cong\downarrow\sigma \\ \mathcal{M}(\langle a\rangle\sigma; b) & \xrightarrow[F]{} & \mathcal{N}(\langle Fa\rangle\sigma; Fb). \end{array}$$

**Definition 2.1.8.** Let $\mathcal{O}$ be a $C$-operad and a $\mathcal{M}$ be a $C$-multicategory. A symmetric algebra in $\mathcal{M}$ over $\mathcal{O}$ is a symmetric multifunctor $\mathcal{O} \to \mathcal{M}$.

Symmetric algebras are usually called algebras, but we add the adjective symmetric to distinguish them from pseudo symmetric algebras, which will be defined later.

**Example 2.1.9.** Since their introduction by May [May72], operads have been used to characterize certain categories as the categories of symmetric algebras over a certain operad. For example, symmetric algebras over $Comm$ in **Set** are commutative monoids. Symmetric algebras over $\Sigma_*$ in **Set** are associative monoids. Symmetric algebras over the Barrat-Eccles operad $E\Sigma_*$ in **Cat** are precisely permutative categories [May74].

Next we define composition of $C$-multifunctors.

**Definition 2.1.10.** We define the *horizontal composition* of $C$-multifunctors in the following way.

- Let $F\colon \mathcal{M} \to \mathcal{N}$, and $G\colon \mathcal{N} \to \mathcal{Q}$ be $C$-multifunctors, we define the $C$-multifunctor $GF\colon \mathcal{M} \to \mathcal{Q}$ on objects as the composition

  $$\mathrm{Ob}(\mathcal{M}) \xrightarrow{F} \mathrm{Ob}(\mathcal{N}) \xrightarrow{G} \mathrm{Ob}(\mathcal{Q}),$$

  and its component functors for $\langle a\rangle \in \mathrm{Ob}(\mathcal{M})^n$, $b \in \mathrm{Ob}(\mathcal{M})$ as the composite

  $$\mathcal{M}(\langle a\rangle; b) \xrightarrow{F} \mathcal{N}(\langle Fa\rangle; Fb) \xrightarrow{G} \mathcal{Q}(\langle GFa\rangle; GFb).$$

- The identity $C$-multifunctor $1_{\mathcal{M}}\colon \mathcal{M} \to \mathcal{M}$ is defined as the identity assignment on objects with the identity functors as component functors.

Next we define 2-cells between $C$-multifunctors. These will be the 2-cells of a 2-category with 0-cells $C$-multicategories and 1-cells $C$-multifunctors.

**Definition 2.1.11.** [Yau24b, Def. 3.2.5] For (symmetric) $C$-multifunctors $F, G\colon \mathcal{M} \to \mathcal{N}$, we define a *$C$-multinatural transformation* $\theta\colon F \Rightarrow G$ as the data of a component morphism $\theta_a\colon 1 \to \mathcal{N}(Fa, Ga)$ in $C$ for each $a \in \mathrm{Ob}(\mathcal{M})$ subject to the commutativity of the following diagram in $C$ for each $\langle a\rangle \in \mathrm{Ob}(\mathcal{M})^n$ and $b \in \mathrm{Ob}(\mathcal{M})$,

$$
\begin{array}{ccccc}
 & 1 \otimes \mathcal{M}(\langle a\rangle; b) & \xrightarrow{\theta_b \otimes F} & \mathcal{N}(Fb; Gb) \otimes \mathcal{N}(\langle Fa\rangle; Fb) & \\
\overset{\cong}{\nearrow} & & & & \overset{\gamma}{\searrow} \\
\mathcal{M}(\langle a\rangle, b) & & & & \mathcal{N}(\langle Fa\rangle; Gb). \\
\underset{\cong}{\searrow} & & & & \underset{\gamma}{\nearrow} \\
 & \mathcal{M}(\langle a\rangle; b) \otimes \bigotimes_{j=1}^n 1 & \xrightarrow{G \otimes \bigotimes \theta_{a_j}} & \mathcal{N}(\langle Ga\rangle; Gb) \otimes \bigotimes_{j=1}^n \mathcal{N}(Fa_j; Ga_j) &
\end{array}
$$

We define the identity multinatural transformation $1_F\colon F \to F$ as having component $(1_F)_a = 1_{Fa}$ for $a$ an object of $\mathcal{M}$.

**Remark 2.1.12.** When $C = \mathbf{Cat}$, and given $F, G\colon \mathcal{M} \to \mathcal{N}$ $\mathbf{Cat}$-multifunctors and the data of a 1-ary 1-cell $\theta_a\colon Fa \to Ga$ for each $a \in \mathrm{Ob}(M)$, the commutativity of the diagram in the previous definition means that for every $n \geq 0$, $\langle a\rangle \in \mathrm{Ob}(\mathcal{M})^n$, $b \in \mathrm{Ob}(\mathcal{M})$ and each 1-cell $f\colon \langle a\rangle \to b$,

$$\gamma(Gf; \langle \theta_{a_j}\rangle) = \gamma(\theta_b; Ff) \tag{2.1.5}$$

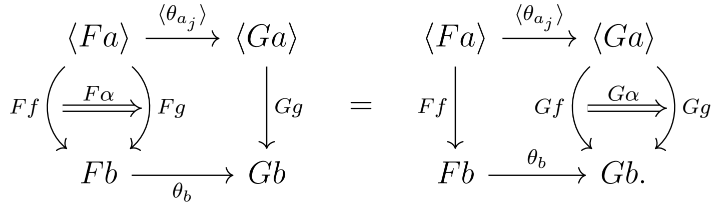
holds in $\mathcal{N}(\langle Fa\rangle; Gb)$ and that, for every 2-cell $\alpha\colon f \to g$ in $\mathcal{M}(\langle a\rangle; b)(f, g)$,

$$\gamma(G\alpha; \langle 1_{\theta_{a_j}}\rangle) = \gamma(1_{\theta_b}; F\alpha) \tag{2.1.6}$$

in $\mathcal{N}(\langle Fa\rangle; Gb)$. We can express (2.1.5) diagrammatically as the commutativity of the square

$$
\begin{array}{ccc}
\langle Fa\rangle & \xrightarrow{\langle \theta_{a_j}\rangle} & \langle Ga\rangle \\
{\scriptstyle Ff}\downarrow & & \downarrow{\scriptstyle Gf} \\
Fb & \xrightarrow{\theta_b} & Gb,
\end{array}
$$

where the composition of adjacent 1-cells is done through $\gamma$ and a square represents an equality between composite 1-cells. In the same fashion, and using (2.1.5), we can express (2.1.6) as the equality of multicategorical pasting diagrams



Here the concatenation of adjacent 2-cells is done through $\gamma$, and an arrow labeled with the 1-cell $h$ is interpreted as the 2-cell $1_h : h \to h$. For example, the left hand side diagram represents $\gamma(1_{\theta_b}, F\alpha)$ while the right hand side represents $\gamma(G\alpha, \langle\theta_{\alpha_j}\rangle)$. The empty squares represent equalities between composite 1-cells.

Next, we define horizontal and vertical compositions of $C$-multinatural transformations.

**Definition 2.1.13.** [Yau24b, Def. 3.2.7]

Suppose given $\theta\colon F \Rightarrow G$, $\zeta\colon G \Rightarrow H$ $C$-multinatural transformations with $F, G, H\colon \mathcal{M} \to \mathcal{N}$ $C$-multifunctors. The *vertical composition* $\zeta\theta\colon F \Rightarrow H$ is defined as having as component at each $a \in \mathrm{Ob}(\mathcal{M})$ $(\zeta\theta)_a$, the composite

$$1 \xrightarrow{\cong} 1 \otimes 1 \xrightarrow{\zeta_a \otimes \theta_a} \mathcal{N}(Ga; Ha) \otimes \mathcal{N}(Fa; Ga) \xrightarrow{\gamma} \mathcal{N}(Fa; Ha).$$

Suppose that $\theta\colon F \Rightarrow G$ and $\zeta\colon F' \Rightarrow G'$ are $C$-multinatural transformations with $F, G\colon \mathcal{M} \to \mathcal{N}$ and $F', G'\colon \mathcal{N} \to \mathcal{Q}$ $C$-multifunctors. The *horizontal composition* $\zeta * \theta\colon F'F \Rightarrow G'G$ is defined as the $C$-multinatural transformation with component at each $a \in \mathrm{Ob}(\mathcal{M})$, given by the composite

$$\begin{array}{ccccc} 1 & \xrightarrow{\qquad\qquad(\zeta*\theta)_a\qquad\qquad} & & & \mathcal{Q}(F'Fa; G'Ga) \\ \downarrow\cong & & & & \uparrow\gamma \\ 1 \otimes 1 & \xrightarrow{\zeta_{Ga} \otimes \theta_a} & \mathcal{Q}(F'Ga; G'Ga) \otimes \mathcal{N}(Fa; Ga) & \xrightarrow{1 \otimes F'} & \mathcal{Q}(F'Ga; G'Ga) \otimes \mathcal{Q}(F'Fa; F'Ga). \end{array}$$

**Remark 2.1.14.** When $C = \mathbf{Cat}$ and given $\theta\colon F \Rightarrow G$, $\zeta\colon G \Rightarrow H$ **Cat**-multinatural transformations with $F, G, H\colon \mathcal{M} \to \mathcal{N}$ $C$-multifunctors and $a \in \mathrm{Ob}(\mathcal{M})$,

$$(\zeta\theta)_a = \gamma(\zeta_a, \theta_a.) \tag{2.1.7}$$

On the other hand, if $\theta\colon F \Rightarrow G$ and $\zeta\colon F' \Rightarrow G'$ are **Cat**-multinatural transformations with $F, G\colon \mathcal{M} \to \mathcal{N}$ and $F', G'\colon \mathcal{N} \to \mathcal{Q}$ **Cat**-multifunctors,

$$(\zeta * \theta)_a = \gamma(\zeta_{Ga}; F'\theta_a). \tag{2.1.8}$$

Yau proves in [Yau24b] that Definitions 2.1.2, 2.1.7, 2.1.10 and 2.1.13 assemble together to give the 2-category $C$-**Multicat**, with 0-cells consisting of $C$-multicategories, 1-cells symmetric $C$-multifunctors, and 2-cells $C$-multinatural transformations.

There is a non symmetric variant where we drop the requirement that the $C$-multifunctors preserve the symmetric group action, as well as dropping the coherence axioms related to the symmetric group action, but we won't refer to this 2-category again.
For the rest of the article we fix our symmetric monoidal category $C$ to be **Cat**, with the symmetric monoidal structure induced by products. In this context we can define a pseudo symmetric variant of this 2-category, namely $\mathbf{Cat\text{-}Multicat}^{\mathbf{ps}}$ using the 2-categorical structure of **Cat**. The objects of $\mathbf{Cat\text{-}Multicat}^{\mathbf{ps}}$ are still **Cat**-multicategories, but the 1-cells are pseudo symmetric **Cat**-multifunctors: **Cat**-multifunctors where we only require that they preserve the symmetric group action up to coherent isomorphisms.

**Definition 2.1.15.** [Yau24b, Def. 4.1.1] Suppose that $\mathcal{M}, \mathcal{N}$ are **Cat**-multicategories. A *pseudo symmetric* **Cat**-*multifunctor* $F\colon \mathcal{M} \to \mathcal{N}$ consists of the following data:

- A function on object sets $F\colon \mathrm{Ob}(\mathcal{M}) \to \mathrm{Ob}(\mathcal{N})$.
- For each $\langle a \rangle \in \mathrm{Ob}(\mathcal{M})^n$ and $b \in \mathrm{Ob}(\mathcal{M})$, a component functor

$$\mathcal{M}(\langle a \rangle; b) \xrightarrow{F} \mathcal{N}(\langle Fa \rangle; Fb).$$

- For each $\sigma \in \Sigma_n$, $\langle a \rangle \in \mathrm{Ob}(\mathcal{M})^n$, $b \in \mathrm{Ob}(\mathcal{M})$, a natural isomorphism $F_{\sigma,\langle a \rangle,b}$

$$\begin{array}{ccc} \mathcal{M}(\langle a \rangle; b) & \xrightarrow{F} & \mathcal{N}(\langle Fa \rangle; Fb) \\ {\scriptstyle\sigma}\downarrow & \overset{\cong}{\Nearrow}{\scriptstyle F_{\sigma,\langle a \rangle,b}} & \downarrow{\scriptstyle\sigma} \\ \mathcal{M}(\langle a \rangle\sigma; b) & \xrightarrow[F]{} & \mathcal{N}(\langle Fa \rangle\sigma; Fb). \end{array}$$

  When $\langle a \rangle$ and $b$ are clear from the context we write simply $F_\sigma$, and if $f \in \mathrm{Ob}(\mathcal{M}(\langle a \rangle, b))$ we will denote by $F_{\sigma,\langle a \rangle,b;f} = F_{\sigma;f}\colon F(f\sigma) \to F(f)\sigma$ the 2-cell in $\mathcal{N}(\langle Fa \rangle\sigma; Fb)$ corresponding to the component of $F_\sigma$ at $f$. Naturality for $F_\sigma$ means that given $\alpha\colon f \to g$ a 2-cell in $\mathcal{M}(\langle a \rangle; b)(f, g)$, the following diagram commutes in $\mathcal{N}(\langle Fa \rangle\sigma; b)$ :

$$\begin{array}{ccc} F(f\sigma) & \xrightarrow{F_{\sigma;f}} & F(f)\sigma \\ {\scriptstyle F(\alpha\sigma)}\downarrow & & \downarrow{\scriptstyle (F\alpha)\sigma} \\ F(g\sigma) & \xrightarrow[F_{\sigma;g}]{} & F(g)\sigma. \end{array} \tag{2.1.9}$$

These data are subject to the same axioms of unit and composition preservation (2.1.4) as a symmetric **Cat**-multifunctor, but we replace the symmetric group action preservation axiom by the following four axioms.

- **Unit permutation:** Let $n \geq 0$, $\langle a \rangle \in \mathrm{Ob}(\mathcal{M})^n$ and $b \in \mathrm{Ob}(\mathcal{M})$, then

$$F_{\mathrm{id}_n, \langle a \rangle, b} = 1_F. \tag{2.1.10}$$

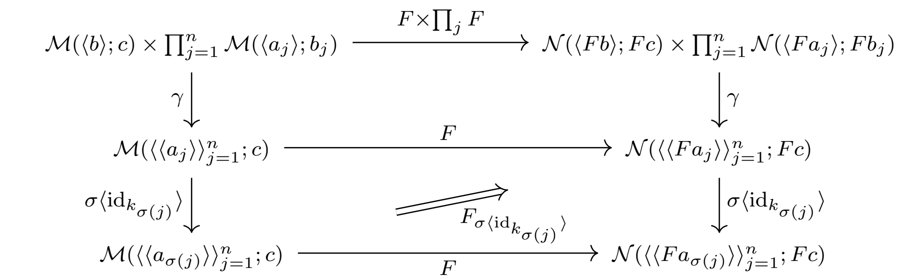
- **Product permutation:** This axiom expresses the coherence of the natural isomorphisms $F_\sigma$, for varying $\sigma$, with respect to the symmetric group action. Let $n \geq 0$, $\langle a \rangle \in \mathrm{Ob}(\mathcal{M})^n$, $b \in \mathrm{Ob}(M)$ and $\sigma, \tau \in \Sigma_n$. Then, the following equality of pasting diagrams holds.

$$\begin{array}{ccc} \mathcal{M}(\langle a \rangle; b) & \xrightarrow{F} & \mathcal{N}(\langle Fa \rangle; Fb) \\ \sigma \downarrow & \overset{F_\sigma}{\Rightarrow} & \downarrow \sigma \\ \mathcal{M}(\langle a \rangle \sigma; b) & \xrightarrow[F]{} & \mathcal{N}(\langle Fa \rangle \sigma; Fb) \\ \tau \downarrow & \overset{F_\tau}{\Rightarrow} & \downarrow \tau \\ \mathcal{M}(\langle a \rangle \sigma\tau; b) & \xrightarrow[F]{} & \mathcal{N}(\langle Fa \rangle \sigma\tau; Fb) \end{array} \quad = \quad \begin{array}{ccc} \mathcal{M}(\langle a \rangle; b) & \xrightarrow{F} & \mathcal{N}(\langle Fa \rangle; Fb) \\ \sigma\tau \downarrow & \overset{F_{\sigma\tau}}{\Rightarrow} & \downarrow \sigma\tau \\ \mathcal{M}(\langle a \rangle \sigma\tau; b) & \xrightarrow[F]{} & \mathcal{N}(\langle Fa \rangle \sigma\tau; Fb). \end{array}$$

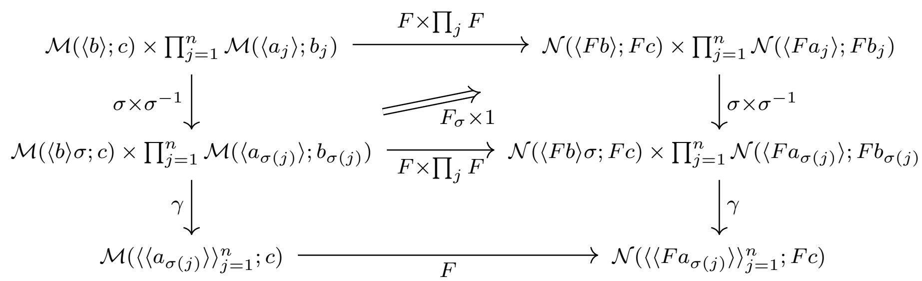
  Thus, for every 1-cell $f \in \mathrm{Ob}(\mathcal{M}(\langle a \rangle; b))$, the following diagram of 2-cells commutes in $\mathcal{N}(\langle Fa \rangle; Fb)$:

$$\begin{array}{ccccc} & & F(f\sigma)\tau & & \\ & \overset{F_{\tau; f\sigma}}{\nearrow} & & \overset{(F_{\sigma; f})\tau}{\searrow} & \\ F(f\sigma\tau) & & \xrightarrow[F_{\sigma\tau; f}]{} & & F(f)\sigma\tau. \end{array} \tag{2.1.11}$$

- **Top equivariance:** For every $c \in \mathrm{Ob}(\mathcal{M})$, $n \geq 0$, $\langle b \rangle = \langle b_j \rangle_{j=1}^n \in \mathrm{Ob}(M)^n$, $k_j \geq 0$ for $1 \leq j \leq n$, and $\langle a_j \rangle = \langle a_{j,i} \rangle_{i=1}^{k_j} \in \mathrm{Ob}(\mathcal{M})^{k_j}$ for $1 \leq j \leq n$, and $\sigma \in \Sigma_n$, the following two pasting diagrams are equal.



$$\parallel$$



Here $\sigma\langle \mathrm{id}_{k_{\sigma(j)}}\rangle = \sigma\langle \mathrm{id}_{k_{\sigma(1)}}, \ldots, \mathrm{id}_{k_{\sigma(n)}}\rangle$. This means that for 1-cells $f \in \mathrm{Ob}(\mathcal{M}(\langle b\rangle; c))$ and $g_j \in \mathrm{Ob}(\mathcal{M}(\langle a_j\rangle; b_j))$ for $1 \le j \le n$,

$$F_{\sigma\langle \mathrm{id}_{k_{\sigma(j)}}\rangle;\gamma(f;\langle g_j\rangle)} = \gamma\left(F_{\sigma;f}; \langle 1_{Fg_{\sigma(j)}}\rangle_{j=1}^{n}\right). \tag{2.1.12}$$

The domains and codomains of these pasting diagrams are equal by top equivariance in $\mathcal{M}$ and $\mathcal{N}$, and the fact that $F$ preserves $\gamma$ implies the commutativity of the empty rectangles, see [Yau24b].

- **Bottom Equivariance:** For every $c \in \mathrm{Ob}(\mathcal{M})$, $n \geq 0$, $\langle b\rangle = \langle b_j\rangle_{j=1}^{n} \in \mathrm{Ob}(M)^n$, $k_j \geq 0$ for $1 \le j \le n$, and $\langle a_j\rangle = \langle a_{j,i}\rangle_{i=1}^{k_j} \in \mathrm{Ob}(\mathcal{M})^{k_j}$ for $1 \le j \le n$ and $1 \le i \le k_j$, and $\tau_j \in \Sigma_{k_j}$, the following two pasting diagrams are equal.

$$
\begin{array}{ccc}
\mathcal{M}(\langle b\rangle; c) \times \prod_{j=1}^n \mathcal{M}(\langle a_j\rangle; b_j) & \xrightarrow{F\times \prod_j F} & \mathcal{N}(\langle Fb\rangle; Fc) \times \prod_{j=1}^n (\langle Fa_j\rangle; Fb_j) \\
\gamma \downarrow & & \downarrow \gamma \\
\mathcal{M}(\langle\langle a_j\rangle\rangle_{j=1}^n; c) & \xrightarrow{F} & \mathcal{N}(\langle\langle Fa_j\rangle\rangle_{j=1}^n; Fc) \\
\mathrm{id}_n\langle\tau_j\rangle \downarrow & \overset{}{\Rightarrow}_{F_{\mathrm{id}_n\langle\tau_i\rangle}} & \downarrow \mathrm{id}_n\langle\tau_j\rangle \\
\mathcal{M}(\langle\langle a_j\rangle\tau_j\rangle_{j=1}^n; c) & \xrightarrow[F]{} & \mathcal{N}(\langle\langle Fa_j\rangle\tau_j\rangle_{j=1}^n; Fc)
\end{array}
$$

$$\|$$

$$
\begin{array}{ccc}
\mathcal{M}(\langle b\rangle; c) \times \prod_{j=1}^n \mathcal{M}(\langle a_j\rangle; b_j) & \xrightarrow{F\times \prod_j F} & \mathcal{N}(\langle Fb\rangle; Fc) \times \prod_{j=1}^n \mathcal{N}(\langle Fa_j\rangle; Fb_j) \\
\mathrm{id}\times\prod_j \tau_j \downarrow & \Rightarrow_{1\times\prod_j F_{\tau_j}} & \downarrow \mathrm{id}\times\prod_j \tau_j \\
\mathcal{M}(\langle b\rangle; c) \times \prod_{j=1}^n \mathcal{M}(\langle a_j\rangle\tau_j; b_j) & \xrightarrow[F\times\prod_j F]{} & \mathcal{N}(\langle Fb\rangle; Fc) \times \prod_{j=1}^n \mathcal{N}(\langle\langle Fa_j\rangle\tau_j\rangle; Fb_j) \\
\gamma \downarrow & & \downarrow \gamma \\
\mathcal{M}(\langle\langle a_j\rangle\tau_j\rangle_{j=1}^n; c) & \xrightarrow[F]{} & \mathcal{N}(\langle\langle Fa_j\rangle\tau_j\rangle_{j=1}^n; Fc)
\end{array}
$$

This means that for 1-cells $f\colon \langle b\rangle \to c$ and $g_j\colon \langle a_j\rangle \to b_j$ for $1 \le j \le n$,

$$F_{\mathrm{id}_n\langle\tau_j\rangle;\gamma(f;\langle g_j\rangle)} = \gamma(1_{Ff}; \langle F_{\tau_j;g_j}\rangle) \tag{2.1.13}$$

as 2-cells in $\mathcal{N}(\langle\langle Fa_j\rangle\tau_j\rangle; Fc)$. The domain and codomain of these pasting diagrams are equal by bottom equivariance for $\mathcal{M}$ and $\mathcal{N}$, and the preservation of $\gamma$ by $F$ guarantees that the empty squares commute, see [Yau24b].

Next we describe the horizontal composition of 1-cells in the 2-category **Cat-Multicat^ps**.

**Definition 2.1.16.** [Yau24b, Def. 4.1.1] Let $F\colon \mathcal{M} \to \mathcal{N}$, and $G\colon \mathcal{N} \to \mathcal{Q}$ be pseudo symmetric **Cat**-multifunctors. We define the pseudo symmetric functor $GF\colon \mathcal{M} \to \mathcal{Q}$. On objects $GF$ is the composite function $GF\colon \mathrm{Ob}(\mathcal{M}) \to \mathrm{Ob}(\mathcal{Q})$. The composite component functor is given for $\langle a\rangle \in \mathrm{Ob}(\mathcal{M})^n$, and $b \in \mathrm{Ob}(\mathcal{M})$ by the pasting

$$\mathcal{M}(\langle a\rangle; b) \xrightarrow{F} \mathcal{N}(\langle Fa\rangle; b) \xrightarrow{G} \mathcal{Q}(\langle GFa\rangle; GFb).$$

The symmetry isomorphisms are given for each $\sigma \in \Sigma_n$, $\langle a\rangle \in \mathrm{Ob}(\mathcal{M})$, and $b \in \mathrm{Ob}(\mathcal{M})$ by

$$
\begin{array}{ccccc}
\mathcal{M}(\langle a\rangle; b) & \xrightarrow{F} & \mathcal{N}(\langle Fa\rangle; Fb) & \xrightarrow{G} & \mathcal{Q}(\langle GFa\rangle; GFb) \\
\sigma\downarrow & \overset{F_{\sigma,\langle a\rangle,b}}{\nearrow\!\!\!\nearrow} & \downarrow\sigma & \overset{G_{\sigma,\langle Fa\rangle,Fb}}{\nearrow\!\!\!\nearrow} & \downarrow\sigma \\
\mathcal{M}(\langle a\rangle\sigma; b) & \xrightarrow{F} & \mathcal{N}(\langle Fa\rangle\sigma; fb) & \xrightarrow{G} & \mathcal{Q}(\langle GFa\rangle\sigma; GFb).
\end{array}
$$

That is, for each 1-cell $f\colon \langle a\rangle \to b$, the $f$ component of $GF_\sigma$ is given by the composite

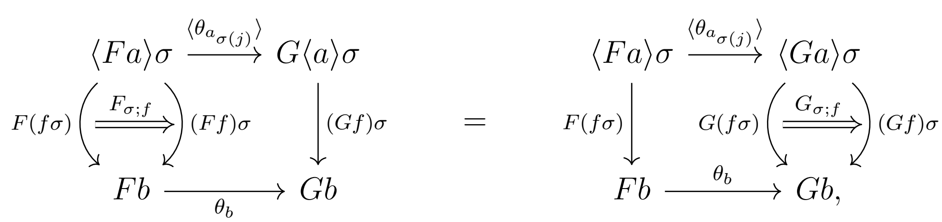
$$
\begin{array}{ccc}
 & G((Ff)\sigma) & \\
\overset{G(F_{\sigma;f})}{\nearrow} & & \overset{G_{\sigma;Ff}}{\searrow} \\
GF(f\sigma) & \xrightarrow[(GF)_{\sigma;f}]{} & (GFf)\sigma.
\end{array}
\tag{2.1.14}
$$

Next we define the 2-cells of the category $\mathbf{Cat}\text{-}\mathbf{Multicat}^{\mathbf{ps}}$.

**Definition 2.1.17.** [Yau24b, Def. 4.2.1] Suppose that $F, G\colon \mathcal{M} \to \mathcal{N}$ are pseudo symmetric **Cat**-multifunctors. A *pseudo symmetric* **Cat**-multinatural transformation $\theta\colon F \Rightarrow G$ is the data of a component 1-cell $\theta_a : Fa \to Ga$ for each $a \in \mathrm{Ob}(\mathcal{M})$ subject to axioms (2.1.5), (2.1.6) and the following extra axiom. For each $n \geq 0$, $\langle a\rangle \in \mathrm{Ob}(\mathcal{M})^n$, $b \in \mathrm{Ob}(M)$, object $f \in \mathrm{Ob}(\mathcal{M}(\langle a\rangle; b))$, and permutation $\sigma \in \Sigma_n$, the following arrow equality holds in the category $\mathcal{N}(\langle Fa\rangle\sigma; b)$,

$$
\gamma\left(1_{\theta_b}; F_{\sigma;f}\right) = \gamma\left(G_{\sigma;f}; \langle 1_{\theta_{a_{\sigma(j)}}}\rangle\right). \tag{2.1.15}
$$

This can also be expressed diagrammatically as the equality of multicategorical pasting diagrams



where the diagrams are interpreted as in Remark 2.1.12, the squares commuting by (2.1.5) and top and bottom equivariance for $\mathcal{N}$, see [Yau24b].

We define the vertical and horizontal composition of pseudo symmetric **Cat**-multinatural transformations in the same way that we did for symmetric ones, through diagrams (2.1.7) and (2.1.8).

It is a theorem of Yau [Yau24b] that the data we have just defined gives the structure of a 2-category, namely $\mathbf{Cat}\text{-}\mathbf{Multicat}^{\mathbf{ps}}$. Definition 2.2.8 says that we can describe this 2-category solely in terms of symmetric **Cat**-multifunctors and symmetric **Cat**-multinatural transformations.

## 2.2 Equivalent definition of Pseudo Symmetry

To prove our first result we use finite products in the category **Cat-Multicat**. Having just the 1-categorical structure in mind, the products in **Cat-Multicat** are given in the following way. If $\mathcal{M}$ and $\mathcal{N}$ are two **Cat**-multicategories, then $\mathcal{M} \times \mathcal{N}$ has objects $\mathrm{Ob}(\mathcal{M} \times \mathcal{N}) = \mathrm{Ob}(\mathcal{M}) \times \mathrm{Ob}(\mathcal{N})$. Now, for $n \geq 0$, $\langle a \rangle \in \mathrm{Ob}(\mathcal{M})^n$, $\langle c \rangle \in \mathrm{Ob}(\mathcal{N})^n$, $b \in \mathrm{Ob}(\mathcal{M})$, and $d \in \mathrm{Ob}(\mathcal{N})$, we define

$$\mathcal{M} \times \mathcal{N}(\langle (a, c) \rangle; (b, d)) = \mathcal{M}(\langle a \rangle; b) \times \mathcal{N}(\langle c \rangle; d).$$

The composition $\gamma$ of $\mathcal{M} \times \mathcal{N}$, as well as the $\Sigma_*$ action and the multicategorical units, are defined componentwise. Next we define the pseudo symmetric multifunctor $\eta_{\mathcal{M}}$ appearing in the statement of 1.0.1.

**Definition 2.2.1.** Let $\mathcal{M}$ be a **Cat**-multicategory. We define the pseudo symmetric **Cat**-multifunctor $\eta_{\mathcal{M}} \colon \mathcal{M} \to \mathcal{M} \times E\Sigma_*$ which, when there is no room for confusion, we will denote $\eta$. For an object $a \in \mathrm{Ob}(\mathcal{M})$ as $\eta(a) = (a, *)$. We will abuse notation and denote the object $(a, *)$ of $\mathcal{M} \times E\Sigma_*$ as $a$.
For $n \geq 0$, $\langle a \rangle \in \mathrm{Ob}(\mathcal{M})^n$ and $b \in \mathrm{Ob}(\mathcal{M})$ we need to define a functor $\eta \colon \mathcal{M}(\langle a \rangle; b) \to \mathcal{M}(\langle a \rangle; b) \times E\Sigma_n$. For a 1-cell $f \colon \langle a \rangle \to b$, we define

$$\eta(f) = (f, \mathrm{id}_n) \in \mathrm{Ob}(\mathcal{M}(a; b) \times E\Sigma_n).$$

Similarly, for a 2-cell $\alpha \colon f \to g$ in $\mathcal{M}(\langle a \rangle; b)$,

$$\eta(\alpha) = (\alpha, 1_{\mathrm{id}_n}) \in \mathcal{M}(\langle a \rangle; b) \times E\Sigma_n((f, \mathrm{id}_n)), (g, \mathrm{id}_n).$$

Next, we define the components of the pseudo symmetry isomorphisms. For $\sigma, \tau \in \Sigma_n$ we will denote from here on by $E_\sigma^\tau$ the unique arrow $\sigma \to \tau$ in $E\Sigma_n$. For $\sigma \in \Sigma_n$, $\langle a \rangle \in \mathrm{Ob}(\mathcal{M})^n$, and $b \in \mathrm{Ob}(\mathcal{M})$ we need to define a natural isomorphism $\eta_{\sigma, \langle a \rangle, b} \colon (\eta \circ \sigma) \to (\sigma \circ \eta)$ that fits in the following diagram

$$\begin{array}{ccc} \mathcal{M}(\langle a \rangle; b) & \xrightarrow{\eta} & \mathcal{M}(\langle a \rangle; b) \times E\Sigma_n \\ {\scriptstyle \sigma}\downarrow & \overset{\cong}{\underset{\eta_{\sigma,\langle a \rangle,b}}{\nearrow}} & \downarrow{\scriptstyle \sigma \times \sigma} \\ \mathcal{M}(\langle a \rangle \sigma, b) & \xrightarrow[\eta]{} & \mathcal{M}(\langle a \rangle \sigma; b) \times E\Sigma_n. \end{array}$$

The isomorphism $\eta_{\sigma,\langle a\rangle,b}$ is defined for every 1-cell $f\colon \langle a\rangle \to b$ as the 2-cell

$$\eta_{\sigma;f} = (1_{f\sigma}, E^{\sigma}_{\mathrm{id}})\colon (f\sigma, \mathrm{id}_n) \to (f\sigma, \sigma).$$

**Lemma 2.2.2.** *Let $\mathcal{M}$ be a* **Cat***-multicategory, then $\eta_{\mathcal{M}}\colon \mathcal{M} \to \mathcal{M} \times E\Sigma_*$ is pseudo symmetric.*

*Proof.* We start from a non symmetric multifunctor $\eta\colon \mathcal{M} \to \mathcal{M} \times E\Sigma_*$ that is the identity on the first coordinate and the multicategorical unit in the second coordinate. As a non symmetric multifunctor, $\eta$ preserves units and $\gamma$ composition. We need to show that $\eta$ is a pseudo symmetric **Cat**-multifunctor. The naturality of $\eta_{\sigma;f}$ follows from the commutativity of the following diagram for any 2-cell $\alpha\colon f \to g$:

$$\begin{array}{ccc} (f\sigma, \mathrm{id}_n) & \xrightarrow{(1_{f\sigma}, E^{\sigma}_{\mathrm{id}_n})} & (f\sigma, \sigma) \\ {\scriptstyle(\alpha\sigma, 1_{\mathrm{id}_n})}\downarrow & & \downarrow{\scriptstyle(\alpha\sigma, 1_{\sigma})} \\ (g\sigma, \mathrm{id}_n) & \xrightarrow[(1_{g\sigma}, E^{\sigma}_{\mathrm{id}_n})]{} & (g\sigma, \sigma). \end{array}$$

Next we focus on the coherence axioms. The unit permutation axiom (2.1.10) holds since, for all $\langle a\rangle \in \mathrm{Ob}(\mathcal{M})^n$, $b \in \mathrm{Ob}(\mathcal{M})$, and $f\colon \langle a\rangle \to b$,

$$\eta_{\mathrm{id}_n;f} = (1_{f\mathrm{id}_n}, E^{\mathrm{id}_n}_{\mathrm{id}_n}) = (1_f, 1_{\mathrm{id}_n}) = 1_{(f,\mathrm{id}_n)} = 1_{\eta(f)}.$$

Let $\langle a\rangle, b$ and $f$ be as before, the product permutation axiom (2.1.11) holds again by definition. Indeed, for $\tau, \sigma \in \Sigma_n$, we have

$$\eta_{\sigma\tau;f} = (1_{f\sigma\tau}, E^{\sigma\tau}_{\mathrm{id}}) = (1_{f\sigma\tau}, E^{\sigma\tau}_{\tau}) \circ (1_{f\sigma\tau}, E^{\tau}_{\mathrm{id}_n}) = (\eta_{\sigma;f}\tau) \circ \eta_{\tau;f\sigma}.$$

For Top Equivariance (2.1.12), suppose that $c \in \mathrm{Ob}(\mathcal{M})$, $n \geq 1$, $\langle b\rangle = \langle b_j\rangle_{j=1}^n \in \mathrm{Ob}(\mathcal{M})^n$, $k_j \geq 0$ for $1 \leq j \leq n$, $\langle a_j\rangle = \langle a_{j,i}\rangle_{i=1}^{k_j} \in \mathrm{Ob}(\mathcal{M})^{k_j}$ for $1 \leq j \leq n$, $\sigma \in \Sigma_n$, $f \in \mathrm{Ob}(\mathcal{M}(\langle b\rangle; c))$, and $g_j \in \mathrm{Ob}(\mathcal{M}(\langle a_j\rangle; b_j))$. We have that

$$\begin{aligned} \gamma(\eta_{\sigma;f}; \langle 1_{i(g_{\sigma(j)})}\rangle) &= \gamma((1_{f\sigma}, E^{\sigma}_{\mathrm{id}}); \langle (1_{g_{\sigma(j)}}, 1_{\mathrm{id}_{k_{\sigma(j)}}})\rangle) \\ &= \left( (\gamma(1_{f\sigma}; 1_{g_{\sigma(j)}}), \gamma\left( E^{\sigma}_{\mathrm{id}}; E^{\mathrm{id}_{k_{\sigma(j)}}}_{\mathrm{id}_{k_{\sigma(j)}}} \right) \right) \\ &= \left( 1_{\gamma(f;\langle g_{\sigma(j)}\rangle)}, E^{\sigma\langle \mathrm{id}_{k_{\sigma(j)}}\rangle}_{\mathrm{id}\langle \mathrm{id}_{k_{\sigma(j)}}\rangle} \right) \end{aligned}$$

$$= \left(1_{\gamma(f;\langle g_j\rangle)\sigma\langle \mathrm{id}_{k_{\sigma(j)}}\rangle}, E_{\mathrm{id}_k}^{\sigma\langle \mathrm{id}_{k_{\sigma(j)}}\rangle}\right)$$
$$= \eta_{\sigma\langle \mathrm{id}_{k_{\sigma(j)}}\rangle;\gamma(f;\langle g_j\rangle)}.$$

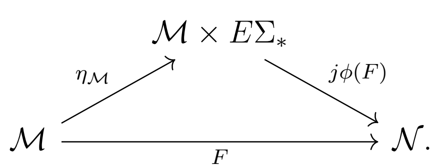
For Bottom Equivariance, let $c$, $n$, $\langle b\rangle$, $k_j$ for $1 \le j \le n$, $\langle a_j\rangle$ for $1 \le j \le n$, $f$ and $g_j$ be as above and let $\tau_j \in \Sigma_{k_j}$ for $1 \le j \le n$. We also let $k = \sum_{j=1}^{n} k_j$. Bottom Equivariance (2.1.13) for $i$ is

$$\begin{aligned}
\gamma\left(1_{if}; \langle \eta_{\tau_j;g_j}\rangle\right) &= \gamma\left((1_f, 1_{\mathrm{id}_n}); \langle (1_{g_j\tau_j}, E_{\mathrm{id}_{k_j}}^{\tau_j})\rangle\right)\\
&= \left(\gamma(1_f; 1_{g_j\tau_j}), 1_{\mathrm{id}_n}\langle E_{\mathrm{id}_{k_j}}^{\tau_j}\rangle\right)\\
&= \left(1_{\gamma(f;\langle g_j\tau_j\rangle)}, E_{\mathrm{id}_k}^{\mathrm{id}_n\langle\tau_j\rangle}\right)\\
&= \left(1_{\gamma(f;\langle g_j\rangle)\mathrm{id}_n\langle\tau_j\rangle}, E_{\mathrm{id}_k}^{\mathrm{id}_n\langle\tau_j\rangle}\right)\\
&= \eta_{\mathrm{id}\langle\tau_j\rangle,\gamma(f;\langle g_j\rangle)}.
\end{aligned}$$

Thus, we conclude that $\eta\colon \mathcal{M} \to \mathcal{M}\times E\Sigma_*$ is a pseudo symmetric **Cat**-multifunctor. ■

Recall that $j\colon$ **Cat-Multicat** $\to$ **Cat-Multicat$^{\mathbf{ps}}$** denotes the inclusion functor. We are ready to present a proof of 1.0.1.

**Theorem 2.2.3.** *Let $\mathcal{M}$ and $\mathcal{N}$ be a **Cat**-multicategories and $F\colon \mathcal{M} \to \mathcal{N}$ a pseudo symmetric* **Cat***-multifunctor. There exists a unique symmetric* **Cat***-multifunctor $\phi(F)\colon \mathcal{M}\times E\Sigma_* \to \mathcal{N}$ such that the following diagram commutes:*




*That is, $F = j\phi(F)\circ\eta_{\mathcal{M}}$ in* **Cat-Multicat$^{\mathbf{ps}}$**.

*Proof of Theorem 1.0.1.* For uniqueness, suppose that $\phi(F)\colon \mathcal{M}\times E\Sigma_* \to \mathcal{N}$ is a symmetric **Cat**-multifunctor satisfying $F = (j\phi(F))\circ\eta$. We will abuse notation and write $j\phi(F) = \phi(F)$. We will prove there is a unique way of defining $\phi(F)$. At the level of the objects of the multicategory we must have $\phi(F)(a,*) = \phi(F)\circ\eta(a) = F(a)$ for each $a \in \mathrm{Ob}(\mathcal{M})$. Next, we show that there is a unique way of defining each component functor of $\phi(F)$. For this let $\langle a\rangle \in \mathrm{Ob}(\mathcal{M})^n$, $b \in \mathrm{Ob}(\mathcal{M})$, and consider the functor $\phi(F)\colon \mathcal{M}(\langle a\rangle; b) \times E\Sigma_n \to \mathcal{N}(\langle Fa\rangle; Fb)$. If $f\colon \langle a\rangle \to b$ is a 1-cell and $\sigma \in \Sigma_n$, we must have that

$$\begin{aligned}\phi(F)(f,\sigma) &= \phi(F)((f\sigma^{-1}, \mathrm{id}_n)\sigma)\\ &= \phi(F)((f\sigma^{-1}, \mathrm{id}_n))\sigma\\ &= \phi(F)\circ\eta(f\sigma^{-1})\sigma\\ &= F(f\sigma^{-1})\sigma, \end{aligned} \tag{2.2.1}$$

where in the second equality we used that $\phi(F)$ is symmetric. So the values of the component functors of $\phi(F)$ on $n$-ary 1-cells are uniquely determined by $F$. In exactly the same fashion, for $\langle a\rangle, b$ and $\sigma$ as before, $f, g\colon \langle a\rangle \to b$, and $\alpha\colon f \to g$ a 2-cell,

$$\phi(F)(\alpha, 1_\sigma) = F(\alpha\sigma^{-1})\sigma. \tag{2.2.2}$$

Finally, if $f, \sigma$ are as before and $\tau \in \Sigma_n$, we get that

$$\begin{aligned}\phi(F)(1_f, E^\tau_\sigma) &= \phi(F)(1_{f\sigma^{-1}}\sigma, E^{\tau\sigma^{-1}}_{\mathrm{id}}\sigma)\\ &= \phi(F)((1_{f\sigma^{-1}}, E^{\tau\sigma^{-1}}_{\mathrm{id}}))\sigma\\ &= \phi(F)(\eta_{\tau\sigma^{-1};f\tau^{-1}})\sigma\\ &= (\phi(F)\circ\eta_{\tau\sigma^{-1};f\tau^{-1}})\sigma\\ &= (F_{\tau\sigma^{-1};f\tau^{-1}})\sigma. \end{aligned} \tag{2.2.3}$$

We have used the definition of composition of pseudo symmetric **Cat**-multifunctors (2.1.14) where we see $\phi(F)$ trivially as a pseudo symmetric functor. For $\langle a\rangle, b, f, g, \alpha, \sigma$, and $\tau$ as before, we can write the morphism $(\alpha\colon f \to g, E^\tau_\sigma)$ in $\mathcal{M}(\langle a\rangle; b) \times \Sigma_n$ as $(1_y, E^\tau_\sigma) \circ (f, 1_\sigma)$. Since both $\phi(F)(1_y, E^\tau_\sigma)$ and $\phi(F)(f, 1_\sigma)$ are uniquely determined by $F$, we conclude that the component functors of $\phi(F)$ are uniquely determined. We have proven the uniqueness of $\phi(F)$.

Next we prove the existence of $\phi(F)$. By uniqueness, we have no choice but to define $\phi(F)(b, *) = Fb$ for any $b \in \mathrm{Ob}(M)$. Likewise, for $\langle a\rangle \in \mathrm{Ob}(\mathcal{M})^n$ and $b \in \mathrm{Ob}(\mathcal{M})$, uniqueness forces the definition of the component functor $\phi(F)\colon \mathcal{M}(\langle a\rangle; b) \times \Sigma_n \to \mathcal{N}(\langle Fa\rangle; b)$. For $f\colon \langle a\rangle \to b$, a 1-cell in $\mathcal{M}(\langle a\rangle; b)$ and $\sigma \in \Sigma_n$ we define

$$\phi(F)(f,\sigma) = F(f\sigma^{-1})\sigma \tag{2.2.4}$$

as in (2.2.1). For a 2-cell $\alpha\colon f \to g$ in $\mathcal{M}(\langle a\rangle; b)(f, g)$, we define

$$\phi(F)(\alpha, 1_\sigma) = F(\alpha\sigma^{-1})\sigma \tag{2.2.5}$$

as in (2.2.2). For $\tau \in \Sigma_n$ we define

$$\phi(F)(1_f, E^\tau_\sigma) = (F_{\tau\sigma^{-1};f\tau^{-1}})\,\sigma \tag{2.2.6}$$

as in (2.2.3).

We still have to prove that $\phi(F) : \mathcal{M}(\langle a\rangle; b)\times\Sigma_n$ is well defined and extend our definition to all 2-cells. Notice that for a 1-cell $f\colon \langle a\rangle \to b$ our definition is ambiguous for the identity arrow $(1_f, 1_\sigma)$ since both (2.2.5) and (2.2.6) apply. However, $\phi(F)$ is well defined in this case since $F$ is a functor componentwise and so, it preserves identities. Explicitly,

$$F(1_f\sigma^{-1})\sigma = F(1_{f\sigma^{-1}})\sigma = 1_{F(f\sigma^{-1})}\sigma = 1_{F(f\sigma^{-1})\sigma},$$

and

$$(F_{\sigma\sigma^{-1},f\sigma^{-1}})\sigma = F_{\mathrm{id}_n,f\sigma^{-1}}\sigma = 1_{F(f\sigma^{-1})}\sigma = 1_{F(f\sigma^{-1})\sigma}.$$

So, our definition is so far unambiguous and $\phi(F)$ preserves identities. We go on to extend the definition of $\phi(F)$ to the rest of the arrows. For $\alpha\colon f \to g$ 2-cell in $\mathcal{M}(\langle a\rangle, b)$ and $\sigma, \tau$ in $\Sigma_n$, we define $\phi(F)(\alpha, E^\tau_\sigma)\colon F(f\sigma^{-1})\sigma \to F(g\tau^{-1})\tau$ by

$$\begin{aligned}\phi(F)(\alpha, E^\tau_\sigma) =&\phi(F)(1_g, E^\tau_\sigma)\circ\phi(F)(\alpha, 1_\sigma)\\ =&\phi(F)(\alpha, 1_\tau)\circ\phi(F)(1_f, E^\tau_\sigma).\end{aligned} \tag{2.2.7}$$

The last equality together with the preservation of identities already proven implies that our definition is unambiguous. This equality holds since,

$$\begin{aligned}\phi(F)(1_g, E^\tau_\sigma)\circ\phi(F)(\alpha, 1_\sigma) &= (F_{\tau\sigma^{-1};g\tau^{-1}})\,\sigma\circ F(\alpha\sigma^{-1})\sigma\\ &= \left(F_{\tau\sigma^{-1};g\tau^{-1}}\circ F(\alpha\sigma^{-1})\right)\sigma\\ &= \left(F(\alpha\tau^{-1})\tau\sigma^{-1}\circ F_{\tau\sigma^{-1};f\tau^{-1}}\right)\sigma\\ &= F(\alpha\tau^{-1})\tau\circ(F_{\tau\sigma^{-1};f\tau^{-1}})\,\sigma\\ &= \phi(F)(\alpha, 1_\tau)\circ\phi(F)(1_f, E^\tau_\sigma).\end{aligned}$$

The third equality holds since it is precisely the commutativity of the following diagram:

$$\begin{array}{ccc} F(f\tau^{-1}\tau\sigma^{-1}) & \xrightarrow{F_{\tau\sigma^{-1};f\tau^{-1}}} & F(f\tau^{-1})\tau\sigma^{-1} \\ {\scriptstyle F(\alpha\tau^{-1}\tau\sigma^{-1})}\downarrow & & \downarrow{\scriptstyle (F\alpha\tau^{-1})\tau\sigma^{-1}} \\ F(g\tau^{-1}\tau\sigma^{-1}) & \xrightarrow[F_{\tau\sigma^{-1};g\tau^{-1}}]{} & F(g\tau^{-1})\tau\sigma^{-1}. \end{array} \tag{2.2.8}$$

This diagram commutes since it is an instance of the pseudo symmetry naturality coherence axiom for $F$, (2.1.9). Next, we check that the defined assignments give a functor $\phi(F)\colon \mathcal{M}(\langle a\rangle; b) \times E\Sigma_n \to \mathcal{N}(\langle Fa\rangle; b)$. The fact that $\phi(F)$ preserves identities was already proven. We prove functoriality in the second variable first. For $f\colon \langle a\rangle \to b$ 1-cell, $\sigma, \tau$, and $\rho$ in $\Sigma_n$,

$$\begin{aligned} \phi(F)(1_f, E^{\rho}_{\tau}) \circ \phi(F)(1_f, E^{\tau}_{\sigma}) &= (F_{\rho\tau^{-1}; f\rho^{-1}}\tau) \circ (F_{\tau\sigma^{-1}; f\tau^{-1}}\sigma) \\ &= \big((F_{\rho\tau^{-1}; f\rho^{-1}})\,\tau\sigma^{-1} \circ F_{\tau\sigma^{-1}; f\tau^{-1}}\big)\,\sigma \\ &= (F_{\rho\sigma^{-1}; f\rho^{-1}})\,\sigma \\ &= \phi(F)(1_f, E^{\rho}_{\sigma}). \end{aligned} \tag{2.2.9}$$

Here the third equality holds by (2.1.11), which implies the commutativity of the following diagram:

$$\begin{array}{ccccc} & & F(f\rho^{-1}\rho\tau^{-1})\tau\sigma^{-1} & & \\ & \overset{F_{\tau\sigma^{-1}; f\rho^{-1}\rho\tau^{-1}}}{\nearrow} & & \overset{(F_{\rho\tau^{-1}; f\rho^{-1}})\tau\sigma^{-1}}{\searrow} & \\ F(f\rho^{-1}\rho\tau^{-1}\tau\sigma^{-1}) & & \xrightarrow[F_{\rho\tau^{-1}\tau\sigma^{-1}; f\rho^{-1}}]{} & & F(f\rho^{-1})\rho\tau^{-1}\tau\sigma^{-1}. \end{array} \tag{2.2.10}$$

On the other hand, if $\alpha\colon f \to g$ and $\beta\colon g \to h$ are 2-cells in $\mathcal{M}(\langle a\rangle; b)$, and $\sigma \in \Sigma_n$ we have that

$$\phi(F)(\beta, 1_\sigma) \circ \phi(F)(\alpha, 1_\sigma) = \phi(F)(\beta\alpha, 1_\sigma). \tag{2.2.11}$$

The functoriality of $\phi(F)$ follows from a straightforward argument by eqs. (2.2.9) and (2.2.11) together with the exchange property (2.2.7).

The next step is to prove that the component functors give rise to a symmetric **Cat**-multifunctor $\phi(F)\colon \mathcal{M} \times E\Sigma_* \to \mathcal{N}$. First, notice that $\phi(F)$ preserves units since, for $a \in \mathrm{Ob}(\mathcal{M})$ $\phi(F)(1_a, \mathrm{id}_1) = F(1_a \mathrm{id}_1^{-1})\mathrm{id}_1 = F(1_a) = 1_{Fa}$, since $F$ itself preserves units. Next we prove that $\phi(F)$ preserves the $\Sigma_n$-action. For $n \geq 0$, $\langle a\rangle \in \mathrm{Ob}(\mathcal{M})^n$, $b \in \mathrm{Ob}(\mathcal{M})$, and $\sigma \in \Sigma_n$, we show that the following diagram commutes in **Cat** :

$$\begin{array}{ccc} \mathcal{M}(\langle a\rangle; b) \times E\Sigma_n & \xrightarrow{\phi(F)} & \mathcal{N}(\langle Fa\rangle; Fb) \\ \big\downarrow{\scriptstyle \sigma} & & \big\downarrow{\scriptstyle \sigma} \\ \mathcal{M}(\langle a_j\rangle\sigma; b) \times E\Sigma_n & \xrightarrow[\phi(F)]{} & \mathcal{N}(\langle Fa\rangle\sigma; Fb). \end{array}$$

For this we don't need any of the pseudo symmetry axioms for $F$. For 1-cells $(f\colon \langle a\rangle \to b, \tau)$ of $\mathcal{M}(\langle a\rangle; b) \times E\Sigma_n$,

$$\begin{aligned}\phi(F)(f,\tau)\sigma &= (F(f\tau^{-1})\tau)\sigma\\ &= F(f\tau^{-1})\tau\sigma\\ &= F(f\sigma(\tau\sigma)^{-1})\tau\sigma\\ &= \phi(F)((f\sigma,\tau\sigma)))\\ &= \phi(F)((f,\tau)\sigma).\end{aligned}$$

A similar calculation works for 2-cells of the form $(\alpha\colon f \to g, 1_\tau)$ in $\mathcal{M}(\langle a\rangle; b) \times E\Sigma_n$. For morphisms of the form $(1_f, E^\rho_\tau)$ in $\mathcal{M}(\langle a\rangle; b) \times E\Sigma_n$,

$$\begin{aligned}(\phi(F)(1_f, E^\rho_\tau))\sigma &= (F_{\rho\tau^{-1};f\rho^{-1}}\tau)\sigma\\ &= F_{\rho\tau^{-1};f\rho^{-1}}(\tau\sigma)\\ &= F_{\rho\sigma(\tau\sigma)^{-1};f\sigma(\rho\sigma)^{-1}}(\tau\sigma)\\ &= \phi(F)(1_{f\sigma}, E^{\rho\sigma}_{\tau\sigma})\\ &= \phi(F)((1_f, E^\rho_\tau)\sigma).\end{aligned}$$

By functoriality of $\phi(F)$ and $\sigma$ we conclude that $\phi(F)$ preserves the action of the symmetric group.
The only step we are missing to finish proving that $\phi(F)$ defines a **Cat**-multifunctor is the preservation of $\gamma$. Let $c \in \mathrm{Ob}(\mathcal{M})$, $n \geq 0$, $\langle b\rangle \in \mathrm{Ob}(\mathcal{M})^n$, $k_j \geq 0$ for $1 \leq j \leq n$, $\langle a_j\rangle = \langle a_{j,i}\rangle_{i=1}^{k_j}$ for $1 \leq j \leq n$. Set $k = \sum_{j=1}^n k_j$. As usual $\langle a\rangle = \langle a_j\rangle = \langle\langle a_{j,i}\rangle_{i=1}^{k_j}\rangle_{j=1}^n$ denotes the concatenation of the $a_j$'s. We will prove that the following square is commutative:

$$\begin{array}{ccc}\mathcal{M}(\langle b\rangle; c) \times E\Sigma_n \times \prod_{j=1}^n \mathcal{M}(\langle a_j\rangle; b_j) \times E\Sigma_{k_j} & \xrightarrow{\phi(F)\times\prod\phi(F)} & \mathcal{N}(\langle Fb\rangle; Fc) \times \prod_{j=1}^n \mathcal{N}(\langle Fa_j\rangle; Fb_j)\\ \big\downarrow{\scriptstyle\gamma} & & \big\downarrow{\scriptstyle\gamma}\\ \mathcal{M}(\langle a\rangle; c) \times E(\Sigma_k) & \xrightarrow[\phi(F)]{} & \mathcal{N}(\langle Fa\rangle; Fc).\end{array} \tag{2.2.12}$$

The commutativity of this diagram at the level of 1-cells will follow from top and bottom equivariance for $\mathcal{M}$ and $\Sigma_*$, as well as the fact that $F$ preserves $\gamma$. Let $f\colon \langle b\rangle \to c$, $\sigma \in \Sigma_n$, and $g_j\colon \langle a_j\rangle \to b_j$ and $\tau_j \in \Sigma_{k_j}$ for $1 \leq j \leq n$. We have that

$$\gamma(\phi(F)(f,\sigma), \langle\phi(F)(g_j,\tau_j)\rangle) = \gamma(F(f\sigma^{-1})\sigma, \langle F(g_j\tau_j^{-1})\tau_j\rangle)$$

$$
\begin{aligned}
&= \gamma\left( (F(f\sigma^{-1}), \left\langle F\left( g_{\sigma^{-1}(j)} \tau^{-1}_{\sigma^{-1}(j)} \right) \right\rangle \right) \sigma\langle \tau_j \rangle \\
&= F\left( \gamma\left( f\sigma^{-1}, \left\langle g_{\sigma^{-1}(j)} \tau^{-1}_{\sigma^{-1}(j)} \right\rangle \right) \right) \sigma\langle \tau_j \rangle \\
&= F\left( \gamma(f, \langle g_j \rangle)(\sigma\langle \tau_j \rangle)^{-1} \right) \sigma\langle \tau_j \rangle \\
&= \phi(F)(\gamma(f, \langle g_j \rangle), \sigma\langle \tau_j \rangle) \\
&= \phi(F)(\gamma((f, \sigma), \langle g_j, \tau_j \rangle)).
\end{aligned}
$$

We have proven that our diagram is commutative at the level of 1-cells. For the morphisms we will consider again morphisms that change the first variable only and morphisms that change the second variable only separately.

For 2-cells that change the first variable only, the commutativity of our diagram follows in the same way as it did for 1-cells. We consider two cases for 2-cells that change the second variable. For 2-cells of the form $((1_f, E^{\tau}_{\sigma}), \langle 1_{g_j}, 1_{\rho_j} \rangle)$ where $f\colon \langle b \rangle \to c$, $\sigma, \tau \in \Sigma_n$, and $g_j \in \mathrm{Ob}(\mathcal{M}(\langle a_j \rangle; b_j))$ and $\rho_j \in \Sigma_{k_j}$ for $1 \leq j \leq n$, we have that

$$
\begin{aligned}
&\gamma\left( \phi(F)(1_f, E^{\tau}_{\sigma}) \left\langle \phi(F)(1_{g_j}, 1_{\rho_j}) \right\rangle \right) \\
=&\gamma\left( (F_{\tau\sigma^{-1}; f\tau^{-1}})\sigma, \left\langle 1_{F(g_j \rho_j^{-1})\rho_j} \right\rangle \right) \\
=&\gamma\left( F_{\tau\sigma^{-1}; f\tau^{-1}}, \left\langle 1_{F\left( g_{\sigma^{-1}(j)} \rho^{-1}_{\sigma^{-1}(j)} \right)} \right\rangle \right) \sigma\langle \rho_j \rangle \\
=&F_{\tau\sigma^{-1}\left\langle \mathrm{id}_{k_{\sigma^{-1}(j)}} \right\rangle; \gamma\left( f\tau^{-1}\left\langle g_{\tau^{-1}(j)} \rho^{-1}_{\tau^{-1}(j)} \right\rangle \right)} \sigma\langle \rho_j \rangle \\
=&F_{\tau\langle \rho_j \rangle (\sigma\langle \rho_j \rangle)^{-1}; \gamma(f, \langle g_j \rangle)(\tau\langle \rho_j \rangle)^{-1}} \sigma\langle \rho_j \rangle \\
=&\phi(F)(1_{\gamma(f, \langle g_j \rangle)}, E^{\tau\langle \rho_j \rangle}_{\sigma\langle \rho_j \rangle}) \\
=&\phi(F)(\gamma(1_f, \langle 1_{g_j} \rangle), \gamma(E^{\tau}_{\sigma}, \langle 1_{\rho_j} \rangle)).
\end{aligned}
$$

The above equalities follow from our definitions, top and bottom equivariance in $\mathcal{M}, \mathcal{N}$, and $E\Sigma_*$ except the third equality which follows from top equivariance for $F$ (2.1.12). Next, let's consider two cells of the form $((1_f, 1_\sigma), \langle 1_{g_j}, E^{\nu_j}_{\rho_j} \rangle)$ where $f\colon \langle b \rangle \to c$, $\sigma \in \Sigma_n$, and $g_j \in \mathrm{Ob}(\mathcal{M}(\langle a_j \rangle; b_j))$ and $\rho_j, \nu_j \in \Sigma_{k_j}$ for $1 \leq j \leq n$. We get that

$$
\begin{aligned}
&\gamma\left( \phi(F)(1_f, 1_\sigma), \phi(F)\left\langle (1_{g_j}, E^{\nu_j}_{\rho_j}) \right\rangle \right) \\
=&\gamma\left( 1_{F(f\sigma^{-1})\sigma}, \left( F_{\nu_j \rho_j^{-1}; g_j \nu_j^{-1}} \right) \rho_j \right) \\
=&\gamma\left( 1_{F(f\sigma^{-1})}, \left\langle F_{\nu_{\sigma^{-1}(j)} \rho^{-1}_{\sigma^{-1}(j)}; g_{\sigma^{-1}(j)} \nu^{-1}_{\sigma^{-1}(j)}} \right\rangle \right) \sigma\langle \rho_j \rangle
\end{aligned}
$$

$$
\begin{aligned}
&= F_{\mathrm{id}_n \left\langle \nu_{\sigma^{-1}(j)} \rho^{-1}_{\sigma^{-1}(j)} \right\rangle ; \gamma\left( f\sigma^{-1}, \left\langle g_{\sigma^{-1}(j)} \nu^{-1}_{\sigma^{-1}(j)} \right\rangle \right)}^{\sigma\langle \rho_j \rangle} \\
&= F_{\sigma\langle \nu_j \rangle (\sigma\langle \rho_j \rangle)^{-1} ; \gamma(f, \langle g_j \rangle)(\rho\langle \nu_j \rangle)^{-1}} \sigma\langle \rho_j \rangle \\
&= \phi(F) \left( 1_{\gamma(f, \langle g_j \rangle)}, \left\langle E^{\sigma\langle \nu_j \rangle}_{\sigma\langle \rho_j \rangle} \right\rangle \right) \\
&= \phi(F) \left( \gamma \left( (1_f, 1_\sigma), \left\langle \left( 1_{g_j}, E^{\nu_j}_{\rho_j} \right) \right\rangle \right) \right).
\end{aligned}
$$

The third equality above follows from the bottom equivariance axiom for $F$ (2.1.13) and the rest by our definitions as well as top and bottom equivariance for $\mathcal{M}, \mathcal{N}$, and $E\Sigma_*$.

By functoriality of $\gamma$ and $\phi(F)$, and since every morphism in the source category can be written as a composite of arrows for which we already proved the commutativity of (2.2.12), we can conclude that the square (2.2.12) is commutative.

We are almost done, we just have to prove that our definition of $\phi(F)$ gives us $F = \phi(F) \circ \eta$ in **Cat-Multicat$^{\mathbf{ps}}$**. This is clear for objects of the multicategory $\mathcal{M}$. For each $n \geq 0$, $\langle a \rangle \in \mathrm{Ob}(\mathcal{M})^n$, $b \in \mathrm{Ob}(\mathcal{M})$, and $f \colon \langle a \rangle \to b$,

$$\phi(F) \circ \eta(f) = \phi(F)(f, \mathrm{id}_n) = F(f \mathrm{id}_n^{-1}) \mathrm{id}_n = F(f).$$

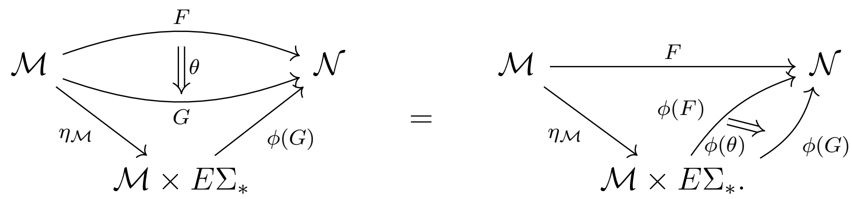
Similarly for $\alpha \colon f \to g$ a 2-cell in $\mathcal{M}(\langle a \rangle; b)$. Finally, we just need to prove that $(\phi(F) \circ \eta)_{\sigma, \langle a_i \rangle, b} = F_{\sigma, \langle a_i \rangle, b}$ for any $\sigma \in \Sigma_n$. Let $f \colon \langle a \rangle \to b$ be a 1-cell. Since $\phi(F)$ is symmetric,

$$(\phi(F)\eta)_{\sigma; f} = \phi(F)(\eta_{\sigma; f}) = \phi(F)(1_{f\sigma}, E_{\mathrm{id}^\sigma}) = F_{\sigma(\mathrm{id})^{-1}; f\sigma\sigma^{-1}} = F_{\sigma; f},$$

where we have used the notation introduced just before (2.1.9). We have proven that $j\phi(F) \circ \eta = F$. This finishes our proof. ■

Similarly, pseudo symmetric **Cat**-multinatural transformations between $F$ and $G$ correspond to symmetric **Cat**-multinatural transformations between $\phi(F)$ and $\phi(G)$.

**Lemma 2.2.4.** *Let $\mathcal{M}, \mathcal{N}$ be* **Cat***-multicategories with $F, G \colon \mathcal{M} \to \mathcal{N}$ pseudo symmetric* **Cat***-multifunctors and $\theta \colon F \to G$ a pseudo symmetric* **Cat***-multinatural transformation. There exists a unique symmetric* **Cat***-multinatural transformation $\phi(\theta) \colon \phi(F) \to \phi(G)$ such that $\phi(\theta) * 1_{\eta_{\mathcal{M}}} = \theta$ in* **Cat-Multicat$^{\mathbf{ps}}$***. That is, the following pasting diagram equality holds in* **Cat-Multicat$^{\mathbf{ps}}$** :




*Proof.* We prove uniqueness first. Suppose $\phi(\theta)$ is a symmetric **Cat**-multinatural transformation $\phi(\theta)\colon \phi(F) \to \phi(G)$ such that $\phi(\theta) * 1_\eta = \theta$. Any object of $\mathcal{M} \times E\Sigma_*$ takes the form $(a, *)$ for some object $a$ of $\mathcal{M}$, with $i(a) = (a, *)$. By definition,

$$\theta_a = \gamma(\phi(\theta)_{\eta a}, \phi(F)((1_\eta)_a)) = \gamma(\phi(\theta)_{\eta a}, 1_{Fa})) = \phi(\theta)_{\eta a}.$$

Since all objects of the **Cat**-multifunctor $\mathcal{M} \times E\Sigma_*$ are of the form $\eta a$ for some object $a$ of $\mathcal{M}$, this is the only possible way of defining such **Cat**-multinatural transformation $\phi(\theta)$. Next, we check that by defining $\phi(\theta)_{(a,*)} = \theta_a$ for $a \in \mathrm{Ob}(\mathcal{M})$, we in fact get a symmetric **Cat**-multinatural transformation $\phi(\theta)\colon \phi(F) \to \phi(G)$. Let $n \geq 0$, $\langle a \rangle \in \mathrm{Ob}(\mathcal{M})^n$, $b \in (\mathrm{Ob}(\mathcal{M})^n)$, $f\colon \langle a \rangle \to b$, and $\sigma \in \Sigma_n$, then

$$\begin{aligned}
\gamma(\phi(G)(f,\sigma); \langle \phi(\theta)_{(a_j,*)} \rangle) &= \gamma\left(G(f\sigma^{-1})\sigma; \langle \theta_{a_j} \rangle\right) \\
&= \gamma\left(G(f\sigma^{-1}); \left\langle \theta_{a_{\sigma^{-1}(j)}} \right\rangle\right)\sigma \\
&= \gamma(\theta_b; F(f\sigma^{-1}))\sigma \\
&= \gamma(\theta_b; F(f\sigma^{-1})\sigma) \\
&= \gamma\left(\phi(\theta)_{(b,*)}, \phi(F)(f,\sigma)\right)
\end{aligned}$$

Where we have used top and bottom equivariance, as well as the **Cat**-multinaturality of $\theta$. Now we need to prove **Cat**-multinaturality of $\phi(\theta)$ for 2-cells. As before, the case where the 2-cell changes just the first variable is similar to what was done for 1-cells. Now, if $\langle a \rangle, b, f$ are as before and $E^\tau_\sigma$ is a morphism in $E\Sigma_n$, $(1_f, E^\tau_\sigma)$ is a morphism in $\mathcal{M}(\langle a \rangle; b) \times E\Sigma_n$, and

$$\begin{aligned}
\gamma\left(\phi(G)(1_f, E^\tau_\sigma); \left\langle 1_{\phi(\theta)_{(a_j,*)}} \right\rangle\right) &= \gamma\left((G_{\tau\sigma^{-1};f\tau^{-1}})\sigma; \langle 1_{\theta_{a_j}} \rangle\right) \\
&= \gamma\left(G_{\tau\sigma^{-1};f\tau^{-1}}; \left\langle 1_{\theta_{a_{\sigma^{-1}(j)}}} \right\rangle\right)\sigma \\
&= \gamma\left(1_{\theta_b}; F_{\tau\sigma^{-1};f\tau^{-1}}\right)\sigma \\
&= \gamma\left(1_{\phi(\theta)_{(b,*)}}; \phi(F)(1_f, E^\tau_\sigma)\right).
\end{aligned}$$

In the third equality we have used pseudo symmetric **Cat**-multinaturality for $\theta$. In conclusion, by componentwise functoriality of $\gamma, \phi(F)$ and $\phi(G)$ we conclude that **Cat**-multinaturality holds for $\phi(\theta)$ at the 2-cell level finishing the proof of the lemma. ■

Furthermore, Theorem 2.2.3 and Lemma 2.2.4 together give the following isomorphism.

**Corollary 2.2.5.** *If $\mathcal{M}, \mathcal{N}$ are* **Cat** *multicategories, then there is an isomorphism of small categories*

$$\mathbf{Cat}\text{-}\mathbf{Multicat}^{\mathbf{ps}}(\mathcal{M}, \mathcal{N}) \cong \mathbf{Cat}\text{-}\mathbf{Multicat}(\mathcal{M} \times E\Sigma_*, \mathcal{N}).$$

*Proof.* Recalling the definitions from the two previous results, we define

$$\phi\colon \mathbf{Cat}\text{-}\mathbf{Multicat}^{\mathbf{ps}}(\mathcal{M}, \mathcal{N}) \to \mathbf{Cat}\text{-}\mathbf{Multicat}(\mathcal{M} \times E\Sigma_*, \mathcal{M}) \tag{2.2.13}$$

for pseudo symmetric **Cat**-multifunctors as in Theorem 2.2.3 and for pseudo symmetric **Cat**-multinatural transformations as in Lemma 2.2.4.
It is immediate from the definitions that $\phi$ is a functor. Indeed, if $\alpha\colon F \to G$ and $\beta\colon G \to H$ are pseudo symmetric **Cat**-multinatural transformations with $F, G, H\colon \mathcal{M} \to \mathcal{N}$

$$\phi(\beta * \alpha)_{(c,*)} = (\beta * \alpha)_c = \gamma(\beta_c, \alpha_c) = \gamma(\phi(\beta)_{(c,*)}, \phi(\alpha)_{(c,*)}) = (\phi(\beta) * \phi(\alpha))_{(c,*)}$$

We can define the inverse of $\phi$, $\eta^*$, as the composite

$$\begin{array}{ccc} \mathbf{Cat}\text{-}\mathbf{Multicat}(\mathcal{M} \times E\Sigma_*, \mathcal{N}) & \xrightarrow{j} & \mathbf{Cat}\text{-}\mathbf{Multicat}^{\mathbf{ps}}(\mathcal{M} \times E\Sigma_*, \mathcal{N}) \\ & \searrow^{\eta^*} & \downarrow \eta^*_{\mathcal{M}} \\ & & \mathbf{Cat}\text{-}\mathbf{Multicat}^{\mathbf{ps}}(\mathcal{M}, \mathcal{N}). \end{array} \tag{2.2.14}$$

Finally, the existence part of Theorem 2.2.3 and Lemma 2.2.4, implies that $\eta^* \circ \phi$ is the identity of $\mathbf{Cat}\text{-}\mathbf{Multicat}^{\mathbf{ps}}(\mathcal{M}, \mathcal{N})$, while the uniqueness part of both results implies that $\phi \circ \eta^*$ is the identity of $\mathbf{Cat}\text{-}\mathbf{Multicat}(\mathcal{M} \times E\Sigma_*, \mathcal{N})$. ■

The two previous results hint at the existence of a 2-adjunction between the 2-inclusion $j\colon \mathbf{Cat}\text{-}\mathbf{Multicat} \to \mathbf{Cat}\text{-}\mathbf{Multicat}^{\mathbf{ps}}$ and the 2-functor which we define next.

**Definition 2.2.6.** We define the 2-functor $\psi\colon \mathbf{Cat}\text{-}\mathbf{Multicat}^{\mathbf{ps}} \to \mathbf{Cat}\text{-}\mathbf{Multicat}$ as follows. For a **Cat**-multicategory $\mathcal{M}$, $\psi\mathcal{M} = \mathcal{M} \times E\Sigma_*$. For $\mathcal{M}, \mathcal{N}$ **Cat**-multicategories, we define the component functor $\psi$ as the composite

$$\begin{array}{ccc}
\textbf{Cat-Multicat}^{\textbf{ps}}(\mathcal{M},\mathcal{N}) & \xrightarrow{\eta_{\mathcal{N}*}} & \textbf{Cat-Multicat}^{\textbf{ps}}(\mathcal{M},\mathcal{N}\times E\Sigma_*) \\
& \searrow^{\psi} & \downarrow \phi \\
& & \textbf{Cat-Multicat}(\mathcal{M}\times E\Sigma_*,\mathcal{N}\times E\Sigma_*).
\end{array}$$

Thus, by Theorem 2.2.3 if $F\colon \mathcal{M}\to\mathcal{N}$ is a pseudo symmetric **Cat**-multifunctor, then $\psi F\colon \mathcal{M}\times E\Sigma_*\to\mathcal{N}\times E\Sigma_*$ is the unique symmetric **Cat**-multifunctor which makes the diagram

$$\begin{array}{ccc}
\mathcal{M} & \xrightarrow{\eta_M} & \mathcal{M}\times E\Sigma_* \\
{\scriptstyle F}\downarrow & & \downarrow{\scriptstyle j\psi F} \\
\mathcal{N} & \xrightarrow[\eta_{\mathcal{N}}]{} & \mathcal{N}\times E\Sigma_*
\end{array} \tag{2.2.15}$$

commute in **Cat-Multicat**$^{\textbf{ps}}$. Similarly, by Lemma 2.2.4, for $\theta\colon F\to G$ a pseudo symmetric **Cat**-multinatural transformation between $F,G\colon \mathcal{M}\to\mathcal{N}$ pseudo symmetric **Cat**-multifunctors, $\psi\theta\colon \psi F\to\psi G$ is the unique symmetric **Cat**-multinatural transformation such that the equality of pasting diagrams

$$\begin{array}{ccc}
\mathcal{M} & \xrightarrow{\eta_{\mathcal{M}}} & \mathcal{M}\times E\Sigma_* \\
{\scriptstyle F}\big(\overset{\theta}{\Longrightarrow}\big){\scriptstyle G} & & \downarrow{\scriptstyle j\psi F} \\
\mathcal{N} & \xrightarrow[\eta_{\mathcal{N}}]{} & \mathcal{N}\times E\Sigma_*
\end{array}
\quad = \quad
\begin{array}{ccc}
\mathcal{M} & \xrightarrow{\eta_{\mathcal{M}}} & \mathcal{M}\times E\Sigma_* \\
{\scriptstyle F}\downarrow & & {\scriptstyle j\psi F}\big(\overset{j\psi\theta}{\Longrightarrow}\big){\scriptstyle j\psi G} \\
\mathcal{N} & \xrightarrow[\eta_{\mathcal{N}}]{} & \mathcal{N}\times E\Sigma_*
\end{array} \tag{2.2.16}$$

holds in **Cat-Multicat**$^{\textbf{ps}}$.

**Theorem 2.2.7.** *There is a 2-adjunction*

$$\textbf{Cat-Multicat}^{\textbf{ps}} \underset{j}{\overset{\psi}{\rightleftarrows}} \textbf{Cat-Multicat} \qquad (\psi \dashv j)$$

*where $j$ is the inclusion 2-functor.*

*Proof.* Following Corollary 2.2.5, we define the unit of the adjunction as the strict 2-natural transformation $\eta\colon 1_{\textbf{Cat-Multicat}^{\textbf{ps}}}\to j\psi$ having component $\eta_{\mathcal{M}}$ at a **Cat**-multicategory $\mathcal{M}$. We also define the counit of the adjunction $\pi\colon \psi j\to 1_{\textbf{Cat-Multicat}}$ as having component at a **Cat**-multicategory $\mathcal{M}$ the projection $\pi_M\colon \mathcal{M}\times E\Sigma_*\to\mathcal{M}$.

The fact that $\eta$ defines a strict 2-natural transformation follows directly from (2.2.15) and (2.2.16). To prove that the data of $\pi$ defines a strict 2-natural transformation we need to prove that given $F\colon \mathcal{M} \to \mathcal{N}$ symmetric **Cat**-multifunctor, the following diagram commutes:

$$\begin{array}{ccc} \mathcal{M} \times E\Sigma_* & \xrightarrow{\pi_{\mathcal{M}}} & \mathcal{M} \\ {\scriptstyle \psi j F}\downarrow & & \downarrow{\scriptstyle F} \\ \mathcal{N} \times E\Sigma_* & \xrightarrow[\pi_{\mathcal{N}}]{} & \mathcal{N}. \end{array}$$

Indeed, we prove that $\psi j F = F \times 1_{E\Sigma_*}$. By (2.2.15), it suffices to show that the following diagram commutes in **Cat-Multicat**$^{\mathbf{ps}}$:

$$\begin{array}{ccc} \mathcal{M} & \xrightarrow{\eta_{\mathcal{M}}} & \mathcal{M} \times E\Sigma_* \\ {\scriptstyle jF}\downarrow & & \downarrow{\scriptstyle j(F\times 1)} \\ \mathcal{N} & \xrightarrow[\eta_{\mathcal{N}}]{} & \mathcal{N} \times E\Sigma_*. \end{array} \tag{2.2.17}$$

It is clear that this diagram commutes at the level of objects, 1-cells, and 2-cells of the multicategory. The pseudo symmetry isomorphisms of both composites also agree. Indeed, for $f\colon \langle a \rangle \to b$ a 1-cell of $\mathcal{M}$ and $\sigma \in \Sigma_n$, by (2.1.14), we get that

$$\begin{aligned} (j(F\times 1)\eta_{\mathcal{M}})_{\sigma;f} &= j(F\times 1)_{\sigma;\eta_{\mathcal{M}}(f)} \circ j(F\times 1)(\eta_{\mathcal{M}\sigma;f}) \\ &= (1_{(Ff)\sigma}, 1_{\sigma}) \circ (1_{(Ff)\sigma}, E^{\sigma}_{\mathrm{id}}) \\ &= (1_{(Ff)\sigma}, E^{\sigma}_{\mathrm{id}}) \circ (1_{(Ff)\sigma}, 1_{\sigma}) \\ &= \eta_{\mathcal{N}\sigma;Ff} \circ \eta_{\mathcal{N}}(jF_{\sigma;f}) \\ &= (\eta_{\mathcal{N}} \circ jF)_{\sigma;f}. \end{aligned}$$

To finish proving the 2-naturality of $\pi_{\mathcal{M}}$, we need to prove that given $\mathcal{M}, \mathcal{N}$ **Cat**-multicategories, $F, G\colon \mathcal{M} \to \mathcal{N}$ **Cat**-multifunctors and a **Cat**-multinatural transformation $\theta\colon F \to G$, the following equality of pasting diagrams holds in **Cat-Multicat**:

$$\begin{array}{ccc} \mathcal{M} \times E\Sigma_* & \xrightarrow{\pi_{\mathcal{M}}} & \mathcal{M} \\ {\scriptstyle j(F\times 1)} \big( \overset{\psi j\theta}{\Longrightarrow} \big) {\scriptstyle j(G\times 1)} & & \downarrow{\scriptstyle G} \\ \mathcal{N} \times E\Sigma_* & \xrightarrow[\pi_{\mathcal{N}}]{} & \mathcal{N} \end{array} \quad = \quad \begin{array}{ccc} \mathcal{M} \times E\Sigma_* & \xrightarrow{\pi_{\mathcal{M}}} & \mathcal{M} \\ {\scriptstyle j(F\times 1)}\downarrow & & {\scriptstyle F} \big( \overset{\theta}{\Longrightarrow} \big) {\scriptstyle G} \\ \mathcal{N} \times E\Sigma_* & \xrightarrow[\pi_{\mathcal{N}}]{} & \mathcal{N}. \end{array}$$

In turn, the last equality of pasting diagrams holds since $\psi j\theta = j(\theta \times 1)$. To see this, by (2.2.16), we must show the following equality of pasting diagrams in **Cat-Multicat**$^{\mathbf{ps}}$ :

$$\begin{array}{ccc} \mathcal{M} & \xrightarrow{\eta_{\mathcal{M}}} & \mathcal{M} \times E\Sigma_* \\ {\scriptstyle jF} \big( \overset{j\theta}{\Longrightarrow} \big) {\scriptstyle jG} & & \downarrow{\scriptstyle j(G\times 1)} \\ \mathcal{N} & \xrightarrow[\eta_{\mathcal{N}}]{} & \mathcal{N} \times E\Sigma_* \end{array} \quad = \quad \begin{array}{ccc} \mathcal{M} & \xrightarrow{\eta_{\mathcal{M}}} & \mathcal{M} \times E\Sigma_* \\ {\scriptstyle jF}\downarrow & & {\scriptstyle j(F\times 1)} \big( \overset{j(\theta\times 1)}{\Longrightarrow} \big) {\scriptstyle j(G\times 1)} \\ \mathcal{N} & \xrightarrow{\eta_{\mathcal{N}}} & \mathcal{N} \times E\Sigma_*. \end{array} \tag{2.2.18}$$

To check that this equality holds let $a \in \mathrm{Ob}(\mathcal{M})$. We get, by (2.1.8), that

$$\begin{aligned}(1_{\eta_{\mathcal{N}}} * j\theta)_a &= \gamma\left(1_{\eta_{\mathcal{N}}(jGa)}; \eta_{\mathcal{N}}(\theta_a)\right)\\ &= \gamma\left((1_{Ga}, 1_{\mathrm{id}}); (\theta_a, 1_{\mathrm{id}})\right)\\ &= \gamma\left((\theta_a, 1_{\mathrm{id}}); (1_{Fa}, 1_{\mathrm{id}})\right)\\ &= \gamma\left(j(\theta \times 1)_{\eta_{\mathcal{N}}(a)}; j(F \times 1)(1_{\eta_{\mathcal{M}}(a)})\right)\\ &= (j(\theta \times 1) * \eta_{\mathcal{M}})_a.\end{aligned}$$

Thus, $\eta$ and $\pi$ are strict 2-natural transformations and we just need to prove that they satisfy the triangle identities. To prove that the identity $(1_j * \pi)(\eta * 1_j) = 1_j$ holds we need to prove that for $\mathcal{M}$ a **Cat**-multicategory the diagram

$$\begin{array}{ccccc} & & \mathcal{M} \times E\Sigma_* & & \\ & \overset{\eta_{\mathcal{M}}}{\nearrow} & & \overset{j\pi_{\mathcal{M}}}{\searrow} & \\ \mathcal{M} & & \xrightarrow[1_{\mathcal{M}}]{} & & \mathcal{M} \end{array}$$

commutes in **Cat-Multicat^ps**. This is clear at the level of objects, $n$-ary 1-cells and $n$-ary 2-cells. The pseudo symmetry isomorphisms of both pseudo symmetric **Cat**-multifunctors also agree since, for $f\colon \langle a \rangle \to b$ an $n$-ary 1-cell of $\mathcal{M}$ and $\sigma \in \Sigma_n$, we obtain, by (2.1.14),

$$((j\pi_{\mathcal{M}}) \circ \eta_{\mathcal{M}})_{\sigma;f} = (j\pi_{\mathcal{M}})_{\sigma;\eta_{\mathcal{M}}(f)} \circ j\pi_{\mathcal{M}}(\eta_{M\,\sigma;f}) = 1_{f\sigma} = 1_{\mathcal{M}\sigma;f}.$$

The other triangle identity is $(\pi * 1_\psi)(1_\psi * \eta) = 1_\psi$. To check it, we must prove that, given a **Cat**-multicategory $\mathcal{M}$, the composite

$$\mathcal{M} \times E\Sigma_* \xrightarrow{\psi\eta_{\mathcal{M}}} \mathcal{M} \times E\Sigma_* \times E\Sigma_* \xrightarrow{\pi_{\mathcal{M}\times E\Sigma_*}} \mathcal{M} \times E\Sigma_*$$

agrees with $1_{\mathcal{M}\times E\Sigma_*}$. This holds since, if $\Delta\colon E\Sigma_* \to E\Sigma_* \times E\Sigma_*$ denotes the diagonal map, then $\psi(\eta_{\mathcal{M}}) = 1_{\mathcal{M}} \times \Delta$. To see this, notice that by (2.2.15) all we need is to prove that the following diagram is commutative:

$$\begin{array}{ccc} \mathcal{M} & \xrightarrow{\eta_{\mathcal{M}}} & \mathcal{M} \times E\Sigma_* \\ {\scriptstyle \eta_{\mathcal{M}}}\downarrow & & \downarrow{\scriptstyle j(1\times\Delta)} \\ \mathcal{M} \times E\Sigma_* & \xrightarrow[\eta_{\mathcal{M}\times E\Sigma_*}]{} & \mathcal{M} \times E\Sigma_* \times E\Sigma_*. \end{array} \qquad (2.2.19)$$

Now, the previous diagram is evidently commutative at the level of objects, 1-cells, and 2-cells. The diagram also commutes at the level of pseudo symmetry isomorphisms since, for $f\colon \langle a \rangle \to b$ an $n$-ary 1-cell in $\mathcal{M}$ and $\sigma \in \Sigma_n$,

$$(\eta_{\mathcal{M}\times E\Sigma_*} \circ \eta_{\mathcal{M}})_{\sigma;f} = \eta_{\mathcal{M}\times E\Sigma_*\,\sigma;\eta_{\mathcal{M}}(f)} \circ \eta_{\mathcal{M}\times E\Sigma_*}(\eta_{M\,\sigma;f})$$

$$
\begin{aligned}
&= (1_{f\sigma}, 1_\sigma, E^\sigma_{\mathrm{id}}) \circ (1_{f\sigma}, E^\sigma_{\mathrm{id}}, 1_{\mathrm{id}}) \\
&= (1_{f\sigma}, 1_\sigma, 1_\sigma) \circ (1_{f\sigma}, E^\sigma_{\mathrm{id}}, E^\sigma_{\mathrm{id}}) \\
&= j(1 \times \Delta)_{\sigma;\eta_{\mathcal{M}}(f)} \circ j(1 \times \Delta)(\eta_{\mathcal{M}\sigma;f}) \\
&= (j(1 \times \Delta) \circ \eta_{\mathcal{M}})_{\sigma;f}\,.
\end{aligned}
$$

We conclude that the triangle identities are satisfied and thus we get the desired 2-adjunction. ■

We can use this 2-adjunction to describe the 2-category **Cat-Multicat^ps** in terms of symmetric **Cat**-multifunctors and symmetric **Cat**-multinatural transformations alone, thus upgrading the functors $\phi$ from Corollary 2.2.5 to an isomorphism of 2-categories.

**Definition 2.2.8.** The 2-category **D** has **Cat**-multicategories as objects. For $\mathcal{M}, \mathcal{N}$ **Cat**-multicategories, the category of morphisms between $\mathcal{M}$ and $\mathcal{N}$ is

$$\mathbf{D}(\mathcal{M}, \mathcal{N}) = \mathbf{Cat}\text{-}\mathbf{Multicat}(\mathcal{M} \times E\Sigma_*, \mathcal{N}).$$

In particular, vertical composition of 2-cells is defined as in **Cat-Multicat**. For $F\colon \mathcal{M} \times E\Sigma_* \to \mathcal{N}$ and $G\colon \mathcal{N} \times E\Sigma_* \to \mathcal{Q}$ symmetric **Cat**-multifunctors, the composition $G \circ F$ is defined as the composite

$$\mathcal{M} \times E\Sigma_* \xrightarrow{1\times\Delta} \mathcal{M} \times E\Sigma_* \times E\Sigma_* \xrightarrow{F\times 1} \mathcal{N} \times E\Sigma_* \xrightarrow{G} \mathcal{Q}$$

in **Cat-Multicat**. Similarly, for $F, J\colon \mathcal{M} \times E\Sigma_* \to \mathcal{N}$, $G, K\colon \mathcal{N} \times E\Sigma_* \to \mathcal{Q}$ symmetric **Cat**-multifunctors and $\theta\colon F \to J$, $\zeta\colon G \to K$ **Cat**-multinatural transformations, $\zeta * \theta$ is defined as the pasting

$$\mathcal{M} \times E\Sigma_* \xrightarrow{1\times\Delta} \mathcal{M} \times E\Sigma_* \times E\Sigma_* \underset{J\times 1}{\overset{F\times 1}{\underset{\Downarrow \theta\times 1}{\rightrightarrows}}} \mathcal{N} \times E\Sigma_* \underset{K}{\overset{G}{\underset{\zeta\Downarrow}{\rightrightarrows}}} \mathcal{Q}$$

in **Cat-Multicat**.

The previous definition makes **D** into a 2-category and the functors $\phi$, and $\eta^*$ from Corollary 2.2.5 into the components of isomorphisms of 2-categories.

**Theorem 2.2.9.** *The data of the previous definition defines a 2-category* **D** *isomorphic to* **Cat-Multicat^ps**.

*Proof.* The (horizontal) composition functors are defined so that $\phi$ and $\eta^*$ become the componentwise functors of a 2-category isomorphism between $\mathbf{D}$ and $\mathbf{Cat\text{-}Multicat^{ps}}$. More precisely, for $\mathcal{M}, \mathcal{N}$ and $\mathcal{Q}$ $\mathbf{Cat}$-multicategories, we will prove that the $\mathbf{D}$ composition functor defined, $\circ' \colon \mathbf{D}(\mathcal{N}, \mathcal{Q}) \times \mathbf{D}(\mathcal{M}, \mathcal{N}) \to \mathbf{D}(\mathcal{M}, \mathcal{Q})$, makes the following diagram commute, where $\circ$ denotes the horizontal composition functor of $\mathbf{Cat\text{-}Multicat^{ps}}$:

$$\begin{array}{ccc}
\mathbf{D}(\mathcal{N}, \mathcal{Q}) \times \mathbf{D}(\mathcal{M}, \mathcal{N}) & \xrightarrow{\quad \circ' \quad} & \mathbf{D}(\mathcal{M}, \mathcal{Q}) \\
{\scriptstyle \eta* \times \eta*}\big\downarrow & & \big\uparrow{\scriptstyle \phi} \\
\mathbf{Cat\text{-}Multicat^{ps}}(\mathcal{N}, \mathcal{Q}) \times \mathbf{Cat\text{-}Multicat^{ps}}(\mathcal{M}, \mathcal{N}) & \xrightarrow[\circ]{} & \mathbf{Cat\text{-}Multicat^{ps}}(\mathcal{M}, \mathcal{Q}).
\end{array} \tag{2.2.20}$$

Let $G \colon \mathcal{N} \times \mathcal{Q}$ and $F \colon \mathcal{M} \times E\Sigma_* \to \mathcal{N}$ be symmetric $\mathbf{Cat}$-multifunctors. The commutativity of (2.2.20) for $(G, F)$ reduces to the commutativity of the following diagram by Theorem 2.2.3:

$$\begin{array}{ccccc}
\mathcal{M} & \xrightarrow{\eta_{\mathcal{M}}} & \mathcal{M} \times E\Sigma_* & & \\
{\scriptstyle \eta_{\mathcal{M}}}\big\downarrow & & \big\downarrow{\scriptstyle j(1\times\Delta)} & & \\
\mathcal{M} \times E\Sigma_* & \xrightarrow{\eta_{\mathcal{M}\times E\Sigma_*}} & \mathcal{M} \times E\Sigma_* \times E\Sigma_* & & \\
{\scriptstyle jF}\big\downarrow & & \big\downarrow{\scriptstyle j(F\times 1)} & & \\
\mathcal{N} & \xrightarrow{\eta_N} & \mathcal{N} \times E\Sigma_* & \xrightarrow{jG} & \mathcal{Q}.
\end{array}$$

This diagram in turn is commutative by (2.2.17) and (2.2.19). Now, if $F, G$ are as before, $J \colon \mathcal{M} \times E\Sigma_*$ and $K \colon \mathcal{N} \times E\Sigma_* \to \mathcal{Q}$ are symmetric $\mathbf{Cat}$-multifunctors, and $\theta \colon F \to J$, $\zeta \colon G \to K$ are $\mathbf{Cat}$-multinatural transformations, by Lemma 2.2.4, the commutativity of (2.2.20) for $(\zeta, \theta)$ reduces to the equality of pasting diagrams:

$$\begin{array}{ccc}
\mathcal{M} & \xrightarrow{\eta_{\mathcal{M}}} & \mathcal{M} \times E\Sigma_* \\
{\scriptstyle \eta_{\mathcal{M}}}\big\downarrow & & \big\downarrow{\scriptstyle j(1\times\Delta)} \\
\mathcal{M} \times E\Sigma_* & \xrightarrow{\eta_{\mathcal{M}\times E\Sigma_*}} & \mathcal{M} \times E\Sigma_* \times E\Sigma_* \\
{\scriptstyle jF}\big\downarrow & & {\scriptstyle j(F\times 1)}\big\downarrow \overset{j(\theta\times 1)}{\Longrightarrow} \big\downarrow{\scriptstyle j(J\times 1)} \\
\mathcal{N} & \xrightarrow{\eta_N} & \mathcal{N} \times E\Sigma_* \\
 & & {\scriptstyle jG}\big\downarrow \overset{j\zeta}{\Longrightarrow} \big\downarrow{\scriptstyle jK} \\
 & & \mathcal{Q}
\end{array}
\quad = \quad
\begin{array}{ccc}
\mathcal{M} & \xrightarrow{\eta_{\mathcal{M}}} & \mathcal{M} \times E\Sigma_* \\
{\scriptstyle \eta_{\mathcal{M}}}\big\downarrow & & \big\downarrow{\scriptstyle j(1\times\Delta)} \\
\mathcal{M} \times E\Sigma_* & \xrightarrow{\eta_{\mathcal{M}\times E\Sigma_*}} & \mathcal{M} \times E\Sigma_* \times E\Sigma_* \\
{\scriptstyle jF}\big\downarrow \overset{j\theta}{\Longrightarrow} \big\downarrow{\scriptstyle jJ} & & \big\downarrow{\scriptstyle j(J\times 1)} \\
\mathcal{N} & \xrightarrow{\eta_N} & \mathcal{N} \times E\Sigma_* \\
 & & {\scriptstyle jG}\big\downarrow \overset{j\zeta}{\Longrightarrow} \big\downarrow{\scriptstyle jK} \\
 & & \mathcal{Q}.
\end{array}$$

This equality holds by (2.2.18) and makes implicit use of (2.2.17) and (2.2.19). We can thus define $\phi$: $\mathbf{Cat\text{-}Multicat^{ps}} \to \mathbf{D}$ in objects as the identity map, and do the same for $\eta^*$: $\mathbf{D} \to \mathbf{Cat\text{-}Multicat^{ps}}$, with the component functors given for $\mathcal{M}$ and $\mathcal{N}$ multicategories by (2.2.13) and (2.2.14) respectively. By (2.2.20) and the fact that $\phi$ and $\eta^*$ are componentwise isomorphisms, $\phi$ and $\eta$ preserve vertical composition of 2-cells and horizontal composition of 1-cells and 2-cells. The fact that $\mathbf{Cat\text{-}Multicat^{ps}}$ is a 2-category implies that $\mathbf{D}$ is a 2-category. This further turns $\phi$ and $\eta^*$ into isomorphisms of 2-categories. ■

## 2.3 Applications to inverse $K$-heory

We use our understanding of pseudo symmetric **Cat**-multifunctors to show that they preserve certain $E_n$-algebras for $n = 1, 2, 3, ..., \infty$. First we define $E_n$ **Cat**-operads.

**Definition 2.3.1.** For $n = 1, ..., \infty$, an $E_n$ **Cat**-operad is a **Cat**-operad that becomes a topological $E_n$-operad (in the sense of [May72]) after applying the classifying space functor. A topological $E_n$-operad is one that has the same $\Sigma$-equivariant homotopy type as the little $n$-cubes operad.

**Example 2.3.2.** An example of an $E_\infty$ **Cat**-operad is $E\Sigma_*$. There are also examples of $E_n$ **Cat**-operads for each $n = 1, 2, \ldots$ in [Ber96] and [BFSV03], which furthermore have a free action of the symmetric group. Importantly, symmetric algebras over topological $E_n$-operads are grouplike $n$-fold loop spaces. Symmetric algebras over the $E_n$ **Cat**-operads in [BFSV03] are $n$-fold monoidal categories, with the group completion of the classifying space of an $n$-monoidal category being an example of an $n$-fold loop space.

**Definition 2.3.3.** Let $\mathcal{M}$ be a **Cat**-multicategory and $\mathcal{O}$ a **Cat**-operad. A pseudo symmetric algebra in $\mathcal{M}$ over $\mathcal{O}$ is a pseudo symmetric **Cat**-multifunctor $\mathcal{O} \to \mathcal{M}$. For $n \in \{1, 2, \ldots, \infty\}$, a symmetric $E_n$-algebra (respectively a pseudo symmetric $E_n$-algebra) in $\mathcal{M}$ is a symmetric algebra (respectively a pseudo symmetric algebra) over an $E_n$-operad.

**Lemma 2.3.4.**

1. *Let $\mathcal{O}$ be a $\Sigma$-free $E_n$* **Cat***-operad. Then $\mathcal{O} \times E\Sigma_*$ is an $E_n$* **Cat***-operad.*

2. *Pseudo symmetric $E_n$-algebras over $\Sigma$-free $E_n$ **Cat**-operads are symmetric $E_n$-algebras for $n = 1, 2, \ldots, \infty$.*

*Proof.* Let $\mathcal{O}$ be a $\Sigma$-free **Cat**-operad. We will show that $\mathcal{O} \times E\Sigma_*$ is componentwise $\Sigma$-equivariantly homotopy equivalent to $\mathcal{O}$ (after taking nerves), that is, for each $n \geq 0$, we will show that the projection $\mathcal{O}(n) \times E\Sigma_n \to \mathcal{O}(n)$ induces a $\Sigma_n$ equivariant homotopy equivalence on classifying spaces. Since $B(\mathcal{O}(n) \times E\Sigma_n)$ and $B(\mathcal{O}(n))$ are $\Sigma_n$-CW complexes we must show that for subgroups $H \leq \Sigma_n$, the projection induces homotopy equivalences $B\left(\mathcal{O}(n) \times \Sigma_n\right)^H \to B\left(\mathcal{O}(n) \times \Sigma_n\right)^H$. Since the action of $\Sigma_n$ on both $\mathcal{O}(n) \times E\Sigma_n$ and $\mathcal{O}(n)$ is free, the fixed point map is either empty when $H$ is non-trivial or the projection $B\left(\mathcal{O}(n)\right) \times B\left(E\Sigma_n\right) \to B\left(\mathcal{O}(n)\right),$ which is a homotopy equivalence since $B\left(E\Sigma_n\right)$ is contractible. ■

**Example 2.3.5.** If $\mathcal{O}$ is **Cat**-operad and $\mathcal{M}$ is a **Cat**-multicategory, the pseudo symmetric algebras over $\mathcal{O}$ agree with symmetric algebras over the operad $\mathcal{O} \times E\Sigma_*$. For example, while algebras over the commutative operad $\{*\}$ in $\mathcal{M}$ are the commutative monoids in $\mathcal{M}$, pseudo symmetric algebras over $\{*\}$ in $\mathcal{M}$ are precisely algebras over the Barratt-Eccles operad and thus, $E_\infty$-algebras. Similarly, pseudo symmetric algebras over the $E_\infty$ **Cat**-operad $E\Sigma_*$, which are defined in [Yau24b] as pseudo symmetric $E_\infty$-algebras in $\mathcal{M}$, are algebras over $E\Sigma_* \times E\Sigma_* = E(\Sigma_* \times \Sigma_*)$ which is still an $E_\infty$ **Cat**-operad, and thus, they are still $E_\infty$-algebras in the sense defined above. Thus, we have the following result.

**Remark 2.3.6.** We remind the reader that $\Sigma$-freedom is not a serious restriction since there are $E_n$-operads in **Cat**, like those in [Ber96] and [BFSV03] which are free. As a corollary, we conclude that pseudo symmetric **Cat**-multifunctors preserve certain $E_n$-algebras.

**Corollary 2.3.7.** *Let $\mathcal{M}$ and $\mathcal{N}$ be **Cat**-multicategories and $F\colon \mathcal{M} \to \mathcal{N}$ be a pseudo symmetric **Cat**-multifunctor, then:*

1. *$F$ sends commutative monoids in $\mathcal{M}$ to $E_\infty$-algebras in $\mathcal{N}$.*

2. *$F$ preserves $E_n$-algebras parameterized by free **Cat**-operads.*

We conclude our paper by applying our understanding of pseudo symmetric **Cat**-multifunctors to multifunctorial inverse $K$-theory. In [JY22], Johnson and Yau define Mandell's inverse $K$-theory multifunctor $\mathcal{P}$ as well as the **Cat**-multicategories that are its domain ($\Gamma$-categories) and target (permutative categories). Yau proves in [Yau24b] that $\mathcal{P}$ is pseudo symmetric. We refer the interested reader [Yau24b] of which the following theorem is one of the main results.

**Theorem 2.3.8.** *[Yau24b] Mandell's inverse $K$-theory functor is a pseudo symmetric* **Cat***-multifunctor* $\mathcal{P}\colon \Gamma\text{-}\mathbf{Cat} \to \mathbf{PermCat}^{\mathbf{sg}}$.

As a consequence, $\mathcal{P}$ sends commutative monoids to $E_\infty$-algebras and preserves $E_n$-algebras parameterized by free $E_n$-operads, as was stated in Corollary 1.0.2.

CHAPTER 3

# COHERENCE FOR SYMMETRIC PSEUDO COMMUTATIVE MONADS

We will prove a coherence result for symmetric, pseudo commutative, strong 2-monads. In this Chapter, we assume that $\mathcal{K}$ is a 2-category with finite products which we will denote by $\times$, with 1 denoting the empty product in $\mathcal{K}$. We will by $\rho\colon 1 \times - \to 1_{\mathcal{K}}$ and $\lambda\colon - \times 1 \to 1_{\mathcal{K}}$ the natural isomorphisms comming from the monoidal structure in $\mathcal{K}$ induced by products. As Hyland and Power, we believe what we do to work as well in a symmetric monoidal 2-category in general.

In Section 3.1, we define pseudo commutative, strong 2-monads $T\colon \ \to \mathcal{K}$ following [HP02]. We also define the multicategory $T\text{-}Alg$ when $T$ is symmetric. This allows us to extend the free $T$-algebra 2-functor $T : \mathcal{K} \to \mathcal{K}$ to a non-symmetric multifunctor $T : \mathcal{K} \to T\text{-}Alg$ in Section 3.2. We finish by proving that this multifunctor is pseudo symmetric in Section 3.3. This implies that the free functor for each of the pseudo commutative operads defined in [GMMO23] and also considered in [Yau24a] is pseudo symmetric.

## 3.1 Symmetric pseudo commutative 2-monads

**Definition 3.1.1.** [Koc70] Suppose that $T\colon \mathcal{K} \to \mathcal{K}$ is a 2-functor. A *strength* $t$ on $T$ is the data of a (strict) 2-natural transformation (see [YJ21]) with source

$$\mathcal{K} \times \mathcal{K} \xrightarrow{1_{\mathcal{K}} \times T} \mathcal{K} \times \mathcal{K} \xrightarrow{\times} \mathcal{K},$$

and target

$$\mathcal{K} \times \mathcal{K} \xrightarrow{\times} \mathcal{K} \xrightarrow{T} \mathcal{K}.$$

The component of $t$ at $(A, B) \in \mathrm{Ob}(\mathcal{K} \times \mathcal{K})$, will be denoted by $t_{A,B}\colon A \times TB \to T(A \times B)$ or just $t$ when there is no room for confusion. These data are required to satisfy the following axioms:

- **Unity:** the triangle

$$\begin{array}{ccc} 1 \times TA & \xrightarrow{t_{1,A}} & T(1 \times A) \\ & \searrow^{\lambda} & \downarrow{\scriptstyle T\lambda} \\ & & TA. \end{array}$$

commutes for all $A \in \mathrm{Ob}(\mathcal{K})$.

- **Associativity:** the triangle

$$\begin{array}{ccc} A \times B \times TC & \xrightarrow{1_A \times t_{B,C}} & A \times T(B \times C) \\ & \searrow^{t_{A\times B,C}} & \downarrow t_{A,B\times C} \\ & & T(A \times B \times C) \end{array}$$

commutes for every $A, B \in \mathrm{Ob}(\mathcal{K})$.

In this case we say that $T \colon \mathcal{K} \to \mathcal{K}$ is *strong* with strength $t$.

**Remark 3.1.2.** Suppose that $T \colon \mathcal{K} \to \mathcal{K}$ is a strong 2-functor. The following notation is introduced in [HP02]. For $n \geq 2$, $t_i^n$ will denote the natural isomorphism having as component at $(A_1, \ldots, A_n) \in \mathrm{Ob}(\mathcal{K}^n)$, the 1-cell

$$\begin{array}{ccc} A_1 \times \cdots \times A_{i-1} \times TA_i \times A_{i+1} \times \cdots \times A_n & \xrightarrow{t^n_{i\ A_1,\ldots,A_n}} & T(A_1 \cdots \times A_n) \\ \cong\downarrow & & \uparrow T\cong \\ A_1 \times \cdots \times A_{i-1} \times A_{i+1} \times \cdots \times A_n \times TA_i & \xrightarrow[t]{} & T(A_1 \times \cdots \times A_n \times A_i). \end{array}$$

We will denote $t^n_{i\ A_1,\ldots,A_n} = t_i$ when there is no room for confusion. Notice that $t = t_2^2$. In [HP02], $t_1^2$ is also called $t^*$. We will write our arrows in terms of $t_1^2$ and $t_2^2$ when possible. We notice that the associativity axiom implies that $t_i^n$ can be written in many different ways using the $t_i^k$ for $k < n$. For example, one can prove by induction that the triangle

$$\begin{array}{ccc} A_1 \times \cdots \times A_{i-1} \times TA_i \times A_{i+1} \times \cdots \times A_n & \xrightarrow{1_{A_1 \times \cdots \times A_{i-1}} \times t_1} & A_1 \times \cdots \times A_{i-1} \times T(A_i \times \cdots \times A_n) \\ & \searrow^{t^n_{i\ A_1,\ldots,A_n}} & \downarrow t_2 \\ & & T(A_1 \times \cdots \times A_n) \end{array}$$

commutes, as well as the triangle

$$\begin{array}{ccc} A_1 \times \cdots \times A_{i-1} \times TA_i \times A_{i+1} \times \cdots \times A_n & \xrightarrow{t_2 \times 1_{A_{i+1} \times \cdots \times A_n}} & T(A_1 \times \cdots \times A_i) \times A_{i+1} \cdots \times A_n \\ & \searrow^{t^n_{i\ A_1,\ldots,A_n}} & \downarrow t_1 \\ & & T(A_1 \times \cdots \times A_n). \end{array}$$

**Definition 3.1.3.** Let $(T\colon \mathcal{K} \to \mathcal{K}, \eta\colon 1_{\mathcal{K}} \to T, \mu\colon T^2 \to T)$ be a 2-monad. That is, $T$ is a strict 2-functor and $\eta, \mu$ are strict 2-natural transformations satisfying the usual triangle identities (see [YJ21]). We say that $(T, \eta, \mu, t)$ is a *strong 2-monad* with strength $t$, if $T\colon \mathcal{K} \to \mathcal{K}$ is strong with strength $t$ as a 2-functor and $\eta, \mu$ and $t$ are compatible in the sense that, for every $A, B \in \mathrm{Ob}(\mathcal{K})$, the diagram

$$\begin{array}{ccc} A \times B & \xrightarrow{1\times\eta} & A \times TB \\ & \searrow^{\eta} & \downarrow{\scriptstyle t} \\ & & T(A \times B) \end{array}$$

commutes, as well as the diagram

$$\begin{array}{ccccc} A \times T^2B & & \xrightarrow{1\times\mu} & & A \times TB \\ {\scriptstyle t}\downarrow & & & & \downarrow{\scriptstyle t} \\ T(A \times TB) & \xrightarrow[Tt]{} & T^2(A \times B) & \xrightarrow[\mu]{} & T(A \times B). \end{array}$$

**Definition 3.1.4.** [Koc70] A strong 2-monad $(T, \eta, \mu)$ is called commutative when the diagram

$$\begin{array}{ccccc} TA \times TB & \xrightarrow{t_1} & T(A \times TB) & \xrightarrow{Tt_2} & T^2(A \times B) \\ {\scriptstyle t_2}\downarrow & & & & \downarrow{\scriptstyle \mu} \\ T(TA \times B) & \xrightarrow[Tt_1]{} & T^2(A \times B) & \xrightarrow[\mu]{} & T(A \times B) \end{array}$$

commutes for every $A, B \in \mathrm{Ob}(\mathcal{K})$.

**Remark 3.1.5.** Suppose that $(T, \eta, \mu, t)$ is a strong 2-monad. Then, $T$ can be regarded as a monoidal 2-functor in two different ways. In each case, the unitary component is given by $\eta_1\colon 1 \to T1$. The binary components are given by the two 1-cells that form the boundary of the previous diagram. For each of these ways of seeing $T$ as a monoidal 2-functor, $\eta$ is a monoidal 2-natural transformation. It is proven in [Koc70] that $T$ is commutative if and only if $T$ is a monoidal 2-monad (i.e., $\mu$ is a monoidal 2-natural transformation).

There are a lot of examples of strong 2-monads which are non-commutative, but that are commutative up to coherent natural isomorphism, these are called pseudo commutative monads and we will defined them next. The examples include the 2-monads $T\colon \mathbf{Cat} \to \mathbf{Cat}$ given by the free construction for symmetric stric monoidal categories, symmetric monoidal categories, categories with finite products, categories with finite coproducts, etc. A longer list is included in [HP02]. More examples come from pseudo commutative operads as defined by Corner and Gurski [CG23]. These are operads whose associated monads are pseudo commutative. Guillou, Merling, May and Osorno [GMMO23] prove that chaotic operads are pseudo commutative.

**Definition 3.1.6.** [HP02, Def. 5] A strong 2-monad $(T, \eta, \mu)$ is called *pseudo-commutative* with *pseudocommutativity* $\Gamma$ if there exists an invertible modification with components

$$\begin{array}{ccccc} TA \times TB & \xrightarrow{t_1} & T(A \times TB) & \xrightarrow{Tt_2} & T^2(A \times B) \\ {\scriptstyle t_2}\downarrow & & \overset{\Gamma_{A,B}}{\Longleftarrow} & & \downarrow{\scriptstyle \mu} \\ T(TA \times B) & \xrightarrow{Tt_1} & T^2(A \times B) & \xrightarrow{\mu} & T(A \times B) \end{array}$$

such that the following axioms are satisfied. We will write $\Gamma$ instead of $\Gamma_{A,B}$ when $A$ and $B$ are clear from the context.

1. $\Gamma_{A\times B,C} \circ (t_{2A,B} \times 1_{TC}) = t_{2A,B\times C} \circ (1_A \times \Gamma_{B,C})$, i.e., the following pasting diagram equality holds:

$$\begin{array}{c} A \times TB \times TC \xrightarrow{t_2 \times 1} T(A \times B) \times TC \overset{\Gamma}{\Downarrow} T(A \times B \times C) \\ \| \\ A \times TB \times TC \overset{1\times\Gamma}{\Downarrow} A \times T(B \times C) \xrightarrow{t_2} T(A \times B \times C). \end{array} \tag{3.1.1}$$

2. $\Gamma_{A,B\times C} \circ (1_{TA} \times t_{2B,C}) = \Gamma_{A\times B,C} \circ (t_{1A,B} \times 1_{TC})$, i.e., the following equality holds:

$$\begin{array}{c} TA \times B \times TC \xrightarrow{1\times t_2} TA \times T(B \times C) \overset{\Gamma}{\Downarrow} T(A \times B \times C) \\ \| \\ TA \times B \times TC \xrightarrow{t_1 \times 1} T(A \times B) \times TC \overset{\Gamma}{\Downarrow} T(A \times B \times C). \end{array} \tag{3.1.2}$$

3. $\Gamma_{A,B\times C} \circ (1_{TA} \times t_{1B,C}) = t_{1A\times B,C} \circ (\Gamma_{A,B} \times 1_C)$, i.e., the following whiskering equality holds:

$$\begin{array}{ccc} TA \times TB \times C \xrightarrow{1\times t_1} TA \times T(B \times C) & \overset{\Gamma \Downarrow}{\rightrightarrows} & T(A \times B \times C) \\ & \| & \\ TA \times TB \times C \overset{\Gamma\times 1 \Downarrow}{\rightrightarrows} T(A \times B) \times C & \xrightarrow{t_1} & T(A \times B \times C). \end{array} \tag{3.1.3}$$

4. $\Gamma_{A,B} \circ (\eta_A \times 1_{TB})$ is an identity 2-cell. That is, the following whiskering is an identity:

$$A \times TB \xrightarrow{\eta\times 1} TA \times TB \overset{\Gamma \Downarrow}{\rightrightarrows} T(A \times B). \tag{3.1.4}$$

5. $\Gamma_{A,B} \circ (1_{TA} \times \eta_B)$ is an identity 2-cell, that is, the following whiskering is an identity:

$$TA \times B \xrightarrow{1\times \eta} TA \times TB \overset{\Gamma \Downarrow}{\rightrightarrows} T(A \times B). \tag{3.1.5}$$

6. The whiskering

$$T^2A \times TB \xrightarrow{\mu\times 1} TA \times TB \overset{\Gamma \Downarrow}{\rightrightarrows} T(A \times B)$$

is equal to the pasting

$$
\begin{tikzcd}
T^2A \times TB \arrow[r, "t_1"] \arrow[d, "t_2"'] & T(TA \times TB) \arrow[r, "Tt_1"] \arrow[d, "Tt_2"] & T^2(A \times TB) \arrow[r, "T^2t_2"] \arrow[dl, Rightarrow, "T\Gamma"'] & T^3(A \times B) \arrow[d, "T\mu"] \\
T(T^2A \times B) \arrow[d, "Tt_1"'] & T^2(TA \times B) \arrow[r, "T^2t_1"] \arrow[d, "\mu"] \arrow[l, Rightarrow, "\Gamma"'] & T^3(A \times B) \arrow[r, "T\mu"] \arrow[d, "\mu"] & T^2(A \times B) \arrow[d, "\mu"] \\
T^2(TA \times B) \arrow[r, "\mu"'] & T(TA \times B) \arrow[r, "Tt_1"'] & T^2(A \times B) \arrow[r, "\mu"'] & T(A \times B).
\end{tikzcd}
\tag{3.1.6}
$$

7. The whiskering

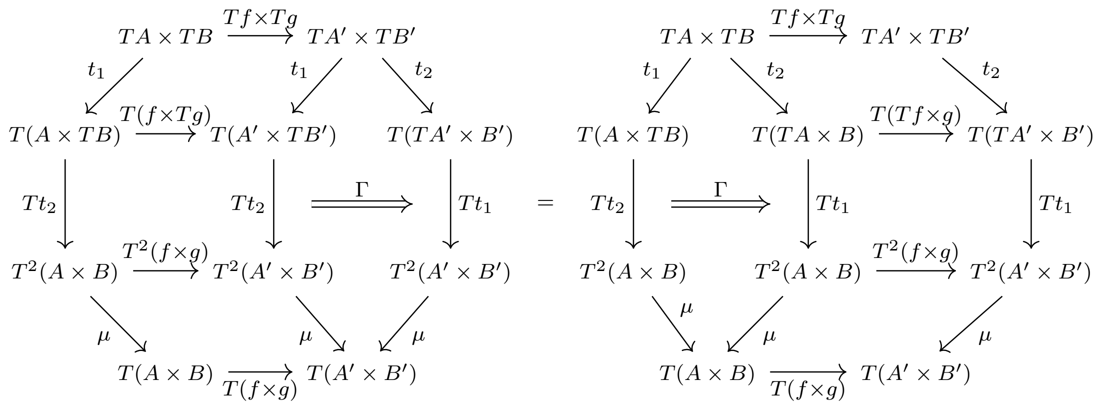
$$
\begin{tikzcd}
TA \times T^2B \arrow[r, "1\times\mu"] & TA \times TB \arrow[r, bend left, ""{name=U}] \arrow[r, bend right, ""'{name=D}] & T(A \times B)
\arrow[Rightarrow, from=U, to=D, "\Gamma"']
\end{tikzcd}
$$

is equal to the pasting

$$
\begin{tikzcd}
TA \times T^2B \arrow[r, "t_1"] \arrow[d, "t_2"'] & T(A \times T^2B) \arrow[r, "Tt_2"] & T^2(A \times TB) \arrow[d, "\mu"] \arrow[dl, Rightarrow, "\Gamma"'] \\
T(TA \times TB) \arrow[r, "Tt_1"'] \arrow[d, "Tt_2"'] & T^2(A \times TB) \arrow[r, "\mu"'] \arrow[d, "T^2t_2"] & T(A \times TB) \arrow[d, "Tt_2"] \\
T^2(TA \times B) \arrow[d, "T^2t_1"'] & T^3(A \times B) \arrow[r, "\mu"] \arrow[d, "T\mu"] \arrow[dl, Rightarrow, "T\Gamma"] & T^2(A \times B) \arrow[d, "\mu"] \\
T^3(A \times B) \arrow[r, "T\mu"'] & T^2(A \times B) \arrow[r, "\mu"'] & T(A \times B)
\end{tikzcd}
\tag{3.1.7}
$$

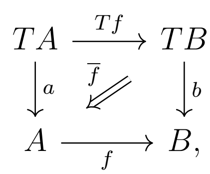
**Remark 3.1.7.** The fact that the source and target of the equal whiskering and pasting diagrams in the previous list of axioms are the same follows from the definition of 2-strong monad. In other words, the pseudo commutativity axioms don't introduce new relations among 1-cells.

A modification is more than a mere collection of 2-cells (see [JY22]). For $\Gamma$ to be a modification we need that given $f\colon A \to A'$ and $g\colon B \to B'$ in $\mathcal{K}$, the following equality of pasting diagrams holds:



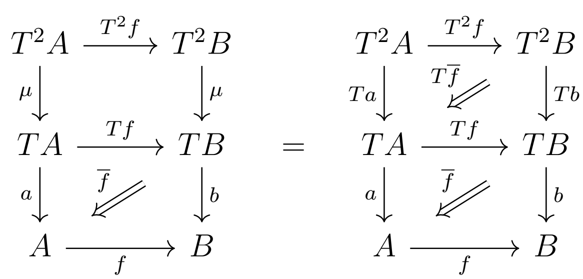

Following Blackwell, Kelly and Power [BKP02], we now define ,for any 2-monad $T\colon \mathcal{K} \to \mathcal{K}$, the 2-category $T$-$Alg$ of $T$-algebras and pseudo morphisms.

**Definition 3.1.8.** [BKP02, Def. 1.2] Let $T\colon \mathcal{K} \to \mathcal{K}$ be a 2-monad. The 2-category $T$-$Alg$ has strict $T$-algebras as 0-cells. A 1-cell $f$ between $T$-algebras $(A, a\colon TA \to A)$ and $(B, b\colon TB \to B)$, also called a strong morphism of $T$-algebras in [JY22], consists of a 1-cell $f\colon A \to B$ in $\mathcal{K}$, together with an invertible 2-cell



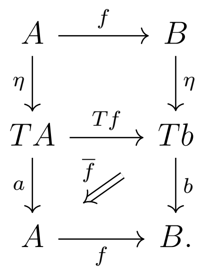

subject to the following axioms.

1. The equality of pasting diagrams

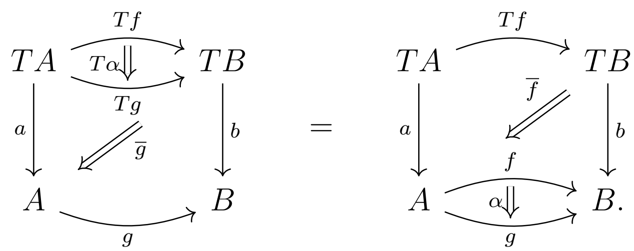



holds.

2. The following pasting diagram equals the identity of $f\colon A \to B$ :




A 2-cell in $T$-$Alg$ between 1-cells $(f, \bar{f}), (g, \bar{g}) \colon A \to B$ is a 2-cell $\alpha \colon f \to g$ in $\mathcal{K}$ such that the following diagram commmutes:




Hyland and Power [HP02] extend Blackwell, Kelly and Power's 2-categorical construction to provide a non symmetric **Cat**-multicategory whose underlying 2-category is $T$-$Alg$. If $T$ is a pseudo commutative 2-monad, the **Cat**-multicategory $T$-$Alg$ is symmetric. When $\mathcal{K} = \mathbf{Cat}$ and $T$ is accesible, Bourke proves [Bou02] that the **Cat**-multicategory structure can be seen to arise from a monoidal bicategory structure on $T$-$Alg$. Guillou, May, Merling and Osorno [GMMO23] specialize this definition to define a multicategory $\mathcal{O}$-$Alg$ for $\mathcal{O}$ a pseudo commutative operad. To be able to define the multicategory $T$-$Alg$, we need to prove a coherence result.

**Definition 3.1.9.** Suppose $T \colon \mathcal{K} \to \mathcal{K}$ is a pseudo-commutative 2-monad, $n \geq 2$ and $1 \leq i < j \leq n$. We define a modification from $\mu \circ Ttj \circ t_i$ to $\mu \circ Tt_i \circ t_j$ as follows. Suppose $A_1, \ldots, A_n$ objects of $\mathcal{K}$, we define the component 2-cell of our modificiation in

$$\mathcal{K}(A_1 \times \cdots \times A_{i-1} \times TA_i \times A_{i+1} \times \cdots \times A_{j-1} \times TA_j \times A_{j+1} \times \cdots \times A_n, T(A_1 \times \cdots \times A_n))$$

in the following way. In principle there are various ways of doing this. Consider a partition $K$ of the symbols $A_1, \ldots, TA_i, \ldots, TA_j, \ldots, A_n$ into 4 subsets $K_1, K_2, K_3, K_4$ obtained by placing 3 bars in between symbols such that:

- $K_2$ contains $TA_i$, and

- $K_3$ contains $TA_j$.

We will represent $K$ in the following way:

$$\underbrace{\cdots \times \cdots}_{K_1} \mid \underbrace{\cdots \times TA_i \times \cdots}_{K_2} \mid \underbrace{\cdots \times TA_j \times \cdots}_{K_3} \mid \underbrace{\cdots \times \cdots}_{K_4}.$$

For such a partition $K$, we can define the 2-cell $\Gamma^K_{i,j}$ as the whiskering

$$\begin{array}{c} A_1 \times \cdots \times TA_i \times \cdots \times TA_j \times \cdots \times A_n \\ \Big\downarrow = \\ \underbrace{\cdots \times \cdots}_{K_1} \mid \underbrace{\cdots \times TA_i \times \cdots}_{K_2} \mid \underbrace{\cdots \times TA_j \times \cdots}_{K_3} \mid \underbrace{\cdots \times \cdots}_{K_4} \\ \Big\downarrow 1\times t^{|K_2|}_{i-|K_1|}\times t^{|K_3|}_{j-|K_1|-|K_2|}\times 1 \\ \cdots \times \cdots \times T(\cdots \times A_i \times \cdots) \times T(\cdots \times A_j \times \cdots) \times \cdots \times \cdots \\ \Big\Downarrow 1\times\Gamma\times 1 \\ \underbrace{\cdots \times \cdots}_{K_1} \times T(\cdots \times A_i \times \cdots \times A_j \times \cdots) \times \underbrace{\cdots \times \cdots}_{K_4} \\ \Big\downarrow t^n_{|K_1|+1} \\ T(A_1 \times \cdots \times A_n). \end{array}$$

**Example 3.1.10.** For $n = 3$, $i = 2$, and $j = 3$, we have 2 possible partitions:

$$K = A_1 \mid TA_2 \mid TA_3 \mid, \text{ and } K' = \mid A_1TA_2 \mid TA_3 \mid .$$

By (3.1.1), we get that $\Gamma^K_{2,3} = \Gamma^{K'}_{2,3}$. For $n = 3$, $i = 1$ and $j = 3$, there are again two partitions:

$$H = \mid TA_1A_2 \mid TA_3 \mid, \text{ and } H' = \mid TA_1 \mid A_2TA_3 \mid,$$

and they induce the same 2-cell $\Gamma^H_{1,3} = \Gamma^{H'}_{1,2}$ by (3.1.2). Similarly for $n = 3$, $i = 1$ and $j = 2$, we have the two partitions

$$J = \mid TA_1 \mid TA_2A_3 \mid, \text{ and } J' = \mid TA_1 \mid TA_2 \mid A_3,$$

with $\Gamma^J_{1,2} = \Gamma^{J'}_{1,2}$ by (3.1.3). In general, we have the following.

**Theorem 3.1.11.** *[HP02, Thm. 5] Suppose $(T, \eta, \mu, t, \Gamma)$ is a pseudo commutative strong 2-monad. The three strength axioms imply that given $n \geq 2$, and $1 \leq i < j \leq$*

*$n$, any two partitions $K$ and $K'$ as in Definition 3.1.9 induce the same 2-cell. That is,*

$$\Gamma^{K}_{i,j} = \Gamma^{K'}_{i,j}.$$

*Proof.* The previous example generalizes and will allow us to change our partition without changing the induced 2-cell using three moves. Let $K$ be a partition of $A_1, \ldots, TA_i, \ldots, TA_j, \ldots, A_n$ as in Definition 3.1.9. The following hold:

(i) If $K_1$ ends by $A_p$, i.e.

$$K = \underbrace{\cdots \times \cdots \times A_p}_{K_1} \mid \underbrace{\times \cdots \times TA_i \times \cdots}_{K_2} \mid \underbrace{\cdots \times TA_j \times \cdots}_{K_3} \mid \underbrace{\cdots \times \cdots}_{K_4},$$

and $K'$ is obtained from $K$ by moving the first bar one spot to the left, i.e.

$$K' = \underbrace{\cdots \times \cdots \times}_{K_1} \mid \underbrace{A_p \times \cdots \times TA_i \times \cdots}_{K'_2} \mid \underbrace{\cdots \times TA_j \times \cdots}_{K'_3} \mid \underbrace{\cdots \times \cdots}_{K_4},$$

then $\Gamma^{K}_{i,j} = \Gamma^{K'}_{i,j}$ by (3.1.1).

(ii) If $K_2$ ends by $A_p$, that is

$$K = \underbrace{\cdots \times \cdots}_{K_1} \mid \underbrace{\cdots \times TA_i \times \cdots \times A_p}_{K_2} \mid \underbrace{\times \cdots \times TA_j \times \cdots}_{K_3} \mid \underbrace{\cdots \times \cdots}_{K_4}.$$

and $K'$ is obtained from $K$ by moving the second bar one spot to the left, i.e.,

$$K' = \underbrace{\cdots \times \cdots}_{K_1} \mid \underbrace{\cdots \times TA_i \times \cdots \times}_{K'_2} \mid \underbrace{A_p \times \cdots \times TA_j \times \cdots}_{K'_3} \mid \underbrace{\cdots \times \cdots}_{K_4},$$

then $\Gamma^{K}_{i,j} = \Gamma^{K'}_{i,j}$ by (3.1.2).

(iii) If $K_3$ ends by $A_p$, i.e.,

$$K = \underbrace{\cdots \times \cdots}_{K_1} \mid \underbrace{\cdots \times TA_i \times \cdots}_{K_2} \mid \underbrace{\cdots \times TA_j \times \cdots \times A_p}_{K_3} \mid \underbrace{\times \cdots \times \cdots}_{K_4}.$$

and $K'$ is obtained from $K$ by moving the third bar one spot to the left, i.e.

$$K' = \underbrace{\cdots \times \cdots}_{K_1} \mid \underbrace{\cdots \times TA_i \times \cdots}_{K_2} \mid \underbrace{\cdots \times TA_j \times \cdots \times}_{K'_3} \mid \underbrace{A_p \times \cdots}_{K'_4},$$

then $\Gamma^{K}_{i,j} = \Gamma^{K'}_{i,j}$ by (3.1.3).

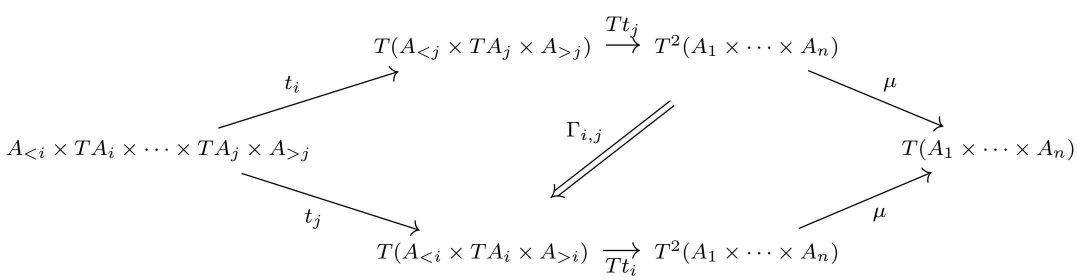
Finally, we notice that any partition $K'$ can be obtained from the partition

$$K = A_1 \times \cdots \times A_{i-1} \mid \times TA_i \times \cdots \times \mid TA_j \times \cdots \times A_n \mid$$

by making some number of moves (i), (ii) and (iii), and so $\Gamma^K_{i,j} = \Gamma^{K'}_{i,j}$. ■

**Definition 3.1.12.** Let $(T, \eta, \mu, t, \Gamma)$ be a pseudo commutative, strong 2-monad, $n \geq 2$, $1 \leq i < j \leq n$ and $A_1, \ldots, A_n$ objects of $\mathcal{K}$ we define the unique 2-cell in the previous theorem as $\Gamma_{i,j}$. That is, if $K$ is a partition as in Definition 3.1.9, then $\Gamma_{i,j} = \Gamma^K_{i,j}$

**Remark 3.1.13.** To save some space in the following definitions we will denote the product $A_1 \times \cdots \times A_{i-1}$ as $A_{<i}$. When considering a product $A_1 \times \cdots \times A_n$ we will also write $A_{>i} = A_{i+1} \times \cdots \times A_n$.
The 2-cell $\Gamma_{i,j}$ defined in the previous theorem fits in the following diagram by the $\mu$ axiom for strong monads in Definition 3.1.3:




Next, we define the **Cat**-multicategory $T$-$Alg$, whose underlying 2-category is $T$-$Alg$ from Definition 3.1.8. In Definition 3.1.14 we define the 2-cells of $T$-$Alg$, in Definition 3.1.15 we define the 2-cells in $T$-$Alg$, and in Definition 3.1.16 we define the composition in $T$-$Alg$.

**Definition 3.1.14.** [HP02, Def. 10] Let $(T, \eta, \mu, t, \Gamma)$ be a pseudo commutative, strong 2-monad. The $n$-ary 1-cells of the **Cat**-multicategory $T$-$Alg$ are defined as follows. When $n = 0$, and $B$ is a $T$-algebra, we define the category $T$-$Alg(-; B)$ as $\mathcal{K}(1, B)$.
Suppose that $(A_i, a_i \colon TA_i \to A_i)$ for $1 \leq i \leq n$ and $(B, b \colon TB \to B)$ are $T$-algebras. An $n$-ary 1-cell of $T$-$Alg$, $\langle A_1 \times \cdots \times A_n \rangle \to B$ is the data of a 1-cell $h \colon A_1 \times \cdots \times A_n \to B$ in $\mathcal{K}$, together with 2-cells $h_i$ for $1 \leq i \leq n$ fitting in the square:

$$
\begin{array}{ccccc}
A_{<i} \times TA_i \times A_{>i} & \xrightarrow{t_i} & T(A_1 \times \cdots \times A_n) & \xrightarrow{Th} & TB \\
{\scriptstyle 1\times a_i \times 1}\big\downarrow & & \overset{\Longleftarrow}{\scriptstyle h_i} & & \big\downarrow{\scriptstyle b} \\
A_1 \times \cdots \times A_n & & \xrightarrow[h]{\qquad\qquad\qquad} & & B.
\end{array}
$$

These data have to satisfy the following axioms.

- $\eta$ axiom: The following pasting diagram is the identity of $h\colon A_1 \times \cdots \times A_n \to B$.

$$
\begin{array}{ccccc}
 & & A_{<i} \times A_i \times A_{>i} & \xrightarrow{h} & B \\
 & \swarrow{\scriptstyle 1\times\eta\times 1} & \big\downarrow{\scriptstyle \eta} & & \big\downarrow{\scriptstyle \eta} \\
A_{<i} \times TA_i \times A_{>i} & \xrightarrow{t_i} & T(A_1 \times \cdots \times A_n) & \xrightarrow[Th]{} & TB \\
{\scriptstyle 1\times a_i \times 1}\big\downarrow & & \overset{\Longleftarrow}{\scriptstyle h_i} & & \big\downarrow{\scriptstyle b} \\
A_1 \times \cdots \times A_n & & \xrightarrow[h]{\qquad\qquad\qquad} & & B.
\end{array}
\tag{3.1.8}
$$

- $\mu$ axiom: The pasting diagrams

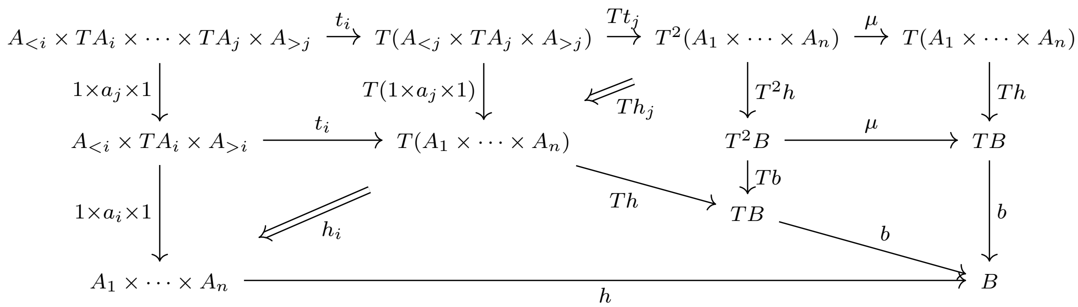
$$
\begin{array}{ccccccc}
A_{<i} \times T^2A_i \times A_{>i} & \xrightarrow{t_i} & T(A_{<i} \times TA_i \times A_{>i}) & \xrightarrow{Tt_i} & T^2(A_1 \times \cdots \times A_n) & \xrightarrow{T^2h} & T^2B \\
{\scriptstyle 1\times\mu\times 1}\big\downarrow & & & & \big\downarrow{\scriptstyle \mu} & & \big\downarrow{\scriptstyle \mu} \\
A_{<i} \times TA_i \times A_{<i} & & \xrightarrow{\qquad t_i \qquad} & & T(A_1 \times \cdots \times A_n) & \xrightarrow{Th} & TB \\
{\scriptstyle 1\times a_i \times 1}\big\downarrow & & \overset{\Longleftarrow}{\scriptstyle h_i} & & & & \big\downarrow{\scriptstyle b} \\
A_1 \times \cdots \times A_n & & & \xrightarrow[h]{\qquad\qquad\qquad} & & & B
\end{array}
\tag{3.1.9}
$$

and,

$$
\begin{array}{ccccccc}
A_{<i} \times T^2A_i \times A_{>i} & \xrightarrow{t_i} & T(A_{<i} \times TA_i \times A_{>i}) & \xrightarrow{Tt_i} & T^2(A_1 \times \cdots \times A_n) & \xrightarrow{T^2h} & T^2B \\
{\scriptstyle 1\times Ta_i\times 1}\big\downarrow & & {\scriptstyle T(1\times a_i\times 1)}\big\downarrow & & \overset{\Longleftarrow}{\scriptstyle Th_i} & & \big\downarrow{\scriptstyle Tb} \\
A_{<i} \times TA_i \times A_{>i} & \xrightarrow{t_i} & T(A_1 \times \cdots \times A_n) & & \xrightarrow[Th]{\qquad\qquad} & & TB \\
{\scriptstyle 1\times a_i \times 1}\big\downarrow & & \overset{\Longleftarrow}{\scriptstyle h_i} & & & & \big\downarrow{\scriptstyle b} \\
A_1 \times \cdots \times A_n & & & \xrightarrow[h]{\qquad\qquad\qquad} & & & B
\end{array}
\tag{3.1.10}
$$

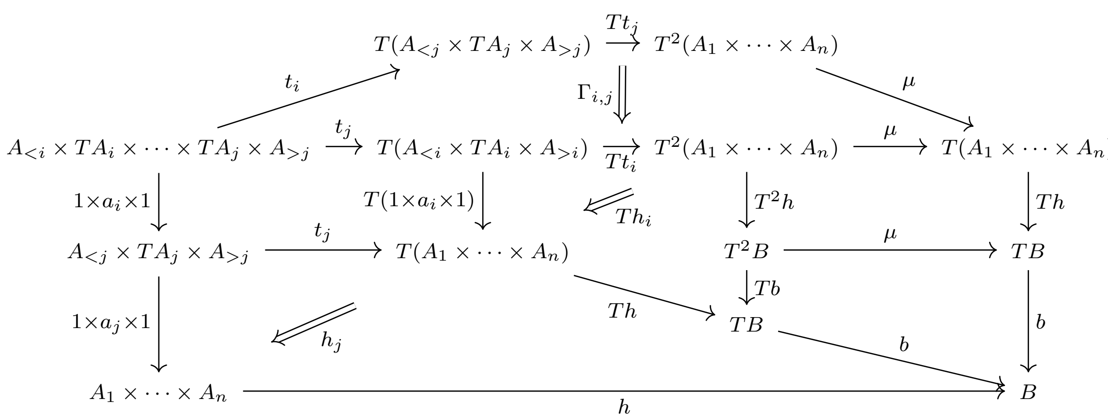
are equal.

- Coherence: For $i < j$, the pasting diagrams



(3.1.11)

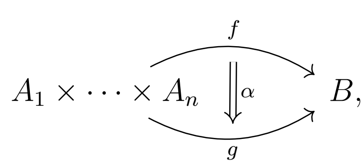
and,



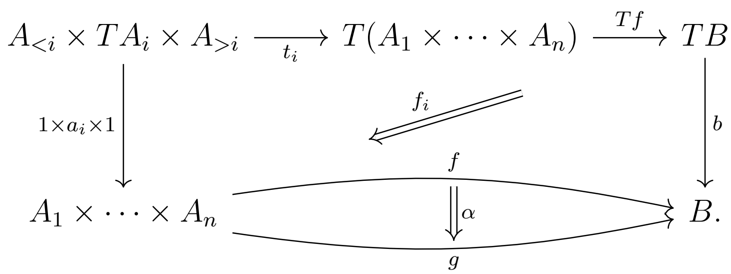
(3.1.12)

are equal.

**Definition 3.1.15.** [HP02, Def. 10] Let $(T, \eta, \mu, t, \Gamma)$ be a pseudo commutative, strong 2-monad. We define the 2-cells of $T\text{-}Alg$ as follows. Suppose that $(A_i, a_i\colon TA_i \to A_i)$ and $(B, b\colon TB \to B)$ are $T$-algebras for $1 \leq i \leq n$, and that $(f, \langle f_i \rangle)$ and $(g, \langle g_i \rangle)$ are 1-cells in $T\text{-}Alg(\langle A_1, \ldots, A_n \rangle, B)$ . A 2-cell $\alpha\colon f \to g$ in $T\text{-}Alg$ is the datum of a 2-cell in $\mathcal{K}$

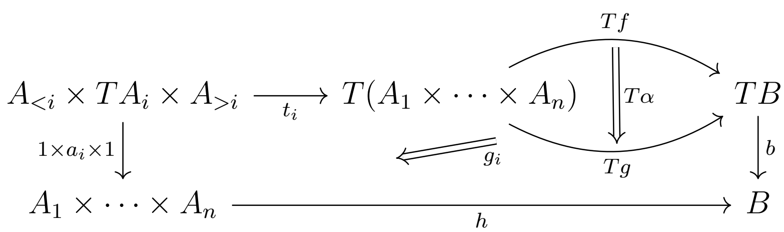


subject to the equality, for $i < n$, of the pasting diagrams



(3.1.13)

and,



(3.1.14)

are equal. Vertical composition of 2-cells in $T\text{-}Alg$ is given by vertical composition in $\mathcal{K}$.

Next we define the $\gamma$ composition in $T\text{-}Alg$.

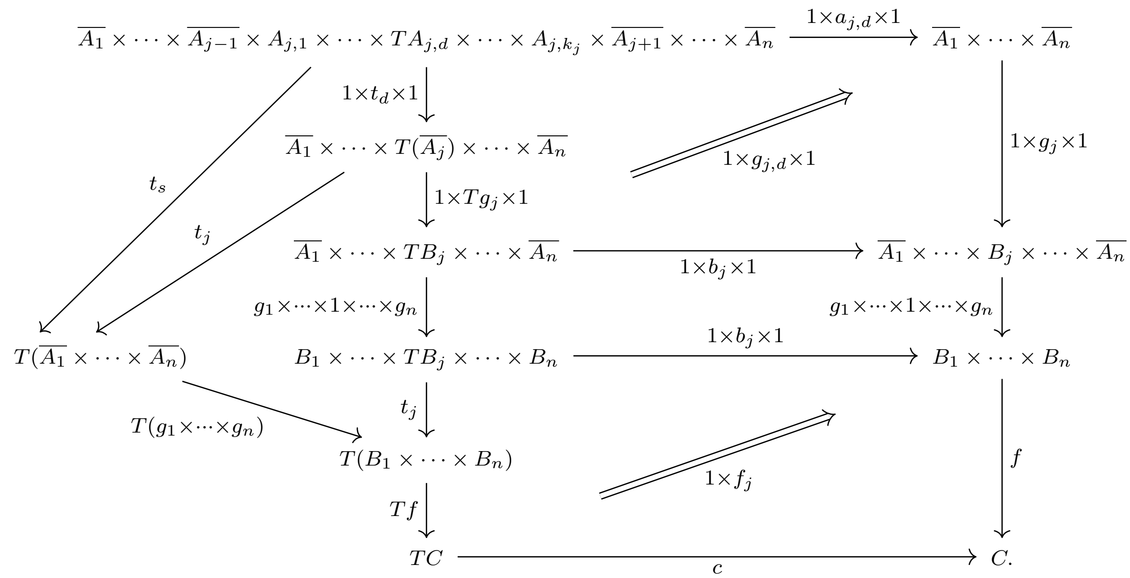
**Definition 3.1.16.** Let $(T, \eta, \mu, t, \Gamma)$ be a pseudo commutative, strong 2-monad. For $C \in \mathrm{Ob}(T\text{-}Alg)$, $n \geq 0$, $\langle B \rangle = \langle B_j \rangle_{j=1}^n \in \mathrm{Ob}(T\text{-}Alg)^n$, $k_j \geq 0$ for $1 \leq j \leq n$, and $\langle A_j \rangle = \langle A_{j,i} \rangle_{i=1}^{k_j} \in \mathrm{Ob}(T\text{-}Alg)^{k_j}$ for $1 \leq j \leq n$, we define

$$T\text{-}Alg(\langle B \rangle; C) \times \prod_{j=1}^{n} T\text{-}Alg(\langle A_j \rangle; B_j) \xrightarrow{\gamma} T\text{-}Alg(\langle A \rangle; C)$$

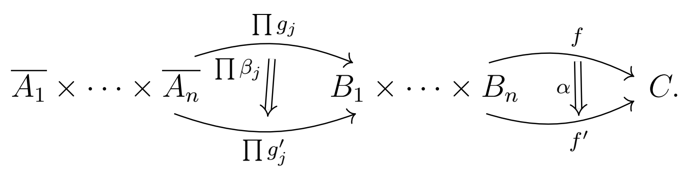
as follows. Let $(f, f_j) \colon B_1 \times \cdots \times B_n \to C$ and $(g_j, g_{ji}) \colon A_{j,1} \times \cdots \times A_{j,k_j} \to B_j$ 1-cells of $T$-algebras. We define their $\gamma$ composition as the $\mathcal{K}$ 1-cell

$$\overline{A_1} \times \cdots \times \overline{A_n} \xrightarrow{\prod g_i} B_1 \times \cdots \times B_n \xrightarrow{f} C$$

where $\overline{A_j}$ denotes $\prod_{i=1}^{k_j} A_{j,i}$. Any number between $s$ with $1 \leq s \leq \sum_{j=1}^{n} k_j$ can be uniquely written as $s = d + \sum_{t<j} k_t$ where $d < k_j$. We define $\gamma(\langle g_j \rangle, f)_s$ as the pasting



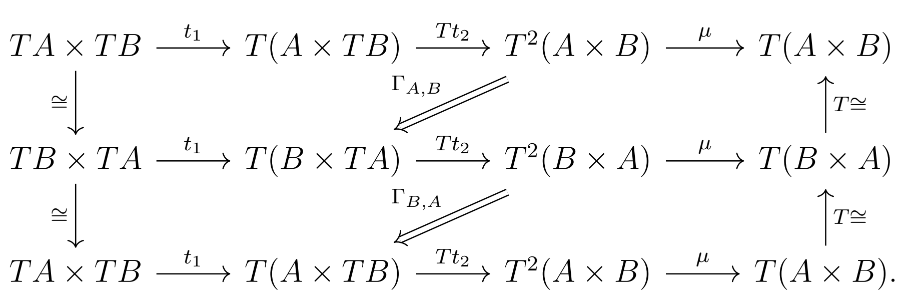

The multilinear composition for 2-cells is defined in the following way. Suppose that $(f, f_j), (f', f_j)\colon \langle B_1, \ldots, B_n \rangle \to C$ and $(g_j, g_{ji}), (g'_j, g'_{ji})\colon \langle A_{j,1}, \ldots, A_{j,k_j} \rangle \to B_j$ are 1-cells in $T$-$Alg$ and $\alpha\colon f \to f', \beta_j\colon g_j \to g'_j$ 2-cells in $T$-$Alg$. Then, the component 2-cell of $\gamma(\alpha; \beta_1, \ldots, \beta_n)$ is the pasting




One can easily check that this composition is well defined.

By imposing an extra condition we on $T$ we can turn $T$-$Alg$ into symmetric **Cat**-multicategory.

**Definition 3.1.17.** A pseudo commutative, strong 2-monad $(T, \eta, \mu, \Gamma)$ is called *symmetric* if for all $A, B$ objects of $\mathcal{K}$, the following pasting diagram equals the identity of the 1-cell $TA \times TB \xrightarrow{t_1} T(A \times TB) \xrightarrow{Tt_2} T^2(A \times B) \xrightarrow{\mu} T(A \times B)$:

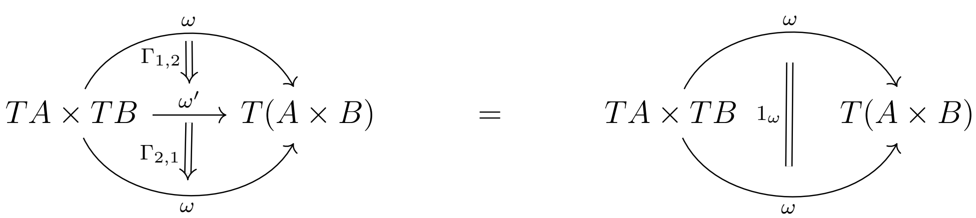


In general, for $1 \leq i < j \leq n$ we can define 2-cells $\Gamma_{j,i}$ that are inverses to the $\Gamma_{i,j}$ from Theorem 3.1.11.

**Definition 3.1.18.** Let $T\colon \mathcal{K} \to \mathcal{K}$ be a pseudo commutative 2-monad and $A_1, \ldots, A_n$ objects of $\mathcal{K}$. Let $K$ be a partition as in Definition 3.1.9. We define $\Gamma_{j,i}$ as the whiskering

$$
\begin{array}{c}
A_1 \times \cdots \times TA_i \times \cdots \times TA_j \times \cdots \times A_n \\
\Big\downarrow{\scriptstyle =} \\
\underbrace{\cdots \times \cdots}_{K_1} \mid \underbrace{\cdots \times TA_i \times \cdots}_{K_2} \mid \underbrace{\cdots \times TA_j \times \cdots}_{K_3} \mid \underbrace{\cdots \times \cdots}_{K_4} \\
\Big\downarrow{\scriptstyle \cong} \\
\underbrace{\cdots \times \cdots}_{K_1} \mid \underbrace{\cdots \times TA_j \times \cdots}_{K_3} \mid \underbrace{\cdots \times TA_i \times \cdots}_{K_2} \mid \underbrace{\cdots \times \cdots}_{K_4} \\
\Big\downarrow{\scriptstyle 1 \times t_{j-|K_1|} \times t_{i-|K_1|-|K_3|} \times 1} \\
\underbrace{\cdots \times \cdots}_{K_1} \times T(\cdots \times A_j \times \cdots) \times T(\cdots \times A_i \times \cdots) \times \cdots \underbrace{\times \cdots}_{K_4} \\
\Big\Downarrow{\scriptstyle 1 \times \Gamma \times 1} \\
\underbrace{\cdots \times \cdots}_{K_1} \times T(\cdots \times A_j \times \cdots \times A_i \times \cdots) \times \underbrace{\cdots \times \cdots}_{K_4} \\
\Big\downarrow{\scriptstyle t_{|K_1|+1}} \\
T(\cdots \times A_j \times \cdots \times A_i \times \cdots) \\
\Big\downarrow{\scriptstyle T\cong} \\
T(A_1 \times \cdots \times A_n).
\end{array}
$$

**Remark 3.1.19.** Notice that $\Gamma_{j,i}$ is independent of the partition by Theorem 3.1.11. The symmetry axiom can thus be written as $\Gamma_{1,2} = \Gamma_{2,1}^{-1}$. If we write $\Gamma_{1,2}\colon \omega \to \omega'$, then the symmetry axiom takes the form



**Lemma 3.1.20.** *Let $T$ be a symmetric, strong, pseudo commutative 2-monad. Let $0 \leq i < j \leq n$. Then $\Gamma_{j,i} = \Gamma_{i,j}^{-1}$.*

By the invertibility of $\Gamma$, the inverse pseudo commutativity 2-cells $\Gamma^{-1}$ satisfy analogous properties to the axioms in Definition 3.1.6. When using this properties we will refer the reader to Definition 3.1.6.

Next we define the symmetric **Cat**-multicategorical structure on $T$-$Alg$ for $T$ symmetric. This is the definition of Hyland and Power [HP02], which agrees with the one given in [GMMO23] for pseudo commutative operads.

**Definition 3.1.21.** [HP02, Prop. 18] Let $(T, \eta, \mu, t, \Gamma)$ be a symmetric, pseudo commutative, strong 2-monad. We give $T$-$Alg$ the structrue of a symmetric **Cat**-multicategory by defining the action of the symmetric group. For $A_1, \ldots, A_n, B$ objects of $\mathcal{K}$, and $\sigma \in \Sigma_n$, define

$$T\text{-}Alg(A_1, \ldots, A_n; B) \xrightarrow{\sigma} T\text{-}Alg(A_{\sigma(1)}, \ldots, A_{\sigma(n)}; B),$$

in the following way. If $(h, h_i)\colon \langle A_1, \ldots, A_n \rangle \to B$ is a 1-cell in $T$-$Alg$, we define the 1-cell component of $h\sigma$ in $\mathcal{K}$ as

$$A_{\sigma(1)} \times \cdots \times A_{\sigma(n)} \xrightarrow{\sigma} A_1 \times \cdots \times A_n \xrightarrow{h} B.$$

We define $h\sigma_i$ as the pasting

$$\begin{array}{ccccccc}
A_{\sigma(1)} \times \cdots \times TA_{\sigma(i)} \times \cdots \times A_{\sigma(n)} & \xrightarrow{\sigma} & A_1 \times \cdots \times TA_{\sigma(i)} \times \cdots \times A_n & \xrightarrow{t_{\sigma(i)}} & T(A_1 \times \cdots \times A_n) & \xrightarrow{h} & TB \\
\downarrow {\scriptstyle 1 \times a_{\sigma(i)} \times 1} & & \downarrow {\scriptstyle 1 \times a_{\sigma(i)} \times 1} & \swarrow {\scriptstyle h^n_{\sigma(i)}} & & & \downarrow {\scriptstyle b} \\
A_{\sigma(1)} \times \cdots \times A_{\sigma(n)} & \xrightarrow[\sigma]{} & A_1 \times \cdots \times A_n & & \xrightarrow[h]{} & & B.
\end{array}$$

Similarly, for $\alpha\colon f \to g$ 2-cell in $T$-$Alg(A_1, \ldots, A_n; B)(f, g)$, $\alpha\sigma$ is defined as having component 2-cell

$$A_{\sigma(1)} \times \cdots \times A_{\sigma(n)} \xrightarrow{\sigma} A_1 \times \cdots \times A_n \underset{g}{\overset{f}{\underset{\alpha \Downarrow}{\rightrightarrows}}} B.$$

**Remark 3.1.22.** To prove that given a 1-cell $h\colon \langle A_1, \ldots, A_n \rangle \to B$ in $T$-$Alg$, $h\sigma\colon \langle A_{\sigma(1)}, \ldots, A_{\sigma_n} \rangle \to B$ is indeed a 1-cell in $T$-$Alg$ we need the symmetry axiom. The $\eta$ and $\mu$ axioms for $h\sigma$ follow from the same axioms for $h$. To prove coherence one can prove that given $0 \leq i < j \leq n$, if $h$ satisfies coherence, then so does

$h\sigma_{i,j}$. Here, $\sigma_{i,j} \in \Sigma_n$ is the transposition that permutes $i$ and $j$. Coherence for $h\sigma_{i,j}$ follows from coherence for $h$ together with Lemma 3.1.20.

## 3.2 The free $T$-algebra functor as a multifunctor

We embark now on the proof of the main theorem of this section. Recall that for $(T, \eta, \mu, t, \Gamma)$ a pseudo commutative, strong 2-monad and $A \in \mathcal{K}$, $(TA, \mu : T^2A \to TA)$ is a $T$-algebra and can be thought of as the free $T$ algebra generated by $A$. This defines a 2-functor $T\colon \mathcal{K} \to T\text{-}Alg$ [BKP02] that, as we show, can be extended to a pseudo symmetric multifunctor when $T$ is symmetric.

**Remark 3.2.1.** Notice that for $(T, \eta, \mu, t, \Gamma)$ a pseudo commutative, strong 2-monad, and $A_1, A_2 \in \mathcal{K}$, $\Gamma$ fits in the following diagram:

$$\begin{array}{ccc} TA_1 \times TA_2 & \xrightarrow{\omega_{A_1,A_2}} & T(A_1 \times A_2) \\ {\scriptstyle \cong}\downarrow & \overset{\Longleftarrow}{\scriptstyle \Gamma_{A_1,A_2}} & \uparrow{\scriptstyle T\cong} \\ TA_2 \times TA_1 & \xrightarrow[\omega_{A_2,A_1}]{} & T(A_2 \times A_1). \end{array}$$

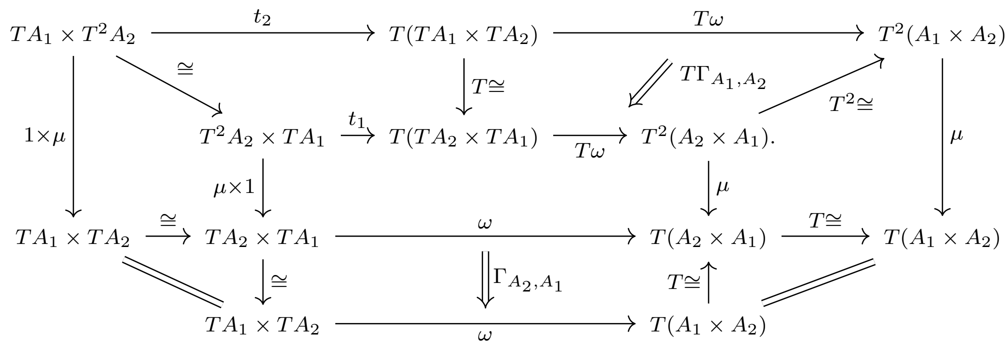
**Definition 3.2.2.** Let $(T, \eta, \mu, t, \Gamma)$ be a symmetric, pseudo commutative, strong 2-monad. Given $A_1, A_2 \in \mathrm{Ob}(\mathcal{K})$ we define the 2-ary 1-cell in $T\text{-}Alg$, $\omega = \omega_{A_1,A_2}\colon \langle TA_1, TA_2\rangle \to T(A_1 \times A_2)$ as follows. The component 1-cell $\omega_{A_1,A_2}$ is the composite

$$TA_1 \times TA_2 \xrightarrow{t_1} T(A_1 \times TA_2) \xrightarrow{Tt_2} T^2(A_1 \times A_2) \xrightarrow{\mu} T(A_1 \times A_2).$$

We can take the 2-cell $\omega_1$ to be the identity since the following diagram commutes by definition of $t_1$ and naturality of $\mu$:

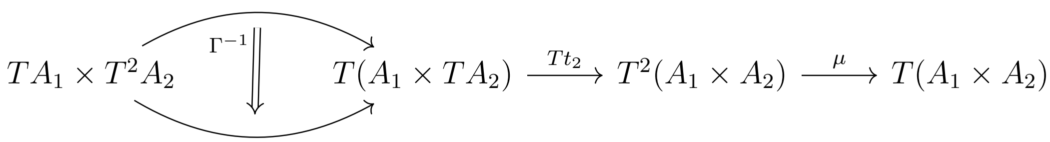
$$\begin{array}{ccccc} TTA_1 \times TA_2 & \xrightarrow{t_1} & T(TA_1 \times TA_2) & \xrightarrow{T\omega} & T^2(A_1 \times TA_2) \\ {\scriptstyle \mu\times 1}\downarrow & & & & \downarrow{\scriptstyle \mu} \\ TA_1 \times A_2 & & \xrightarrow[\omega]{\qquad\qquad} & & T(A_1 \times A_2). \end{array}$$

We define $\omega_2$ as being the following pasting diagram:

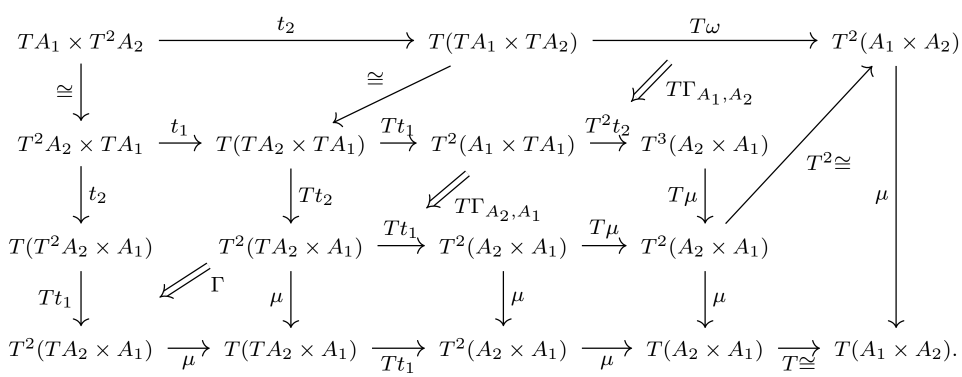


**Remark 3.2.3.** The previous definition generalizes the definitions of [GMMO23] for the case of pseudo commutative operads.

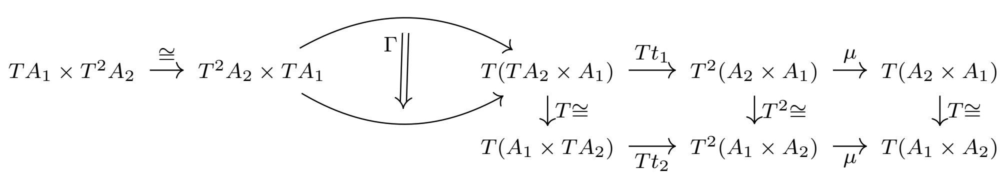
**Lemma 3.2.4.** *Let $(T, \eta, \mu, t, \Gamma)$ be a symmetric, pseudo commutative, strong 2-monad. For $A_1, A_2$ be objects of $\mathcal{K}$, $\omega_2$ equals the whiskering*



*Proof.* By applying the strength $\mu$-axiom (7) in Definition 3.1.6, we get that $\omega_2$ equals the pasting



By symmetry $\omega_2$ agrees with the whiskering



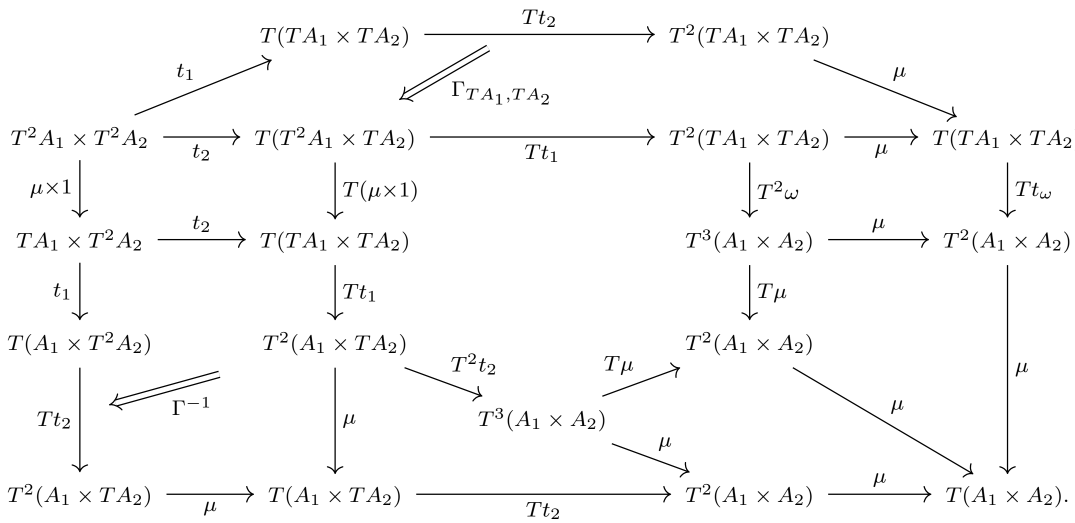
By symmetry, the last whiskering equals the one in the lemma. ∎

**Lemma 3.2.5.** *Let $A_1, A_2$ be objects of $\mathcal{K}$. Then*

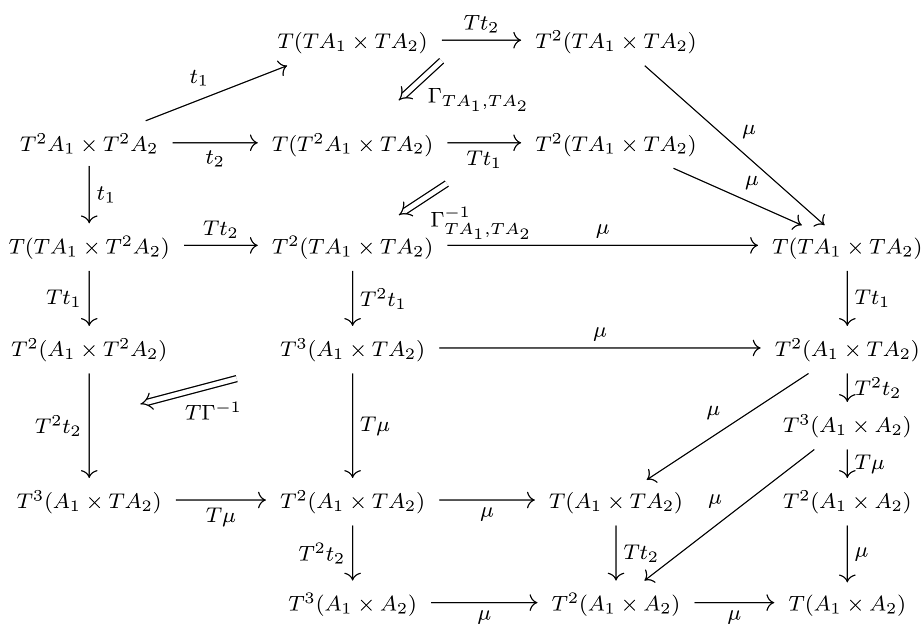
$$\omega \colon \langle TA_1, TA_2 \rangle \to T(A_1 \times A_2)$$

*is a 2-ary 1-cell in $T$-Alg.*

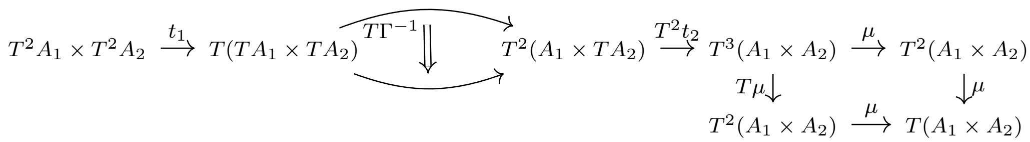
*Proof.* First we tackle coherence. By Lemma 3.2.4, we can write the diagram (3.1.12) for $\omega$ as



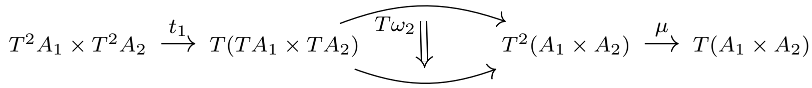
By (6) in Definition 3.1.6, this equals the pasting



The 2-cells $\Gamma_{TA_1,TA_2}$ and its inverse cancel out, so the previous pasting diagram equals the whiskering

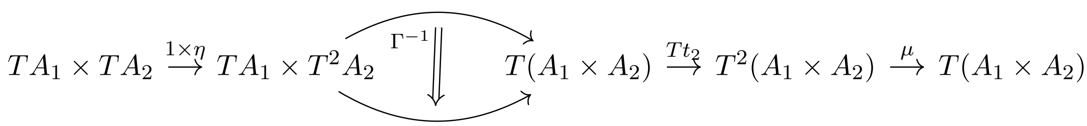



which, by Lemma 3.2.4 equals the whiskering



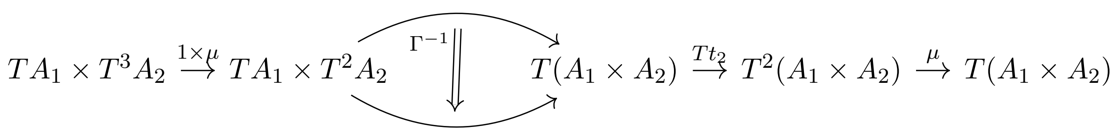

This is precisely (3.1.11).

Now we tackle the $\eta$ and $\mu$ axioms. For $i = 1$ there is nothing to prove since $\omega_1$ is the identity. For the $\eta$ axiom for $i = 2$, by Lemma 3.2.4, we need to prove that the whiskering

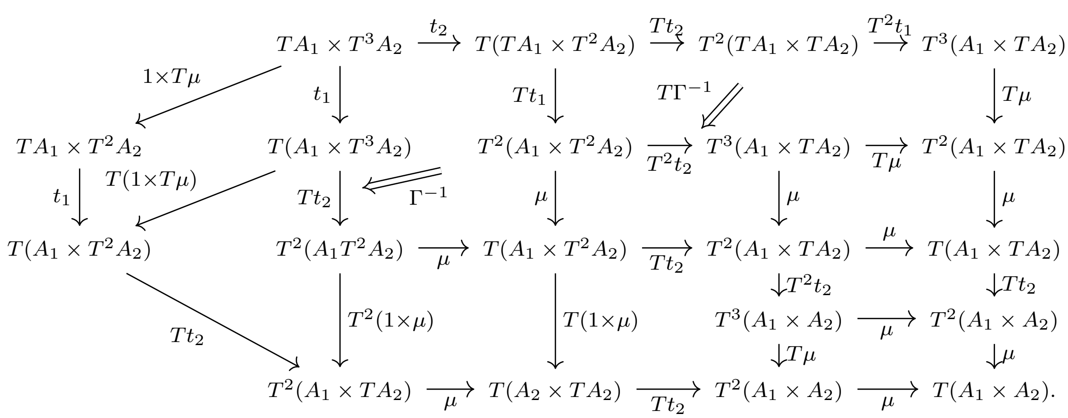



is an identity, but this follows at once from 5 in Definition 3.1.6.

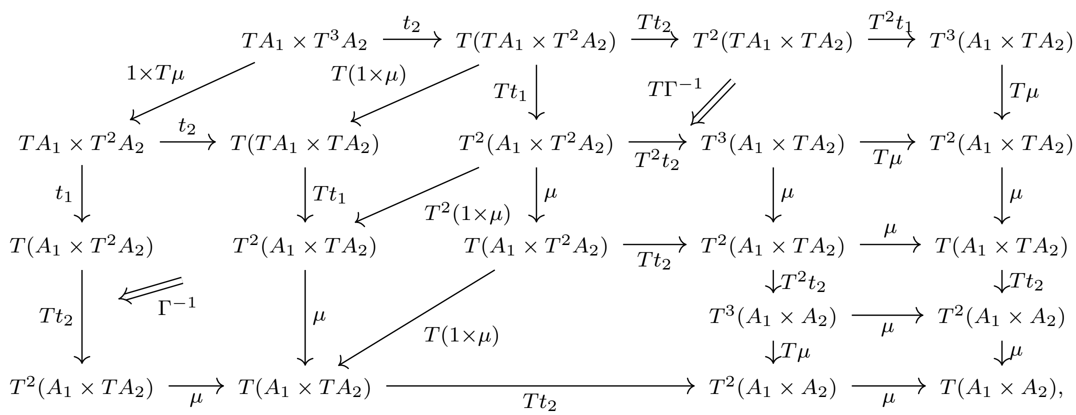
Let's prove that $\omega$ satisfies the $\mu$ axiom for $i = 2$. We start from pasting (3.1.9), which by Lemma 3.2.4 we can express as the whiskering




By (7) in Definition 3.1.6, this pasting equals



Here we decorated the diagram coming from (3.1.7) with some extra commutative squares that do not change the pasting. By using that $\Gamma^{-1}$ is a modification, we get that the previous pasting equals



The last pasting equals (3.1.10) for $\omega$ by two applications of Lemma 3.2.4 and a change in 1-cells. The previous diagram has the correct source 1-cell since the 1-cells $\mu(T\mu)(T^2t_2)$ and $\mu(Tt_2)\mu\colon T^2(A_1 \times TA_2) \to T(A_1 \times A_2)$ are equal.

■

**Lemma 3.2.6.** *Let* $(T, \eta, \mu, t, \Gamma)$ *be a symmetric, pseudo commutative, strong 2-monad. Let* $A, B, C$ *be objects of* $\mathcal{K}$, *then*

$$\gamma(\omega_{A,B\times C}; 1_A, \omega_{B,C}) = \gamma(\omega_{A\times B,C}; \omega_{A,B}, 1_C),$$

*that is, the following multicategorical diagram commutes*

$$\begin{array}{ccc} \langle TA, TB, TC\rangle & \xrightarrow{\langle \omega_{A,B}, 1_C\rangle} & \langle T(A\times B), TC\rangle \\ {\scriptstyle \langle 1_A, \omega_{B,C}\rangle}\downarrow & & \downarrow{\scriptstyle \omega_{A\times B,C}} \\ \langle TA, T(B\times C)\rangle & \xrightarrow[\omega_{A,B\times C}]{} & T(A\times B\times C). \end{array}$$

*Proof.* First of all by associativity of $t$, the strength axioms, the monad axioms and naturality of various 2-natural transformations, the corresponding 1-cells $TA \times TB \times TC \to T(A \times B \times C)$ are equal. We must show that the 2-cell constrains are equal, i.e., $\gamma(\omega; \omega, 1)_i = \gamma(\omega; 1, \omega)_i$ for $i = 1, 2, 3$. For $i = 1$ this follows since both $\gamma(\omega; \omega, 1)_1$ and $\gamma(\omega; 1, \omega)_1$ are identities. For $i = 2$, we have from Definition 3.1.16 and by Lemma 3.2.4 that $\gamma(\omega; \omega, 1)_2$ is the 2-cell

$$
\begin{array}{c}
TA \times T^2B \times TC \\
\big\downarrow \overset{\Gamma^{-1}\times 1}{\Longrightarrow} \big\downarrow \\
T(A \times TB) \times TC \xrightarrow{Tt_2 \times 1} T^2(A \times B) \times TC \xrightarrow{\mu \times 1} T(A \times B) \times TC \xrightarrow{t_1} T(A \times B \times TC) \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \searrow^{\omega} \qquad \big\downarrow{\scriptstyle \mu \circ Tt_3} \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad T(A \times B \times C).
\end{array}
$$

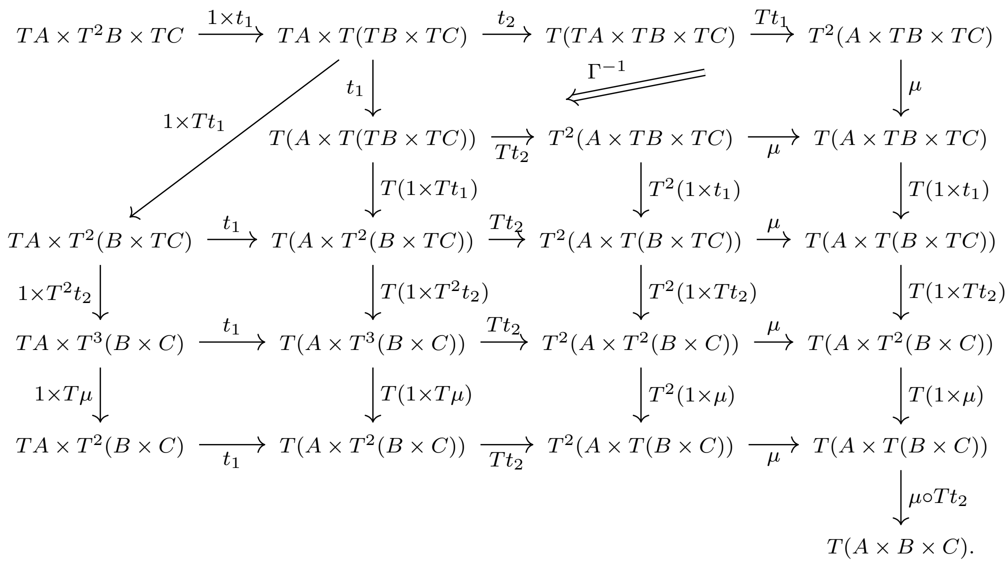
We can decorate this pasting with some extra commutative squares

$$
\begin{array}{ccccccc}
TA \times T^2B \times TC & & & & & & \\
\big\downarrow \overset{\Gamma^{-1}\times 1}{\Longrightarrow} \big\downarrow & & & & & & \\
T(A \times TB) \times TC & \xrightarrow{Tt_2 \times 1} & T^2(A \times B) \times TC & \xrightarrow{\mu \times 1} & T(A \times B) \times TC & \xrightarrow{t_1} & T(A \times B \times TC) \\
{\scriptstyle t_1}\big\downarrow & & \big\downarrow{\scriptstyle t_1} & & & \nearrow^{\mu} & \big\downarrow{\scriptstyle \mu \circ Tt_3} \\
T(A \times TB \times TC) & \xrightarrow[T(t_2 \times 1)]{} & T(T(A \times B) \times TC) & \xrightarrow[Tt_1]{} & T^2(A \times B \times TC) & & T(A \times B \times C),
\end{array}
$$

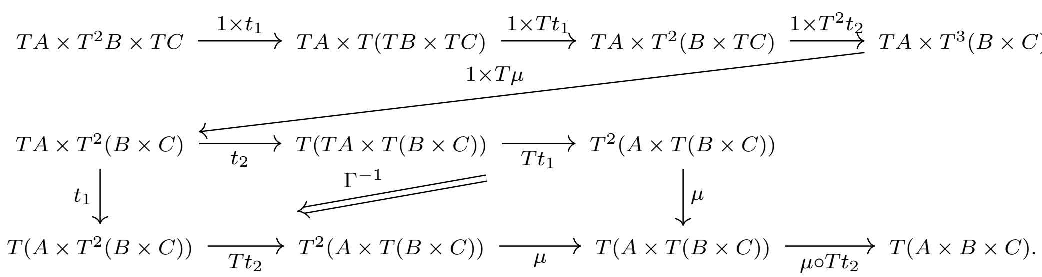
so that we can apply (3) in Definition 3.1.6, to get

$$
\begin{array}{ccccccc}
TA \times T^2B \times TC & \xrightarrow{1 \times t_1} & TA \times T(TB \times TC) & \xrightarrow{t_2} & T(TA \times TB \times TC) & \xrightarrow{Tt_1} & T^2(A \times TB \times TC) \\
& & {\scriptstyle t_1}\big\downarrow & & \overset{\Gamma^{-1}}{\Longleftarrow} & & \big\downarrow{\scriptstyle \mu} \\
& & T(A \times T(TB \times TC)) & \xrightarrow[Tt_2]{} & T^2(A \times TB \times TC) & \xrightarrow{\mu} & T(A \times TB \times TC) \\
& & & & & & {\scriptstyle \mu \circ Tt_3 \circ \mu \circ Tt_1 \circ T(t_2 \times 1)}\big\downarrow \\
& & & & & & T(A \times B \times C).
\end{array}
$$

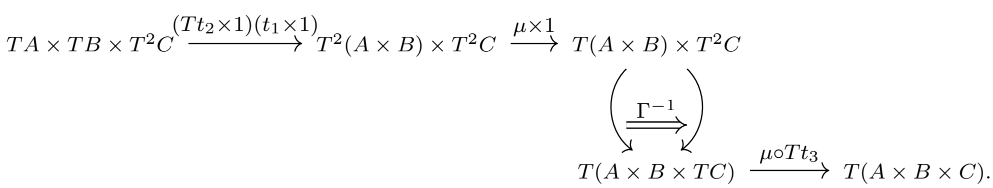
Since $\mu \circ Tt_3 \circ \mu \circ Tt_1 \circ T(t_2 \times 1)$ equals $\mu \circ Tt_2 \circ T(1 \times \mu) \circ T(1 \times Tt_2) \circ T(1 \times t_1)$ as 1-cells $T(A \times TB \times TC) \to T(A \times B \times C)$ by associativity of $t$, strength axioms for $\mu$ and monad axioms for $\mu$, we can write the previous pasting as



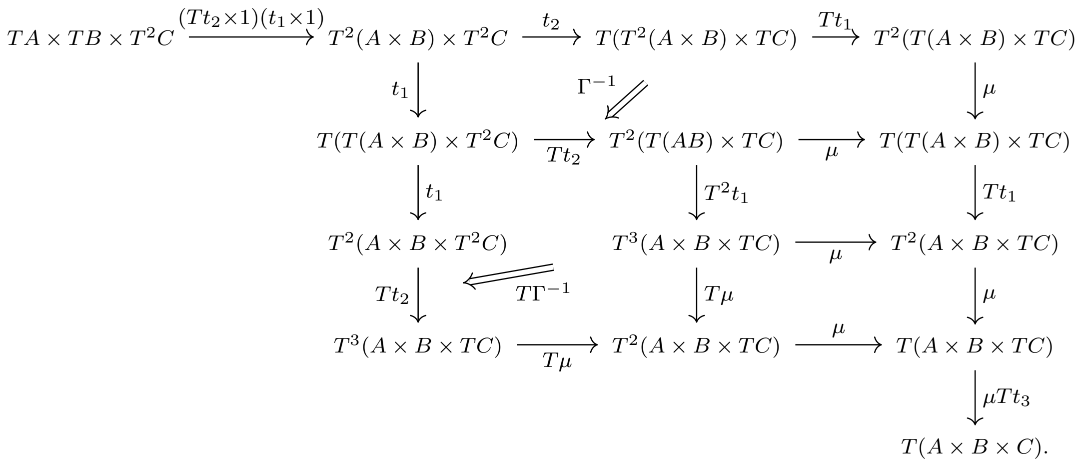

Since $\Gamma^{-1}$ is a modification, our diagram equals




By an application of Lemma 3.2.4, the previous whiskering is precisely $\gamma(\omega; 1, \omega)_2$. Let's now prove that $\gamma(\omega; \omega, 1)_3 = \gamma(\omega; 1, \omega)_3$. Definition 3.1.16 and an application of Lemma 3.2.4 give us that $\gamma(\omega; \omega, 1)_3$ is the whiskering

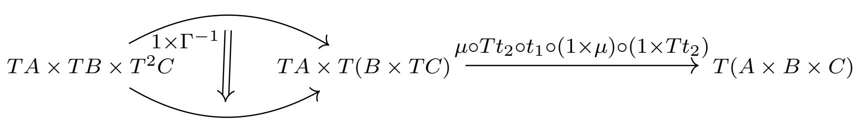



By an application of (6) in Definition 3.1.6, we then have that $\gamma(\omega; \omega, 1)_3$ equals



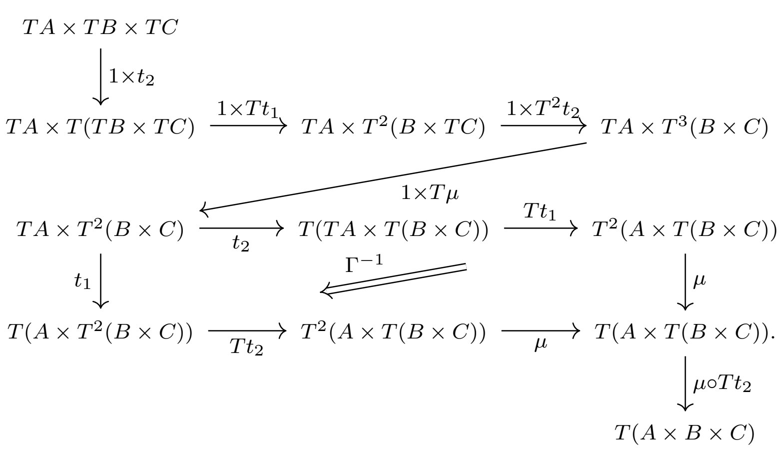

(3.2.1)

Now, by Definition 3.1.16 and two applications of Lemma 3.2.4, $\gamma(\omega; 1, \omega)_3$ is the vertical composition of the whiskering



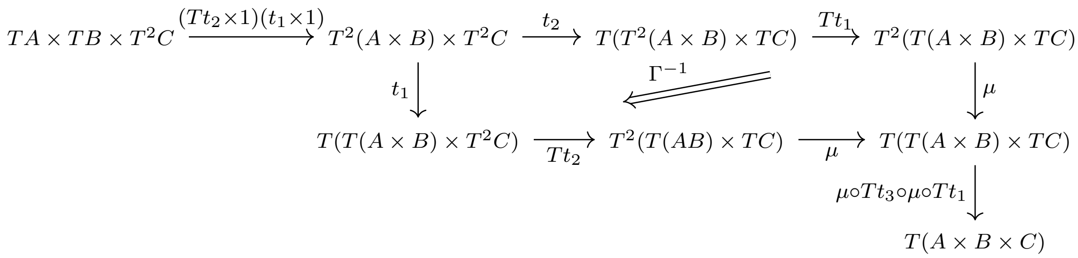

(3.2.2)

with the whiskering



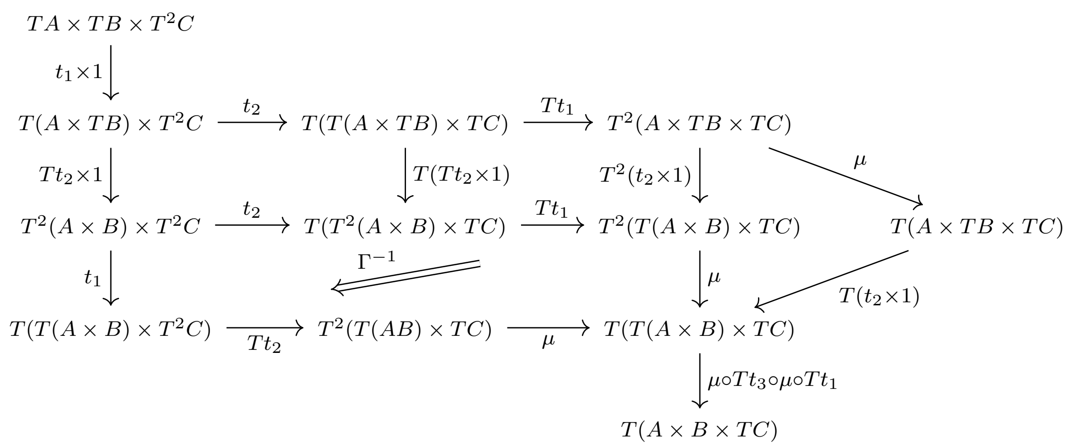

(3.2.3)

We will show that diagram (3.2.3) equals the whiskering

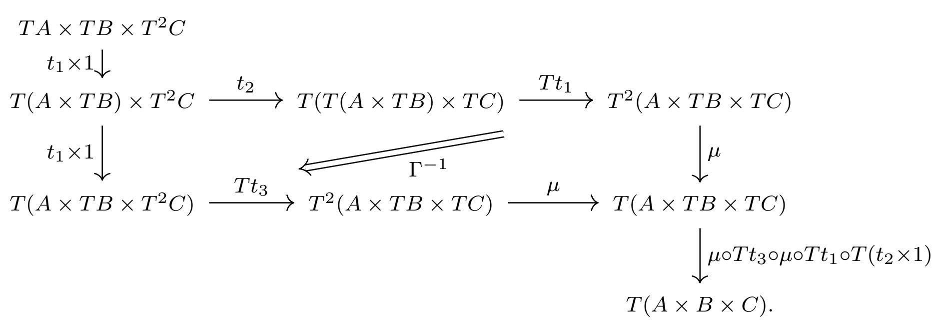



(3.2.4)

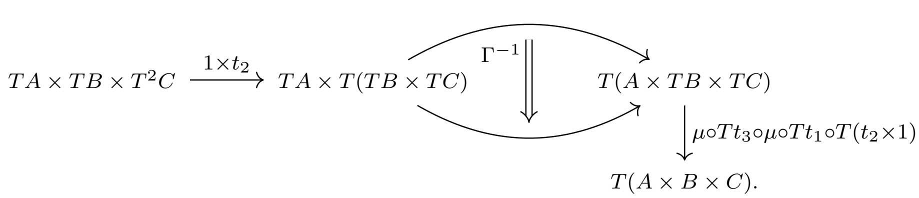
which comes from diagram (3.2.1), as well as an analogous statement for (3.2.2). We can decorate the pasting diagram (3.2.4) with some extra commutative squares




so that we can apply the fact that $\Gamma^{-1}$ is a modification to get

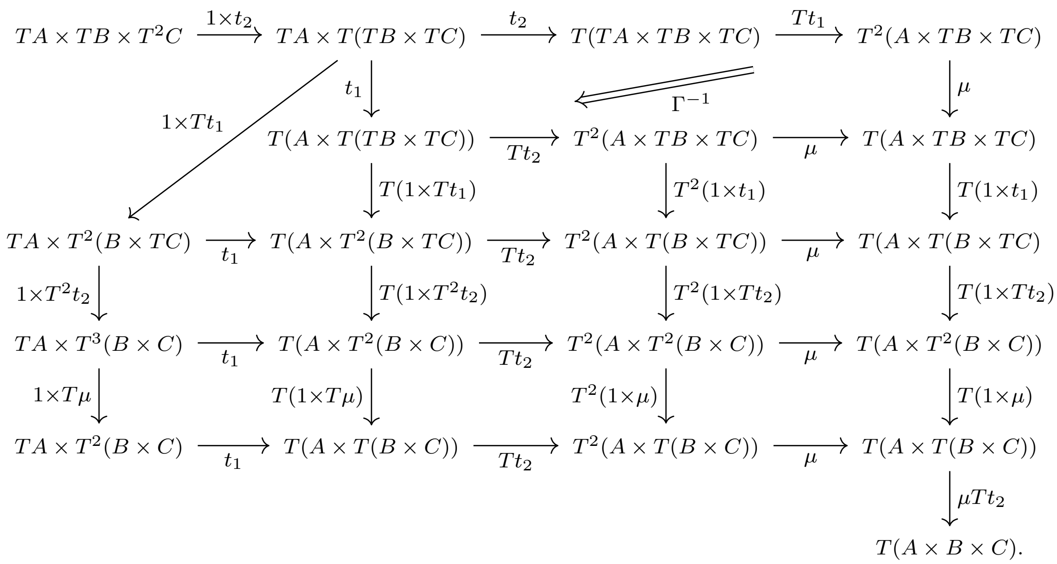



By (2) in Definition 3.1.6 this pasting becomes

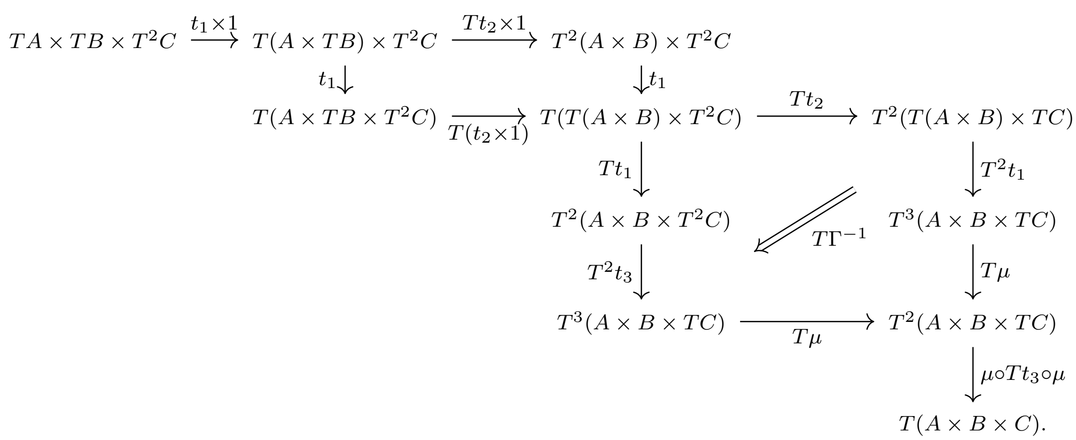



(3.2.5)

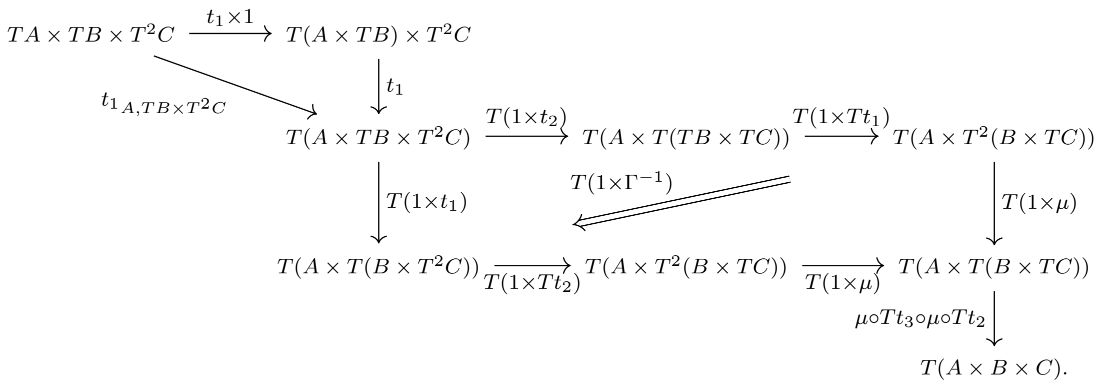
Next, we decorate diagram (3.2.5) with some commutative squares without altering the pasting. We are using that $\mu \circ Tt_3 \circ \mu \circ Tt_1 \circ T(t_2 \times 1)$ is equal to $\mu \circ Tt_2 \circ T(1 \times \mu) \circ T(1 \times Tt_2) \circ T(1 \times t_1)$ as 1-cells from $T(A \times TB \times TC)$ to $T(A \times B \times C)$.



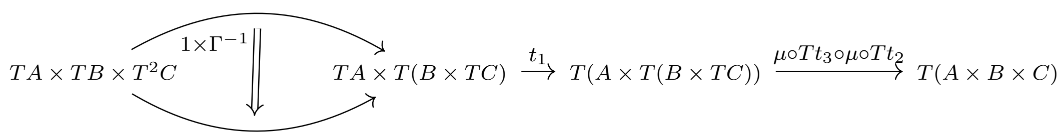

We then get (3.2.3) since $\Gamma^{-1}$ is a modification. To finish the proof we just have to show that diagram (3.2.2) equals the following whiskering coming from (3.2.1)




We can apply (1) in Definition 3.1.6 to get




Since $t_1$ is a 2-natural transformation, we get that the last pasting equals the whiskering




This equals (3.2.2) since $\mu \circ Tt_3 \circ \mu \circ Tt_2 \circ t_1 = \mu \circ Tt_2 \circ t_1 \circ (1 \times \mu) \circ (1 \times Tt_2)$ as 1-cells from $TA \times T(B \times TC)$ to $T(A \times B \times C)$. ■

**Lemma 3.2.7.** *Let $(T, \eta, \mu, t, \Gamma)$ be a symmetric, pseudo commutative, strong 2-monad, then $\omega$ is 2-natural in the following sense:*

1. *For $f_1 \colon A_1 \to B_1$ and $f_2 \colon A_2 \to B_2$ in $\mathcal{K}$,*

$$\gamma(T(f_1 \times f_2); \omega_{A_1,A_2}) = \gamma(\omega_{B_1,B_2}; Tf_1, Tf_2).$$

*That is, the following multicategorical diagram commutes:*

$$\begin{array}{ccc} \langle TA_1, TA_2\rangle & \xrightarrow{\omega} & T(A_1 \times A_2) \\ {\scriptstyle \langle Tf_1, Tf_2\rangle}\downarrow & & \downarrow{\scriptstyle T(f_1\times f_2)} \\ \langle TB_1, TB_2\rangle & \xrightarrow[\omega]{} & T(B_1 \times B_2). \end{array}$$

2. *For 2-cells $\alpha_1 \colon f_1 \to g_1$ in $\mathcal{K}(A_1, B_1)$ and $\alpha_2 \colon f_2 \to g_2$ in $\mathcal{K}(A_2, B_2)$*

$$\gamma(T(\alpha_1 \times \alpha_2); 1_{\omega_{A_1,A_2}}) = \gamma(1_{\omega_{B_1,B_2}}; T\alpha_1, T\alpha_2).$$

*That is, the multicategorical pasting*

$$\begin{array}{ccc} \langle TA_1, TA_2\rangle & \xrightarrow{\ \omega\ } & T(A_1\times A_2) \\ {\scriptstyle\langle Tf_1,Tf_2\rangle}\ \Big\downarrow \ \xRightarrow{\langle T\alpha_1, T\alpha_2\rangle}\ \Big\downarrow\ {\scriptstyle\langle Tg_1,Tg_2\rangle} & & \Big\downarrow {\scriptstyle T(g_1\times g_2)} \\ \langle TB_1, TB_2\rangle & \xrightarrow[\ \omega\ ]{} & T(B_1\times B_2). \end{array}$$

*equals*

$$\begin{array}{ccc} \langle TA_1, TA_2\rangle & \xrightarrow{\ \omega\ } & T(A_1\times A_2) \\ {\scriptstyle\langle Tf_1,Tf_2\rangle}\ \Big\downarrow & & {\scriptstyle T(f_1\times f_2)}\ \Big\downarrow \ \xRightarrow{T(\alpha_1\times\alpha_2)}\ \Big\downarrow\ {\scriptstyle T(g_1\times g_2)} \\ \langle TB_1, TB_2\rangle & \xrightarrow[\ \omega\ ]{} & T(B_1\times B_2). \end{array}$$

*Proof.* For part (1), the corresponding 1-cells of $\gamma(T(f_1 \times f_2); \omega)$ and $\gamma(\omega; Tf_1, Tf_2)$ are equal since $\omega\colon TA_1 \times TA_2 \to T(A_1 \times A_2)$ equals $\mu \circ Tt_2 \circ t_1$, a composition of 2-natural transformations. The 1-cells $\gamma(T(f_1 \times f_2); \omega)_1$ and $\gamma(\omega; Tf_1, Tf_2)_1$ are equal since both are identity 1-cells, with $Tf_1$ and $Tf_2$ being strict maps of $T$-algebras. Let's show that $\gamma(T(f_1 \times f_2); \omega)_2 = \gamma(\omega; Tf_1, Tf_2)_2$. By a double application of Definition 3.1.6, $\gamma(\omega; Tf_1, Tf_2)_2$ equals

$$TA_1 \times T^2A_2 \xrightarrow{Tf_1\times T^2 f_2} TB_1 \times T^2B_2 \ \underset{\Downarrow\,\Gamma^{-1}}{\rightrightarrows}\ T(B_1\times TB_2) \xrightarrow{\mu\circ Tt_2} T(B_1\times B_2).$$

Since $\Gamma^{-1}$ is a modification this whiskering can be writen as

$$TA_1 \times T^2A_2 \ \underset{\Downarrow\,\Gamma^{-1}}{\rightrightarrows}\ T(A_1\times TB_2) \xrightarrow{T(f_1\times Tf_2)} T(B_1\times TB_2) \xrightarrow{\mu\circ Tt_2} T(B_1\times B_2),$$

and since $\mu$ and $Tt_2$ are 2-natural, this equals

$$TA_1 \times T^2A_2 \ \underset{\Downarrow\,\Gamma^{-1}}{\rightrightarrows}\ T(A_1\times TB_2) \xrightarrow{\mu\circ Tt_2} T(A_1\times A_2) \xrightarrow{T(f_1\times f_2)} T(B_1\times B_2).$$

An application of Lemma 3.2.4 gives us that this is exactly $\gamma(T(f_1 \times f_2), \omega)_2$.
Part (2) follows from the 2-naturality of $t_1, Tt_2$ and $\mu$. ∎

**Definition 3.2.8.** Let $(T, \eta, \mu, t, \Gamma)$ be a symmetric, pseudo commutative, strong 2-monad. For $A_1, \ldots, A_n \in \mathrm{Ob}(\mathcal{K})$, we define

$$\omega_n = \omega_{A_1,\ldots,A_n} \colon \langle TA_1, \ldots, TA_n \rangle \to T(A_1 \times \cdots \times A_n)$$

in $T$-$Alg$ by recursion in the following way.

- For $n = 0$, $\omega_0 \colon 1 \to T1$ is $\eta_1 \colon 1 \to T1$.
- For $n = 1$ $\omega_1 \colon TA_1 \to TA_1$ is the identity $1_{TA_1}$.
- For $n = 2$, $\omega_2$ is $\omega_{A_1,A_2} \colon \langle TA_1, TA_2 \rangle \to T(A_1 \times TA_2)$ from Lemma 3.2.5.
- For $n \geq 3$ $\omega_n = \gamma(\omega_2; \omega_{n-1}, \omega_1)$

**Corollary 3.2.9.** *Let $(T, \eta, \mu, t, \Gamma)$ be a symmetric, pseudo commutative, strong 2-monad, and $A_1, \ldots, A_n$ objects of $\mathcal{K}$. For $n \geq 3$,*

$$\omega_n = \gamma(\omega_2; \omega_{n-1}, \omega_1) = \gamma(\omega_2; \omega_1, \omega_{n-1})$$

It follows by a straightforward induction that $\omega_n$ is natural in the following sense. We will denote $\omega_n$ as $\omega$ when there is no room for confusion.

**Lemma 3.2.10.** *Let $(T, \eta, \mu, t, \Gamma)$ be a symmetric, pseudo commutative, strong 2-monad. For any $n$, $\omega_n$ is natural in the following sense:*

*(1) Suppose $f_i \colon A_i \to B_i$ are 1-cells in $\mathcal{K}$ for $1 \leq i \leq n$, then*

$$\gamma(T(f_1 \times \cdots \times f_n); \omega_{A_1,\ldots,A_n}) = \gamma(\omega_{B_1,\ldots,B_n}; Tf_1, \ldots, Tf_n).$$

*That is, the following multicategorical diagram commutes:*

$$\begin{array}{ccc} \langle TA_1, \ldots, TA_n \rangle & \xrightarrow{\omega} & T(A_1 \times \cdots \times A_n) \\ {\scriptstyle \langle Tf_1,\ldots,Tf_n \rangle}\downarrow & & \downarrow{\scriptstyle T(f_1 \times \cdots \times f_n)} \\ \langle TB_1, \ldots, TB_n \rangle & \xrightarrow[\omega]{} & T(B_1 \times \cdots \times B_n). \end{array}$$

*(2) Suppose $\alpha_i \colon f_i \to g_i$ are 2-cells in $\mathcal{K}(A_i, B_i)$. Then*

$$\gamma(T(\alpha_1 \times \cdots \times \alpha_n); 1_{\omega_{A_1,\ldots,A_n}}) = \gamma(1_{\omega_{B_1,\ldots,B_n}}; T\alpha_1, \ldots, T\alpha_n).$$

*That is, the multicategorical pasting*

$$
\begin{array}{ccc}
\langle TA_1, \dots, TA_2 \rangle & \xrightarrow{\ \omega\ } & T(A_1 \times \cdots \times A_n) \\
\langle Tf_1,\dots,Tf_n\rangle \Big\downarrow \overset{\langle T\alpha_1,\dots,T\alpha_n\rangle}{\Longrightarrow} \Big\downarrow \langle Tg_1,\dots,Tg_n\rangle & & \Big\downarrow T(g_1\times\cdots\times g_n) \\
\langle TB_1, \dots, TB_n \rangle & \xrightarrow[\ \omega\ ]{} & T(B_1 \times \cdots \times B_n).
\end{array}
$$

*equals the pasting*

$$
\begin{array}{ccc}
\langle TA_1, \dots, TA_n \rangle & \xrightarrow{\ \omega\ } & T(A_1 \times \cdots \times A_n) \\
\langle Tf_1,\dots,Tf_n\rangle \Big\downarrow & & T(f_1\times\cdots\times f_n) \Big\downarrow \overset{T(\alpha_1\times\cdots\times\alpha_n)}{\Longrightarrow} \Big\downarrow T(g_1\times\cdots\times g_n) \\
\langle TB_1, \dots, TB_n \rangle & \xrightarrow[\ \omega\ ]{} & T(B_1 \times \cdots \times B_n).
\end{array}
$$

Next, we define the free algebra **Cat**-multifunctor $T : \mathcal{K} \to T\text{-}Alg$.

**Definition 3.2.11.** Let $T$ be a symmetric, pseudo commutative, strong 2-monad. We define the multifunctor $T\colon \mathcal{K} \to T\text{-}Alg$ as follows:

- $T$ is already defined on objects and since $(TA, \mu : T^2A \to TA)$ is a $T$ algebra for $A \in \mathrm{Ob}(\mathcal{K})$.

- For $n = 0$, $T\colon \mathcal{K}(1, A) \to T\text{-}Alg(1, TA)$ is defined as the composition

$$
\begin{array}{ccccc}
 & & \mathcal{K}(T1, TA) & & \\
 & \overset{T}{\nearrow} & & \overset{\eta_1^*}{\searrow} & \\
\mathcal{K}(1, A) & & \xrightarrow[\quad T \quad]{} & & T\text{-}Alg(1, TA) = \mathcal{K}(1, TA)
\end{array}
$$

  where $\eta_1\colon 1 \to T1$.

- For $n = 1$, we define $T\colon \mathcal{K}(A, B) \to T\text{-}Alg(TA, TB)$ as sending $f\colon A \to B$ to $TF\colon TA \to TB$ with $(Tf)_1$ being an identity. Similarly a 2-cell $\alpha\colon f \to g$ in $\mathcal{K}(A, B)$ is sent to $T\alpha$.

- For $n \geq 2$, we define $T\colon \mathcal{K}(A_1 \times \cdots \times A_n, B) \to T\text{-}Alg(TA_1, \cdots, TA_n; TB)$ as the composition

$$
\begin{array}{ccccc}
 & & T\text{-}Alg(T(A_1 \times \cdots \times TA_n), TB) & & \\
 & \overset{T}{\nearrow} & & \overset{\omega_n^*}{\searrow} & \\
\mathcal{K}(A_1 \times \cdots \times A_n, B) & & \xrightarrow[\quad T \quad]{} & & T\text{-}Alg(TA_1, \dots, TA_n; TB)
\end{array}
$$

**Theorem 3.2.12.** *Let $(T, \eta, \mu, t, \Gamma)$ be a symmetric, pseudo commutative, strong 2-monad. Then $T\colon \mathcal{K} \to T$-Alg is a non-symmetric* **Cat***-multifunctor.*

*Proof.* It is clear from the definition that $T$ preserves identities. Preservation of $\gamma$ by $T$ follows at once from Lemma 3.2.10 and Lemma 3.2.6. ∎

## 3.3 Pseudo symmetry of the free $T$-algebra multifunctor

Next, we define the pseudo symmetry isomorphisms. We do this in a recursive way, starting with the non trivial element of $\Sigma_2$. From here on $\sigma_i$ will denote the transposition in $\Sigma_n$ that permutes $i$ and $i+1$.

**Definition 3.3.1.** Let $(T, \eta, \mu, t, \Gamma)$ be a symmetric, pseudo commutative, strong 2-monad. For $A, B \in \mathcal{K}$ we define $\omega'\colon \langle TA, TB\rangle \to T(A \times B)$ as the image of $\omega$ trhough the composition

$$T\text{-}Alg(TB, TA; T(B \times A)) \xrightarrow{\sigma_1} T\text{-}Alg(TA, TB; T(B \times A)) \xrightarrow{T\cong_*} T\text{-}Alg(TA, TB; T(A \times B)).$$

**Lemma 3.3.2.** *For $\omega'$ in the previous definition, its component is*

$$TA \times TB \xrightarrow{t_2} T(TA \times TB) \xrightarrow{Tt_1} T^2(A \times B) \xrightarrow{\mu} T(A \times B).$$

*The 2-cell $\omega'_1$ equals*

$$T^2A \times TB \underset{\longrightarrow}{\overset{\longrightarrow}{\Downarrow \Gamma}} T(TA \times B) \xrightarrow{Tt_1} T^2(A \times B) \xrightarrow{\mu} T(A \times B),$$

*and $\omega'_2$ is an identity 1-cell.*

*Proof.* Since $\omega_1$ is an identity 2-cell we get that $\omega'_2$ is as well. On the other hand, by Lemma 3.2.4, $\omega'_1$ can be written as

$$T^2A \times B \xrightarrow{\cong} TB \times T^2A \underset{\longrightarrow}{\overset{\longrightarrow}{\Downarrow \Gamma^{-1}}} T(B \times TA) \xrightarrow{T\cong \circ \mu \circ Tt_2} T(A \times B).$$

By naturality of $\mu$ and definition of $t_1$ we can write this whiskering as

$$T^2A \times B \xrightarrow{\cong} TB \times T^2A \underset{\longrightarrow}{\overset{\longrightarrow}{\Downarrow \Gamma^{-1}}} T(B \times TA) \xrightarrow{T\cong} T(TA \times B) \xrightarrow{\mu \circ Tt_1} T(A \times B).$$

By Definition 3.1.17 we have that $\omega'_1$ agrees with the whiskering in the statment of the lemma. ■

**Lemma 3.3.3.** *Let $(T, \eta, \mu, t, \Gamma)$ be a symmetric, pseudo commutative, strong 2-monad. For $A, B$ objects of $\mathcal{K}$, there is a 2-cell*

$$\langle TA, TB \rangle \underset{\omega'}{\overset{\omega}{\Downarrow \Gamma_{A,B}}} T(A \times B)$$

*in the multicategory $T$-Alg with component 2-cell $\Gamma_{A,B}$.*

*Proof.* We need to prove that diagrams (3.1.13) and (3.1.14) for $\Gamma_{A,B}$ are equal for $i = 1, 2..$ For $i = 1$ this diagram (3.1.13) takes the form

$$T^2A \times TB \xrightarrow{\mu \times 1} TA \times TB \underset{\longrightarrow}{\overset{\longrightarrow}{\Downarrow \Gamma}} T(A \times B).$$

Now, by Lemma 3.3.2, the diagram (3.1.14) for $i = 1$ agrees exactly with diagram (3.1.6) and we are done by (6) in Definition 3.1.6. For $i = 2$ the diagram (3.1.13) is, by an application of (3.1.7) and Lemma 3.2.4,

$$\begin{array}{ccccc}
TA \times T^2B & & \xrightarrow{\omega'_{A,B}} \;\; \Swarrow \Gamma^{-1}_{A,TB} \;\; \xrightarrow{\omega_{A,B}} & & T(A \times TB) \\
\downarrow t_2 & & & & \\
T(TA \times TB) & \underset{Tt_1}{\longrightarrow} \; \Leftarrow \Gamma_{A,TB} & T^2(A \times TB) & \xrightarrow{\mu} & T(A \times TB) \\
\downarrow Tt_2 & & \downarrow T^2t_2 & & \downarrow Tt_2 \\
T^2(TA \times B) & \Swarrow T\Gamma_{A,B} & T^3(A \times B) & \xrightarrow{\mu} & T^2(A \times B) \\
\downarrow T^2t_1 & & \downarrow T\mu & & \downarrow \mu \\
T^3(A \times B) & \xrightarrow[T\mu]{} & T^2(A \times B) & \xrightarrow[\mu]{} & T(A \times B).
\end{array}$$

This diagram is equal to the diagram (3.1.14) for $i = 2$ which is the whiskering

$$TA \times T^2B \xrightarrow{t_2} T(TA \times TB) \underset{\longrightarrow}{\overset{\longrightarrow}{\Downarrow T\Gamma}} T^2(A \times B) \xrightarrow{\mu} T(A \times B).$$

■

Next, we define recursively the pseudo symmetry isomorphisms for $T$.

**Definition 3.3.4.** Let $(T, \eta, \mu, t, \Gamma)$ be a symmetric, pseudo commutative, strong 2-monad. Let $A, B$ and $C$ be objects of $\mathcal{K}$. We define the natural transformation

$$\begin{array}{ccc} \mathcal{K}(A \times B, C) & \xrightarrow{T} & T\text{-}Alg(TA, TB; TC) \\ \sigma_1 \downarrow & \overset{T_{\sigma_1}}{\Nearrow} & \downarrow \sigma_1 \\ \mathcal{K}(B \times A, C) & \xrightarrow[T]{} & T\text{-}Alg(TB, TA; C), \end{array}$$

as having component $T_{\sigma;f}$ for $f\colon A \times B \to C$, the whiskering in the multicategory $T$-$Alg$

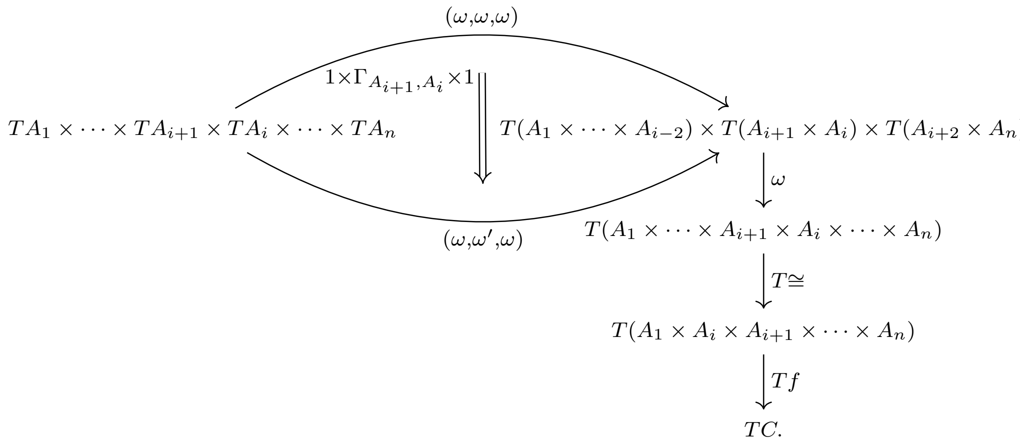
$$\langle TB, TA \rangle \underset{\omega'}{\overset{\omega}{\Downarrow \Gamma_{B,A}}} T(B \times A) \xrightarrow{T\cong} T(A \times B) \xrightarrow{Tf} TC.$$

The fact that $T_{\sigma_1}$ is in fact a natural transformation follows from the exchange property in the 2-category $\mathcal{K}$.

**Definition 3.3.5.** Let $\sigma_i$ the transposition that interchanges $i$ and $i+1$ in $\Sigma_n$ for $n \geq 3$. We define the natural transformation $T_{\sigma_i}$

$$\begin{array}{ccc} \mathcal{K}(A_1 \times \cdots \times A_n, C) & \xrightarrow{T} & T\text{-}Alg(TA_1, \ldots, TA_n; TC) \\ \sigma_i \downarrow & \overset{T_{\sigma_i}}{\Nearrow} & \downarrow \sigma_i \\ \mathcal{K}(A_1 \times \cdots \times A_{i+1} \times A_i \times \cdots \times A_n, C) & \xrightarrow[T]{} & T\text{-}Alg(TA_1, \ldots, TA_{i+1}, TA_i, \ldots, TA_n; C) \end{array}$$

as follows. For $f\colon A_1 \times \cdots \times A_n \to C$ the 2-cell $T_{\sigma_i;f}$ is




The fact that this is well defined comes from the associativity of $\omega$ (Lemma 3.2.6), and the fact that $T_{\sigma_i}$ is in fact a natural transformation follows from the exchange rule in $\mathcal{K}$.

Next, we prove that this defines $T_\sigma$ for every $\sigma \in \Sigma_n$ and every $n$ by using that the symmetric group $\Sigma_n$ is generated by the consecutive transpositions $\sigma_1, \ldots, \sigma_{n-1}$ subject to the relations:

(a) $\sigma_i^2 = \mathrm{id}$

(b) $\sigma_i\sigma_j = \sigma_j\sigma_i$ for $|i-j| > 1$.

(c) $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$.

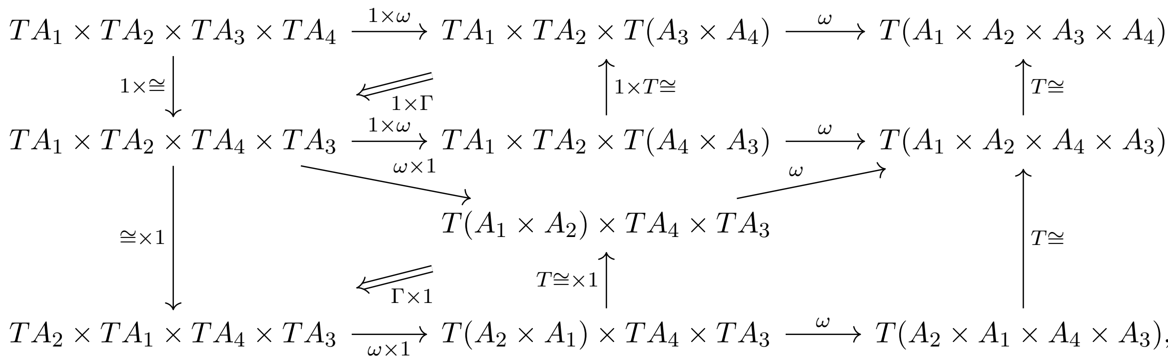
The relations between the different $T_{\sigma_i}$ will follow from relations between 2-cells in $T$-*Alg* which can be proven in $\mathcal{K}$. The relations in $\mathcal{K}$ can be proven even when $T$ is not symmetric, except for the relation induced by $\sigma_i\sigma_i = \mathrm{id}$. The following follows (in fact, it is equivalent to) symmetry for $T$.

**Lemma 3.3.6.** *Suppose that $T$ is a symmetric, pseudo commutative, strong 2-monad. Then the following pasting diagram is the identity:*

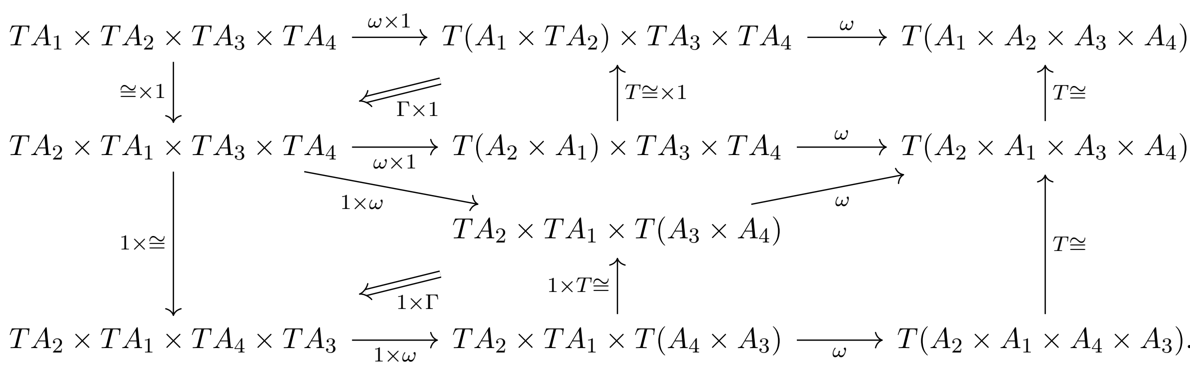
$$\begin{array}{ccc}
\mathcal{K}(A_1 \times \cdots \times \times A_n, C) & \xrightarrow{\quad T \quad} & T\text{-}Alg(TA_1, \ldots, TA_n; TC) \\
\downarrow{\scriptstyle \sigma_i} & \overset{T_{\sigma_i}}{\Longrightarrow} & \downarrow{\scriptstyle \sigma_i} \\
\mathcal{K}(A_1 \times \cdots \times A_{i+1} \times A_i \times \cdots \times A_n, C) & \xrightarrow[T]{} & T\text{-}Alg(TA_1, \ldots, TA_{i+1}, TA_i, \ldots, TA_n; C) \\
\downarrow{\scriptstyle \sigma_i} & \underset{T_{\sigma_i}}{\Longrightarrow} & \downarrow{\scriptstyle \sigma_i} \\
\mathcal{K}(A_1 \times \cdots \times A_n, C) & \xrightarrow[\quad T \quad]{} & T\text{-}Alg(TA_1, \ldots, TA_n; TC).
\end{array}$$

The following holds even in the absence of symmetry.

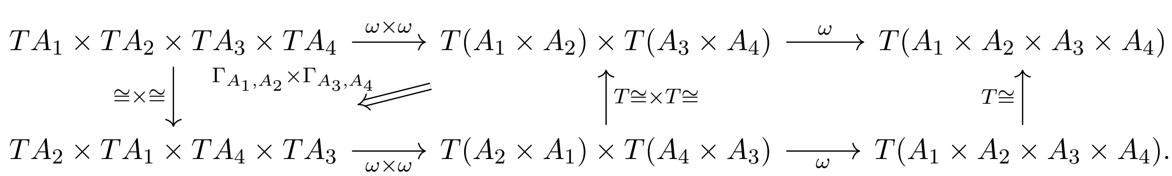
**Lemma 3.3.7.** *Suppose that* $(T, \eta, \mu, t, \Gamma)$ *is a pseudo commutative, strong 2-monad. Then the pasting diagram*




*equals the pasting*




*Proof.* Both pastings are equal to the pasting




■

When $T$ is symmetric, a slight generalization of the previous lemma can be interpreted as follows. To save space we will write $\overline{TA} = TA_1 \times \cdots \times TA_n$ and $\overline{TA}\sigma = TA_{\sigma(1)} \times \cdots \times TA_{\sigma(n)}$ when $\sigma \in \Sigma_n$ and $A_1, \ldots, A_n$ are objects of $\mathcal{K}$

**Lemma 3.3.8.** *Supposet $T$ is a symmetric, pseudo commutative, strong 2-monad, $n \geq 3$ and $1 \leq i < i+2 \leq j \leq n-1$. Let $A_1, \ldots, A_n, C$ be objects of $\mathcal{K}$. Then, the pasting*

$$\begin{array}{ccc} \mathcal{K}(\overline{TA}, C) & \xrightarrow{\quad T \quad} & T\text{-}Alg(\langle TA\rangle; C) \\ {\scriptstyle\sigma_i}\downarrow & \overset{T\sigma_i}{\Swarrow} & \downarrow{\scriptstyle\sigma_i} \\ \mathcal{K}(\overline{TA}\sigma_i, C) & \xrightarrow[T]{\quad} & T\text{-}Alg(\langle TA\rangle\sigma_i; C) \\ {\scriptstyle\sigma_j}\downarrow & \overset{T\sigma_j}{\Swarrow} & \downarrow{\scriptstyle\sigma_j} \\ \mathcal{K}(\overline{TA}\sigma_i\sigma_j, C) & \xrightarrow[T]{\quad} & T\text{-}Alg(\langle TA\rangle\sigma_i\sigma_j; C), \end{array}$$

*equals the pasting*

$$\begin{array}{ccc} \mathcal{K}(\overline{TA}, C) & \xrightarrow{\quad T \quad} & T\text{-}Alg(\langle TA\rangle; C) \\ {\scriptstyle\sigma_j}\downarrow & \overset{T\sigma_j}{\Swarrow} & \downarrow{\scriptstyle\sigma_j} \\ \mathcal{K}(\overline{TA}\sigma_j, C) & \xrightarrow[T]{\quad} & T\text{-}Alg(\langle TA\rangle\sigma_j; C) \\ {\scriptstyle\sigma_i}\downarrow & \overset{T\sigma_i}{\Swarrow} & \downarrow{\scriptstyle\sigma_i} \\ \mathcal{K}(\overline{TA}\sigma_j\sigma_i, C) & \xrightarrow[T]{\quad} & T\text{-}Alg(\langle TA\rangle\sigma_j\sigma_i; C), \end{array}$$

Next, we focus on the Yang-Banxter equation. First we prove the following lemma that we will also need later. It doesn't require symmetry.

**Lemma 3.3.9.** *Let $(T, \eta, \mu, t, \Gamma)$ be a pseudo comutative, strong 2-monad, and $A_1, A_2, A_3$ objects of $\mathcal{K}$. Then,*

1. *The pasting diagram*

$$\begin{array}{ccccc} TA_1 \times TA_2 \times TA_3 & \xrightarrow{\omega\times 1} & T(A_1\times A_2)\times TA_3 & \xrightarrow{\omega} & T(A_1\times A_2\times A_3) \\ {\scriptstyle\cong\times 1}\downarrow & \overset{\Gamma\times 1}{\Swarrow} & \uparrow{\scriptstyle T(\cong\times 1)} & & \uparrow{\scriptstyle T\cong\times 1} \\ TA_2 \times TA_1 \times TA_3 & \xrightarrow[\omega\times 1]{} & T(A_2\times A_1)\times TA_3 & \xrightarrow[\omega]{} & T(A_2\times A_1\times A_3) \\ & \searrow{\scriptstyle 1\times\omega} & & \nearrow{\scriptstyle\omega} & \\ {\scriptstyle 1\times\cong}\downarrow & & TA_2\times T(A_1\times A_3) & & \uparrow{\scriptstyle T(1\times\cong)} \\ & \overset{1\times\Gamma}{\Swarrow} & \uparrow{\scriptstyle 1\times T\cong} & & \\ TA_2 \times TA_3 \times TA_1 & \xrightarrow[1\times\omega]{} & TA_2\times T(A_3\times A_1) & \xrightarrow[\omega]{} & T(A_2\times A_3\times A_1) \end{array} \tag{3.3.1}$$

*equals the whiskering*

$$TA_1 \times TA_2 \times TA_3 \xrightarrow{1\times\omega} TA_1 \times T(A_2 \times A_3) \begin{array}{c} \xrightarrow{\quad\omega\quad} \\ \Gamma \Downarrow \\ \xrightarrow[\quad\omega'\quad]{} \end{array} T(A_1 \times A_2 \times A_3).$$

2. *The pasting diagram*

$$\begin{array}{ccccc}
TA_1 \times TA_2 \times TA_3 & \xrightarrow{1\times\omega} & TA_1 \times T(A_2 \times A_3) & \xrightarrow{\omega} & T(A_1 \times A_2 \times A_3) \\
{\scriptstyle 1\times\cong}\downarrow & \overset{1\times\Gamma}{\swarrow\!\!\swarrow} & \uparrow{\scriptstyle 1\times T\cong} & & \uparrow{\scriptstyle T\cong} \\
TA_1 \times TA_3 \times TA_2 & \xrightarrow[1\times\omega]{} & TA_1 \times T(A_3 \times A_2) & \xrightarrow[\omega]{} & T(A_1 \times A_3 \times A_2) \\
 & {\scriptstyle \omega\times 1}\searrow & & \nearrow{\scriptstyle \omega} & \\
{\scriptstyle \cong\times 1}\downarrow & & T(A_1 \times A_3) \times TA_2 & & \uparrow{\scriptstyle T\cong} \\
 & \overset{\Gamma\times 1}{\swarrow\!\!\swarrow} & \uparrow{\scriptstyle T\cong\times 1} & & \\
TA_3 \times TA_1 \times TA_2 & \xrightarrow[\omega\times 1]{} & T(A_3 \times A_1) \times TA_2 & \xrightarrow[\omega]{} & T(A_3 \times A_1 \times A_2)
\end{array} \qquad (3.3.2)$$

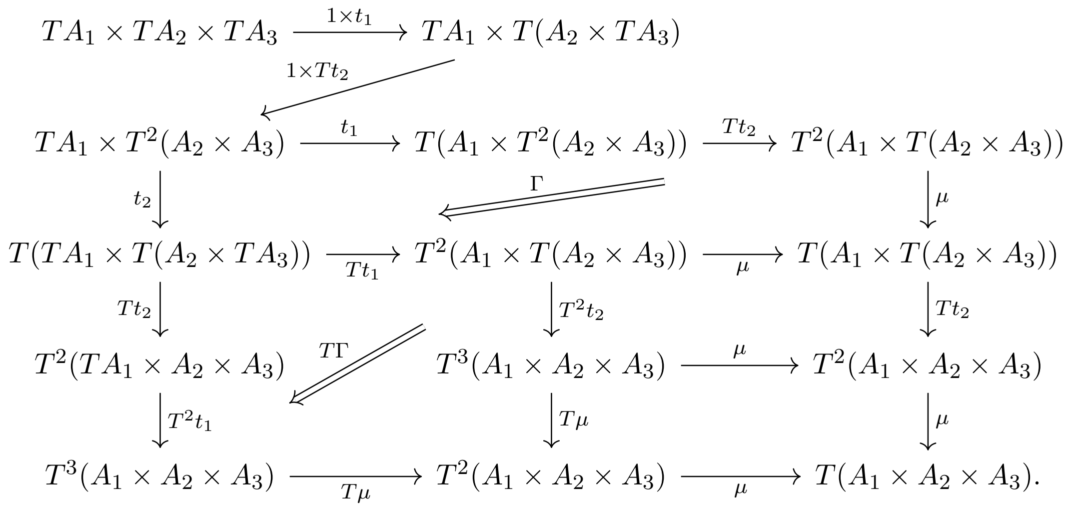
*equals the whiskering*

$$TA_1 \times TA_2 \times TA_3 \xrightarrow{\omega\times 1} T(A_1 \times A_2) \times TA_3 \begin{array}{c} \xrightarrow{\quad\omega\quad} \\ \Gamma \Downarrow \\ \xrightarrow[\quad\omega'\quad]{} \end{array} T(A_1 \times A_2 \times A_3).$$

*Proof.* For part (1) we start from

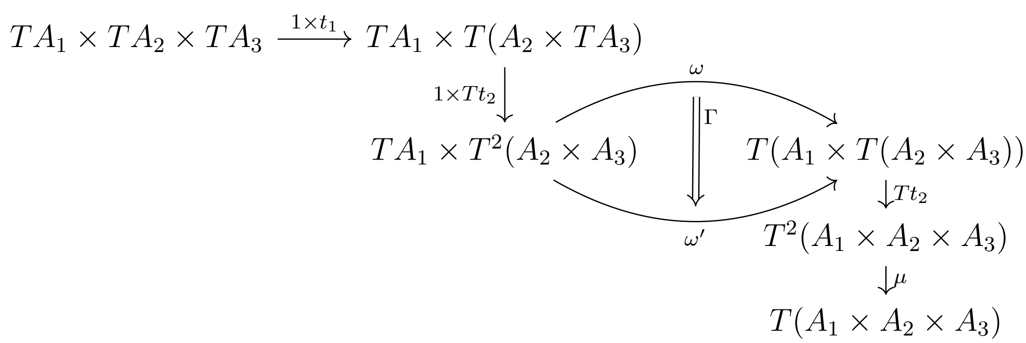
$$\begin{array}{lll}
TA_1 \times TA_2 \times TA_3 \xrightarrow{1\times t_1} TA_1 \times T(A_2 \times TA_3) & & \\
\qquad\qquad {\scriptstyle 1\times Tt_2}\swarrow & & \\
TA_1 \times T^2(A_2 \times A_3) \xrightarrow[1\times\mu]{} TA_1 \times T(A_2 \times A_3) & \begin{array}{c} \xrightarrow{\quad\omega\quad} \\ \Downarrow \Gamma \\ \xrightarrow[\quad\omega'\quad]{} \end{array} & T(A_1 \times A_2 \times A_3).
\end{array}$$

By (7) in Definition 3.1.6, the previous whiskering equals the pasting diagram



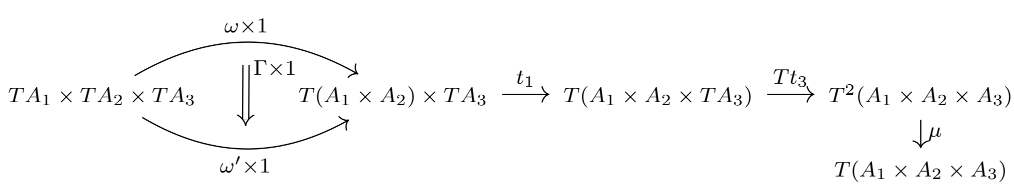

(3.3.3)

First we will prove that the whiskering

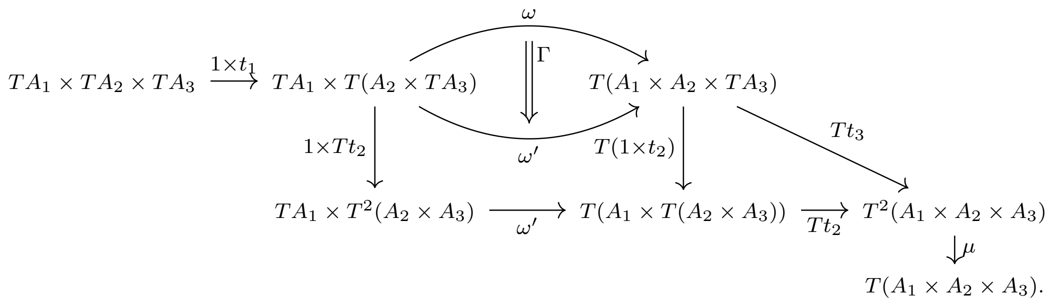



(3.3.4)

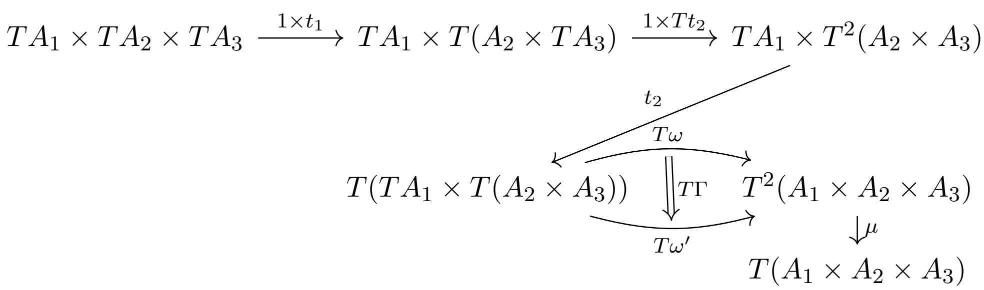
coming from the previous diagram equals the whiskering




coming from diagram (3.3.1). By (3) in Definition 3.1.6, the previous whiskering equals

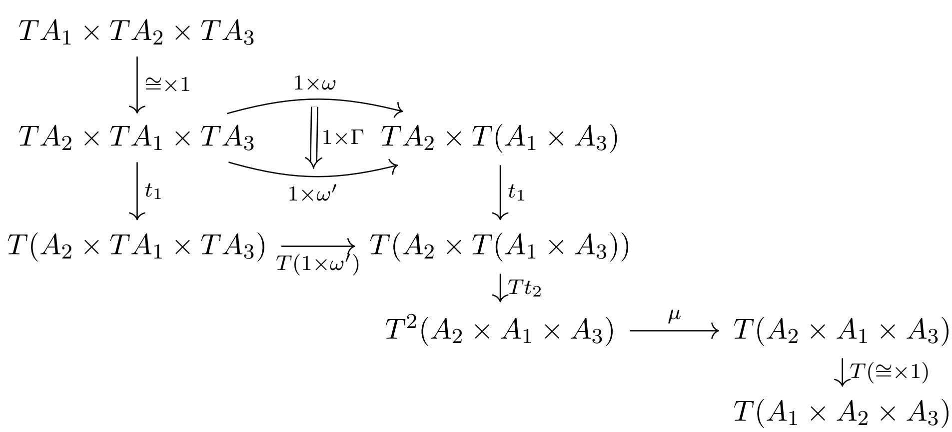



Since Γ is a modification, the previous diagram equals (3.3.4). To finish part (1), we will prove that the whiskering

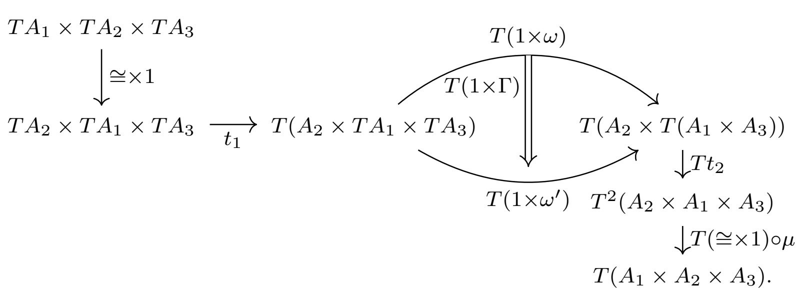



(3.3.5)

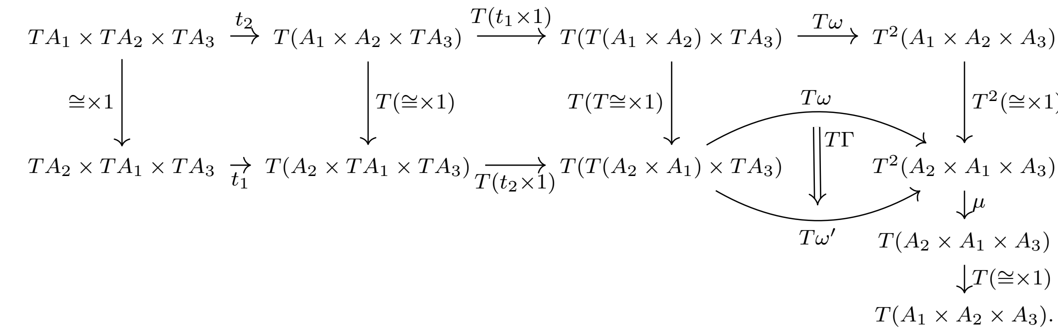
coming from (3.3.3) equals the whiskering



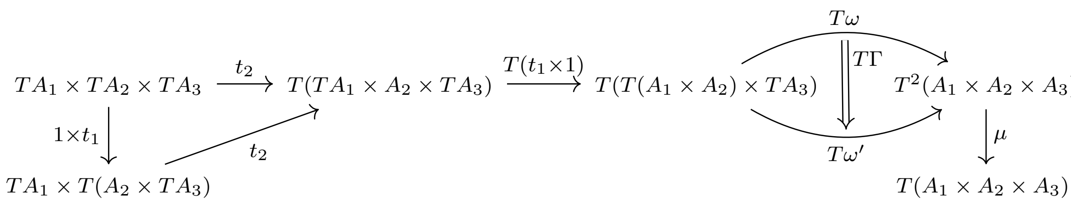

coming from (3.3.1). By 2-naturality of $t_1$, the previous diagram equals



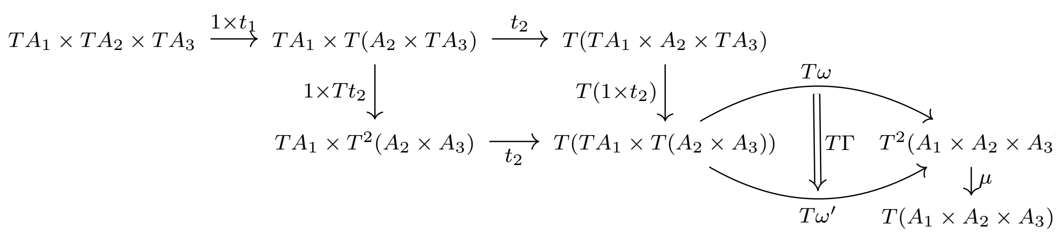
By (1) in Definition 3.1.6, this whiskering equals




Since $\Gamma$ is a modificiation we can write the previous whiskering as




By (2) in Definition 3.1.6, we get

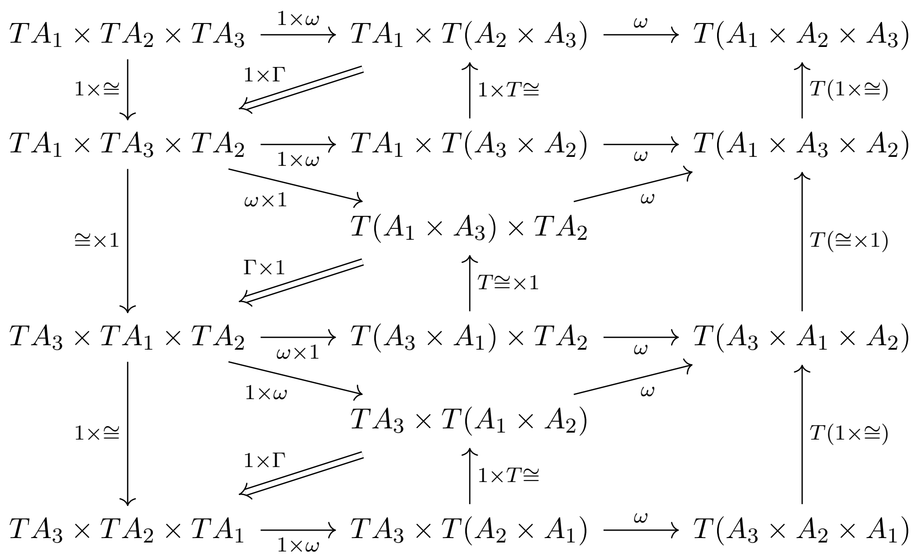



which is precisely (3.3.5). We have proven part (1). Part (2) is proven in a similar fashion. ∎

The next Lemma is the Yang-Baxter equation for non-symmetryc pseudo commutative, strong 2-monads. Part (3) is called the Associativity Equation in [HP02].

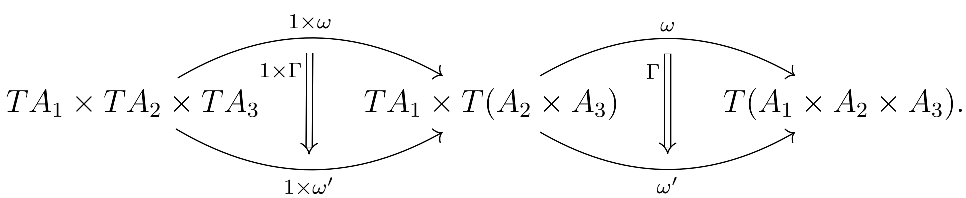
**Lemma 3.3.10.** *Let $T$ be a pseudo commutative, strong, 2-monad. Then:*

1. *The pasting*

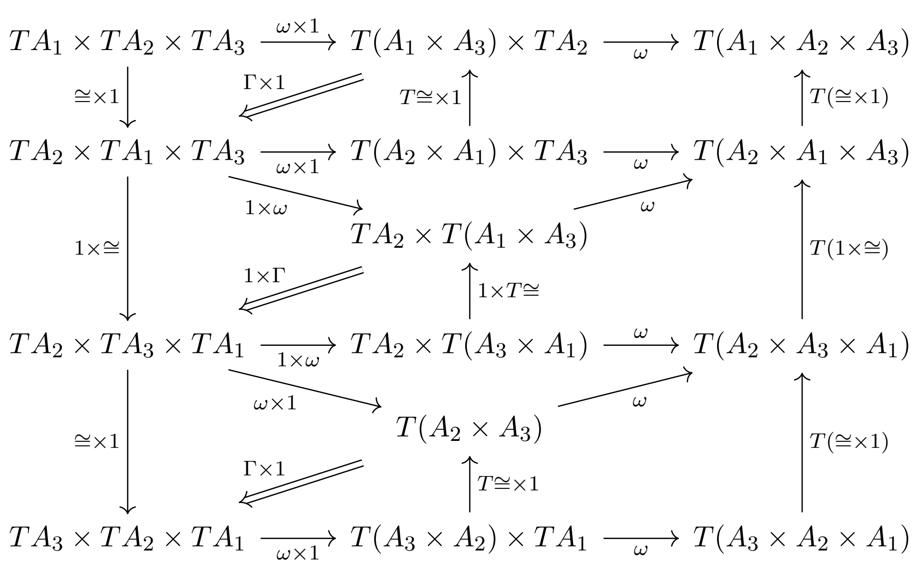



(3.3.6)

*equals the horizontal composite*

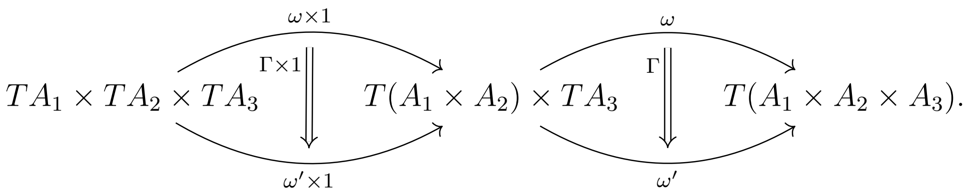



*2. The pasting*




(3.3.7)

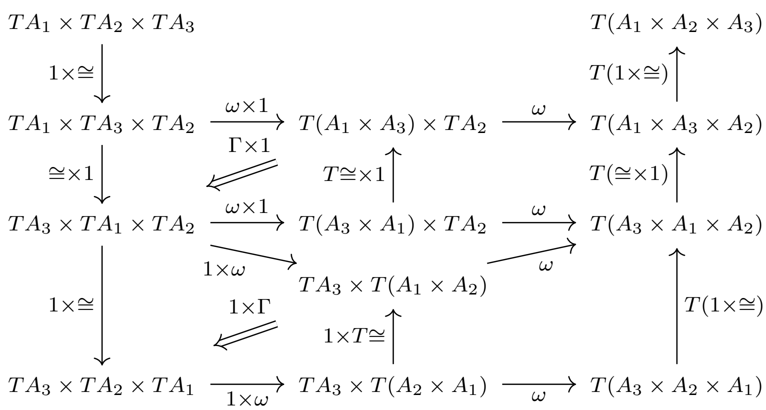
*equals the whiskering*



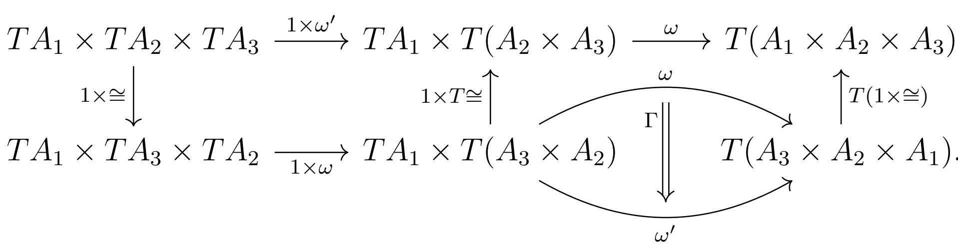
(3.3.8)

3. *The pastings and horizontal composites in (1) and (2) are equal.*

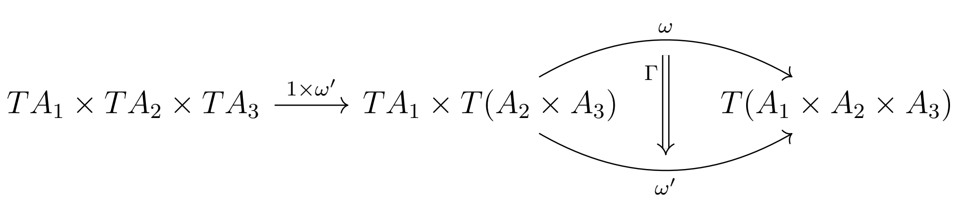
*Proof.* For (1), notice that by the Lemma 3.3.9, the pasting diagram



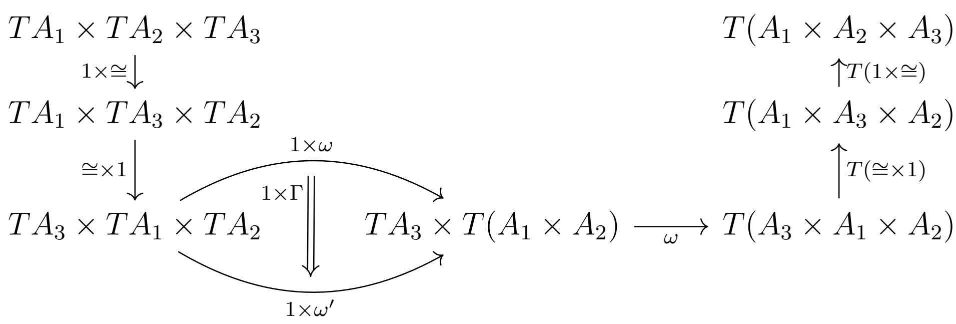
equals the whiskering



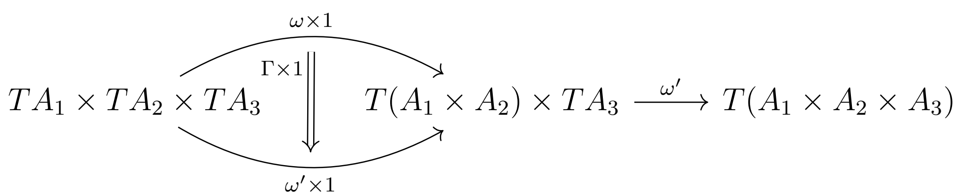
Since $\Gamma$ is a modification the last whiskering equals



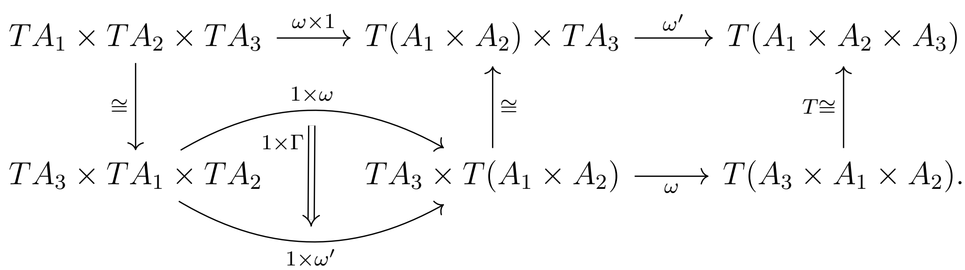
Part (1) follows from this and part (2) is proven similarly. To prove part (3) we will prove that diagrams (3.3.6), and (3.3.8) are equal. By (2) in Lemma 3.3.9, we are reduced to proving that the whiskerings




and,



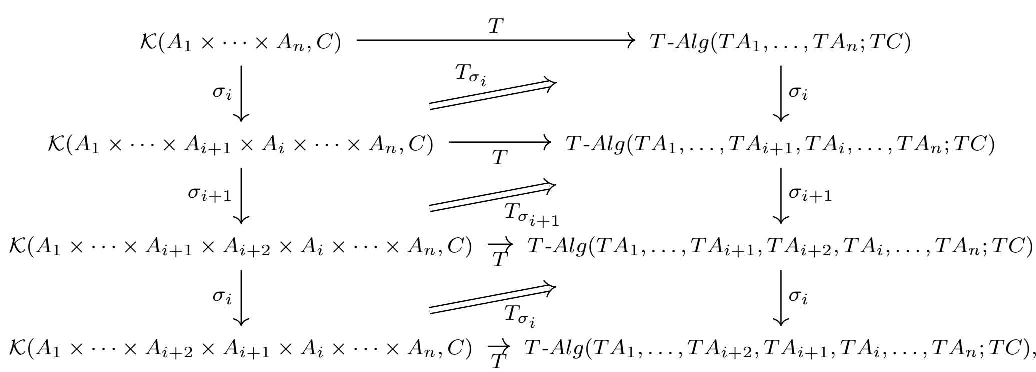

are equal. This holds since both whiskerings are equal to




■

In the presence of symmetry, we can give (a slight generalization of) the previous lemma the following interpretation.

**Lemma 3.3.11.** *Suppose that $(T, \eta, \mu, t, \Gamma)$ is a symmetric, pseudo commutative, strong 2-monad. Then the pasting diagram*




*equals the pasting diagram*

$$
\begin{array}{ccc}
\mathcal{K}(A_1 \times \cdots \times A_n, C) & \xrightarrow{\quad T \quad} & T\text{-}Alg(TA_1, \dots, TA_n; TC) \\
\sigma_{i+1}\big\downarrow & \overset{T_{\sigma_{i+1}}}{\Longrightarrow} & \big\downarrow\sigma_{i+1} \\
\mathcal{K}(A_1 \times \cdots \times A_{i+2} \times A_{i+1} \times \cdots \times A_n, C) & \xrightarrow[T]{} & T\text{-}Alg(TA_1, \dots, TA_{i+2}, TA_{i+1}, \dots, TA_n; TC) \\
\sigma_i\big\downarrow & \underset{T_{\sigma_i}}{\Longrightarrow} & \big\downarrow\sigma_i \\
\mathcal{K}(A_1 \times \cdots \times A_{i+2} \times A_i \times A_{i+1} \times \cdots \times A_n, C) & \xrightarrow[T]{} & T\text{-}Alg(TA_1, \dots, TA_{i+2}, TA_i, TA_{i+1}, \dots, TA_n; TC) \\
\sigma_{i+1}\big\downarrow & \underset{T_{\sigma_{i+1}}}{\Longrightarrow} & \big\downarrow\sigma_{i+1} \\
\mathcal{K}(A_1 \times \cdots \times A_{i+2} \times A_{i+1} \times A_i \times \cdots \times A_n, C) & \xrightarrow[T]{} & T\text{-}Alg(TA_1, \dots, TA_{i+2}, TA_{i+1}, TA_i, \dots, TA_n; TC),
\end{array}
$$

The three previous lemmas give us the following.

**Theorem 3.3.12.** *Suppose that $(T, \eta, \mu, t, \Gamma)$ is a symmetric, pseudo commutative strong to monad and let $A_1, \dots, A_n, C$ be objects of $\mathcal{K}$. The transformations $T_{\sigma_i}$ for $1 \leq i \leq n-1$ assemble together to give, for $\sigma \in \Sigma_n$, a unique transformation*

$$
\begin{array}{ccc}
\mathcal{K}(A_1 \times \cdots \times A_n, C) & \xrightarrow{\quad T \quad} & T\text{-}Alg(TA_1, \dots, TA_n; TC) \\
\sigma\big\downarrow & \underset{T_{\sigma}}{\Longrightarrow} & \big\downarrow\sigma \\
\mathcal{K}(A_{\sigma(1)} \times \cdots \times A\sigma(n)) & \xrightarrow[T]{} & T\text{-}Alg(TA_{\sigma(1)}, \dots, TA_{\sigma(n)}; TC).
\end{array}
$$

*These satisfy the unit and the product permutation axiom in Definition 2.1.15.*

We are just missing the top and bottom equivariance axioms to prove that our functor $T : \mathcal{K} \to T\text{-}Alg$ is pseudo symmetric. When $T$ is a pseudo commutative, strong 2-monad that fails to be symmetric, we can still give Lemma 3.3.10 an interpretation using the Bruhat order of the symmetric group $\Sigma_n$ on generators $\sigma_i$ for $1 \leq i \leq n-1$.

**Definition 3.3.13.** Let $\Sigma_n$ be the symmetric group with generators $\{\sigma_i\}_{1 \leq i < n}$ and presentation:

- $\sigma_i \sigma_i = 1$,
- $\sigma_i \sigma_j = \sigma_j \sigma_i$ if $|i - j| \geq 2$
- $\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$.

The length of a permutation $\sigma \in \Sigma_n$, $\ell(\sigma)$, is the number of inversions of $\sigma$, i.e., the number of couples $(i, j)$ such that $1 \leq i < j \leq n$ and $\sigma(i) < \sigma(j)$. This agrees with the length of a minimal word for $\sigma$ in the previous presentation [BB05, Prop. 1.5.2.]. The weak right order on $\Sigma_n$ [BB05, Def. 3.1.1.]is the partial order on $\Sigma_n$

generated by declaring that $\sigma < \sigma\sigma_i$ when $\ell(\sigma) < \ell(\sigma\sigma_i)$ [BB05, p. 66]. This only happens when none of the reduced words for $\sigma$ end in $\sigma_i$. The bottom of this order is the identity and the top is the reverse order permutation.

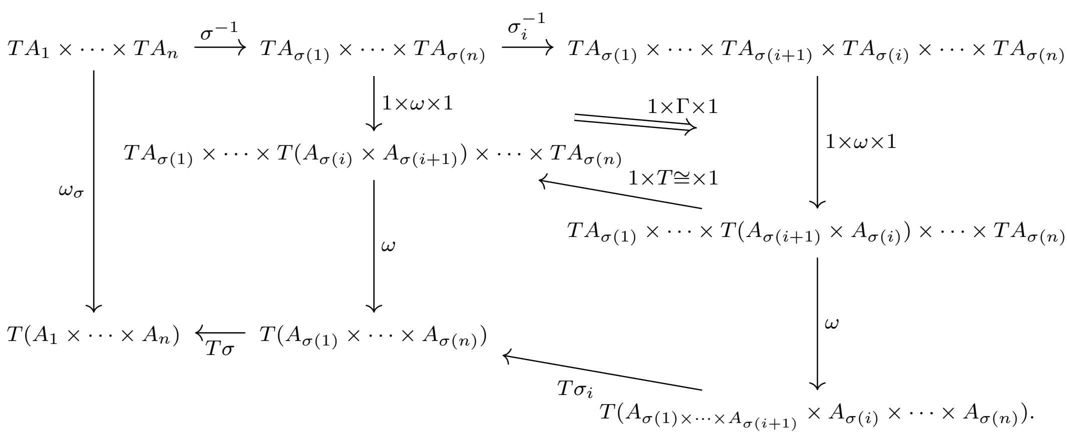
**Remark 3.3.14.** Let $(T, \eta, \mu, t, \Gamma)$ be a pseudo commutative, strong 2-monad and $A_1, \ldots, A_n$ objects of $\mathcal{K}$. We have the 1-cell

$$\omega\colon A_1 \times \cdots \times A_n \longrightarrow T(A_1 \times \cdots \times A_n).$$

Although we don't have a symmetric **Cat**-multicategory, we still have a 1-cell $\omega_\sigma$ (it is called $t_\sigma$ on [HP02]):

$$\begin{array}{ccc} TA_1 \times \cdots \times TA_n & \xrightarrow{\omega_\sigma} & T(A_1 \times \cdots \times A_n) \\ {\scriptstyle \sigma^{-1}}\downarrow & & \uparrow{\scriptstyle T\sigma} \\ TA_{\sigma(1)} \times \cdots \times TA_{\sigma(n)} & \xrightarrow[\omega]{} & T(A_{\sigma(1)} \times \cdots \times A_{\sigma(n)}). \end{array}$$

When $\ell(\sigma) < \ell(\sigma\sigma_i)$, we can define a 2-cell $\omega_\sigma \to \omega_{\sigma\sigma_i}$ as




(3.3.9)

Thus we have a 2-cell $\omega_\sigma \to \omega_{\sigma'}$ when $\sigma < \sigma'$ in the weak right order.
Notice that our definition gives a 2-cell $\omega_\sigma \to \omega_{\sigma\sigma_i}$ even when $\sigma < \sigma\sigma_i$ is false in the weak right order, but we avoid considering these cells since, in the absence of symmetry, $\omega_\sigma \to \omega_{\sigma\sigma_i} \to \omega_{\sigma\sigma_i\sigma_i}$ may not be the identity.
By Lemmas 3.3.7 and 3.3.10, there is functor

$$\Omega : B_n \to \mathcal{K}(TA_1 \times \cdots \times TA_n, T(A_1 \times \cdots \times A_n)),$$

with $\Omega(1) = \omega$, $\Omega(\sigma) = \omega_\sigma$, and such that when $\sigma < \sigma\sigma_i$ in $B_n$, $\Omega(\sigma < \sigma\sigma_i)$ is the 2-cell (3.3.9). We believe this to be the coherence statement that Hyland and Power refer to in [HP02].

To finish proving our coherence theorem, that is, that $T : \mathcal{K} \to T\text{-}Alg$ is pseudo symmetric we need to prove top and bottom invariance axioms for $T$ in Definition 2.1.15. First we will prove top equivariance for $\sigma_i$, for which we will need the following.

**Lemma 3.3.15.** *Let $(T, \eta, \mu, t, \Gamma)$ be a symmetric, pseudo commutative, strong 2-monad. Suppose $A_{i,1}, \ldots, A_{i,k_i}$ for $1 \leq i \leq n$ and $C$ be objects of $\mathcal{K}$. The component of the natural transformation*

$$
\begin{array}{ccc}
\mathcal{K}(\overline{A_1} \times \cdots \times \overline{A_n}, C) & \xrightarrow{\quad T \quad} & T\text{-}Alg(\langle TA_1 \rangle, \ldots, \langle TA_n \rangle, TC) \\
{\scriptstyle \sigma_i \langle id_{k_{\sigma_i(j)}} \rangle_{j=1}^n} \big\downarrow & \nearrow\!\!\!\nearrow \; {\scriptstyle T_{\sigma_i \langle id_{k_{\sigma_i(j)}} \rangle_{j=1}^n}} & \big\downarrow {\scriptstyle \sigma_i \langle id_{k_{\sigma_i(j)}} \rangle_{j=1}^n} \\
\mathcal{K}(\overline{A_1} \times \cdots \times \overline{A_{i+1}} \times \overline{A_i} \times \cdots \times \overline{A_n}, C) & \xrightarrow[T]{} & T\text{-}Alg(\langle TA_{<i} \rangle, \langle TA_{i+1} \rangle, \langle TA_i \rangle, \ldots, \langle TA_n \rangle; C)
\end{array}
$$

*at $h \colon \overline{A_1} \times \cdots \times \overline{A_n} \to C$ in $\mathcal{K}$ is*

$$
\begin{array}{ccc}
\overline{TA_1} \times \cdots \times \cdots \times \overline{TA_{i+1}} \times \overline{TA_i} \times \cdots \times \overline{TA_n} & \xrightarrow{1 \times \cong \times 1} & \overline{TA_1} \times \cdots \times \overline{TA_i} \times \overline{TA_{i+1}} \times \cdots \times \overline{TA_n} \\
{\scriptstyle 1 \times \omega \times \omega \times 1} \big\downarrow & & \big\downarrow {\scriptstyle 1 \times \omega \times \omega \times 1} \\
\overline{TA_1} \times \cdots \times T(\overline{A_{i+1}}) \times T(\overline{A_i}) \times \cdots \times \overline{TA_n} & \xrightarrow{1 \times \cong \times 1} & \overline{TA_1} \times \cdots \times T(\overline{A_i}) \times T(\overline{A_{i+1}}) \times \cdots \times \overline{TA_n} \\
{\scriptstyle 1 \times \omega \times 1} \big\downarrow & \nearrow\!\!\!\nearrow \; {\scriptstyle 1 \times \Gamma \times 1} & \big\downarrow {\scriptstyle 1 \times \omega \times 1} \\
\overline{TA_1} \times \cdots \times T(\overline{A_{i+1}} \times \overline{A_i}) \times \cdots \times \overline{TA_n} & \xleftarrow[1 \times T(\cong) \times 1]{} & \overline{TA_1} \times \cdots \times T(\overline{A_i} \times \overline{A_{i+1}}) \times \cdots \times \overline{TA_n} \\
{\scriptstyle \omega} \big\downarrow & & \\
T(\overline{A_1} \times \overline{A_{i+1}} \times \overline{A_i} \times \cdots \times \overline{A_n}) & & \\
\big\downarrow {\scriptstyle T(1 \times \cong \times 1)} & & \\
T(\overline{A_1} \times \cdots \times \overline{A_n}) & & \\
\big\downarrow {\scriptstyle Th} & & \\
TC. & &
\end{array}
$$

*Here, we wrote $\overline{A_i}$ instead of $\prod_{j=1}^{k_i} A_{i,j}$ and $\overline{TA_i}$ instead of $\prod_{j=1}^{k_i} TA_{i,j}$.*

*Proof.* We prove this by induction on $k_i$ and $k_{i+1}$. For $k_i = k_{i+1} = 1$ this is just Definition 3.3.5. Next we induct on $k_{i+1}$ assuming $k_i = 1$. In this case we can write $\sigma_i \langle \mathrm{id}_{k_{\sigma_i(j)}} \rangle$ as the composition

$$\overline{A_1} \times \cdots \times \overline{A_{i-1}} \times A_{i+1,1} \times A_{i+1,2} \times \cdots \times A_{i+1,k_{i+1}} \times A_i \times \overline{A_{i+2}} \times \cdots \times \overline{A_n}$$

$$\Big\downarrow \sigma_{i+1}\langle \mathrm{id}_{k_1},\ldots,\mathrm{id}_{k_{i-1}},\mathrm{id}_1,\mathrm{id}_{k_{i+1}-1},\mathrm{id}_1,\mathrm{id}_{k_{i+2}},\ldots,\mathrm{id}_{k_n}\rangle$$

$$\overline{A_1} \times \cdots \times \overline{A_{i-1}} \times A_{i+1,1} \times A_i \times A_{i+1,2} \times \cdots \times A_{i+1,k_{i+1}} \times \overline{A_{i+2}} \times \cdots \times \overline{A_n}$$

$$\Big\downarrow \sigma_{1+\sum_{t=1}^{i-1} k_t}$$

$$\overline{A_1} \times \cdots \times \overline{A_{i-1}} \times A_i \times \overline{A_{i+1}} \times \overline{A_{i+2}} \times \cdots \times \overline{A_n}.$$

After applying the inductive hypothesis to the permutation

$$\sigma_{i+1}\langle \mathrm{id}_{k_1},\ldots,\mathrm{id}_{k_{i-1}},\mathrm{id}_1,\mathrm{id}_{k_{i+1}-1},\mathrm{id}_1,\mathrm{id}_{k_{i-2}},\ldots,\mathrm{id}_{k_n}\rangle,$$

Definition 3.3.5 to $\sigma_{1+\sum_{t=1}^{i-1} k_t}$ and the product axiom, we get the result for $\sigma_i\langle \mathrm{id}_{k_{\sigma_i(j)}}\rangle$ by an application of (1) in Lemma 3.3.9. By induction, the result holds for any $k_{i+1}$ and $k_i = 1$.

We finish by induction on $k_i$, proving that the result holds for all $k_{i+1}$. We have proven that this is true for $k_i = 1$. For the inductive step we can write $\sigma_i\langle \mathrm{id}_{k_{\sigma_i(j)}}\rangle$ as

$$\overline{A_1} \times \cdots \times \overline{A_{i-1}} \times \overline{A_{i+1}} \times A_{i,1} \times \cdots \times A_{i,k_i-1} \times A_{i,k_i} \times \overline{A_{i+2}} \times \cdots \times \overline{A_n}$$

$$\Big\downarrow \sigma_i\langle \mathrm{id}_{k_1},\ldots,\mathrm{id}_{k_{i-1}},\mathrm{id}_{k_{i+1}},\mathrm{id}_{k_i-1},\mathrm{id}_1,\mathrm{id}_{k_{i+2}},\ldots,\mathrm{id}_{k_n}\rangle$$

$$\overline{A_1} \times \cdots \times \overline{A_{i-1}} \times A_{i,1} \times \cdots \times A_{i,k_i-1} \times \overline{A_{i+1}} \times A_{i,k_i} \times \overline{A_{i+2}} \times \cdots \times \overline{A_n}$$

$$\Big\downarrow \sigma_{i+1}\langle \mathrm{id}_{k_1},\ldots,\mathrm{id}_{k_{i-1}},\mathrm{id}_{k_i-1},\mathrm{id}_{k_{i+1}},\mathrm{id}_1,\mathrm{id}_{k_{i+2}},\ldots,\mathrm{id}_{k_n}\rangle$$

$$\overline{A_1} \times \cdots \times \overline{A_i} \times \overline{A_{i+1}} \times \cdots \times \overline{A_i}.$$

After applying the inductive hypothesis to the permutation

$$\sigma_i\langle \mathrm{id}_{k_1},\ldots,\mathrm{id}_{k_{i-1}},\mathrm{id}_{k_{i+1}},\mathrm{id}_{k_i-1},\mathrm{id}_1,\mathrm{id}_{k_{i+2}},\ldots,\mathrm{id}_{k_n}\rangle,$$

the already proven to the permutation

$$\sigma_{i+1}\langle \mathrm{id}_{k_1},\ldots,\mathrm{id}_{k_{i-1}},\mathrm{id}_{k_i-1},\mathrm{id}_{k_{i+1}},\mathrm{id}_1,\mathrm{id}_{k_{i+2}},\ldots,\mathrm{id}_{k_n}\rangle,$$

and the product axiom, we get our result by an application of (2) in Lemma 3.3.9.

■

**Lemma 3.3.16.** *Suppose $(T, \eta, \mu, t, \Gamma)$ is a symmetric, pseudo commutative, strong 2-monad. Let $n \geq 2$ and $1 \leq i \leq n-1$, and consider the* **Cat***-multifunctor $T : \mathcal{K} \to T\text{-}Alg$. Then, the top equivariance axiom in Definition 2.1.15 holds for $\sigma_i\langle id_{k_{\sigma_i(j)}}\rangle_{j=1}^n$, that is, For every $C \in Ob(\mathcal{K})$, $\langle B\rangle = \langle B_j\rangle_{j=1}^n \in Ob(\mathcal{K})^n$, $k_j \geq 0$ for $1 \leq j \leq n$, and $\langle A_j\rangle = \langle A_{j,i}\rangle_{i=1}^{k_j} \in Ob(\mathcal{K})^{k_j}$ for $1 \leq j \leq n$, the pasting diagram*

$$
\begin{array}{ccc}
\mathcal{K}\left(\prod \overline{B}; C\right) \times \prod_j \mathcal{K}\left(\prod \overline{A_j}, B_j\right) & \xrightarrow{T \times \prod T} & T\text{-}Alg(\langle FB\rangle; FC) \times \prod_j T\text{-}Alg\left(\langle FA_j\rangle; FB_j\right) \\
\downarrow \gamma & & \downarrow \gamma \\
\mathcal{K}\left(\prod_j \overline{A_j}, C\right) & \xrightarrow{T} & T\text{-}Alg\left(\langle\langle FA_{j,l}\rangle\rangle_{j=1}^{n}; Fc\right) \\
\downarrow \sigma_i\langle id_{k_{\sigma_i(j)}}\rangle & \Rightarrow T_{\sigma_i\langle id_{k_{\sigma_i(j)}}\rangle} & \downarrow \sigma_i\langle id_{k_{\sigma_i(j)}}\rangle \\
\mathcal{K}\left(\overline{A_{<i}} \times \overline{A_{i+1}} \times \overline{A_i} \times \overline{A_{>i+1}}; C\right) & \xrightarrow[T]{} & T\text{-}Alg\left(\langle TA_{j<i}\rangle, \langle TA_{i+1}\rangle, \langle TA_i\rangle, \langle TA_{j>i+1}\rangle; TC\right)
\end{array}
$$

*equals the pasting diagram*

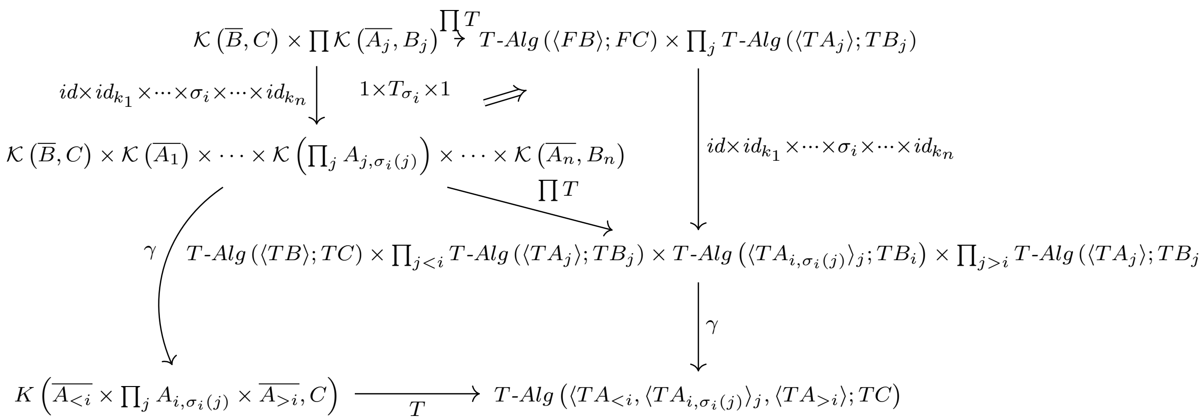
$$
\begin{array}{ccc}
\mathcal{K}\left(\overline{B}; C\right) \times \prod_j \mathcal{K}\left(\overline{A_j}; B_j\right) & \xrightarrow{T \times \prod T} & T\text{-}Alg(\langle FB\rangle; FC) \times \prod_j T\text{-}Alg\left(\langle FA_j\rangle; FB_j\right) \\
\downarrow \sigma_i \times \sigma_i^{-1} & \Rightarrow T_{\sigma_i} \times 1 & \downarrow \sigma_i \times \sigma_i^{-1} \\
\mathcal{K}\left(B_{<i} \times B_{i+1} \times B_i \times B_{>i+1}; C\right) \times \prod_j \mathcal{K}\left(\overline{A_{\sigma_i(j)}}; B_{\sigma_i(j)}\right) & \xrightarrow{T \times \prod T} & T\text{-}Alg(\langle TB\rangle\sigma_i; TC) \times \prod_j T\text{-}Alg(\langle TA_{\sigma_i(j)}\rangle; TB_{\sigma_i(j)}) \\
\downarrow \gamma & & \downarrow \gamma \\
\mathcal{K}\left(\overline{A_{<i}} \times \overline{A_{i+1}} \times \overline{A_i} \times \overline{A_{>i+1}}; C\right) & \xrightarrow[T]{} & T\text{-}Alg\left(\langle TA_{j<i}\rangle, \langle TA_{i+1}\rangle, \langle TA_i\rangle, \langle TA_{j>i+1}\rangle; TC\right).
\end{array}
$$

*Proof.* The lemma follows at once from Lemma 3.3.15 Definition 3.3.5, and (1) in Lemma 3.3.9. ■

**Lemma 3.3.17.** *Suppose $(T, \eta, \mu, t, \Gamma)$ is a symmetric, pseudo commutative, strong 2-monad. Let $n \geq 1$ and $1 \leq i \leq n-1$, and consider the* **Cat***-multifunctor $T : \mathcal{K} \to T\text{-}Alg$. Then, the bottom equivariance axiom in Definition 2.1.15 holds for $id_n\langle id_{k_1}, \ldots, \sigma_i, \ldots, id_{k_n}\rangle$ that is, For every $C \in Ob(\mathcal{K})$, $\langle B\rangle = \langle B_j\rangle_{j=1}^{n} \in Ob(K)^n$, $k_j \geq 0$ for $1 \leq j \leq n$, and $\langle A_j\rangle = \langle A_{j,l}\rangle_{l=1}^{k_j} \in Ob(\mathcal{K})^{k_j}$ for $1 \leq j \leq n$, the pasting diagram*

$$
\begin{array}{ccc}
\mathcal{K}\left(\overline{B}, C\right) \times \prod \mathcal{K}\left(\overline{A_j}, B_j\right) & \xrightarrow{\prod T} & T\text{-}Alg\left(\langle FB\rangle; FC\right) \times \prod_j T\text{-}Alg\left(\langle TA_j\rangle; TB_j\right) \\
\downarrow \gamma & & \downarrow \gamma \\
\mathcal{K}\left(\prod_j \overline{A_j}, C\right) & \xrightarrow{T} & T\text{-}Alg\left(\langle\langle TA_j\rangle\rangle_j; TC\right) \\
\downarrow id_n\langle id_{k_1}, \ldots, \sigma_i, \ldots, id_{k_n}\rangle & \Rightarrow T_{id_n\langle id_{k_1}, \ldots, \sigma_i, \ldots, id_{k_n}\rangle} & \downarrow id_n\langle id_{k_1}, \ldots, \sigma_i, \ldots, id_{k_n}\rangle \\
\mathcal{K}\left(\overline{A_{<i}} \times \prod_j A_{i,\sigma_i(j)} \times \overline{A_{>i}}, C\right) & \xrightarrow[T]{} & T\text{-}Alg\left(\langle TA_{<i}, \langle TA_{i,\sigma_i(j)}\rangle_j, \langle TA_{>i}\rangle; TC\right)
\end{array}
$$

*is equal to the pasting*




*Proof.* The lemma follows at once from Definition 3.3.5, and (2) in Lemma 3.3.9. ■

Finally we arrive at the proof of our main theorem.

**Theorem 3.3.18.** *Let $(T, \eta, \mu, t, \Gamma)$ be a symmetric, pseudo commutative, strong 2-monad. The free algebra* **Cat***-multifunctor $T\colon \mathcal{K} \to T\text{-}Alg$ is pseudo symmetric.*

*Proof.* We just need to prove that the bottom and top equivariance axioms hold for $T$. For the top equivariance axiom we notice that given $\sigma, \tau \in \Sigma_n$, and $k_1, \ldots, k_n$, we can write $\sigma\tau\langle \mathrm{id}_{k_{\sigma\tau(1)}}, \ldots, \mathrm{id}_{k_{\sigma\tau(n)}} \rangle$ as the composition

$$\begin{array}{c} \overline{A_{\sigma\tau(1)}} \times \cdots \times \overline{A_{\sigma\tau(n)}} \\ \Big\downarrow{\scriptstyle \tau\langle \mathrm{id}_{k_{\sigma\tau(i)}} \rangle} \\ \overline{A_{\sigma(1)}} \times \cdots \times \overline{A_{\sigma(n)}} \\ \Big\downarrow{\scriptstyle \sigma\langle \mathrm{id}k_{\sigma(i)} \rangle} \\ \overline{A_1} \times \cdots \times \overline{A_n}. \end{array}$$

By an application of the product axiom, if $\sigma\langle \mathrm{id}_{k_{\sigma(i)}} \rangle$ and $\tau\langle \mathrm{id}_{k_{\sigma\tau(i)}} \rangle$ satisfy the top invariance axiom, then so does $\sigma\tau\langle \mathrm{id}_{k_{\sigma\tau(1)}}, \ldots, \mathrm{id}_{k_{\sigma\tau(n)}} \rangle$. We are done by Lemma Lemma 3.3.16.

Similarly, for the bottom equivariance axiom. Given $n$, $k_1, \ldots k_n$ and $\sigma, \tau \in \Sigma_{k_i}$. If the bottom equivariance axiom holds for the permutations $\mathrm{id}_n\langle \mathrm{id}_{k_1}, \ldots, \tau, \ldots \mathrm{id}_{k_n} \rangle$ and $\mathrm{id}_n\langle \mathrm{id}_{k_1}, \ldots, \sigma, \ldots, \mathrm{id}_{k_n} \rangle$, then it also holds for $\mathrm{id}_n\langle \mathrm{id}_{k_1}, \ldots, \sigma\tau, \ldots, \mathrm{id}_{k_n} \rangle$ by an application of the product axiom. By Lemma 3.3.17, we get the bottom equivariance axiom for $\mathrm{id}_n\langle \mathrm{id}_{k_1}, \ldots, \sigma, \ldots, \mathrm{id}_{k_n} \rangle$ for any $\sigma \in \Sigma_{k_i}$. On the other hand, if the bottom equivariance axiom holds for $\mathrm{id}_n\langle\sigma_1,\ldots,\sigma_n\rangle$ and $\mathrm{id}_n\langle\tau_1,\ldots,\tau_n\rangle$, where $\sigma_i,\tau_i \in \Sigma_{k_i}$, then it also holds for $\mathrm{id}_n\langle\sigma_1\tau_1,\ldots,\sigma_n\tau_n\rangle$ by another application of the product axiom. We conclude that $T$ satisfies bottom equivariance and this concludes the proof that $T$ is pseudo symmetric. ■

Since the free functor associated to a pseudo commutative operad is a symmetric, pseudo commutative strong 2-monad, the free functor of the pseudo commutative operads defined in [GMMO23] and considered as well in [Yau24a] is pseudo symmetric.

## REFERENCES CITED


[BFSV03] C. Balteanu, Z. Fiedorowicz, R. Schwänzl, and R. Vogt, *Iterated monoidal categories.* Adv. Math. **176 no. 2** (2003), 277–349.

[BGS20] C. Barwick, S. Glasman, and J. Shah, *Spectral Mackey functors and equivariant algebraic $K$-theory, II*, Tunis. J. Math. **2** (2020), no. 1, 97–146.

[Ber96] C. Berger, *Opérades cellulaires et espaces de lacet itérés*, Annales de l'Institut Fourier **Volume 46** (1996), 1125–1157.

[BB05] A. Björner and F. Brenti, *Combinatorics of Coxeter groups*, *Vol. 231*, Graduate Texts in Mathematics, Springer, New York, 2005, xiv+363.

[BKP02] R. Blackwell, G. M. Kelly, and A. J. Power, *Two-dimensional monad theory*, J. Pure and Appl. Algebra **59 no. 1.** (2002), 1–41.

[Bou02] J. Bourke, *Skew structure in 2-category theory and homotopy theory*, J. Homotopy Relat. Struct. **12 no. 1.** (2002), 31–81.

[BF78] A. K. Bousfield and E. M. Friedlander, *Homotopy theory of* $\Gamma$*-spaces, spectra, and bisimplicial sets*, Geometric applications of homotopy theory (Proc. Conf., Evanston, Ill., 1977), II, Vol. **658**, Lecture Notes in Math. Springer, Berlin-New York, 1978, 80–130.

[Cis99] D.-C. Cisinski, *La classe des morphismes de Dwyer n'est pas stable par retractes*, Cahiers Topologie Géom. Différentielle Catég. **40 no. 3** (1999), 227–231.

[CG23] A. Corner and N. Gurski, *Operads with general groups of equivariance, and some 2-categorical aspects of operads in* **Cat**, 2023, Available at https://arxiv.org/abs/1312.5910.

[EM06] A. D. Elmendorf and M. A. Mandell, *Rings, modules, and algebras in infinite loop space theory*, Adv. Math. **205 no. 1** (2006), 163–228.

[Elm21] A. D. Elmendorf, *Multiplicativity in Mandell's Inverse $K$-Theory*, 2021, Preprint available at https://arxiv.org/pdf/2110.07512.pdf.

[GMMO23] B. J. Guillou, J. P. May, M. Merling, and A. M. Osorno, *Multiplicative equivariant $K$-theory and the Barratt-Priddy-Quillen theorem*, Adv. Math. **414** (2023), Paper No. 108865, 111 p.

[GJO22] N. Gurski, N. Johnson, and A. Osorno, *The symmetric monoidal 2-category of permutative categories*, 2022, Preprint available at https://arxiv.org/pdf/2211.04464.pdf.

[HP02] M. Hyland and J. Power, *Pseudo-commutative monads and pseudo-closed 2-categories*, J. Pure and Appl. Algebra **175 no. 1-3**, Special Volume celebrating the 70th birthday of Professor Max Kelly (2002), 141–185.

[JY22] N. Johnson and D. Yau, *Multifunctorial Inverse $K$-theory*, Ann. $K$-Theory **7 no. 3** (2022), 507–548.

[Koc70] A. Kock, *Monads on symmetric monoidal closed categories*, Arch. Math. (Basel) **21** (1970), 1–10.

[KMZ24] H. Kong, J. May, and F. Zou, *Group completions and the homotopical monadicity theorem*, 2024, https://arxiv.org/pdf/2402.03649.

[Man10] M. A. Mandell, *An inverse $K$-theory functor*, Doc. Math. **vol 15** (2010), 765–791.

[MT78] J. P. May and R. Thomason, *The uniqueness of infinite loop space machines*, Topology **17 no. 3** (1978), 205–224.

[May72] J. May, *The geometry of iterated loop spaces*, Lecture notes in Mathematics, Springer, Berlin, Heidelberg, 1972.

[May74] ———, *$E_\infty$ spaces, group completions, and permutative categories*, New developments in topology. The Edited and Revised Proceedings of the Symposium on Algebraic Topology, Oxford, June 1972, London Math. Soc. Lecture Note Ser. **11**, Cambridge University Press, London-New York, 1974, 61–93.

[May77] ———, *$E_\infty$ ring spaces and $E_\infty$ ring spectra, Vol. Vol. 577*, Lecture Notes in Mathematics, Springer-Verlag, Berlin-New York, 1977, With contributions by Frank Quinn, Nigel Ray, and Jørgen Tornehave.

[May09] ———, *The construction of $E_\infty$ ring spaces from bipermutative categories*, New topological contexts for Galois theory and algebraic geometry (BIRS 2008), Vol. 16, Geom. Topol. Monogr. Geom. Topol. Publ., Coventry, 2009, 283–330.

[Seg74] G. Segal, *Categories and cohomology theories*, Topology **13** (1974), 293–312.

[Tho80] R. W. Thomason, **Cat** *as a closed model category*, Cahiers Topologie Géom. Différentielle **21 no. 3** (1980), 305–324.

[Tho95] ______, *Symmetric monoidal categories model all connective spectra*, Theory Appl. Categ. **1 no. 3** (1995), 78–118.

[YJ21] D. Yau and N. Johnson, *2-dimensional categories*, Oxford University press, Oxford, 2021.

[Yau24a] D. Yau, *Multifunctorial Equivariant Algebraic K-Theory*, 2024, Submitted. Available at https://arxiv.org/abs/2404.02794.

[Yau24b] ______, *The Grothendieck Construction of Bipermutative-Indexed Categories and Pseudo Symmetric Inverse K-Theory*, Chapman & Hall/CRC, CRC Press, Boca Raton, FL, 2024.