Two Problems in Symmetric Tensor Categories

by

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ABSTRACT

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Fix an algebraically closed field $\mathbf{k}$ of characteristic $p \geq 0$. A *symmetric fusion category* $\mathcal{C}$ over $\mathbf{k}$ is a fusion category endowed with a braiding $c_{X,Y} : X \otimes Y \to Y \otimes X$ such that $c_{Y,X}c_{X,Y} = \mathrm{id}_{X \otimes Y}$ for all $X, Y \in \mathcal{C}$.

The first part of this dissertation focuses on the study of symmetric fusion categories in positive characteristic. We give lower bounds for the rank of a symmetric fusion category in characteristic $p \geq 5$ in terms of p. We also prove that the second Adams operation ψ_2 is not the identity for any non-trivial symmetric fusion category, and that symmetric fusion categories satisfying $\psi_2^a = \psi_2^{a-1}$ for some positive integer a are super-Tannakian. As an application, we classify all symmetric fusion categories of rank 3 and those of rank 4 with exactly two self-dual simple objects.

The second part of this dissertation treats symmetric categories in the context of topological quantum field theories. We construct a family of unoriented 2-dimensional cobordism theories parametrized by certain triples of sequences, and prove that some specializations of these sequences yield equivalences with an exterior product of Deligne categories. It is known that, modding out the category of 2-dimensional oriented cobordisms by the relation that a handle is the identity, and evaluating 2-spheres to t, produces a category equivalent to the Deligne category $\mathbf{Rep}(S_t)$, which generalizes the representation category of the symmetric group S_n from $n \in \mathbb{N}$ to $t \in \mathbf{k}$. We show an analogous story for unoriented 2-dimensional

cobordisms, with a construction that recovers the category $\mathbf{Rep}(S_t \wr \mathbb{Z}_2)$.

This dissertation contains previously published material.

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Chapter 1

Introduction

This chapter contains previously published material. The material in Section 1.2 appeared in [11]. The material in Section 1.3 appeared in [12].

1.1 Symmetric categories

Fix an algebraically closed field $\mathbf{k}$ of characteristic $p \geq 0$. A *braided monoidal category* $\mathcal{C}$ over $\mathbf{k}$ is a monoidal category endowed with a natural isomorphism $c_{X,Y} : X \otimes Y \xrightarrow{\sim} Y \otimes X$, for all $X, Y \in \mathcal{C}$, called *braiding*. Intuitively, this means the category is equipped with an isomorphism that allows us to permute the tensor product of any two objects in the category. That is, the tensor product "commutes" in a categorical sense. Hence this notion is a categorification of that of a commutative monoid.

When this braiding satisfies the condition

$$c_{Y,X} c_{X,Y} = \mathrm{id}_{X \otimes Y},$$

for all $X, Y \in \mathcal{C}$, we say that the category $\mathcal{C}$ is *symmetric*. Note that in the definition of braiding, we get that the tensor product of two objects commutes up to isomorphism. The difference here is that in a symmetric monoidal category, going from $X \otimes Y$ to $Y \otimes X$ and back to $X \otimes Y$ gives an equality with the original. This is why it is said that in a symmetric monoidal category, the tensor product is made to be "as commutative as possible".

The simplest example of a symmetric monoidal category is the category **Vec** of finite-dimensional vector spaces over **k**, with the standard tensor product of vector spaces and braid given by the isomorphism $V \otimes W \xrightarrow{\sim} W \otimes V$, defined on homogeneous elements by $v \otimes w \mapsto w \otimes v$, for all $V, W \in \mathbf{Vec}$ and $v \in V, w \in W$. More generally, for an abelian group G the category $\mathbf{Vec}_{\mathrm{G}}$ of G-graded finite-dimensional vector spaces over **k** is a symmetric monoidal category. For $G = \mathbb{Z}_2$, modifying the braiding $V \otimes W \to W \otimes V$ so that $v \otimes w \mapsto (-1)^{\deg(v)\deg(w)} w \otimes v$, for homogeneous elements $v \in V, w \in W$, we get the symmetric monoidal category **sVec** of *super vector spaces*.

More examples arise from groups in the following way: for any group G the category $\mathbf{Rep}(G)$ of finite-dimensional representations of G over **k** is a symmetric monoidal category, where the tensor product and braiding are induced from **Vec**. In fact, these are all examples of *tensor categories*, which are monoidal categories with an abelian **k**-linear structure compatible with the tensor product. It is a well known result by Deligne [13] that over a field of characteristic zero, every *pre-Tannakian* category under a moderate-growth condition arises from the representation category of an affine group scheme over **sVec**, where by pre-Tannakian we mean a symmetric tensor category wherein all objects have finite length.

However, when we take **k** of characteristic $p > 0$, Deligne's theorem for pre-Tannakian categories of moderate growth is no longer true. Counter-examples come from the symmetric tensor categories $\mathbf{Ver}_p$, called the *Verlinde categories*, which are defined as the semisimplification of the category of tilting modules over SL_2. When $p \geq 5$, these categories have no fiber functor to **Vec** or **sVec**, hence in contrast to what happens in characteristic zero, they cannot be reconstructed from group scheme theory in **sVec**, see [4].

The Verlinde categories are the first term in a nested sequence of *incompressible* categories of moderate growth [4],

$$\mathbf{Ver}_p \subset \mathbf{Ver}_{p^2} \subset \cdots \subset \mathbf{Ver}_{p^n} \subset \ldots$$

By incompressible, we mean a pre-Tannakian category for which any symmetric tensor functor from it is an embedding. As a consequence, incompressible categories cannot be reconstructed from group theory in a smaller symmetric tensor category. In characteristic zero, it is known that **Vec** and **sVec** are the only examples of incompressible categories of moderate growth. On the other hand, in positive characteristic $p > 0$, it is conjectured that incompressible categories of moderate growth are of the form $\mathbf{Ver}_{p^n}$ for some $n \in \mathbb{N}$ [9].

Recently, it was proven in [9] that pre-Tannakian categories of moderate growth admit a symmetric tensor category to some incompressible category of moderate growth. So this gives a version of Deligne's theorem in positive characteristic. Furthermore, it was shown in [10] that assuming a Frobenius-exact condition on a pre-Tannakian category is equivalent to getting a symmetric tensor functor into $\mathbf{Ver}_p$. Thus pre-Tannakian categories of moderate growth that are Frobenius-exact can be constructed from group schemes in the Verlinde category. In particular, this is the case of *fusion categories*, which fall under the moderate-growth and Frobenius-exact conditions. The first part of this work will focus on the study of symmetric fusion categories in the positive characteristic case. We give more details of the context and results obtained in this direction in Section 1.2.

There are also examples of symmetric monoidal categories which do not satisfy the moderate-growth condition. This is the case for the category $\mathbf{Rep}(S_t)$ introduced by Deligne in [14], which interpolates the category $\mathbf{Rep}(S_n)$ of finite-dimensional $\mathbf{k}$-representations of the symmetric group S_n from $n \in \mathbb{N}$ to any parameter $t \in \mathbf{k}$. In particular, it is known that this category is equivalent to a category constructed in purely topological terms, giving an explicit relation between $\mathbf{Rep}(S_t)$ and the category of oriented 2-dimensional cobordisms. In the second part of this thesis, we further study symmetric monoidal categories in the context of 2-dimensional unoriented cobordisms, and prove a related equivalence with the category $\mathbf{Rep}(S_t \wr \mathbb{Z}_2)$, which interpolates the representation category of the wreath product group $S_n \wr \mathbb{Z}_2$ from $n \in \mathbb{N}$ to any parameter $t \in \mathbf{k}$. In the last part of this Introduction, Section 1.3, we give a brief description of the context and our results in this direction.

1.2 In positive characteristic

A tensor category that is semisimple and has finitely many simples is called a *fusion category*. In particular, fusion categories in characteristic zero satisfy the moderate-growth condition. A well-known theorem by Deligne [13] implies that a symmetric fusion category in characteristic 0 is *super-Tannakian*, that is, admits a symmetric tensor functor to the category sVec of super vector spaces. As a consequence of this theorem, a symmetric fusion category over a field of characteristic 0 is equivalent to the category $\mathbf{Rep}_{\mathbf{k}}(G, z)$ of finite-dimensional representations of a finite group G. Here $z \in G$ is a central element of order 2 that modifies the braiding, see [13, Section 8.19]. This result gives a classification of symmetric fusion categories in characteristic zero in terms of group data.

An important result by Victor Ostrik in [37] gives a new version of Deligne's theorem for the case of symmetric fusion categories in positive characteristic. He proved that any symmetric fusion category $\mathcal{C}$ in characteristic $p > 0$ admits a Verlinde fiber functor, that is, a $\mathbf{k}$-linear exact symmetric tensor functor

$$F : \mathcal{C} \to \mathbf{Ver}_p \,.$$

As a consequence, any $\mathbf{k}$-linear symmetric fusion category is equivalent to the category $\mathbf{Rep}_{\mathbf{Ver}_p}(G, \epsilon)$ of representations of some finite group scheme G in $\mathbf{Ver}_p$ [37, Corollary 1.6]. However, this statement does not give an explicit classification for $p \geq 5$, since the classification of finite group schemes G in $\mathbf{Ver}_p$ such that $\mathbf{Rep}_{\mathbf{Ver}_p}(G, \epsilon)$ is semisimple is not known, even when ϵ is trivial.

When $p > 0$, Nagata [16, IV, 3.6] and Masuoka [32] give a classification of finite group schemes G in $\mathbf{Vec}$ and sVec, respectively, such that $\mathbf{Rep}_{\mathbf{k}}(G)$ is semisimple. This yields a reasonable classification of symmetric fusion categories in the super-Tannakian case. Note that when $\mathrm{char}(\mathbf{k}) = 2$ or 3, symmetric fusion categories over $\mathbf{k}$ are Tannakian and super-Tannakian, respectively, so we know their classification.

In this work we will focus on the non super-Tannakian case. We will approach the classification of symmetric fusion categories in positive characteristic by rank, i.e., by the number of simple objects. Here is our first result.

Theorem 3.1.1. *Let $p \geq 5$. If $\mathcal{C}$ is a non super-Tannakian symmetric fusion category, then*

$$\mathrm{rank}(\mathcal{C}) \geq \frac{p-1}{2}.$$

We note that the statement above does not hold for super-Tannakian categories. For example, for $p \geq 3$ the category $\mathbf{Rep}(\mathbb{Z}_2)$ is semisimple and has rank 2 which is strictly less than $\frac{p-1}{2}$ for $p > 5$.

Note that equality in Theorem 3.1.1 is achieved by $\mathbf{Ver}_p^+$, the fusion subcategory of $\mathbf{Ver}_p$ generated by simple objects of odd index, see Section 2.3.1. In characteristic 5, it is known that the equality is only achieved by $\mathbf{Ver}_5^+$, see [21, 4.6].

Question 1.2.1. Let $p > 5$ and $\mathcal{C}$ a symmetric fusion category of rank $\frac{p-1}{2}$. If $\mathcal{C}$ is not super-Tannakian, is it true that $\mathcal{C} \cong \mathbf{Ver}_p^+$?

We give a positive answer for Question 1.2.1 for the case $p = 7$ in Theorem 4.1.2.

We also know that there exist non super-Tannakian symmetric fusion categories of rank $\frac{p+3}{2}$. In fact, let $\delta \in \mathbf{k}$ and consider the Karoubian envelope $\underline{\mathrm{Rep}}(O(\delta))$ of the *Brauer category* as defined in [13, Section 9.3]. Let $\underline{\mathrm{Rep}}^{\mathrm{ss}}(O(\delta))$ denote the semisimplification of $\underline{\mathrm{Rep}}(O(\delta))$, i.e., the quotient of $\underline{\mathrm{Rep}}(O(\delta))$ by the tensor ideal of negligible morphisms, see e.g. [13, Section 6.1]. It turns out that when $\delta = -1$ this category contains a symmetric subcategory equivalent to $\mathbf{Rep}(\mathbb{Z}_2)$. The symmetric fusion category obtained by de-equivariantization by $\mathbb{Z}_2$ of the neutral component of the standard $\mathbb{Z}_2$-grading of $\underline{\mathrm{Rep}}^{\mathrm{ss}}(O(-1))$ has rank $\frac{p+3}{2}$ [36].

We thus have examples of non-super-Tannakian symmetric fusion categories in ranks $\frac{p-1}{2}$ and $\frac{p+3}{2}$. A natural question follows.

Question 1.2.2. Are there non super-Tannakian symmetric fusion categories of rank $\frac{p+1}{2}$?

For $p \geq 5$, the category $\mathbf{Ver}_p$ has precisely four fusion subcategories: $\mathrm{Vec}, \mathrm{sVec}, \mathbf{Ver}_p^+$ and $\mathbf{Ver}_p$, see [37, Proposition 3.3]. Thus, if $\mathcal{C}$ is not super-Tannakian, its Verlinde functor $F : \mathcal{C} \to \mathbf{Ver}_p$ is either surjective or its image is $\mathbf{Ver}_p^+$. Our next result gives an improvement on the bound for the former case.

Theorem 3.1.4. *Let $p \geq 5$ and let $\mathcal{C}$ be a symmetric fusion category with Verlinde fiber functor $F\colon \mathcal{C} \to \mathbf{Ver}_p$. If F is surjective then*

$$\operatorname{rank}(\mathcal{C}) \geq p - 1.$$

The main tool in the proofs of Theorems 3.1.1 and 3.1.4 is Galois theory.

Another useful tool for the classification of symmetric fusion categories in positive characteristic is the *second Adams operation*. Let $p \neq 2$. For a symmetric fusion category $\mathcal{C}$ with Grothendieck ring $\mathcal{K}(\mathcal{C})$, the second Adams operation is the ring endomorphism $\psi_2 : \mathcal{K}(\mathcal{C}) \to \mathcal{K}(\mathcal{C})$ given by

$$\psi_2(X) = S^2(X) - \Lambda^2(X),$$

for all X in $\mathcal{C}$, see [21].

Theorem 3.2.6. *Let $p > 2$ and let $\mathcal{C}$ be a non-super-Tannakian symmetric fusion category. If the Adams operation $\psi_2 : \mathcal{K}(\mathcal{C}) \to \mathcal{K}(\mathcal{C})$ satisfies $\psi_2^a = \psi_2^b$ for some $a, b \in \mathbb{Z}_{\geq 0}$, then $2^a \equiv \pm 2^b \mod p$.*

Corollary 3.2.7. *Let $p > 2$ and let $\mathcal{C}$ be a symmetric fusion category. If $\psi_2^a = \psi_2^{a-1}$ for some $a \geq 1$, then $\mathcal{C}$ is super-Tannakian.*

The following comes as a consequence.

Theorem 3.2.9. *Let $p \neq 2$. If $\mathcal{C}$ is a non-trivial symmetric fusion category then ψ_2 is not the identity.*

We apply the second Adams operation to the problem of classification of symmetric fusion categories of low rank in positive characteristic. In [21], the second Adams operation was employed to give a complete clasification for rank 2. We classify symmetric fusion categories of rank 3, and symmetric fusion categories of rank 4 with exactly two self-dual simple objects, see Theorems 4.1.2 and 4.2.7, respectively. We also note that by Theorem 3.1.1 non super-Tannakian symmetric fusion categories of rank 4 are only possible in characteristic $p = 5$ or 7.

Even though our results show that the second Adams operation is non-trivial for non-trivial symmetric fusion categories, we note that it is useful for the classification problem but definitely not sufficient on its own, see Remark 4.2.3.

1.3 In the TQFTs context

Let $\mathbf{k}$ be an algebraically closed field. Symmetric monoidal functors from the category Cob_n of oriented n-dimensional cobordisms into a symmetric monoidal category $\mathcal{C}$ are known as n-dimensional $\mathcal{C}$-valued topological quantum field theories (TQFTs) [2]. In the 2-dimensional oriented case, the theory of tensor categories has been employed to understand and build examples of TQFTs, see e.g. [26, 27, 39] and references therein. A special feature of Cob_2 is that it admits a description by generators and relations, which provides an algebraic understanding of topological quantum field theories in the 2-dimensional case. Explicitly, it is well known (see e.g. [39, Theorem 0.1]) that 2-dimensional TQFTs $\mathrm{Cob}_2 \to \mathcal{C}$ are in bijection with commutative Frobenius algebras in $\mathcal{C}$.

The second part of this work is devoted to the study of *unoriented* 2-dimensional TQFTs, that is, symmetric monoidal functors from the category UCob_2 of unoriented 2-dimensional cobordisms into a symmetric monoidal category. As in the oriented case, UCob_2 admits a description by generators and relations. We will use the one given in [41], see also [1] for a related construction. Explicitly, the generators are given by the cup, cap, pair of pants, reverse pair of pants, and twist cobordisms, same as in Cob_2, plus two extra ones, representing the orientation reversing diffeomorphism of the circle, and the Möebius band or *crosscap*. The presence of these two extra generators results in a connection with *extended* Frobenius algebras, which are Frobenius algebras with additional structure [41].

A family of $\mathbf{k}$-linear 2-dimensional TQFTs, one for each rational sequence α in $\mathbf{k}$, was introduced by Khovanov and Sazdanovic in [28]. One can "linearize" the category Cob_2 using the sequence α by allowing $\mathbf{k}$-linear combinations of cobordisms and evaluating closed connected oriented surfaces of genus g to α_g. This results in the $\mathbf{k}$-linear monoidal category VCob_α of *viewable cobordisms* [28, 27]. It can then be quotiented by a tensor ideal defined using the sequence α to produce the category SCob_α of *skein* cobordisms, with objects non-negative integers, and hom spaces $\mathrm{Hom}_{\mathrm{SCob}_\alpha}(n, m)$ given by linear combinations of oriented cobordisms from n to m circles. See also [6] for a related construction concerning r-spin TQFTs, which generalizes some of the results in [27].

When $\mathbf{k}$ has characteristic zero, 2-dimensional cobordisms can be understood in purely algebraic terms using the Deligne category $\mathbf{Rep}(S_t)$. Introduced in [14], $\mathbf{Rep}(S_t)$ interpolates the categories $\mathbf{Rep}(S_n)$ of finite-dimensional $\mathbf{k}$-representations of the symmetric group S_n from $n \in \mathbb{N}$ to any parameter $t \in \mathbf{k}$. It was observed by Comes in [7, Section 2.2] that, quotienting Cob_2 by the tensor ideal arising from the relation that a handle is the identity and evaluating 2-spheres to t, induces an equivalence with $\mathbf{Rep}(S_t)$. In terms of the Khovanov-

Sazdanovic construction, if we specialize

$$\alpha = (t, t, \dots),$$

we get an equivalence

$$\mathrm{Cob}_\alpha \cong \mathbf{Rep}(S_t),$$

where Cob_α denotes the pseudo-abelian envelope of SCob_α. That is, this construction recovers the Deligne category $\mathbf{Rep}(S_t)$ from Cob_2, and so the categories SCob_α give generalizations of the Deligne categories.

The aim of our work is to show an analogous story in the unoriented case. It turns out that, in characteristic zero, unoriented 2-dimensional cobordisms can also be understood in purely algebraic terms. We show a connection to the category $\mathbf{Rep}(S_t \wr \mathbb{Z}_2)$, which interpolates the category $\mathbf{Rep}(S_n \wr \mathbb{Z}_2)$ of finite dimensional representations of the wreath product $S_n \wr \mathbb{Z}_2$, see [29, 34]. Our two main results, Theorems I and II, describe this connection explicitly.

Following the construction in the oriented case, see [26, 27], we define a family of 2-dimensional unoriented TQFTs parametrized by sequences α, β and γ. We still evaluate orientable closed surfaces of genus g to α_g. However, the presence of the two extra sequences β and γ is due to the existence of unorientable surfaces: unorientable surfaces with one or two crosscaps and genus g are evaluated to β_g and γ_g, respectively. We thus start by constructing the category $\mathrm{VUCob}_{\alpha,\beta,\gamma}$ of viewable unoriented cobordisms, obtained by linearizing UCob_2 and evaluating unoriented closed surfaces via α, β and γ. When these sequences satisfy certain conditions, we can quotient $\mathrm{VUCob}_{\alpha,\beta,\gamma}$ by the tensor ideal generated by the *handle relation* associated to these sequences, see Section 5.3. The resulting category $\mathrm{SUCob}_{\alpha,\beta,\gamma}$

has objects non-negative integers and morphisms given by linear combinations of unoriented cobordisms from n to m circles, with up to a certain number of handles. Hence the morphism spaces $\mathrm{Hom}_{\mathrm{SUCob}_{\alpha,\beta,\gamma}}(n, m)$ are finite dimensional. We denote the pseudo-abelian closure of $\mathrm{SUCob}_{\alpha,\beta,\gamma}$ by $\mathrm{UCob}_{\alpha,\beta,\gamma}$, see Table 1.1.

Assume from now on that $\mathbf{k}$ has characteristic zero. For a spherical category $\mathcal{C}$, we denote by $\underline{\mathcal{C}}$ its quotient by negligible morphisms. Our first main result, stated below, considers theories that evaluate unorientable closed surfaces to zero and orientable closed surfaces via a geometric progressions.

Theorem I. *Let* $\alpha = (\alpha_0, \lambda\alpha_0, \lambda^2\alpha_0, \ldots)$ *and* $\beta = (0, 0, \ldots) = \gamma$ *be sequences in* $\mathbf{k}$*, for* $\alpha_0, \lambda \in \mathbf{k}^\times$*. We have an equivalence of* $\mathbf{k}$*-linear symmetric monoidal categories*

$$\underline{\mathrm{OCob}_\alpha} \cong \underline{\mathbf{Rep}(S_t \wr \mathbb{Z}_2)},$$

where $t = \frac{\lambda\alpha_0}{2}$*, and* OCob_α *is the quotient of* $\mathrm{SUCob}_{\alpha,\beta,\gamma}$ *by the relation* $\theta = 0$*, where* θ *denotes the crosscap cobordism.*

Corollary 6.4.7. *Let* α, β *and* γ *be as above. If* $\lambda\alpha_0$ *is not a non-negative even integer, then*

$$\mathrm{OCob}_\alpha \cong \mathbf{Rep}(S_t \wr \mathbb{Z}_2).$$

In particular, OCob_α *is semisimple.*

The proofs of Theorem I and Corollary 6.4.7 can be found in Section 6.4.

We state now our second main result in this direction, where we consider theories that evaluate all closed surfaces via geometric progressions, and show their connection with a product of Deligne categories.

Theorem II. *Let $\alpha_0, \beta_0, \gamma_0, \lambda \in \mathbf{k}^\times$, and consider the sequences*

$$\alpha = (\alpha_0, \lambda\alpha_0, \lambda^2\alpha_0, \dots), \quad \beta = (\beta_0, \lambda\beta_0, \lambda^2\beta_0, \dots) \text{ and } \gamma = (\gamma_0, \lambda\gamma_0, \lambda^2\gamma_0, \dots).$$

We have an equivalence

$$\underline{\mathrm{UCob}_{\alpha,\beta,\gamma}} \cong \underline{\mathbf{Rep}(S_t \wr \mathbb{Z}_2)} \boxtimes \underline{\mathbf{Rep}(S_{t_+})} \boxtimes \underline{\mathbf{Rep}(S_{t_-})},$$

where $t = \frac{\lambda}{2}\alpha_0 - \frac{1}{2}\gamma_0$, $t_+ = \frac{\sqrt{\lambda}}{2}\beta_0 + \frac{1}{2}\gamma_0$ and $t_- = -\frac{\sqrt{\lambda}}{2}\beta_0 + \frac{1}{2}\gamma_0$.

Notation	Category
UCob_2	Unoriented 2-dimensional cobordisms.
$\mathrm{VUCob}_{\alpha,\beta,\gamma}$	Viewable unoriented cobordisms. Closed components are evaluated via α, β, γ.
$\mathrm{SUCob}_{\alpha,\beta,\gamma}$	Quotient of $\mathrm{VUCob}_{\alpha,\beta,\gamma}$ by the handle relation.
$\mathrm{UCob}_{\alpha,\beta,\gamma}$	Pseudo-abelian envelope of $\mathrm{SUCob}_{\alpha,\beta,\gamma}$.
SOCob_α	For $\beta = \gamma = (0, \dots)$, quotient of $\mathrm{SUCob}_{\alpha,\beta,\gamma}$ by $\theta = 0$.
OCob_α	Pseudo-abelian envelope of SOCob_α.

Table 1.1: Constructions arising from the unoriented cobordism category

Chapter 2

Preliminaries

This chapter contains previously published material, which appeared in [11] and [12].

2.1 Notational conventions

Throughout this paper $\mathbf{k}$ will denote an algebraically closed field of characteristic $p \geq 0$, unless otherwise stated.

We denote the ring of integers by $\mathbb{Z}$, and by $\mathbb{N}$ the ring of non-negative integers.

For a ring R, we denote by $R_{\mathbb{Q}}$ the scalar extension $R \otimes_{\mathbb{Z}} \mathbb{Q}$. If z is a complex number, we denote by $\mathbb{Q}(z)$ the field extension generated by z over $\mathbb{Q}$, and by $[\mathbb{Q}(z) : \mathbb{Q}]$ the degree of said extension.

We denote by $\mathbf{Vec}$ the categories of finite dimensional vector spaces, and by $\mathbf{Vec}_{\mathrm{G}}$ the category of finite-dimensional G-graded vector spaces over $\mathbf{k}$. We denote by $\mathbf{sVec}$ the category of finite-dimensional $\mathbb{Z}_2$-graded vector spaces with braiding $V \otimes W \to W \otimes V$ given by $v \otimes w \mapsto (-1)^{\deg(v)\deg(w)} w \otimes v$, for all homogeneous $v \in V, w \in W$. Let also $\mathbf{Rep}(G)$ be the category of finite-dimensional representations of a group G over $\mathbf{k}$.

For a category $\mathcal{C}$, we denote by $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ the space of morphisms $X \to Y$ in $\mathcal{C}$, and by $\mathrm{End}_{\mathcal{C}}(X)$ the space of endomorphisms $X \to X$ in $\mathcal{C}$, for all $X, Y \in \mathcal{C}$.

2.2 Tensor categories background

In this section, we recall some useful definitions regarding monoidal and tensor categories. We refer the reader to [22] for a more detailed background on these topics.

2.2.1 Monoidal categories

Definition 2.2.1. A *monoidal category* is a collection $(\mathcal{C}, \otimes, a, \mathbf{1}, l, r)$, where $\mathcal{C}$ is a category, $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ is a bifunctor, called *tensor product*, $\mathbf{1} \in \mathcal{C}$ is the *unit object*, and $a_{X,Y,Z} : (X \otimes Y) \otimes Z \xrightarrow{\sim} X \otimes (Y \otimes Z)$, $r_X : X \otimes \mathbf{1} \xrightarrow{\sim} X$, y $l_X : \mathbf{1} \otimes X \xrightarrow{\sim} X$ are natural isomorphisms, for all $X, Y, Z \in \mathcal{C}$, such that the following diagrams commute:

$$
\begin{array}{ccccc}
((X \otimes Y) \otimes Z) \otimes W & \xrightarrow{a_{X \otimes Y, Z, W}} & (X \otimes Y) \otimes (Z \otimes W) & \xrightarrow{a_{X,Y,Z \otimes W}} & X \otimes (Y \otimes (Z \otimes W)) \\
\downarrow{\scriptstyle a_{X,Y,Z} \otimes id_W} & & & & \downarrow{\scriptstyle id_X \otimes a_{Y,Z,W}} \\
(X \otimes (Y \otimes Z)) \otimes W & & \xrightarrow[a_{X, Y \otimes Z, W}]{\qquad\qquad\qquad} & & X \otimes ((Y \otimes Z) \otimes W),
\end{array}
$$

$$
\begin{array}{ccc}
(X \otimes \mathbf{1}) \otimes Y & \xrightarrow{a_{X,\mathbf{1},Y}} & X \otimes (\mathbf{1} \otimes Y) \\
 & {\scriptstyle r_X \otimes id_Y} \searrow \quad \swarrow {\scriptstyle id_X \otimes l_Y} & \\
 & X \otimes Y. &
\end{array}
$$

We will call $a : (- \otimes -) \otimes - \xrightarrow{\sim} - \otimes (- \otimes -)$ the *associativity isomorphism* of $\mathcal{C}$. A *monoidal* *subcategory* of $\mathcal{C}$ is a monoidal category $(\mathcal{D}, \otimes, a, \mathbf{1}, l, r)$, where $\mathcal{D} \subset \mathcal{C}$ is a subcategory such that $\mathbf{1} \in \mathcal{D}$, and is closed under the tensor product.

Example 2.2.2. The category **Vec** is a monoidal category. More generally, for a commutative ring with unity R, the category **R-Mod** of left R-modules is monoidal, with $\otimes = \otimes_R$ and unit $\mathbf{1} = R$.

For a group G, the category $\mathbf{Rep}(G)$ is also monoidal, where $\otimes$ is the tensor product of representations and the unit is the trivial representation.

Example 2.2.3. For a group G, consider the category $\mathbf{Vec}_{\mathrm{G}}$ of finite-dimensional G-graded vector spaces. That is, objects are vector spaces V with a decomposition $V = \bigoplus_{g \in G} V_g$, and morphisms are linear transformations that preserve the grading. This is a monoidal category, with tensor product given by

$$(V \otimes W)_g = \bigoplus_{x,y \in G \ : \ xy = g} V_x \otimes W_y,$$

unit object δ_1 defined by $(\delta_1)_1 = \mathbf{k}$ and $(\delta_1)_g = 0$ for $g \neq 1$, and associativity given by the identity.

More generally, let $\omega : G \times G \times G \to \mathbf{k}^\times$ be a 3-cocycle, that is, a function satisfying

$$\omega(g_1 g_2, g_3, g_4)\omega(g_1, g_2, g_3 g_4) = \omega(g_1, g_2, g_3)\omega(g_1, g_2 g_3, g_4)\omega(g_2, g_3, g_4),$$

for all g_1, g_2, g_3, g_4 in G. We obtain the monoidal category $\mathbf{Vec}_{\mathrm{G}}^\omega$ modifying the associativity isomorphism in $\mathbf{Vec}_{\mathrm{G}}$ in the following way. For $U, V, W \in \mathbf{Vec}_{\mathrm{G}}$, define $a^\omega_{U,V,W} : (U \otimes V) \otimes W \to U \otimes (V \otimes W)$ as the lineal extension of

$$a^\omega_{U_g, V_h, W_m} := \omega(g, h, m) \, \mathrm{Id}_{U_g \otimes V_h \otimes W_m} : (U_g \otimes V_h) \otimes W_m \to U_g \otimes (V_h \otimes W_m),$$

for all $g, h, m \in G$.

Definition 2.2.4. Given two monoidal categories $(\mathcal{C}, \otimes, a, \mathbf{1}, l, r)$ and $(\mathcal{C}', \otimes', a', \mathbf{1}', l', r')$, a *monoidal functor* from $\mathcal{C}$ to $\mathcal{C}'$ is a triple (F, J, u), where $F : \mathcal{C} \to \mathcal{C}'$ is a functor, $u : \mathbf{1}' \to F(\mathbf{1})$ is an isomorphism, and $J : \otimes' \circ (F \times F) \to F \circ \otimes$ is a natural isomorphism, such that the following diagrams commute,

$$\begin{array}{ccc}
(F(X) \otimes' F(Y)) \otimes' F(Z) & \xrightarrow{a'_{F(X),F(Y),F(Z)}} & F(X) \otimes' (F(Y) \otimes' F(Z)) \\
\downarrow{\scriptstyle J_{X,Y} \otimes' id_{F(Z)}} & & \downarrow{\scriptstyle id_{F(X)} \otimes' J_{Y,Z}} \\
F(X \otimes Y) \otimes' F(Z) & & F(X) \otimes' F(Y \otimes Z) \\
\downarrow{\scriptstyle J_{X \otimes Y, Z}} & & \downarrow{\scriptstyle J_{X, Y \otimes Z}} \\
F((X \otimes Y) \otimes Z) & \xrightarrow[F(a_{X,Y,Z})]{} & F(X \otimes (Y \otimes Z)),
\end{array}$$

$$\begin{array}{ccc}
\mathbf{1}' \otimes' F(X) & \xrightarrow{l'_{F(X)}} & F(X) \\
\downarrow{\scriptstyle u \otimes \mathrm{id}_{F(X)}} & & \downarrow{\scriptstyle F(l_X)^{-1}} \\
F(\mathbf{1}) \otimes' F(X) & \xrightarrow[J_{\mathbf{1},X}]{} & F(\mathbf{1} \otimes X),
\end{array}
\qquad
\begin{array}{ccc}
F(X) \otimes' \mathbf{1}' & \xrightarrow{r'_{F(X)}} & F(X) \\
\downarrow{\scriptstyle id_{F(X)} \otimes u} & & \downarrow{\scriptstyle F(r_X)^{-1}} \\
F(X) \otimes' F(\mathbf{1}) & \xrightarrow[J_{X,\mathbf{1}}]{} & F(X \otimes \mathbf{1}),
\end{array}$$

for all X, Y, Z in $\mathcal{C}$.

A monoidal functor $F : \mathcal{C} \to \mathcal{D}$ is said to be *full* or *faithful* if $\mathrm{Hom}_{\mathcal{C}}(X, Y) \xrightarrow{F} \mathrm{Hom}_{\mathcal{D}}(F(X), F(Y))$ is surjective or injective for all $X, Y \in \mathcal{C}$, respectively, and it is said to be *essentially surjective* if for all $Y \in \mathcal{D}$ there exists $X \in \mathcal{C}$ such that $F(X) \cong Y$. We say that F is an *equivalence* if it is both essentially surjective and fully-faithful.

2.2.2 Rigidity

Let $\mathcal{C}$ be a monoidal category, and let $X \in \mathcal{C}$.

Definition 2.2.5. $\diamondsuit$ A *left dual* of X (if it exists) is a triple $(X^*, \mathrm{ev}_X, \mathrm{coev}_X)$, such that $X^* \in \mathcal{C}$, and the morphisms $\mathrm{ev}_X : X^* \otimes X \to \mathbf{1}$ and $\mathrm{coev}_X : \mathbf{1} \to X \otimes X^*$ satisfy that the following compositions yield the identity of X and X^*, respectively:

$$X \cong \mathbf{1} \otimes X \xrightarrow{\mathrm{coev}_X \otimes \mathrm{id}_X} (X \otimes X^*) \otimes X \xrightarrow{a_{X,X^*,X}} X \otimes (X^* \otimes X) \xrightarrow{\mathrm{id}_X \otimes \mathrm{ev}_X} X \otimes \mathbf{1} \cong X,$$

$$X^* \cong X^* \otimes \mathbf{1} \xrightarrow{\mathrm{id}_{X^*} \otimes \mathrm{coev}_X} X^* \otimes (X \otimes X^*) \xrightarrow{a^{-1}_{X^*,X,X^*}} (X^* \otimes X) \otimes X^* \xrightarrow{\mathrm{ev}_{X^*} \otimes \mathrm{id}_{X^*}} \mathbf{1} \otimes X^* \cong X^*.$$

The morphisms ev_X and coev_X are called *evaluation* and *co-evaluation*, respectively.

$\diamondsuit$ The definition of a *right dual* for X is analogous.

Left and right duals are unique up to unique isomorphism.

Remark 2.2.6. Let $\mathcal{C}$ and $\mathcal{D}$ be monoidal categories, and $F : \mathcal{C} \to \mathcal{D}$ be a monoidal functor. If X has left dual X^*, then $F(X^*)$ is a left dual of $F(X)$. The analogous statement is true for right duals.

Definition 2.2.7. A monoidal category is said to be *rigid* if every object has a left and right dual.

Example 2.2.8. The category **Vec** of finite-dimensional vector spaces over $\mathbf{k}$ is rigid. On the other hand, the category of all vector spaces (that is, including those of infinite dimension) is not rigid.

Similarly, for a group G, the category $\mathbf{Rep}\,G$ of finite-dimensional representations of G over $\mathbf{k}$ is rigid, and so are the categories $\mathbf{Vec}_{\mathrm{G}}$ and $\mathbf{Vec}^{\omega}_{\mathrm{G}}$.

2.2.3 Abelian categories

Definition 2.2.9. $\diamondsuit$ We call a category $\mathcal{C}$ $\mathbf{k}$-*linear* if for each pair of objects $X, Y \in \mathcal{C}$ the set $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ has a $\mathbf{k}$-module structure such that composition of morphisms is $\mathbf{k}$-bilinear.

$\diamondsuit$ We say $\mathcal{C}$ is *additive* if it is $\mathbf{k}$-linear and for every finite sequence of objects $X_1, \ldots, X_n$ in $\mathcal{C}$, there exists their direct sum $X_1 \oplus \cdots \oplus X_n$ in $\mathcal{C}$.

$\diamondsuit$ We say $\mathcal{C}$ is *Karoubian* if it is $\mathbf{k}$-linear and every idempotent $e = e^2 : X \to X$ in $\mathcal{C}$ has a kernel, and hence also a cokernel.

$\diamondsuit$ We say $\mathcal{C}$ is *pseudo-abelian* if it is additive and Karoubian.

$\diamondsuit$ An additive category is called *abelian* if every morphism has a kernel and cokernel, every monomorphism is a kernel, and every epimorphism is a cokernel.

If $\mathcal{C}$ is Karoubian then for any idempotent $e \in \mathrm{End}_{\mathcal{C}}(X)$ there exists its image $eX \in \mathcal{C}$. That is, we have a direct sum decomposition $X \simeq eX \oplus (1-e)X$.

Starting with a $\mathbf{k}$-linear category we can construct new categories by formally adding images of idempotents or direct sum of objects.

Definition 2.2.10. Let $\mathcal{C}$ be a $\mathbf{k}$-linear category.

$\diamondsuit$ We define the *additive envelope* $\mathcal{A}(\mathcal{C})$ as the category with:

- Objects: Finite formal sums $X_1 \oplus \cdots \oplus X_m$ of objects in $\mathcal{C}$.

- Morphisms: For every $X_1 \oplus \dots X_m, Y_1 \oplus \dots Y_n$ in $\mathcal{A}(\mathcal{C})$, let

$$\mathrm{Hom}_{\mathcal{A}(\mathcal{C})}(X_1 \oplus \dots X_m, Y_1 \oplus \cdots \oplus Y_n) := \bigoplus_{i,j} \mathrm{Hom}_{\mathcal{C}}(X_i, Y_j).$$

  Composition of morphisms is given by matrix multiplication.

$\diamondsuit$ We define the *Karoubian envelope* $\mathcal{K}(\mathcal{C})$ of $\mathcal{C}$ as the category with:

- Objects: Pairs (X, e), where X is an object in $\mathcal{C}$ and e is an idempotent in $\mathrm{End}_{\mathcal{C}}(X)$.

- Morphisms: For every $(X, e), (Y, f) \in \mathcal{K}(\mathcal{C})$, let

$$\mathrm{Hom}_{\mathcal{K}(\mathcal{C})}((X, e), (Y, f)) = f \circ \mathrm{Hom}_{\mathcal{C}}(X, Y) \circ e.$$

  Composition of morphisms is as expected.

◇ We define the *pseudo-abelian envelope* of $\mathcal{C}$ as the category $\mathcal{P}(\mathcal{C}) := \mathcal{K}(\mathcal{A}(\mathcal{C}))$.

We note that the operations above do not commute in general. These constructions take a $\mathbf{k}$-linear category and return an additive, Karoubian or pseudo-abelian category, respectively.

2.2.4 Tensor categories

Definition 2.2.11. A *tensor category* $\mathcal{C}$ is a $\mathbf{k}$-linear abelian and monoidal category, such that the tensor bifunctor $\otimes$ is $\mathbf{k}$-bilinear.

Example 2.2.12. For a group G, the categories $\mathbf{Vec}$, $\mathbf{Vec}_{\mathrm{G}}^{\omega}$ and $\mathbf{Rep}(G)$ are tensor categories.

Definition 2.2.13. For tensor categories $\mathcal{C}$ and $\mathcal{D}$, a *tensor functor* $F : \mathcal{C} \to \mathcal{D}$ is a $\mathbf{k}$-linear exact monoidal functor, such that F preserves the unit object.

For tensor categories $\mathcal{C}$ and $\mathcal{D}$, a tensor functor $F : \mathcal{C} \to \mathcal{D}$ is *fully faithful* if the induced function $\mathrm{Hom}_{\mathcal{C}}(X, Y) \to \mathrm{Hom}_{\mathcal{D}}(F(X), F(Y))$ is a bijection, and *essentially surjective* if every object $D \in \mathcal{D}$ is isomorphic to $F(X)$ for some $X \in \mathcal{C}$. We say F is an *equivalence* if it is fully faithful and essentially surjective, and in such case we say the categories $\mathcal{C}$ and $\mathcal{D}$ are equivalent, and denote it by $\mathcal{C} \cong \mathcal{D}$.

2.2.5 Braided categories

Definition 2.2.14. ◇ A *braided monoidal category* is a monoidal category $\mathcal{C}$ endowed with a natural isomorphism $c_{X,Y} : X \otimes Y \to Y \otimes X$, $X, Y \in \mathcal{C}$, called *braiding*, such that the following diagrams commute,

$$\begin{array}{ccccc}
(X \otimes Y) \otimes Z & \xrightarrow{a_{X,Y,Z}} & X \otimes (Y \otimes Z) & \xrightarrow{\sigma_{X,Y\otimes Z}} & (Y \otimes Z) \otimes X \\
\downarrow{\scriptstyle \sigma_{X,Y}\otimes id_Z} & & & & \downarrow{\scriptstyle a_{X,Y,Z}} \\
(Y \otimes X) \otimes Z & \xrightarrow[a_{Y,X,Z}]{} & Y \otimes (X \otimes Z) & \xrightarrow[id_Y \otimes \sigma_{X,Z}]{} & Y \otimes (Z \otimes X),
\end{array}$$

$$\begin{array}{ccccc}
X \otimes (Y \otimes Z) & \xrightarrow{a^{-1}_{X,Y,Z}} & (X \otimes Y) \otimes Z & \xrightarrow{\sigma_{X\otimes Y,Z}} & Z \otimes (X \otimes Y) \\
\downarrow{\scriptstyle id_X \otimes \sigma_{Y,Z}} & & & & \downarrow{\scriptstyle a^{-1}_{Z,X,Y}} \\
X \otimes (Z \otimes Y) & \xrightarrow[a^{-1}_{X,Z,Y}]{} & (X \otimes Z) \otimes Y & \xrightarrow[\sigma_{X,Z}\otimes id_Y]{} & (Z \otimes X) \otimes Y,
\end{array}$$

for all $X, Y, Z \in \mathcal{C}$.

♢ We say a braided monoidal category $\mathcal{C}$ is *symmetric* if

$$c_{Y,X} c_{X,Y} = \mathrm{id}_{X \otimes Y},$$

for all $X, Y \in \mathcal{C}$.

A *symmetric tensor functor* between symmetric tensor categories is a tensor functor compatible with the commutativity isomorphism.

Example 2.2.15. For a group G, the categories **Vec** and $\mathbf{Rep}(G)$ are braided, with braided given by the usual transposition of factors. If G is abelian, the category $\mathbf{Vec}_{\mathrm{G}}$ is also braided. In particular, all these categories are symmetric.

We note that for a monoidal category, having a braiding is an extra structure, not a property. For example, the category $\mathbf{Vec}_{\mathbb{Z}_2}$ can be endowed with two different braidings, yielding non-isomorphic braided categories. We denote by **sVec** the braided category $\mathbf{Vec}_{\mathbb{Z}_2}$ with braiding $V \otimes W \to W \otimes V$ so that $v \otimes w \mapsto (-1)^{\deg(v)\deg(w)} w \otimes v$, for homogeneous elements $v \in V, w \in W$. This is a symmetric monoidal category, called the category of *super vector spaces*.

Definition 2.2.16. We say a symmetric fusion category $\mathcal{C}$ is *Tannakian* (resp., *super-Tannakian*) if it admits a symmetric fiber functor, that is, a symmetric tensor functor $\mathcal{C} \to \mathrm{Vec}$ (resp., $\mathcal{C} \to \mathrm{sVec}$), see [38, 15, 13].

2.2.6 Grothendieck ring

Let $\mathcal{C}$ be a tensor category. An object X in $\mathcal{C}$ is *simple* if 0 and X are its only subobjects, and *semisimple* if it can be written as direct sum of simple objects. If all objects in $\mathcal{C}$ are semisimple, then the category is said to be *semisimple*.

Definition 2.2.17. An object X in $\mathcal{C}$ has *finite length* if there exists a chain

$$0 = X_0 \subset X_1 \subset ... \subset X_{n-1} \subset X_n = X,$$

of objects X_i in $\mathcal{C}$ such that X_i is a subobject of X_{i+1} and X_{i+1}/X_i is simple for all $i = 1, \ldots, n-1$.

A chain like this is called a *Jordan-Hölder series* for X. We say that this series contains a simple object Y with multiplicity m if X_i/X_{i-1} is isomorphic to Y for exactly m distinct values of $i = 1, \ldots, n$. Jordan-Hölder's theorem establishes that if X has finite length, then any filtration of X can be extended into a Jordan-Hölder series for X, and any two series contain each simple object with the same multiplicity.

Let $\mathcal{C}$ be a tensor category in which every object has finite length. We denote by $[X : Y]$ the multiplicity of Y in a Jordan-Hölder series for X, which is well defined by Jordan-Hölder's theorem.

Definition 2.2.18. The *Grothendieck group* $\mathcal{K}(\mathcal{C})$ of $\mathcal{C}$ is the free abelian group generated by the isomorphism classes of simple objects in $\mathcal{C}$.

For every object $X \in \mathcal{C}$, we denote its class in $\mathcal{K}(\mathcal{C})$ also by X, to simplify the notation. Let X_i, $i \in I$, be the (isomorphism classes of) simple objects in $\mathcal{C}$. The tensor product in $\mathcal{C}$ induces a multiplication in $\mathcal{K}(\mathcal{C})$ given by

$$X_i X_j := \sum_{k \in I} [X_i \otimes X_j : X_k] X_k. \tag{2.2.1}$$

This multiplication is associative, and makes $\mathcal{K}(\mathcal{C})$ a $\mathbb{Z}_+$-ring with basis $\{X_i\}_{i \in I}$ and unit $\mathbf{1}$, which we call the *Grothendieck ring* of $\mathcal{C}$. Equation (2.2.1) is often referred to as the *fusion rules* of $\mathcal{C}$.

A tensor functor $F : \mathcal{C} \to \mathcal{D}$ determines a ring homomorphism $\mathcal{K}(\mathcal{C}) \to \mathcal{K}(\mathcal{D})$, which we will also denote by F to simplify the notation.

2.2.7 Fusion categories

Definition 2.2.19. A tensor category $\mathcal{C}$ is said to be *pre-Tannakian* if it satisies:

1. For every pair of objects X, Y in $\mathcal{C}$, $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ is a finite dimensional $\mathbf{k}$-vector space.

2. Every object in $\mathcal{C}$ has finite length.

3. The category $\mathcal{C}$ is rigid.

4. We have $\mathrm{End}_{\mathcal{C}}(\mathbf{1}) \cong \mathbf{k}$.

An object $X \in \mathcal{C}$ in a pre-Tannakian category is called *simple* if its only subobjects are 0 and itself.

Definition 2.2.20. A *fusion category* $\mathcal{C}$ is a pre-Tannakian category such that:

1. $\mathcal{C}$ is semisimple, i.e., every object can be written as a finite direct sum of simple objects, and
2. $\mathcal{C}$ has finitely many simple objects.

We will denote the (finite) set of isomorphism classes of simple objects in $\mathcal{C}$ by $\mathcal{O}(\mathcal{C})$.

Example 2.2.21. The category **Vec** is a fusion category, with a unique (isomorphism class of) simple object, with representative the field $\mathbf{k}$. More generally, for a finite group G, the category $\mathbf{Vec}_G$ is a fusion category, where the simple objects are given by δ_g, with $g \in G$, defined by $(\delta_g)_h = 0$ if $h \neq g$, and $(\delta_g)_g = \mathbf{k}$.

If $|G|$ does not divide char($\mathbf{k}$), then $\mathbf{Rep}(G)$ is also fusion category, where simple objects are the irreducible representations of G.

In a fusion category, the left and right duals of an object are isomorphic. Morever, if X is simple, then so is its dual X^*.

A *fusion subcategory* of a fusion category $\mathcal{C}$ is a full tensor subcategory $\mathcal{C}' \subset \mathcal{C}$, such that if $X \in \mathcal{C}$ is isomorphic to a direct summand of an object of $\mathcal{C}'$, then $X \in \mathcal{C}'$, see [17, 2.1].

Let $X_1, \ldots, X_n$ denote the simple objects in $\mathcal{C}$. Then any other object X in C can be written as $X = \sum_{i=1}^{n} N_i X_i$, where $N_i = \dim(\mathrm{Hom}(X_i, X))$. Hence, we have that

$$X_i \otimes X_j = \sum_{k=1}^{n} N_{ij}^k X_k,$$

for all $i, j = 1, \ldots, n$, where $N_{ij}^k = \dim(\mathrm{Hom}(X_k, X_i \otimes X_j)$. These non-negative integers $\{N_{ij}^k\}_{i,j,k=1,\ldots,n}$ are called the *fusion rules* of $\mathcal{C}$.

For fusion categories $\mathcal{C}$ and $\mathcal{D}$, a *tensor functor* $F : \mathcal{C} \to \mathcal{D}$ is a $\mathbf{k}$-linear exact and faithful monoidal functor, see [22, Definition 4.2.5]. For a tensor functor $F : \mathcal{C} \to \mathcal{D}$, its image $F(\mathcal{C})$ is the fusion subcategory of $\mathcal{D}$ generated by objects $F(X)$, $X \in \mathcal{C}$. The functor F is called *surjective* if $F(\mathcal{C}) = \mathcal{D}$, see [23, 5.7]. Thus a tensor functor is an equivalence if and only if it is both surjective and fully-faithful.

For two fusion categories $\mathcal{C}$ and $\mathcal{D}$, we can define their *external tensor product*, see [35, Section 2.2], which we will denote by $\mathcal{C} \boxtimes \mathcal{D}$.

2.2.8 Frobenius-Perron dimension

Let $\mathcal{C}$ be a fusion category. There is a unique ring homomorphism $\mathrm{FPdim} : \mathcal{K}(\mathcal{C}) \to \mathbb{R}$ called *Frobenius-Perron dimension* such that $\mathrm{FPdim}(X) \geq 1$ for any object $X \neq 0$. For $X \in \mathcal{K}(\mathcal{C})$, $\mathrm{FPdim}(X)$ is given by the maximal non-negative real eigenvalue of the matrix of left multiplication by X. This is well defined thanks to the Frobenius-Perron theorem, since said matrix has non-negative entries. For a simple object $X \in \mathcal{C}$, the *Frobenius-Perron dimension* of X is given by $\mathrm{FPdim}(X)$.

We list some of the properties of the Frobenius-Perron dimension below:

1. $\mathrm{FPdim}(X) \geq 1$ for all $X \neq 0$.

2. The number $\mathrm{FPdim}(X)$ is an algebraic integer, for all $X \in \mathcal{C}$.

3. $\mathrm{FPdim}(\mathbf{1}) = 1$.

4. $\mathrm{FPdim}(X \otimes Y) = \mathrm{FPdim}(X)\,\mathrm{FPdim}(Y)$, for all $X, Y \in \mathcal{C}$.

5. $\mathrm{FPdim}(X) = \mathrm{FPdim}(X^*)$, for all $X \in \mathcal{C}$.

See for example [22, Section 3.3] for details.

We define the *Frobenius-Perron dimension* $\mathrm{FPdim}(\mathcal{C})$ of $\mathcal{C}$ by

$$\mathrm{FPdim}(\mathcal{C}) = \sum_{X \in \mathcal{O}(\mathcal{C})} \mathrm{FPdim}(X)^2.$$

Definition 2.2.22. Let $\mathcal{C}$ be a fusion category.

♢ We say $\mathcal{C}$ is *weakly integral* if $\mathrm{FPdim}(\mathcal{C})$ is an integer.

♢ We say $\mathcal{C}$ is *integral* if $\mathrm{FPdim}(X)$ is an integer for all simple objects X in $\mathcal{C}$. In this case, $\mathrm{FPdim}(\mathcal{C})$ is also an integer.

Let $F : \mathcal{C} \to \mathcal{D}$ be a tensor functor between fusion categories. Then F preserves the Frobenius-Perron dimension. That is, $\mathrm{FPdim}_{\mathcal{D}}(F(X)) = \mathrm{FPdim}_{\mathcal{C}}(X)$f or all X in $\mathcal{C}$, see for example [22, Proposition 3.3.13].

Example 2.2.23. Let G be a finite group such that $|G|$ does not divide the charactersitic of $\mathbf{k}$. The fusion category $\mathbf{Rep}(G)$ is integral, with Frobenius-Perron dimension $\mathrm{FPdim}(\mathcal{C}) = |G|$. In fact, the Frobenius-Perron dimension of a representation is the same as its dimension as a $\mathbf{k}$-vector space.

Example 2.2.24. For G a finite group, all simple objects δ_g, for $g \in G$, in $\mathbf{Vec}_G$ have Frobenius-Perron dimension 1. Hence $\mathrm{FPdim}(\mathbf{Vec}_G) = |G|$.

Definition 2.2.25. Let $\mathcal{C}$ be a fusion category.

♢ An object $X \in \mathcal{C}$ is said to be *invertible* if its evaluation $\mathrm{ev}_X : X^* \otimes X \to \mathbf{1}$ and coevaluation $\mathrm{coev}_X : \mathbf{1} \to X \otimes X^*$ morphisms are isomorphisms.

♢ We say $\mathcal{C}$ is *pointed* if every simple object in $\mathcal{C}$ is invertible.

It is easy to check that an object $X \in \mathcal{C}$ is invertible if and only if $\mathrm{FPdim}(X) = 1$. Hence if an object X is invertible, so is its dual. And if X and Y are invertible, so is $X \otimes Y$.

Example 2.2.26. Let G be a finite group such that $|G|$ does not divide $\mathbf{p} = \mathrm{char}(\mathbf{k})$. Invertible objects in $\mathbf{Rep}(G)$ are one-dimensional representations.

For a finite group G and a 3-cocycle ω, the category $\mathbf{Vec}_G^\omega$ is pointed. In fact, it is well known that every pointed fusion category is of this form.

Definition 2.2.27. The *pointed subcategory* of a fusion category $\mathcal{C}$ is the fusion subcategory $\mathcal{C}_{\mathrm{pt}}$ generated by the invertible objects in $\mathcal{C}$.

2.2.9 Equivariantization by groups

We give here a brief introduction to the construction given by equivariantization by a group, which we will use in Sections 4.1 and 4.2. We refer to [22, Section 2.7] for details.

Let $\mathcal{C}$ be a tensor category, and denote by $\mathrm{Aut}_{\otimes}(\mathcal{C})$ the monoidal category whose objects are the autoequivalences of $\mathcal{C}$, morphisms are natural isomorphisms between functors, and tensor product is given by the composition of functors. Let G be a group, and define $\underline{G}$ as the strict monoidal category with objects the elements in G, morphisms are identities, and the tensor product is given by the multiplication in G.

Definition 2.2.28. An *action* of G on a category $\mathcal{C}$ is given by:

1. For every $g \in G$, a tensor functor $T_g : \mathcal{C} \to \mathcal{C}$;

2. For $g, h \in G$, a monoidal isomorphism $J_{g,h} : T_g T_h \xrightarrow{\sim} T_{gh}$;

3. A monoidal isomoprhism $u : \mathrm{Id}_{\mathcal{C}} \xrightarrow{\sim} T_e$;

such that the following diagrams commute,

$$\begin{array}{ccc} T_gT_hT_k & \xrightarrow{J_{g,h}T_k} & T_{gh}T_k \\ {\scriptstyle T_gJ_{h,k}}\downarrow & & \downarrow{\scriptstyle J_{gh,k}} \\ T_gT_{hk} & \xrightarrow[J_{g,hk}]{} & T_{ghk}, \end{array} \qquad \begin{array}{ccc} T_g & \xrightarrow{T_gu} & T_gT_e \\ {\scriptstyle uT_g}\downarrow & \searrow^{=} & \downarrow{\scriptstyle J_{g,e}} \\ T_eT_g & \xrightarrow[J_{e,g}]{} & T_g, \end{array} \tag{2.2.2}$$

for all $g, h, k \in G$.

Definition 2.2.29. Let $\mathcal{C}$ be a category with an action of a group G.

◇ A *G-equivariant object* in $\mathcal{C}$ is a pair (X, v), where $X \in \mathcal{C}$ and $v = \{v_g : T_g(X) \xrightarrow{\sim} X \mid g \in G\}$, such that the diagram

$$\begin{array}{ccc} T_g(T_h(X)) & \xrightarrow{T_g(v_h)} & T_g(X) \\ {\scriptstyle J_{g,h}(X)}\Big\downarrow & & \Big\downarrow{\scriptstyle v_g} \\ T_{gh}(X) & \xrightarrow[v_{gh}]{} & X \end{array} \tag{2.2.3}$$

commutes for all $g, h \in G$.

◇ A *G-equivariant morphism* $f : (X, v) \to (Y, w)$ is a morphism $f : X \to Y$ in $\mathcal{C}$ such that $fv_g = w_g f$, for all $g \in G$.

◇ The G-equivariantization $\mathcal{C}^G$ of a category $\mathcal{C}$ by a group G is the category of equivariant objects and morphisms in $\mathcal{C}$.

If $\mathcal{C}$ is a fusion category and G a finite group such that $|G|$ does not divide $\operatorname{char}(\mathbf{k})$, $\mathcal{C}^G$ is also a fusion category. Moreover, its Frobenius-Perron dimension satisfies $\operatorname{FPdim} \mathcal{C}^G = |G| \operatorname{FPdim} \mathcal{C}$.

2.2.10 Categorical trace

Let $\mathcal{C}$ be a rigid monoidal category.

Definition 2.2.30. Let $X \in \mathcal{C}$. For an isomorphism $f : X \to X^{**}$, its *left quantum trace* is given by

$$\mathrm{Tr}(f) : \mathbf{1} \xrightarrow{\mathrm{coev}_X} X \otimes X^* \xrightarrow{f \otimes \mathrm{id}_{X^*}} X^{**} \otimes X^* \xrightarrow{\mathrm{ev}_{X^*}} \mathbf{1}. \tag{2.2.4}$$

Its *right quantum trace* is similarly defined. This notion of trace behaves well with direct sums, tensor products and duals, see for example [22, Proposition 4.7.3].

By definition, $\mathrm{Tr}(f) \in \mathrm{End}_{\mathcal{C}}(\mathbf{1})$. Under the assumption $\mathrm{End}_{\mathcal{C}}(\mathbf{1}) \cong \mathbf{k}$, we can identify $\mathrm{Tr}(f)$ with an element in $\mathbf{k}$. In this work, this will be the case for tensor (and hence fusion) categories, which will be used throughout. .

Definition 2.2.31. A *pivotal structure* in $\mathcal{C}$ is an isomorphism of tensor functors $\psi : \mathrm{Id} \to (-)^{**}$, see [3, 22]. That is, for every $X \in \mathcal{C}$ we have a natural isomorphism $\psi_X : X \xrightarrow{\sim} X^{**}$, and these isomorphisms satisfy that

$$\psi_{X \otimes Y} = \psi_X \otimes \psi_Y,$$

for all $X, Y \in \mathcal{C}$.

A category with a pivotal structure is called *pivotal*.

The pivotal structure is called *spherical* if for any such morphism its right trace equals its left trace. A *spherical fusion category* is a fusion category equipped with a spherical structure.

In the case when $\mathcal{C}$ is symmetric, there is a canonical choice of spherical structure given by

$$X \xrightarrow{\mathrm{Id}_X \otimes \mathrm{coev}_{X^*}} X \otimes X^* \otimes X^{**} \xrightarrow{c_{X,X^*} \otimes \mathrm{id}_{X^{**}}} X^* \otimes X \otimes X^{**} \xrightarrow{\mathrm{ev}_X \otimes \mathrm{Id}_{X^{**}}} X^{**},$$

see e.g. [22, Secion 9.9].

2.2.11 Quantum dimension and non-degenerate categories

Let $\mathcal{C}$ be a pivotal tensor category, with pivotal structure given by $\psi_X : X \xrightarrow{\sim} X^{**}$.

Definition 2.2.32. For an object $X \in \mathcal{C}$, its *quantum dimension* $\dim(X)$ with respect to ψ is defined by

$$\dim(X) := \mathrm{Tr}(\psi_X) \in \mathrm{End}(\mathbf{1}) \cong \mathbf{k}.$$

Hence $\dim(X)$ can be thought of as en element in $\mathbf{k}$, for all $X \in \mathcal{C}$. This determines a ring homomorphism $\dim : \mathcal{K}(\mathcal{C}) \to \mathbf{k}$, sending X to $\dim(X)$. By [22, Proposition 4.8.4], if X is simple then $\dim(X) \neq 0$.

Example 2.2.33. In the category $\mathbf{Rep}(G)$ for a group G, the quantum dimension of a representation coincides with its dimension as a vector space.

The *global dimension* $\dim(\mathcal{C}) \in \mathbf{k}$ of a pivotal fusion category $\mathcal{C}$ is defined as

$$\dim(\mathcal{C}) := \sum_{X \in \mathcal{O}(C)} \dim(X) \dim(X^*) \in \mathbf{k}.$$

When $\mathcal{C}$ is spherical, we have that $\dim(X) = \dim(X^*)$ for all $X \in \mathcal{C}$. In particular, for a spherical fusion category (and thus for a symmetric fusion category), $\dim \mathcal{C} = \sum_{X \in \mathcal{O}(\mathcal{C})} \dim(X)^2$.

Remark 2.2.34. We note that the quantum dimension of an object in a pivotal category $\mathcal{C}$ is not necessarily a real number. Hence it is not true in general that $\dim(X) = \mathrm{FPdim}(X)$ for an object $X \in \mathcal{C}$.

Definition 2.2.35. We say a pivotal fusion category $\mathcal{C}$ is *non-degenerate* if $\dim(\mathcal{C}) \neq 0$, see [23, Definition 9.1].

A crucial property of non-degenerate fusion categories is that they can be lifted to characteristic zero, see [18] and [23, Section 9]. It is known that for $p = 0$, any fusion category is non-degenerate, see [23, Theorem 2.3]. We will use these facts repeatedly in Sections 4.1 and 4.2.

2.2.12 Semisimplification

For a spherical category $\mathcal{C}$, we recall the theory of semisimplification, a process through which we obtain a semisimple quotient of $\mathcal{C}$, see [3, 20] for details.

Definition 2.2.36. A *tensor ideal* I in a $\mathbf{k}$-linear monoidal category $\mathcal{C}$ is a collection of subspaces $I(X,Y) \subset \mathrm{Hom}(X,Y)$, for every pair of objects $X, Y \in \mathcal{C}$, closed under composition and tensor product in the following way:

1. For $f \in I(X,Y)$, $g_1 \in \mathrm{Hom}(Y,Z)$ and $g_2 \in \mathrm{Hom}(Z,X)$, the compositions $g_1 \circ f$ and $f \circ g_2$ are in $I(X,Z)$ and $I(Z,Y)$, respectively;

2. For $f \in I(X,Y)$ and $g \in \mathrm{Hom}(Z,W)$, the products $f \otimes g$ and $f \otimes f$ are in $I(X \otimes Z, Y \otimes W)$ and $I(Z \otimes X, W \otimes Y)$, respectively;

for all $X, Y, Z, W \in \mathcal{C}$.

For a tensor ideal I in $\mathcal{C}$, we can define the *quotient* $\mathcal{C}'$ of $\mathcal{C}$ by I, see [20, Section 2.1]. The new category $\mathcal{C}'$ is again a $\mathbf{k}$-linear monoidal category, with:

- Objects are the objects of $\mathcal{C}$, and

- Hom spaces are defined by

$$\mathrm{Hom}_{\mathcal{C}'}(X,Y) := \mathrm{Hom}_{\mathcal{C}}(X,Y)/I(X,Y),$$

for all $X, Y \in \mathcal{C}$.

- Composition of morphisms and tensor product are induced from $\mathcal{C}$.

Definition 2.2.37. [22, Exercise 8.18.9] Let $\mathcal{C}$ be a spherical $\mathbf{k}$-linear monoidal category. We call a morphism $x \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$ *negligible* if, for any $y \in \mathrm{Hom}_{\mathcal{C}}(Y, X)$, we have that $\mathrm{tr}(yx) = 0$.

The set $\mathcal{N}(\mathcal{C})$ of all negligible morphisms in a spherical category $\mathcal{C}$ is a proper tensor ideal, see for example [20, Proposition 2.4]. Moreover, the spherical category category $\bar{\mathcal{C}} := \mathcal{C}/\mathcal{N}(\mathcal{C})$ is semisimple.

Definition 2.2.38. We call $\bar{\mathcal{C}}$ the *semisimplification* of $\mathcal{C}$. The simple objects of $\mathcal{C}$ are the images of the indecomposable objects of $\mathcal{C}$ of nonzero dimension [3, 20]

2.3 In positive characteristic

For this section, we assume $\mathbf{k}$ is an algebraically closed field of characteristic $p > 0$.

2.3.1 Verlinde categories

Let $\mathbb{Z}_p$ denote the cyclic group of p elements, and let σ be a generator σ. We then have an isomorphism of algebras

$$\mathbf{k}[\mathbb{Z}_p] \cong \mathbf{k}[\sigma]/(\sigma^p - 1) \cong \mathbf{k}[\sigma]/(\sigma - 1)^p.$$

Thus isomorphism classes of indecomposable objects in the category $\mathbf{Rep}_{\mathbf{k}}(\mathbb{Z}_p)$ are given by the $\mathbb{Z}_p$-modules $\tilde{L}_s := \mathbf{k}[\sigma]/(1 - \sigma)^s$, for $s \in \mathbb{Z}$ satisfying $1 \leq s \leq p$.

Definition 2.3.1. The *Verlinde category* $\mathbf{Ver}_p$ is the semisimplification of $\mathbf{Rep}(\mathbb{Z}_p)$.

That is, $\mathbf{Ver}_p$ is the symmetric fusion category obtained by quotienting $\mathbf{Rep}_{\mathbf{k}}(\mathbb{Z}_p)$ by the tensor ideal of negligible morphisms, see Section 2.2.12. Since the indecomposable object $\tilde{L}_p$ in $\mathbf{Rep}(\mathbb{Z}_p)$ has quantum dimension 0, simple objects in $\mathbf{Ver}_p$ are the images of the indecomposables $\tilde{L}_s$ for $s = 1, \ldots, p-1$. We denote them by

$$\mathbf{1} = L_1, L_2, \ldots, L_{p-1},$$

respectively. So $\mathbf{Ver}_p$ has rank $p-1$. The Verlinde fusion rules are given by

$$L_r \otimes L_s = \sum_{i=1}^{\min(r,s,p-r,p-s)} L_{|r-s|+2i-1}.$$

In particular, we have that all simple objects in $\mathbf{Ver}_p$ are self-dual.

Example 2.3.2. For $p = 2$, $\mathbf{Ver}_2$ is equivalent to $\mathbf{Vec}$. For $p = 3$, $\mathbf{Ver}_3$ is equivalent to $\mathbf{sVec}$.

For $p = 5$, $\mathbf{Ver}_5$ has 4 simple objects $\mathbf{1} = L_1, L_2, L_3, L_4$, and

$$L_2 \otimes L_2 = \mathbf{1} \oplus L_3, \qquad L_2 \otimes L_3 = L_2 \oplus L_4, \qquad L_2 \otimes L_4 = L_3,$$

$$L_3 \otimes L_3 = \mathbf{1} \oplus L_3, \qquad L_3 \otimes L_4 = L_2, \qquad L_4 \otimes L_4 = \mathbf{1}.$$

From this, we can compute

$$\mathrm{FPdim}(\mathbf{1}) = 1 = \mathrm{FPdim}(L_4), \qquad \text{and} \qquad \mathrm{FPdim}(L_3) = \frac{1+\sqrt{5}}{2} = \mathrm{FPdim}(L_2).$$

Definition 2.3.3. Let $\mathbf{Ver}_p^+$ be the abelian subcategory of $\mathbf{Ver}_p$ generated by L_i for i odd.

By the Verlinde fusion rules, it turns out that $\mathbf{Ver}_p^+$ is a fusion subcategory of $\mathbf{Ver}_p$. Hence $\mathrm{rank}(\mathbf{Ver}_p^+) = \frac{p-1}{2}$. For $p > 2$, the fusion subcategory generated by L_1 and L_{p-1} is tensor equivalent to sVec. Moreover, $\mathbf{Ver}_p$ has exactly 4 fusion subcategories: $\mathbf{Ver}_p, \mathbf{Ver}_p^+, \mathbf{sVec}$ and $\mathbf{Vec}$, and we have an equivalence of categories

$$\mathbf{Ver}_p \cong \mathbf{Ver}_p^+ \boxtimes \mathbf{sVec}, \tag{2.3.1}$$

see [37].

The Verlinde fiber functor

Let $\mathcal{C}$ be a symmetric fusion category over $\mathbf{k}$ of characteristic $p > 0$. The main result of [37] states:

Theorem 1. *[37, Theorem 1.5] There exists a symmetric tensor functor $F : \mathcal{C} \to \mathbf{Ver}_p$.*

The functor F is called the *Verlinde fiber functor*. It is shown in [21, Theorem 2.6] that it is unique up to a non-unique isomorphism of tensor functors.

As a consequence of this theorem, a symmetric fusion category is of the form $\mathbf{Rep}_{\mathbf{Ver}_p}(G, \pi)$, for G a linearly reductive finite group scheme in $\mathbf{Ver}_p$. However, this does not give a complete classification of fusion categories in positive characteristic, as at the moment there are not many known examples of group schemes in $\mathbf{Ver}_p$.

In characteristic 2, the theorem implies that all symmetric fusion categories are Tannakian, as $\mathbf{Ver}_2 = \mathbf{Vec}$. Moreover, we have a classification of finite group schemes G in $\mathbf{Vec}$ such that $\mathbf{Rep}_{\mathbf{k}}(G)$ is semisimple, following a well known theorem by Nagata [16]. Hence we do have classification of symmetric fusion categories in the particular case $p = 2$. Namely, any such category is obtained as the equivariantization of a pointed category associated with an abelian 2-group, by the action of a group of odd order.

Similarly, since $\mathbf{Ver}_3 = \mathbf{sVec}$, the theorem implies that symmetric fusion categories in characteristic 3 are super-Tannakian. In [32], Masuoka provided a classification of finite group schemes G in $\mathbf{sVec}$ such that $\mathbf{Rep}_{\mathbf{k}}(G)$ is semisimple, and so we also have a classification of symmetric fusion categories when $p = 3$.

2.3.2 The second Adams operation

Let $\mathcal{C}$ be a symmetric fusion category over a field of characteristic $p \neq 2$. We recall the definition of the n-th symmetric and exterior powers of an object $X \in \mathcal{C}$, following [22, Definition 9.9.5] and [19, 2.1].

Consider the action of the symmetric group S_n on $X^{\otimes n}$ in $\mathcal{C}$, where

$$S_n \to \mathrm{Aut}_{\mathcal{C}}(X^{\otimes n})$$

$$(i\ i+1) \mapsto \mathrm{id}_{X^{\otimes(i-1)}} \otimes c_{X,X} \otimes \mathrm{id}_{X^{\otimes(n-i-1)}},$$

for all $i = 1, \ldots, n-1$, where $(i\ i+1)$ denotes the transposition $i \mapsto i+1$ in S_n, see [22, Remark 8.2.5].

Definition 2.3.4. ◇ The *n-th symmetric power* $S^n(X)$ of X is the maximal quotient of $X^{\otimes n}$ on which the action of S_n is trivial.

◇ The *n-th exterior power* $\Lambda^n(X)$ of X is the maximal quotient of $X^{\otimes n}$ on which the action of S_n factors through the sign representation.

Definition 2.3.5. The *second Adams operation* $\psi_2 : \mathcal{K}(\mathcal{C}) \to \mathcal{K}(\mathcal{C})$ is defined by

$$\psi_2(X) = S^2(X) - \Lambda^2(X),$$

for all $X \in \mathcal{K}(\mathcal{C})$.

The second Adams operation defines a ring endomorphism of $\mathcal{K}(\mathcal{C})$, see [21, Lemma 4.4].

Example 2.3.6. Let G be an abelian finite group. In $\mathbf{Vec}_G$, $\psi_2(g) = g^2$ for all $g \in G$.

Example 2.3.7. Let G be a finite group such that $p \nmid |G|$. Recall that $\mathcal{K}(\mathbf{Rep}(G))$ can be identified with the group of characters of G. Thus $\psi_2(g) : \mathcal{K}(\mathbf{Rep}(G)) \to \mathcal{K}(\mathbf{Rep}(G))$ maps $\chi(g) \mapsto \chi(g^2)$, for every character χ and all $g \in G$.

Since $X^2 = S^2(X) + \Lambda^2(X)$ for all $X \in \mathcal{C}$, then

$$X^2 \equiv \psi_2(X) \mod 2 \text{ for all } X \in \mathcal{K}(\mathcal{C}).$$

We will use this fact repeatedly throughout this work. We also have that ψ_2 commutes with duality, that is, $\psi_2(X)^* = \psi_2(X^*)$ for all $X \in \mathcal{K}(\mathcal{C})$.

When studying properties of the second Adams operation, we will often look at its scalar extension

$$(\psi_2)_{\mathbb{Q}} := \psi_2 \otimes 1 : \mathcal{K}(\mathcal{C})_{\mathbb{Q}} \to \mathcal{K}(\mathcal{C})_{\mathbb{Q}}.$$

2.4 Unoriented TQFTs

In this last section of the preliminaries, we give context to the study of unoriented tqfts, their relation to extended Froebnius algebras, and show graphical representation of the category $\mathbf{Rep}(S_t \wr \mathbb{G})$ [34], which will be used for our results in later chapters.

2.4.1 The category of unoriented cobordisms

We denote by Cob_n the category of n-dimensional oriented cobordisms. This is a symmetric monoidal category, with monoidal structure given by disjoint union, see for example [30].

Definition 2.4.1. Let UCob_n be the category of *n-dimensional unoriented cobordisms* defined as follows:

♢ Objects: closed $(n-1)$-dimensional smooth manifolds.

♢ Morphisms: given two objects X, Y, we define $\mathrm{Hom}_{\mathrm{UCob}_n}(X, Y)$ to be the space consisting of n-dimensional smooth compact manifolds with boundary M, together with a diffeomorphism of their boundary $\partial M \cong X \sqcup Y$. Such two manifolds are considered to be the same if they differ by a diffeomorphism which induces a diffeomorphism between their boundaries that is isotopic to the identity.

♢ Composition: let $M : X \to Y$ and $N : Y \to Z$ be morphisms in UCob_n. Since $\partial M \cong X \sqcup Y$ and $\partial N \cong Y \sqcup Z$, we have induced diffeomorphisms $M_0 \cong Y$ and $N_0 \cong Y$ for some $M_0 \subseteq \partial M$ and $N_0 \subseteq \partial N$. The composition of M and N is given by glueing M with N via these diffeomorphisms, see [33].

The category UCob_n is a rigid symmetric monoidal category, with monoidal structure induced by the disjoint union.

Example 2.4.2. Let X be a closed $(n-1)$-manifold. Then $\mathrm{Hom}_{\mathrm{UCob}_n}(X, X)$ contains cylinder n-manifolds $M = X \times [0, 1]$.

In the special case where $n = 2$ and $X = S^1$ we have exactly two classes of diffeomorphisms up to isotopy: one which is orientation preserving and another which is orientation reversing. Thus we have exactly two diffeomorphism classes in $\mathrm{Hom}_{\mathrm{UCob}_2}(S^1, S^1)$, pictured below:

Figure 2.1: Two cylinders

From a slightly different point of view, we can consider the cylinders above as morphisms from $0 \to 2$ and $2 \to 0$, instead of $1 \to 1$. This yields two different compositions of cylinders, which result in the Klein bottle and the Torus, respectively:

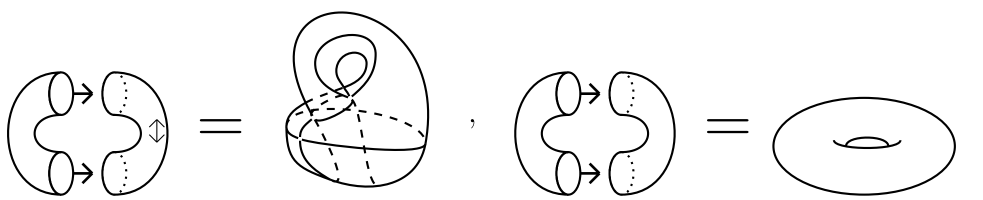


Figure 2.2: Composition of cylinders

Definition 2.4.3. An n-dimensional $\mathcal{C}$-valued *unoriented topological quantum field theory (TQFT)* is a symmetric monoidal functor $\mathrm{UCob}_n \to \mathcal{C}$ for some symmetric monoidal category $\mathcal{C}$.

There is a similar definition of $\mathcal{C}$-valued *oriented* TQFTs $\mathrm{Cob}_n \to \mathcal{C}$, and it is well known they are in bijection with Frobenius algebras in $\mathcal{C}$, see for example [30]. We discuss in the following section the relation between unoriented TQFTs and extended Frobenius algebras.

2.4.2 Extended Frobenius algebras

In this section we recall some algebraic structures in monoidal categories, with the goal of talking about extended Frobenius algebras and their relation to unoriented TQFTs.

Definition 2.4.4. Let $\mathcal{C}$ be a monoidal category.

- An *algebra* in $\mathcal{C}$ is a triple (A, m_A, u_A), consisting of an object $A \in \mathcal{C}$, and morphisms $m_A : A \otimes A \to A$ and $u_A : \mathbb{1} \to A$ in $\mathcal{C}$, called *multiplication* and *unit*, satisfying the following commutative diagrams:

$$\begin{CD} A \otimes A \otimes A @>{m_A \otimes \mathrm{Id}_A}>> A \otimes A \\ @V{\mathrm{Id}_A \otimes m_A}VV @VV{m_A}V \\ A \otimes A @>{m_A}>> A \end{CD} \quad , \quad \begin{CD} \mathbb{1} \otimes A @>{u_A \otimes \mathrm{Id}_A}>> A \otimes A \\ @V{\mathrm{Id}_A \otimes u_A}VV @VV{m_A}V \\ A \otimes A @>{m_A}>> A. \end{CD} \qquad (\mathbb{1} \otimes A \xrightarrow{\sim} A) \tag{2.4.1}$$

  A morphism of algebras (A, m_A, u_A) and (B, m_B, u_B) is a morphism $f : A \to B$ in $\mathcal{C}$ so that $f m_A = m_B(f \otimes f)$ and $f u_A = u_B$.

- A *coalgebra* in $\mathcal{C}$ is a triple $(A, \Delta_A, \epsilon_A)$, consisting of an object $A \in \mathcal{C}$, and morphisms $\Delta_A : A \to A \otimes A$ and $\epsilon_A : A \to \mathbb{1}$ in $\mathcal{C}$, called *comultiplication* and *counit*, satisfying the following commutative diagrams:

$$\begin{CD} A @>{\Delta_A}>> A \otimes A \\ @V{\Delta_A}VV @VV{\Delta_A \otimes \mathrm{Id}_A}V \\ A \otimes A @>{\mathrm{Id}_A \otimes \Delta_A}>> A \otimes A \otimes A \end{CD} \quad , \quad \begin{CD} A @>{\Delta_A}>> A \otimes A \\ @V{\Delta_A}VV @VV{\epsilon_A \otimes \mathrm{Id}_A}V \\ A \otimes A @>{\mathrm{Id}_A \otimes \Delta_A}>> \mathbb{1} \otimes A. \end{CD} \qquad (A \xrightarrow{\sim} \mathbb{1} \otimes A) \tag{2.4.2}$$

  A *morphism of coalgebras* $(A, \Delta_A, \epsilon_A)$ and $(B, \Delta_B, \epsilon_B)$ is a morphism $f : A \to B$ in $\mathcal{C}$ so that $\Delta_B f = (f \otimes f)\Delta_A$ and $\epsilon_B f = \epsilon_A$.

- A *Frobenius algebra* in $\mathcal{C}$ is a 5-tuple $(A, m_A, u_A, \Delta_A, \epsilon_A)$ where (A, m_A, u_A) is an algebra in $\mathcal{C}$ and $(A, \Delta_A, \epsilon_A)$ is a coalgebra in $\mathcal{C}$, so that the following diagram commutes

$$\begin{array}{ccc} A \otimes A & \xrightarrow{\mathrm{Id}_A \otimes \Delta_A} & A \otimes A \otimes A \\ \Big\downarrow{\scriptstyle \Delta_A \otimes \mathrm{Id}_A} & \searrow^{\Delta_A m_A} & \Big\downarrow{\scriptstyle m_A \otimes \mathrm{Id}_A} \\ A \otimes A \otimes A & \xrightarrow{\mathrm{Id}_A \otimes m_A} & A \otimes A. \end{array} \tag{2.4.3}$$

A *morphism of Frobenius algebras* $f : A \to B$ is a both an algebra and coalgebra map.

Definition 2.4.5. Let $\mathcal{C}$ be a braided monoidal category with braiding $c_{X,Y} : X \otimes Y \xrightarrow{\sim} Y \otimes X$ for all $X, Y \in \mathcal{C}$.

- A *commutative algebra* in $\mathcal{C}$ is an algebra (A, m_A, u_A) in $\mathcal{C}$ such that $m_A\ c_{A,A} = m_A$.
- A *commutative Frobenius algebra* in $\mathcal{C}$ is a Frobenius algebra that is commutative as an algebra.

It is well known that commutative Frobenius algebras in a symmetric tensor category $\mathcal{C}$ are in bijection with 2-dimensional oriented TQFTs $\mathrm{Cob}_2 \to \mathcal{C}$ (see e.g. [39, Theorem 0.1]). An analogous correspondence takes place between extended Frobenius algebras and unoriented TQFTs. We give a precise statement in what follows.

Definition 2.4.6. [41] Let $\mathcal{C}$ be a braided monoidal category. An *extended Frobenius algebra* in $\mathcal{C}$ is a 7-tuple $(A, m_A, u_A, \Delta_A, \epsilon_A, \phi_A, \theta_A)$, where $(A, m_A, u_A, \Delta_A, \epsilon_A)$ is a commutative Frobenius algebra in $\mathcal{C}$, $\phi_A : A \to A$ and $\theta_A : \mathbb{1} \to A$ are morphisms of Frobenius algebras, $\phi_A^2 = \mathrm{id}_A$, and the diagrams

$$\begin{array}{ccccc} \mathbb{1} \otimes A & & \xrightarrow{\theta_A \otimes \mathrm{id}_A} & & A \otimes A \\ \Big\downarrow{\scriptstyle \theta_A \otimes 1} & & & & \Big\downarrow{\scriptstyle m_A} \\ A \otimes A & \xrightarrow{m_A} & A & \xrightarrow{\phi_A} & A \end{array} , \qquad \begin{array}{ccccccc} \mathbb{1} & \xrightarrow{u_A} & A & \xrightarrow{\Delta_A} & A \otimes A & \xrightarrow{\phi_A \otimes \mathrm{id}_A} & A \otimes A \\ \Big\downarrow{\scriptstyle \sim} & & & & & & \Big\downarrow{\scriptstyle m_A} \\ \mathbb{1} \otimes \mathbb{1} & \xrightarrow{\theta_A \otimes \theta_A} & A \otimes A & & \xrightarrow{m_A} & & A \end{array} \tag{2.4.4}$$

commute.

Example 2.4.7. Let $n \geq 1$ and let $S_n \wr \mathbb{Z}_2$ denote the wreath product of S_n and $\mathbb{Z}_2$, where S_n acts on $\mathbb{Z}_2^n$ by permutation of indices. Let $\mathcal{C} = \mathbf{Rep}(S_n \wr \mathbb{Z}_2)$. For $\lambda \in \mathbf{k}^\times$, we define $A = A_\lambda \in \mathcal{C}$ to be the algebra of functions on the set $\{x_1, x_{-1}, \ldots, x_n, x_{-n}\}$ over $\mathbf{k}$, with unit and multiplication maps given by

$$u_A : \mathbf{k} \to A, \qquad\qquad m_A : A \otimes A \to A,$$
$$1 \mapsto \sum_{i=1}^{n} (\delta_i + \delta_{-i}) \qquad\qquad f \otimes g \mapsto fg$$

respectively, where $\delta_i(x_j) = \delta_{i,j}$ and $\delta_{-i}(x_{-j}) = \delta_{i,j}$ for all $1 \leq i, j \leq n$. Note that A is in fact in $\mathcal{C}$, since $S_n \wr \mathbb{Z}_2$ acts on $\{x_1, x_{-1}, \ldots, x_n, x_{-n}\}$ by permuting indices, and thus also acts on its algebra of functions A.

We show that $(A, m_A, u_A, \Delta_A, \varepsilon_A, \phi_A, \theta_A)$ is an extended Frobenius algebra in $\mathcal{C}$, where

$$\varepsilon_A : A \to \mathbf{k}, \qquad \Delta_A : A \to A \otimes A, \qquad \phi_A : A \to A, \qquad \theta_A : \mathbf{k} \to A$$
$$f \mapsto \frac{1}{\lambda} \sum_{i=1}^{n} (f(x_i) + f(x_{-i})) \qquad \delta_i \mapsto \lambda \delta_i \otimes \delta_i, \qquad \delta_i \mapsto \delta_{-i}, \qquad 1 \mapsto 0$$
$$\delta_{-i} \mapsto \lambda \delta_{-i} \otimes \delta_{-i}, \qquad \delta_{-i} \mapsto \delta_i,$$

for all $1 \leq i \leq n$ and for all $f, g : \{x_1, x_{-1}, \ldots, x_n, x_{-n}\} \to \mathbf{k}$.

It is easy to check that (A, u_A, m_A) is an algebra and $(A, \varepsilon_A, \Delta_A)$ is a coalgebra in $\mathcal{C}$. We compute

$$(m_A \otimes \mathrm{Id}_A)(\mathrm{Id}_A \otimes \Delta_A)(\delta_i \otimes \delta_i) = \lambda(m_A \otimes \mathrm{Id}_A)(\delta_i \otimes \delta_i \otimes \delta_i) = \lambda \delta_i \otimes \delta_i = \Delta_A m_A(\delta_i \otimes \delta_i), \text{ and}$$
$$(m_A \otimes \mathrm{Id}_A)(\mathrm{Id}_A \otimes \Delta_A)(\delta_i \otimes \delta_j) = \lambda(m_A \otimes \mathrm{Id}_A)(\delta_i \otimes \delta_j \otimes \delta_j) = 0 = \Delta_A m_A(\delta_i \otimes \delta_i),$$

for $i \neq j \in \{1, -1, \ldots, n, -n\}$. Thus $(A, u_A, m_A, \varepsilon_A, \Delta_A)$ is a Frobenius algebra in $\mathcal{C}$, which is clearly commutative. On the other hand, $\phi(\theta_A) = \theta_A$ trivially, and

$$\begin{aligned} m_A(\phi_A \otimes \mathrm{Id})(\Delta_A(1)) &= \lambda m_A(\phi_A \otimes \mathrm{Id}) \left(\sum_{i=1}^{n} (\delta_i \otimes \delta_i) + (\delta_{-i} \otimes \delta_{-i}) \right) \\ &= \lambda m_A \left(\sum_{i=1}^{n} (\delta_{-i} \otimes \delta_i) + (\delta_i \otimes \delta_{-i}) \right) \\ &= 0, \text{ and} \\ m_A(\theta_A \otimes \theta_A) &= m_A(0) = 0. \end{aligned}$$

Hence all the conditions for an extended Frobenius algebra are satisfied, see Definition 2.4.4.

Proposition 2.4.8. *Let $\mathcal{C}$ be a symmetric monoidal category. Isomorphism classes of unoriented 2–dimensional $\mathcal{C}$-valued TQFTs are in bijective correspondence with isomorphism classes of extended Frobenius algebras in $\mathcal{C}$.*

A proof for the Proposition above is given in [41, Proposition 2.9], for the case where $\mathcal{C}$ is the category of modules over a commutative ring R. Their proof works verbatim for the general case.

Explicitly, an extended Frobenius algebra A in a symmetric monoidal category $\mathcal{C}$ gives rise to a symmetric monoidal functor $F_A : \mathrm{UCob}_2 \to \mathcal{C}$ by mapping the circle object in UCob_2 to A in $\mathcal{C}$.

2.4.3 The category $\mathbf{Rep}(S_t \wr G)$

For this section, we assume $\mathbf{k}$ has characteristic zero. We also assume that whenever $\mathcal{C}$ is a $\mathbf{k}$-linear monoidal category, the tensor bifunctor $\otimes$ is $\mathbf{k}$-bilinear.

Fix a finite group G and let $\mathbf{k}[G]$ be the regular representation of G. The symmetric $\mathbf{k}$-linear monoidal category $\mathbf{Rep}(S_t \wr G)$, introduced by Knop in [29], interpolates the category

$\mathbf{Rep}(S_n \wr G)$ of representations of the wreath product $S_n \wr G$ from $n \in \mathbb{N}$ to $t \in \mathbf{k}$. In [34], Mori introduced a 2-functor which sends a $\mathbf{k}$-linear monoidal category $\mathcal{C}$ to a new $\mathbf{k}$-linear monoidal category $\mathrm{S_t}(\mathcal{C})$. When $\mathcal{C} = \mathbf{Rep}(G)$, we have that $S_t(\mathcal{C}) = \mathbf{Rep}(S_t \wr G)$ as defined by Knop, see [34, Remark 4.14].

We will give a brief summary of the construction and graphical description of $S_t(\mathcal{C})$ as presented in [34]. We will use this description later on to prove Theorems I and II.

Definition 2.4.9. Let $I_1, \ldots, I_l$ be finite sets. A *recollement* of $I_1, \ldots, I_l$ is an equivalence relation r on $I_1 \sqcup \ldots \sqcup I_l$ such that for any $k = 1, \ldots, l$ and $i, i' \in I_k$, if $i \sim_r i'$ then $i = i'$.

We denote by $R(I_1, \ldots, I_l)$ the set of recollements of $I_1, \ldots, I_l$. For example, any element in $R(I, J)$ is of the form

$$r = \{\{i, j\}, \ldots, \{i'\}, \ldots, \{j'\}, \ldots\}$$

where $i, i' \in I$ and $j, j' \in J$.

For $\{a_1, \ldots, a_p\} \subseteq \{1, \ldots, l\}$, let $\pi_{a_1 \ldots, a_p} : R(I_1, \ldots, I_l) \to R(I_{a_1}, \ldots, I_{a_p})$ be the map that restricts the equivalence relation $R(I_1, \ldots, I_l)$ to $I_{a_1} \sqcup \cdots \sqcup I_{a_p} \subset I_1 \sqcup \cdots \sqcup I_l$. Given finite sets I, J, K and $r \in R(I, J), s \in R(J, K)$, let

$$R(s \circ r) = \{u \in R(I, J, K) \mid \pi_{1,2}(u) = r, \pi_{2,3}(u) = s\}.$$

Definition 2.4.10. [34, Definition 2.13] Let $\mathcal{C}$ be a $\mathbf{k}$-linear category, and let $t \in \mathbf{k}$. Then $\mathrm{S_t}(\mathcal{C})$ is the pseudo-abelian envelope of the category defined as follows:

♢ **Objects:** Finite families $U_I = (U_i)_{i \in I}$ of objects in $\mathcal{C}$. We denote them by $\langle U_I \rangle_t$.

♢ **Morphisms:** For objects $\langle U_I \rangle_t$ and $\langle V_J \rangle_t$,

$$\mathrm{Hom}_{\mathrm{S_t}(\mathcal{C})}(\langle U_I \rangle_t, \langle V_J \rangle_t) \cong \bigoplus_{r \in R(I,J)} \left(\bigotimes_{(i,j) \in r} \mathrm{Hom}_{\mathcal{C}}(U_i, V_j) \right), \tag{2.4.5}$$

where for each Φ on the right hand side, we denote by $\langle \Phi \rangle_t$ the corresponding morphism in $\mathrm{S_t}(\mathcal{C})$.

$\diamondsuit$ **Composition:** For $\Phi \in \bigotimes_{(i,j)\in r} \mathrm{Hom}_{\mathcal{C}}(U_i, V_j)$ and $\Psi \in \bigotimes_{(j,k)\in s} \mathrm{Hom}_{\mathcal{C}}(V_j, W_k)$, composition is given by

$$\langle \Psi \rangle_t \circ \langle \Phi \rangle_t := \sum_{u \in R(s\circ r)} P_u(t) \langle \Psi \circ_u \Phi \rangle_t,$$

where we denote by $\Psi \circ_u \Phi$ the element obtained by composing terms of $\Phi \otimes \Psi$ using

$$\mathrm{Hom}_{\mathcal{C}}(U_i, V_j) \otimes \mathrm{Hom}_{\mathcal{C}}(V_j, W_k) \to \mathrm{Hom}_{\mathcal{C}}(U_i, W_k),$$

for all $(i, j, k) \in u$, and $P_u(t)$ is the polynomial

$$P_u(t) = \prod_{\#\pi_{1,3}(u) \leq a < \#u} (t - a).$$

The unit object $\mathbb{1}_{\mathrm{S_t}(\mathcal{C})}$ of $\mathrm{S_t}(\mathcal{C})$ is the object corresponding to the empty family.

Remark 2.4.11. In this language, Deligne's category $\mathbf{Rep}(S_t, \mathbb{C})$ as defined in [13] is equivalent to $S_t(\mathrm{Vec})$, see [**?**, Remark 4.14].

Having a monoidal structure in $\mathcal{C}$ induces a monoidal structure in $S_t(\mathcal{C})$, with tensor products defined as follows.

Definition 2.4.12. [34, Definition 4.16] Let $\mathcal{C}$ be a $\mathbf{k}$-linear monoidal category. Define tensor products in $S_t(\mathcal{C})$ by:

- For families $U_I = (U_i)_{i\in I}$ and $V_J = (V_j)_{j\in J}$ in $\mathcal{C}$,

$$\langle U_I \rangle_t \otimes \langle V_J \rangle_t := \bigoplus_{r \in R(I,J)} \langle (U_i \otimes V_j)_{(i,j)\in r} \rangle_t.$$

- For families $U_I = (U_i)_{i\in I}, V_J = (V_j)_{j\in J}, W_K = (W_k)_{k\in K}$ and $X_L = (X_l)_{l\in L}$, and morphisms $\Phi \in \bigotimes_{(i,j)\in r} \mathrm{Hom}_{\mathcal{C}}(U_i, V_j)$ and $\Psi \in \bigotimes_{(k,l)\in s} \mathrm{Hom}_{\mathcal{C}}(W_k, X_l)$,

$$\langle \Phi \rangle_t \otimes \langle \Psi \rangle_t := \sum_{u\in R(r\otimes s)} \langle \Phi \otimes_u \Psi \rangle_t,$$

  where $\Phi \otimes_u \Psi$ is obtained by composing terms of $\Phi \otimes \Psi$ using tensor products

$$\mathrm{Hom}_{\mathcal{C}}(U_i, V_j) \otimes \mathrm{Hom}_{\mathcal{C}}(W_k, X_l) \to \mathrm{Hom}_{\mathcal{C}}(U_i \otimes W_k, V_j \otimes X_l),$$

  for all $(i, k, j, l) \in u$.

Graphical description of $S_t(\mathcal{C})$

When $\mathcal{C}$ is a braided **k**-linear monoidal category, it induces a natural braiding in $\mathcal{S}_t(\mathcal{C})$, see [34, Section 4.5]. In particular, when $\mathcal{C}$ is symmetric then so is $S_t(\mathcal{C})$.

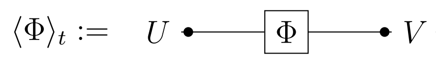
For $\mathcal{C}$ a braided **k**-linear monoidal category, [34] shows a useful graphical description of morphisms in $S_t(\mathcal{C})$. We will use it later on for $\mathcal{C} = \mathbf{Rep}(G)$, as it is easier to work with than the definitions above. We give now a quick summary of this graphical description. We note however that we make a minor change, as we represent morphisms from left to right, rather than from top to bottom.

$\diamondsuit$ Represent objects $\langle U_1 \rangle_t \otimes \cdots \otimes \langle U_l \rangle_t$ by labeled points placed vertically:

$$\begin{array}{cc} U_l & \bullet \\ & \vdots \\ U_2 & \bullet \\ U_1 & \bullet \end{array}$$

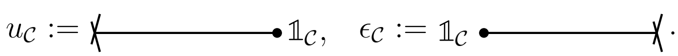
The unit object $\mathbb{1}_{\mathrm{S}_t(\mathcal{C})}$ is represented by "no points". Morphisms between objects of this form are represented by strings which connect points from left to right. Since objects of the form $\langle U_1\rangle_t \otimes \cdots \otimes \langle U_l\rangle_t$ generate $S_t(\mathcal{C})$ as a pseudo-abelian category, it is enough to consider morphisms between them [34, Corollary 4.18].

◇ For each morphism $\Phi : U \to V$ in $\mathcal{C}$, represent the corresponding morphism in $\mathrm{S_t}(\mathcal{C})$ by a string with a label:

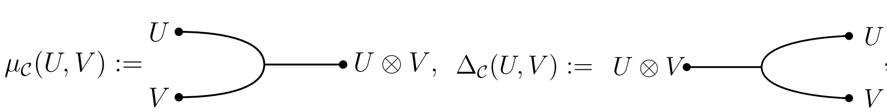


When $\Phi = \mathrm{id}$, we will sometimes omit the label.

◇ Take morphisms $u_{\mathcal{C}} \in \mathrm{Hom}_{S_t(\mathcal{C})}(\mathbb{1}_{\mathrm{S_t}(\mathcal{C})}, \langle \mathbb{1}_{\mathcal{C}}\rangle_t)$ and $\epsilon_{\mathcal{C}} \in \mathrm{Hom}_{S_t(\mathcal{C})}(\langle \mathbb{1}_{\mathcal{C}}\rangle_t, \mathbb{1}_{\mathrm{S_t}(\mathcal{C})})$ which correspond to $\mathrm{Id}_{\mathbb{1}_{\mathcal{C}}}$ via their respective isomorphisms to $\mathrm{Hom}_{\mathcal{C}}(\mathbb{1}_{\mathcal{C}}, \mathbb{1}_{\mathcal{C}})$, and represent them by broken strings:



◇ Since $\langle U \otimes V\rangle_t$ is a direct summand of $\langle U\rangle_t \otimes \langle V\rangle_t$ (see [34]), we have retraction $\mu_{\mathcal{C}}(U,V) : \langle U\rangle_t \otimes \langle V\rangle_t \to \langle U \otimes V\rangle_t$ and section $\Delta_{\mathcal{C}}(U,V) : \langle U \otimes V\rangle_t \to \langle U\rangle_t \otimes \langle V\rangle_t$ morphisms, which we represent by ramifications of strings:



for all $U, V \in \mathcal{C}$.

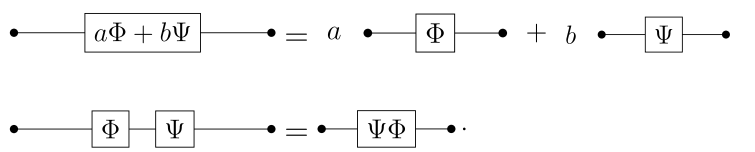
The tensor product of these maps is represented by stackings of diagrams, and composition by connecting them from left to right.

Proposition 2.4.13. *[34, Section 4.6] The morphisms $\langle \Phi\rangle_t$, $\mu_{\mathcal{C}}(U,V)$, $u_{\mathcal{C}}$, $\Delta_{\mathcal{C}}(U,V)$ and $\epsilon_{\mathcal{C}}$, for $U, V \in \mathcal{C}$ and arbitrary $\Phi : U \to V$, generate the category $\mathrm{S_t}(\mathcal{C})$ as a pseudo-abelian* *braided* **k***-linear monoidal category.*

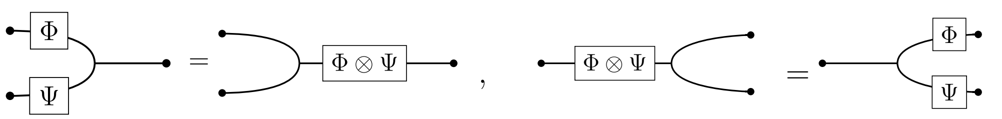
It is shown in [34, Section 4.6] that $S_t(\mathcal{C})$ is defined by generators and relations, with the generators given in the proposition above. We include below the relations, which we will use throughout this work.

<u>Relations:</u>

♢ $\langle\cdot\rangle_t : \mathcal{C} \to S_t(\mathcal{C})$ is **k**-linear and compatible with the composition in $\mathcal{C}$:

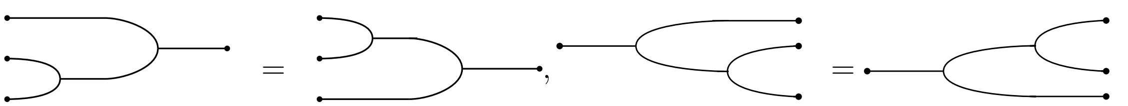


♢ $\mu_{\mathcal{C}} : \langle\cdot\rangle_t \otimes \langle\cdot\rangle_t \to \langle\cdot \otimes \cdot\rangle_t$ and $\Delta_{\mathcal{C}} : \langle\cdot \otimes \cdot\rangle_t \to \langle\cdot\rangle_t \otimes \langle\cdot\rangle_t$ are **k**-linear and compatible with the tensor product in $\mathcal{C}$:

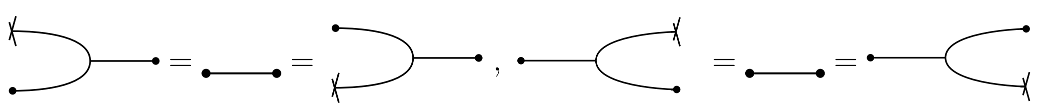


(2.4.6)

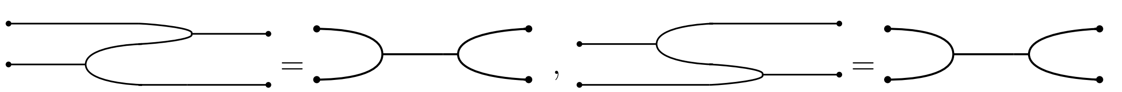
♢ Associativity and coassociativity:



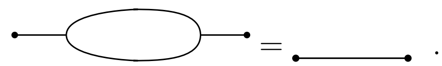
♢ Unitality and counitality:



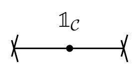
♢ Compatibility between $\mu_{\mathcal{C}}$ and $\Delta_{\mathcal{C}}$:



♢ $\mu_{\mathcal{C}}$ is a retraction and $\Delta_{\mathcal{C}}$ is a section:



♢ The object $\langle \mathbb{1}_{\mathcal{C}} \rangle_t$ is of dimension t:

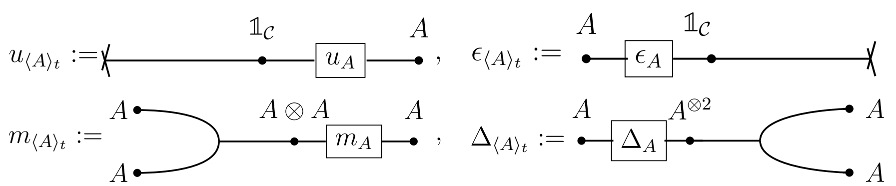


$= t \ \mathrm{id}_{\mathbb{1}}$.

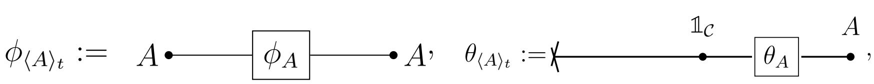
♢ Relations concerning the braiding (see relations (5) and (8) in [34]).

In [**?**, Section 4.6], it is discussed that $C \to \mathrm{S_t}(\mathcal{C})$ is a Frobenius functor [**?**, Definition 4.28]. The following proposition shows (graphically) how to get extended Frobenius algebras in $\mathrm{S_t}(\mathcal{C})$ from extended Frobenius algebras in $\mathcal{C}$.

Proposition 2.4.14. *1. A Frobenius algebra $A \in \mathcal{C}$ induces a Frobenius algebra $\langle A \rangle_t$ in $\mathrm{S_t}(\mathcal{C})$, with structure maps*

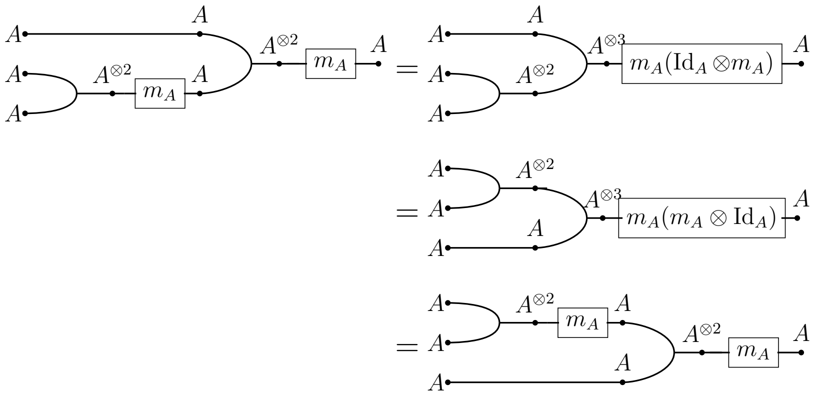


2. An extended Frobenius algebra $A \in \mathcal{C}$ induces an extended Frobenius algebra $\langle A \rangle_t$ in $\mathrm{S_t}(\mathcal{C})$, with multiplication, comultiplication, unit and counit maps as above, and



Proof. Let $A \in \mathcal{C}$ be a Frobenius algebra. We show first that $\langle A \rangle_t$ with the maps given in (1) is a Frobenius algebra in $\mathrm{S_t}(\mathcal{C})$ by graphical calculus.

Associativity:

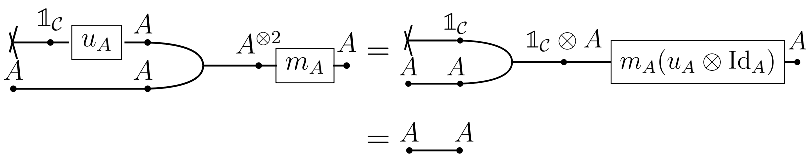


linearity of $\mu_{\mathcal{C}}$,

associativity of $\mu_{\mathcal{C}}$ and m_A,

linearity of $\mu_{\mathcal{C}}$.

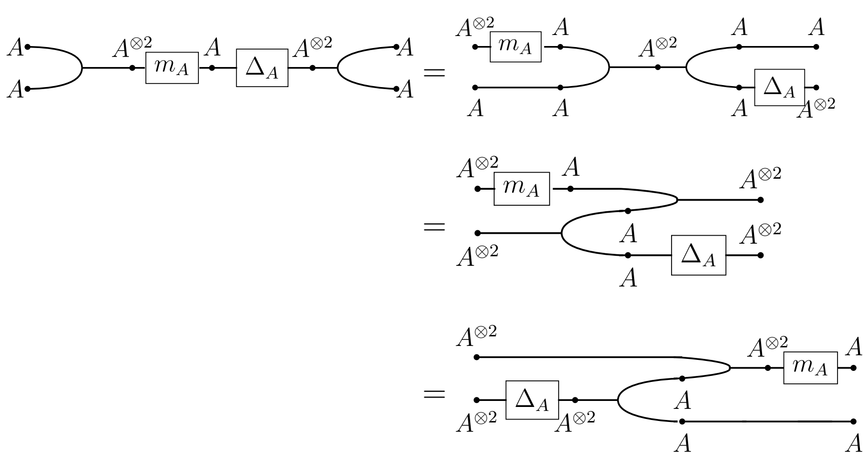
Unitality:



linearity of $\mu_{\mathcal{C}}$, (2.4.7)

unitality of $\mathrm{S_t}(\mathcal{C})$ and u_A. (2.4.8)

This shows that $(\langle A\rangle_t, u_{\langle A\rangle_t}, m_{\langle A\rangle_t})$ is an algebra in $\mathrm{S_t}(\mathcal{C})$. Similarly, coassociativity and counitality of $\mathrm{S_t}(\mathcal{C})$ make $(\langle A\rangle_t, \epsilon_{\langle A\rangle_t}, \Delta_{\langle A\rangle_t})$ a coalgebra. Moreover, by the equalities below



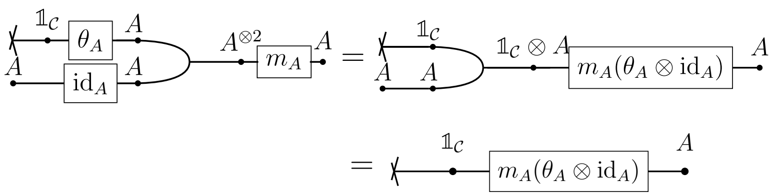
compatibility of $\mu_{\mathcal{C}}$ and $\Delta_{\mathcal{C}}$,

associativity of $\mu_{\mathcal{C}}$ and m_A,

compatibility of $\mu_{\mathcal{C}}$ and $\Delta_{\mathcal{C}}$.

multiplication and comultiplication satisfy diagram (2.4.2) and thus we have a Frobenius algebra structure in $\langle A\rangle_t$.

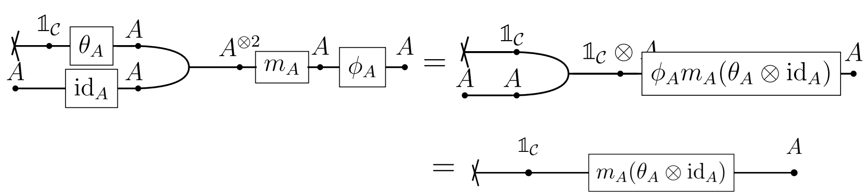
We show now that if $A \in \mathcal{C}$ is an extended Frobenius algebra, then $\langle A\rangle_t$ with maps as in (2) defines an extended Frobenius algebra in $\mathrm{S_t}(\mathcal{C})$. Commutativity follows from commutativity of m_A and $\mu_{\mathcal{C}}$. Since ϕ_A is an involution in $\mathcal{C}$, then $\phi_{\langle A\rangle_t}$ is an involution in $\mathrm{S_t}(\mathcal{C})$. We show below that the first diagram in Equation (2.4.4) commutes. In fact, we arrive at the same equality when computing



linearity of $\mu_{\mathcal{C}}$,

unitality of $\mu_{\mathcal{C}}$.

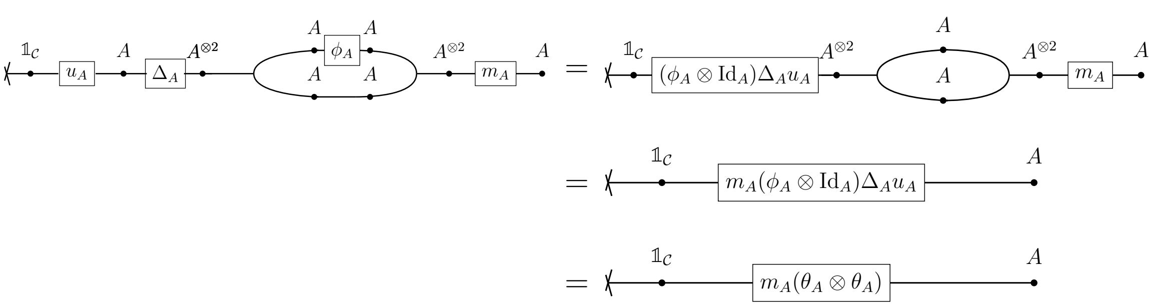
and



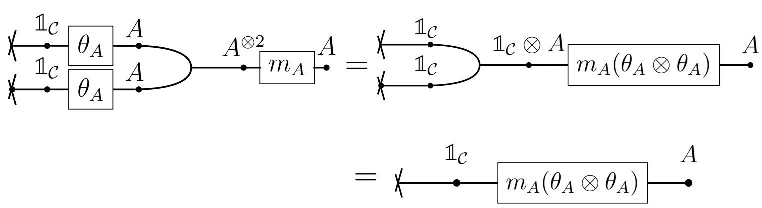
linearity of $\mu_{\mathcal{C}}$,

unitality of $\mu_{\mathcal{C}}$.

Lastly, the second diagram in Equation (2.4.4) also commutes since



is equal to



linearity of $\mu_{\mathcal{C}}$,

unitality of $\mu_{\mathcal{C}}$.

Hence $\langle A\rangle_t$ is an extended Frobenius algebra in $\mathrm{S_t}(\mathcal{C})$. $\square$

Chapter 3

Symmetric fusion categories in positive charactersitic

This chapter contains previously published material, which appeared in [11].

3.1 Bounds for the ranks of symmetric fusion categories

In this section we prove our two main results concerning the ranks of non-super-Tannakian symmetric fusion categories. Throughout this section, we assume $\mathrm{char}(\mathbf{k}) = p \geq 5$. Let $z = e^{2\pi i/p}$ be a primitive p-th root of unity. Recall that we denote by $\mathbb{Q}(z)$ the field extension generated by z over $\mathbb{Q}$.

Let $\mathcal{C}$ be a symmetric fusion category and consider its Verlinde fiber functor $F : \mathcal{C} \to \mathbf{Ver}_p$. Recall that by [37, Proposition 3.3] we have an equivalence of symmetric fusion categories

$$\mathbf{Ver}_p \cong \mathbf{Ver}_p^+ \boxtimes \mathbf{sVec}\,.$$

Consider the monoidal (non symmetric) forgetful functor Forget : $\mathbf{sVec} \to \mathbf{Vec}$. We have a (possibly non symmetric) tensor functor

$$\tilde{F} := (\mathrm{id} \boxtimes \mathrm{Forget}) \circ F : \mathcal{C} \to \mathbf{Ver}_p^+; \tag{3.1.1}$$

we denote also by $\tilde{F}$ the induced ring homomorphism $\mathcal{K}(\mathcal{C}) \to \mathcal{K}(\mathbf{Ver}_p^+)$, and the induced $\mathbb{Q}$-algebra homomorphism $\mathcal{K}(\mathcal{C})_{\mathbb{Q}} \to \mathcal{K}(\mathbf{Ver}_p^+)_{\mathbb{Q}}$. We are interested in studying the image of this map. By [4, Theorem 4.5 (iv)], we have an isomorphism of $\mathbb{Q}$-algebras

$$\mathcal{K}(\mathbf{Ver}_p^+)_{\mathbb{Q}} \cong \mathbb{Q}(z + z^{-1}),$$

and so $\tilde{F}(\mathcal{K}(\mathcal{C})_{\mathbb{Q}})$ is a subalgebra of $\mathbb{Q}(z+z^{-1})$. Since a subalgebra of a finite field extension is a field, then $\tilde{F}(\mathcal{K}(\mathcal{C})_{\mathbb{Q}})$ is a subfield of $\mathbb{Q}(z+z^{-1})$. Hence to study the image of $\mathcal{K}(\mathcal{C})_{\mathbb{Q}}$ under $\tilde{F}$, we start by looking at subfields of $\mathbb{Q}(z+z^{-1})$.

We make the usual identification of the Galois group of $\mathbb{Q}(z)$ with the multiplicative group $\mathbb{Z}_p^\times$. This is a cyclic group with $p-1$ elements, where j acts on $\mathbb{Q}(z)$ by $j \cdot z = z^j$ for all $j \in \mathbb{Z}_p^\times$. We denote the Galois group of the maximal real subextension $\mathbb{Q}(z+z^{-1})$ of $\mathbb{Q}(z)$ by $\mathcal{G}$, which corresponds to the quotient of $\mathbb{Z}_p^\times$ by the subgroup $\{\pm 1\}$. Thus $\mathcal{G}$ is a cyclic group of order $\frac{p-1}{2}$.

By Galois correspondence, subextensions of $\mathbb{Q}(z+z^{-1})$ are in bijection with subgroups of $\mathcal{G}$. That is, for every positive integer k that divides $\frac{p-1}{2}$ there exists a unique subextension A_k of $\mathbb{Q}(z+z^{-1})$ such that $[A_k : \mathbb{Q}] = k$, and its Galois group is exactly the quotient of $\mathcal{G}$ by the unique subgroup H_m of order m, where $mk = \frac{p-1}{2}$. Moreover, every subextension is of this form, and A_k is the set of elements fixed by every element in H_m.

Consider the basis $\{z^i + z^{-i}\}_{i=1}^{\frac{p-1}{2}}$ of $\mathbb{Q}(z + z^{-1})$. Then the group $\mathcal{G}$ (and thus also all subgroups H_m) acts on this set freely and transitively by permutation,

$$a \cdot (z^j + z^{-j}) = z^{aj} + z^{-aj},$$

for all $a \in \mathcal{G}$. So the orbits of the action of H_m on this set have exactly m elements. Let $\mathcal{O}_1, \dots, \mathcal{O}_k$ denote the orbits of the action of H_m, and define

$$x_i := \sum_{z^t + z^{-t} \in \mathcal{O}_i} (z^t + z^{-t}), \tag{3.1.2}$$

so that $\{x_1, \dots, x_k\}$ is a basis of A_k. Without loss of generality, we choose the labelling so that $z + z^{-1} = z^{p-1} + z^{-(p-1)} \in \mathcal{O}_k$,

Theorem 3.1.1. *Let $p \geq 5$. If $\mathcal{C}$ is a non-super-Tannakian symmetric fusion category, then*

$$\operatorname{rank}(\mathcal{C}) \geq \frac{p-1}{2}.$$

Proof. Consider the tensor functor $\tilde{F} : \mathcal{C} \to \mathbf{Ver}_p^+$ as defined in Equation (3.1.1). According to [4, Theorem 4.5 (iv)], under the isomorphism

$$\mathcal{K}(\mathbf{Ver}_p^+)_{\mathbb{Q}} \cong \mathbb{Q}(z + z^{-1}),$$

we have identifications

$$L_{2j+1} = \sum_{l=1}^{j} (z^{2l} + z^{-2l}) + 1, \quad \text{for } j = 0, \dots, (p-3)/2. \tag{3.1.3}$$

Recall that $\tilde{F}(\mathcal{K}(\mathcal{C})_{\mathbb{Q}})$ is a subfield of $\mathcal{K}(\mathbf{Ver}_p^+)_{\mathbb{Q}} \cong \mathbb{Q}(z + z^{-1})$. Now,

$$\operatorname{rank}(\mathcal{K}(\mathcal{C})) \geq \operatorname{rank}(\tilde{F}(\mathcal{K}(\mathcal{C}))) = \dim(\tilde{F}(\mathcal{K}(\mathcal{C})_{\mathbb{Q}})),$$

where $\operatorname{rank}(\mathcal{K}(\mathcal{C}))$ refers to the rank of $\mathcal{K}(\mathcal{C})$ as a free abelian group. Thus we would like to show that $\dim(\tilde{F}(\mathcal{K}(\mathcal{C})_{\mathbb{Q}})) = \frac{p-1}{2}$, or in other words, that $\tilde{F}(\mathcal{K}(\mathcal{C})_{\mathbb{Q}}) = \mathbb{Q}(z+z^{-1})$.

By our discussion at the beginning of this section, we know that $\tilde{F}(\mathcal{K}(\mathcal{C})_{\mathbb{Q}})$ is of the form A_k for some k that divides $\frac{p-1}{2}$, where A_k is the unique subextension of order k. Hence to prove the statement it is enough to show that $k = \frac{p-1}{2}$. Note that $k > 1$, since if $k = 1$ the image of $\tilde{F}$ would be a multiple of the identity and we would have a symmetric tensor functor from $\mathcal{C}$ to **sVec**, which would mean that $\mathcal{C}$ is super-Tannakian.

For objects $X \in \mathcal{C}$, their images under $\tilde{F}$ are objects in $\mathbf{Ver}_p^+$, and thus can be written as $\mathbb{Z}_{\geq 0}$ linear combinations of L_i's with i odd. Then $\tilde{F}(\mathcal{K}(\mathcal{C}))$ has a basis of elements of this form, and thus so does A_k. That is,

$$A_k \text{ has a basis given by } \mathbb{Z}_{\geq 0} \text{ linear combinations of } L_i\text{'s for } i \text{ odd.} \tag{3.1.4}$$

For the sake of contradiction, assume that $k < \frac{p-1}{2}$. We already know that $k > 1$.

Using formula (3.1.3) we compute

$$z^t + z^{-t} \;=\; L_{t+1} - L_{t-1} \quad \text{for } t \text{ even, } 2 \leq t < p-1.$$

On the other hand,

$$L_{p-2} = \sum_{l=1}^{\frac{p-3}{2}} (z^{2l} + z^{-2l}) + 1 = -(z^{p-1} + z^{-(p-1)}),$$

since $\sum_{\substack{i=-(p-1)\\ i \text{ even}}}^{p-1} z^i = 0$. Let $\mathcal{O}_1, \ldots, \mathcal{O}_k$ and $x_1, \ldots, x_k$ be as in (3.1.2). Since we can pick t to be a positive even number for each summand $z^t + z^{-t}$ of x_i (if not, replace t by $-t$ or $t-p$), then we can identify each x_i with a sum of L_s's with multiplicity ± 1, as follows:

$$x_i = \sum_{\substack{z^t+z^{-t} \in \mathcal{O}_i\\ t \text{ even}\\ 2 \leq t < p-1}} (L_{t+1} - L_{t-1}), \text{ for } i \neq k, \quad \text{ and } \quad x_k = -L_{p-2} + \sum_{\substack{z^t+z^{-t} \in \mathcal{O}_k\\ t \text{ even}\\ 2 \leq t < p-1}} (L_{t+1} - L_{t-1}).$$

Let s odd, $1 < s \leq p-2$. Note that L_s appears with nonzero multiplicity in either two basis elements, with multiplicity 1 and -1, respectively, or in none (since it may cancel out with itself). On the other hand, L_1 appears in only one basis element (explicitly, the basis element x_j such that $z^2 + z^{-2} \in \mathcal{O}_j$), with multiplicity -1. We will say L_s is a “positive” summand of x_i if it has multiplicity 1 in x_i, and is a “negative” summand if it has multiplicity -1.

We claim that every x_i has at least one positive and one negative summand. In fact, this is clear for $1 \leq i < k$, since the number of positive summands in x_i is the same as the number of negative summands. Suppose for contradiction that we have $x_k = -L_s$ for some even s, $2 \leq s \leq p-2$. Our assumption $k < \frac{p-1}{2}$ assures that every orbit has at least two elements, so it is not possible to have $x_k = -L_{p-2}$. Since we are assuming x_k has only one negative summand, L_{p-2} must cancel out with a positive summand. This implies $z^3 + z^{-3} = z^{p-3} + z^{-(p-3)} \in \mathcal{O}_k$ as well, and so H_m, the unique subgroup of order m, contains the class $\bar{3}$ of the number $3 \in \mathbb{Z}_p^\times$, see discussion around Equation (3.1.2). Now, either $x_k = -L_{p-4}$, or $-L_{p-4}$ cancels out. In the latter case, we have that $z^5 + z^{-5} = z^{p-5} + z^{-(p-5)} \in \mathcal{O}_k$ and so $\bar{5} \in H_m$. Recursively, we get that $H_m = \{\bar{1}, \bar{3}, \bar{5}, \ldots, \bar{j}\}$ for some odd $3 \leq j \leq p-2$. We claim this contradicts that H_m is a proper subgroup. In fact, since H_m is a subgroup, it must contain the classes of $3l$ for all l odd, $1 \leq l \leq j$. Let $l \in H_m$ such that $3l \leq j < 3(l+2)$. Note that $l+2$ is also in H_m (if not, then $3l \leq j < l+2$, and so $l < 1$, which is not possible). So $3(l+2)$ must also be in H_m. But since $j < 3(l+2)$, it must be the case that $3(l+2) > p$ and $\overline{3(l+2)} = \bar{n}$ for some odd $1 \leq n < j$. So we have the inequalities

$$3l \leq j < p < 3(l+2),$$

which imply $p = 3l + 2$ or $p = 3l + 4$. If $p = 3l + 2$, since H_m contains the classes of all odd elements from 1 to $3l = p - 2$ we get that $|H_m| = \frac{p-1}{2}$, a contradiction. If $p = 3l + 4$, then H_m contains all odd elements from 1 to $3l = p - 4$ (its missing at most one element) and thus again H_m must have all odd elements, a contradiction. Hence x_k has at least one positive and one negative summand.

Our aim is to construct sequences of indexes, alternating between negative and positive summands of different x_i's. We have shown every basis element has at least one positive and one negative summand. With this in mind, we begin the construction of our sequences.

Fix $s_0 \neq 1$ so that L_{s_0} is a positive summand of some x_{j_0}. Since $k > 1$, then there exists $j_1 \neq j_0$ such that L_{s_0} is a negative summand of x_{j_1}. By our preceding discussion, there must exist a positive summand of x_{j_1}. So we can find $s_1 \neq 1$ (since L_1 can only be a negative summand) such that L_{s_1} is a positive summand of x_{j_1}. Thus L_{s_1} is a negative summand of x_{j_2} for some $j_2 \neq j_1$. Again, there exists some $s_2 \neq 1$ such that L_{s_2} is a positive summand of x_{j_2}. Recursively, we can construct sequences of indexes $\{s_t\}$ and $\{j_t\}$ such that $s_t \neq 1$ and $j_t \neq j_{t+1}$ for all t, and L_{s_t} is a positive summand of x_{j_t} and a negative summand of $x_{j_{t+1}}$. Since there are only finitely many $\{x_i\}$, the indexes j_t must repeat at some point. Without loss of generality, assume j_1 is the first one that repeats, so our sequence is $\{j_1, j_2, j_3, \ldots, j_n, j_1, \ldots\}$, for some $n \geq 2$.

Let $y := a_1x_1 + \cdots + a_kx_k$ be an element in A_k that can be written as a positive linear combination of L_t's. We show that y is in the subspace generated by $\{x_i\}_{i \neq j_1, \ldots j_n}$ and $x_{j_1} + \cdots + x_{j_n}$. Since L_{s_1} has multiplicity $a_{j_1} - a_{j_2}$ in y, it must happen that $a_{j_1} \geq a_{j_2}$. Now, L_{s_2} has multiplicity $a_{j_2} - a_{j_3}$ in y, which implies $a_{j_2} \geq a_{j_3}$. Then we can obtain a sequence

$$a_{j_1} \geq a_{j_2} \geq \cdots \geq a_{j_n} \geq a_{j_1},$$

which implies $a_{j_1} = a_{j_2} = \cdots = a_{j_n}$, as desired.

Consequently, elements that can be written as a positive linear combination of L_i's are contained in a subspace of dimension less than k, which contradicts our statement (3.1.4). Hence $k = \frac{p-1}{2}$ and so

$$\operatorname{rank}(\mathcal{K}(\mathcal{C})) \geq \operatorname{rank}(\tilde{F}(\mathcal{K}(\mathcal{C}))) = \dim(\tilde{F}(\mathcal{K}(\mathcal{C})_{\mathbb{Q}})) = \dim(A_k) = \frac{p-1}{2},$$

which finishes the proof. □

Let $\mathcal{C}$ now be a symmetric fusion category with Verlinde fiber functor $F : \mathcal{C} \to \mathbf{Ver}_p$, and suppose F is surjective. We denote also by F the induced $\mathbb{Q}$-algebra homomorphism $\mathcal{K}(\mathcal{C})_{\mathbb{Q}} \to \mathcal{K}(\mathbf{Ver}_p)_{\mathbb{Q}}$. Then $F(\mathcal{K}(\mathcal{C})_{\mathbb{Q}})$ is a subalgebra of $\mathcal{K}(\mathbf{Ver}_p)_{\mathbb{Q}}$, and we will show that it is exactly $\mathcal{K}(\mathbf{Ver}_p)_{\mathbb{Q}}$.

Remark 3.1.2. Consider the $\mathbb{Q}$-algebra $\mathbb{Q}[\mathbb{Z}_2] \cong \mathbb{Q}(\epsilon)/(\epsilon^2 - 1)$. Then we have an isomorphism of $\mathbb{Q}$-algebras

$$\begin{gathered} \mathbb{Q}(z+z^{-1}) \otimes \mathbb{Q}[\mathbb{Z}_2] \xrightarrow{\sim} \mathbb{Q}(z+z^{-1}) \oplus \mathbb{Q}(z+z^{-1}) \\ w \otimes (a+b\epsilon) \mapsto ((a+b)w, (a-b)w), \end{gathered} \tag{3.1.5}$$

for all $w \in \mathbb{Q}(z+z^{-1})$ and $a, b \in \mathbb{Q}$. Recall that by [37, Proposition 3.3] we have an equivalence of symmetric fusion categories

$$\mathbf{Ver}_p \cong \mathbf{Ver}_p^+ \boxtimes \mathbf{sVec}\,.$$

Hence (3.1.5) induces an isomorphism of $\mathbb{Q}$-algebras

$$\mathcal{K}(\mathbf{Ver}_p)_{\mathbb{Q}} \cong \mathcal{K}(\mathbf{Ver}_p^+)_{\mathbb{Q}} \otimes \mathcal{K}(\mathbf{sVec})_{\mathbb{Q}} \cong \mathbb{Q}(z+z^{-1}) \otimes \mathbb{Q}[\mathbb{Z}_2] \xrightarrow{\sim} \mathbb{Q}(z+z^{-1})^{\oplus 2}, \tag{3.1.6}$$

where the second isomorphism is given in [4, Theorem 4.5 (iv)].

By (3.1.6), we can identify $F(\mathcal{K}(\mathcal{C})_{\mathbb{Q}})$ with a $\mathbb{Q}$-subalgebra of $\mathbb{Q}(z+z^{-1})^{\oplus 2}$. Hence we start by looking at subalgebras of $\mathbb{Q}(z+z^{-1})^{\oplus 2}$. Recall we denote by A_k the unique subextension of $\mathbb{Q}(z+z^{-1})$ such that $[A_k : \mathbb{Q}] = k$, see discussion at the beginning of the section.

Lemma 3.1.3. *Subalgebras of $\mathbb{Q}(z+z^{-1})^{\oplus 2}$ of dimension greater than $\frac{p-1}{2}$ are of the form $\mathbb{Q}(z+z^{-1}) \oplus A_k$ or $A_k \oplus \mathbb{Q}(z+z^{-1})$, where k is a positive integer dividing $\frac{p-1}{2}$.*

Proof. Let $\mathcal{A}$ be a subalgebra of $\mathbb{Q}(z+z^{-1})^{\oplus 2}$. Suppose first that $\mathcal{A}$ has no nontrivial idempotents. Note that $\mathbb{Q}(z+z^{-1})^{\oplus 2}$ has no nilpotent elements and thus neither does $\mathcal{A}$. Hence $\mathcal{A}$ is semisimple and so by Artin-Wedderburn's theorem it is isomorphic to a finite product of field extensions over $\mathbb{Q}$. Since $\mathcal{A}$ has no idempotents, this implies that $\mathcal{A}$ is isomorphic to a field extension over $\mathbb{Q}$.

Consider the projection map p from $\mathbb{Q}(z+z^{-1})^{\oplus 2}$ to its first summand, and let q denote its restriction to $\mathcal{A}$. Then $\ker(q) = 0$ or $\mathcal{A}$. If $\ker(q) = \mathcal{A}$ then elements in $\mathcal{A}$ are of the form $(0, a)$, which is only possible for $a = 0$ since $\mathcal{A}$ is a field. Hence if $\mathcal{A} \neq 0$ we must have $\ker(q) = 0$, that is, we have an injective map $A \hookrightarrow \mathbb{Q}(z+z^{-1})$, and so $\dim(\mathcal{A}) \leq \frac{p-1}{2}$.

Suppose now that $\mathcal{A}$ contains a nontrivial idempotent e. Then e is either $(1,0)$ or $(0,1)$, and we have an isomorphism of $\mathbb{Q}$-algebras

$$\mathcal{A} \cong e \cdot \mathcal{A} \oplus (1-e) \cdot \mathcal{A}.$$

Hence $\mathcal{A}$ is a direct sum of $A_k \oplus A_l$ of subalgebras of $\mathbb{Q}(z+z^{-1})$, where $k, l \in \mathbb{Z}_{\geq 0}$ divide $\frac{p-1}{2}$. Lastly, note that if both $k, l < \frac{p-1}{2}$, then

$$\frac{p-1}{2} < \dim(A_k \oplus A_l) = k + l \leq \frac{p-1}{4} + \frac{p-1}{4} = \frac{p-1}{2},$$

a contradiction. Hence we must have either $k = \frac{p-1}{2}$ or $l = \frac{p-1}{2}$, and thus either $A_k = \mathbb{Q}(z+z^{-1})$ or $A_l = \mathbb{Q}(z+z^{-1})$, as desired. □

The proof of the following theorem follows analogous steps as the ones in the proof of Theorem 3.1.1.

Theorem 3.1.4. *Let $p \geq 5$ and let $\mathcal{C}$ be a symmetric fusion category with Verlinde fiber functor $F : \mathcal{C} \to \mathbf{Ver}_p$. If F is surjective, then*

$$\operatorname{rank}(\mathcal{C}) \geq p - 1.$$

Proof. Let $F : \mathcal{C} \to \mathbf{Ver}_p$ be as in the statement. By Equation (3.1.6), we have an isomorphism of $\mathbb{Q}$-algebras

$$\mathcal{K}(\mathbf{Ver}_p)_{\mathbb{Q}} \cong \mathcal{K}(\mathbf{Ver}_p^+)_{\mathbb{Q}} \otimes \mathcal{K}(\mathbf{sVec})_{\mathbb{Q}} \cong \mathbb{Q}(z+z^{-1}) \otimes \mathbb{Q}[\mathbb{Z}_2] \xrightarrow{\sim} \mathbb{Q}(z+z^{-1})^{\oplus 2}, \tag{3.1.7}$$

induced from the $\mathbb{Q}$-algebras isomorphism $\mathbb{Q}(z+z^{-1}) \otimes \mathbb{Q}[\mathbb{Z}_2] \xrightarrow{\sim} \mathbb{Q}(z+z^{-1})^{\oplus 2}$, given in Equation 3.1.5. Hence, under this isomorphism we have identifications

$$\begin{aligned} L_{t+1} - L_{t-1} &= (z^t + z^{-t}, z^t + z^{-t}), \qquad \text{for } t \text{ even, } 1 < t < p-1, \\ L_{t+1} - L_{t-1} &= (z^t + z^{-t}, -(z^t + z^{-t})), \quad \text{for } t \text{ odd, } 1 < t < p-1, \\ -L_{p-2} &= (z^{p-1} + z^{-(p-1)}, z^{p-1} + z^{-(p-1)}), \text{ and} \\ L_2 &= (z + z^{-1}, -(z + z^{-1})). \end{aligned} \tag{3.1.8}$$

Since $\mathcal{C}$ is not super-Tannakian, by the proof of Theorem 3.1.1 we know that the composition

$$\mathcal{K}(\mathcal{C}) \xrightarrow{F} \mathcal{K}(\mathbf{Ver}_p) \cong \mathcal{K}(\mathbf{Ver}_p^+) \boxtimes \mathcal{K}(\mathbf{sVec}) \xrightarrow{\mathrm{id} \boxtimes \mathrm{Forget}} \mathcal{K}(\mathbf{Ver}_p^+),$$

is surjective. Moreover, since we are assuming that $F : \mathcal{C} \to \mathbf{Ver}_p$ is surjective, $F(\mathcal{K}(\mathcal{C}))$ cannot be equal to $\mathcal{K}(\mathbf{Ver}_p^+)$, and so

$$\operatorname{rank}(F(\mathcal{K}(\mathcal{C}))) > \frac{p-1}{2}.$$

This together with Lemma 3.1.3 implies that $F(\mathcal{K}(\mathcal{C})_{\mathbb{Q}})$ is identified with a subalgebra of the form $\mathbb{Q}(z + z^{-1}) \oplus A_k$, for some k that divides $\frac{p-1}{2}$. Note that the rank of $F(\mathcal{K}(\mathcal{C}))$ is equal to the dimension of $F(\mathcal{K}(\mathcal{C})_{\mathbb{Q}})$, and so we want to show that $k = \frac{p-1}{2}$.

For objects $X \in \mathcal{C}$, their images under F are objects in $\mathbf{Ver}_p$, and thus can be written as $\mathbb{Z}_{\geq 0}$ linear combinations of L_t's. Then $F(\mathcal{K}(\mathcal{C})_{\mathbb{Q}}) = \mathbb{Q}(z + z^{-1}) \oplus A_k$ has a basis of elements of this form. That is,

$$\mathbb{Q}(z + z^{-1}) \oplus A_k \text{ has a basis given by } \mathbb{Z}_{\geq 0} \text{ linear combinations of } L_t\text{'s.} \tag{3.1.9}$$

For the sake of contradiction, assume that $k < \frac{p-1}{2}$. Since F is surjective, we already know that $k > 1$.

Let $\mathcal{O}_1, \ldots, \mathcal{O}_k$ and $x_1, \ldots, x_k$ be as in (3.1.2). Without loss of generality, we choose the labelling so that $z + z^{-1} = z^{p-1} + z^{-(p-1)} \in \mathcal{O}_1$. Then

$$\{(z^i + z^{-i}, 0)\}_{i \text{ odd}, \, 1 \leq i \leq p-1} \cup \{(0, x_i)\}_{i=1,\ldots,k},$$

is a basis for $\mathbb{Q}(z + z^{-1}) \oplus A_k$. We start by writing this basis as a linear combination of L_t's, following the identification (3.1.8). Let $1 < t < p-1$ even. Then we have

$$\begin{aligned} 2(z^t + z^{-t}, 0) &= (z^t + z^{-t}, z^t + z^{-t}) + (z^t + z^{-t}, -(z^t + z^{-t})) \\ &= (z^t + z^{-t}, z^t + z^{-t}) + (z^{p-t} + z^{-(p-t)}, -(z^{p-t} + z^{-(p-t)})) \\ &= L_{t+1} - L_{t-1} + L_{p-t+1} - L_{p-t-1}, \end{aligned} \tag{3.1.10}$$

where the last equality is due to (3.1.8), since t is even and $p-t$ is odd. Analogously,

$$2(z^{p-1} + z^{-(p-1)}, 0) = -L_{p-2} + L_2.$$

On the other hand,

$$\begin{aligned} 2(0, z^t + z^{-t}) &= (z^t + z^{-t}, z^t + z^{-t}) - (z^t + z^{-t}, -(z^t + z^{-t})) \\ &= (z^t + z^{-t}, z^t + z^{-t}) - (z^{p-t} + z^{-(p-t)}, -(z^{p-t} + z^{-(p-t)})) \\ &= L_{t+1} - L_{t-1} - L_{p-t+1} + L_{p-t-1}, \text{ and} \end{aligned}$$

$$2(0, z^{p-1} + z^{-(p-1)}) = -L_{p-2} - L_2.$$

Hence

$$\begin{aligned} 2(0, x_i) &= \sum_{\substack{z^t + z^{-t} \in \mathcal{O}_i \\ t \text{ even} \\ 2 \leq t < p-1}} 2\left(0, z^t + z^{-t}\right) \\ &= \sum_{\substack{z^t + z^{-t} \in \mathcal{O}_i \\ t \text{ even} \\ 2 \leq t < p-1}} (L_{t+1} - L_{t-1} - L_{p-t+1} + L_{p-t-1}), \quad \text{for } i \neq 1, \end{aligned} \tag{3.1.11}$$

and

$$2(0, x_1) = \sum_{\substack{z^t + z^{-t} \in \mathcal{O}_1 \\ t \text{ even} \\ 2 \leq t < p-1}} (L_{t+1} - L_{t-1} - L_{p-t+1} + L_{p-t-1}) + (-L_{p-2} - L_2).$$

Consider first the set $\{2(0, x_i)\}_{i=1,\ldots,k}$ of basis elements of A_k. Let $1 < s < p-1$. Note that L_s appears with nonzero multiplicity in either two of these elements, with multiplicity

1 and -1, respectively, or in none (since it may cancel out with itself). On the other hand, L_1 and L_{p-1} appear in only one basis element (explicitly, the basis element $2(0, x_j)$ such that $z^2 + z^{-2} \in \mathcal{O}_j$), both with multiplicity -1. We will say L_s is a "positive" summand of x_i if it has multiplicity 1 in x_i, and is a "negative" summand if it has multiplicity -1. We will also say that L_s is an "odd" summand if $1 \leq s \leq p-1$ is odd, and an "even" summand when s is even.

Our assumption $k < \frac{p-1}{2}$ assures that every orbit has at least two elements. We thus claim that every $2(0, x_i)$ has at least one odd positive summand and one odd negative summand. In fact, this is clear for $i \neq 1$ since the number of odd positive summands in $2(0, x_i)$ is the same as the number of odd negative summands. For $2(0, x_1)$, the argument is the same as the one given in the proof of Theorem 3.1.1.

Our aim is to construct sequences of indexes, alternating between negative odd and positive odd summands of different $2(0, x_i)$'s. We know every basis element has at least one positive and one negative odd summand. With this in mind, we begin the construction of our sequences.

Fix $1 < s_0 < p-1$ so that L_{s_0} is an odd positive summand of some $(0, x_{j_0})$. Since $k > 1$, then there exists $j_1 \neq j_0$ such that L_{s_0} is an odd negative summand of $(0, x_{j_1})$. By our preceding discussion, there must exist an odd positive summand L_{s_1} of x_{j_1}, $s_1 \neq 1$ (L_1 can only be a negative summand). Thus L_{s_1} must be an odd negative summand of some $(0, x_{j_2})$, with $j_2 \neq j_1$. Again, there exists some L_{s_2} odd positive summand of $(0, x_{j_2})$, $s_2 \neq 1$. Recursively, we can construct sequences of indexes $\{s_t\}$ and $\{j_t\}$ such that $1 < s_t < p-1$ is odd, $j_t \neq j_{t+1}$ for all t, and L_{s_t} is an odd positive summand of x_{j_t} and an odd negative summand of $x_{j_{t+1}}$. Since there are only finitely many $\{(0, x_i)\}$, the indexes j_t must repeat at some point. Without loss of generality, assume j_1 is the first one that repeats, so our sequence is $\{j_1, j_2, j_3, \ldots, j_n, j_{n+1} = j_1, \ldots\}$, for some $n \geq 2$.

We now use our sequences of indexes to show that elements of $\mathbb{Q}(z + z^{-1}) \oplus A_k$ that can be written as a positive linear combination of L_t's are contained in a subspace of dimension strictly less than $\dim(\mathbb{Q}(z + z^{-1}) \oplus A_k) = \frac{p-1}{2} + k$.

Consider now the basis

$$\{2(z^i + z^{-i}, 0)\}_{i \text{ odd}, \, 1 \leq i \leq p-1} \cup \{2(0, x_i)\}_{i=1,\ldots,k},$$

of $\mathbb{Q}(z + z^{-1}) \oplus A_k$. Let

$$y := \left(\sum_{i \text{ odd}} a_i 2(z^i + z^{-i}), \sum_{j=1}^{k} b_j 2x_j \right) \in \mathbb{Q}(z + z^{-1}) \oplus A_k, \tag{3.1.12}$$

so that y that can be written as a positive linear combination of L_t's under the identification 3.1.8. We show that y is in the subspace generated by

$$\{2(z^i + z^{-i}, 0)\}_{i \text{ odd}, \, 1 \leq i \leq p-1} \cup \{2(0, x_i)\}_{i \neq j_1, \ldots, j_n} \cup \{2(0, x_{j_1} + \cdots + x_{j_n})\}. \tag{3.1.13}$$

We do this by computing the multiplicities of L_{s_t} and L_{p-s_t} in (3.1.12), for all $t = 1, \ldots, n$. Note that, if L_s is an odd positive (respectively, negative) summand of $2(0, x_i)$, then L_{p-s} is an even positive (respectively, negative) summand of $2(0, x_i)$, see Equation (3.1.11).

Recall that L_{s_1} is an odd positive summand of $2(0, x_{j_1})$, and an odd negative summand of $2(0, x_{j_2})$. Also, L_{s_1} is a positive summand of $2(z^{s_1 - 1} + z^{-(s_1 - 1)}, 0)$ and a negative summand of $2(z^{s_1+1} + z^{-(s_1+1)}, 0)$, see Equation (3.1.10). Hence the multiplicity of L_{s_1} in (3.1.12) under the identifications (3.1.10) and (3.1.11) is

$$b_{j_1} - b_{j_2} + a_{s_1 - 1} - a_{s_1 + 1} \geq 0. \tag{3.1.14}$$

On the other hand, L_{p-s_1} is an even positive summand of $2(0, x_{j_1})$, and an even negative summand of $2(0, x_{j_2})$. But L_{p-s_1} is a negative summand of $2(z^{s_1-1}+z^{-(s_1-1)}, 0)$ and a positive summand of $2(z^{s_1+1} + z^{-(s_1+1)}, 0)$, see Equation (3.1.10). Hence the multiplicity of L_{p-s_1} in (3.1.12) under the identifications (3.1.10) and (3.1.11) is

$$b_{j_1} - b_{j_2} - a_{s_1-1} + a_{s_1+1} \geq 0. \tag{3.1.15}$$

Now, equations (3.1.14) and (3.1.15) imply that

$$b_{j_1} \geq b_{j_2}.$$

Analogously, for $1 \leq i \leq n$ we have that L_{s_i} has multiplicity

$$b_{j_i} - b_{j_{i+1}} + a_{s_i-1} - a_{s_i+1} \geq 0,$$

in (3.1.12), and L_{p-s_i} has multiplicity

$$b_{j_i} - b_{j_{i+1}} - a_{s_i-1} + a_{s_i+1} \geq 0,$$

which implies

$$b_{j_i} \geq b_{j_{i+1}}.$$

Hence, since $j_{n+1} = j_1$, we have that

$$b_{j_1} \geq b_{j_2} \geq \cdots \geq b_{j_n} \geq b_{j_{n+1}} = b_1,$$

which implies $b_{j_1} = b_{j_2} = \cdots = b_{j_n}$, as desired.

Consequently, elements of $\mathbb{Q}(z+z^{-1}) \oplus A_k$ that can be written as a positive linear combination of L_t's are contained in the subspace (3.1.13), which has dimension strictly less than

$\dim(\mathbb{Q}(z+z^{-1}) \oplus A_k)$, since $n \geq 2$. This contradicts (3.1.9), and the contradiction came from assuming $k < \frac{p-1}{2}$. Thus we must have $k = \frac{p-1}{2}$, and so

$$F(\mathcal{K}(\mathcal{C})_{\mathbb{Q}}) \cong \mathbb{Q}(z+z^{-1}) \oplus A_k = \mathbb{Q}(z+z^{-1}) \oplus \mathbb{Q}(z+z^{-1}), \tag{3.1.16}$$

as $\mathbb{Q}$-algebras. Lastly,

$$\begin{aligned} \operatorname{rank}(\mathcal{K}(\mathcal{C})) \geq \operatorname{rank}(F(\mathcal{K}(\mathcal{C}))) &= \dim(F(\mathcal{K}(\mathcal{C})_{\mathbb{Q}})) \\ &= \dim(\mathbb{Q}(z+z^{-1}) \oplus \mathbb{Q}(z+z^{-1})) = p-1, \end{aligned} \tag{3.1.17}$$

which finishes the proof. □

Corollary 3.1.5. *Let $p \geq 5$, and let $\mathcal{C}$ be a symmetric fusion category that is not super Tannakian. Let $F : \mathcal{C} \to \mathbf{Ver}_p$ be the Verlinde fiber functor. Then*

$$F(\mathcal{K}(\mathcal{C})) = \mathcal{K}(\mathbf{Ver}_p) \quad \text{or} \quad F(\mathcal{K}(\mathcal{C})) = \mathcal{K}(\mathbf{Ver}_p^+).$$

In particular,

$$F(\mathcal{K}(\mathcal{C})_{\mathbb{Q}}) \cong \mathbb{Q}(z+z^{-1})^{\oplus 2} \qquad \text{or} \qquad F(\mathcal{K}(\mathcal{C})_{\mathbb{Q}}) \cong \mathbb{Q}(z+z^{-1}).$$

Proof. The image of the functor $F : \mathcal{C} \to \mathbf{Ver}_p$ is a fusion subcategory of $\mathbf{Ver}_p$, thus it can only be $\mathbf{Vec}, \mathbf{sVec}, \mathbf{Ver}_p^+$ or $\mathbf{Ver}_p$. The first two choices are not possible since we are assuming that $\mathcal{C}$ is not super Tannakian.

Suppose first that the image is $\mathbf{Ver}_p$. Then F is surjective, and so by the proof of Theorem 3.1.4 we have that $\operatorname{rank}(F(\mathcal{K}(\mathcal{C})) = p-1$, see Equation (3.1.17), which implies $F(\mathcal{K}(C)) = \mathcal{K}(\mathbf{Ver}_p)$. Also $F(\mathcal{K}(\mathcal{C})_{\mathbb{Q}}) \cong \mathbb{Q}(z+z^{-1})^{\oplus 2}$ by Equation (3.1.16).

Suppose now that the image of F is $\mathbf{Ver}_p^+$. Then the induced ring homomorphism $F : \mathcal{K}(\mathcal{C}) \to \mathcal{K}(\mathbf{Ver}_p)$ has image contained in $\mathcal{K}(\mathbf{Ver}_p^+)$, which implies

$$\operatorname{rank}(F(\mathcal{K}(\mathcal{C})) \leq \operatorname{rank}(\mathcal{K}(\mathbf{Ver}_p^+)) = \frac{p-1}{2}. \tag{3.1.18}$$

On the other hand, by the proof of Theorem 3.1.1, we know that the functor

$$\tilde{F} := (\mathrm{id} \boxtimes \mathrm{Forget}) \circ F : \mathcal{C} \to \mathbf{Ver}_p^+,$$

induces a surjective homomorphism of $\mathbb{Q}$-algebras

$$\tilde{F} : \mathcal{K}(\mathcal{C})_{\mathbb{Q}} \twoheadrightarrow \mathcal{K}(\mathbf{Ver}_p^+)_{\mathbb{Q}} \cong \mathbb{Q}(z + z^{-1}). \tag{3.1.19}$$

Then

$$\frac{p-1}{2} \geq \mathrm{rank}(F(\mathcal{K}(\mathcal{C}))) = \dim(F(\mathcal{K}(\mathcal{C})_{\mathbb{Q}})) \geq \dim(\tilde{F}(\mathcal{K}(\mathcal{C})_{\mathbb{Q}})) = \frac{p-1}{2},$$

where the first inequality is due to Equation (3.1.18) and the last to Equation (3.1.19). This implies that $\mathrm{rank}(F(\mathcal{K}(\mathcal{C}))) = \frac{p-1}{2}$, and thus $F(\mathcal{K}(\mathcal{C})) = \mathcal{K}(\mathbf{Ver}_p^+)$. In particular, this implies that $F(\mathcal{K}(\mathcal{C})_{\mathbb{Q}}) \cong \mathbb{Q}(z + z^{-1})$. □

3.2 Some properties of the Adams operation

Throughout this section, we assume $p > 2$.

3.2.1 Adams operation in $\mathbf{Ver}_p$

Recall that the Adams operation of a symmetric fusion category $\mathcal{C}$ is defined as the ring endomorphism $\psi_2 : \mathcal{K}(\mathcal{C}) \to \mathcal{K}(\mathcal{C})$ given by

$$\psi_2(X) = S^2(X) - \Lambda^2(X),$$

for all $X \in \mathcal{K}(\mathcal{C})$, see Section 2.3.2. We note that, for an object X in $\mathcal{C}$, $\psi_2(X)$ is in $\mathcal{K}(\mathcal{C})$, and thus $\psi_2(X)$ is not necessarily a linear combination (of simple objects) with non-negative coefficients.

In this section we study some properties of the Adams operation in $\mathbf{Ver}_p$. We first give an explicit formula for the second Adams operation on simple objects L_t, $1 \leq t \leq p-1$. We then use this formula to show that if an object X in $\mathbf{Ver}_p$ is fixed by the Adams operation, then X is in the abelian subcategory $\mathbf{Vec}$ generated by $\mathbf{1} = L_1$.

Remark 3.2.1. The image of $\psi_2 : \mathcal{K}(\mathbf{Ver}_p) \to \mathcal{K}(\mathbf{Ver}_p)$ is contained in $\mathcal{K}(\mathbf{Ver}_p^+)$. In fact,

$$L_t^2 = \sum_{s=1}^{\min(t,p-t)} L_{2s-1} = S^2(L_t) + \Lambda^2(L_t),$$

for all $i = 1, \ldots, p-1$, and so the multiplicity of even simples is zero in both $S^2(L_i)$ and $\Lambda^2(L_i)$.

Note that to compute ψ_2 on simple objects L_r of $\mathbf{Ver}_p$, it is enough to compute it for r odd, since

$$\psi_2(L_r) = -\psi_2(L_{p-r}).$$

This follows from

$$\Lambda^2 L_r = L_{p-1}^2 \otimes S^2 L_{p-r} = S^2 L_{p-r},$$

see [21, Proposition 2.4].

Example 3.2.2. In $\mathbf{Ver}_5$,

$$\psi_2(L_3) = L_1 - L_3 = -\psi_2(L_2).$$

In fact, we know that $\psi_2(L_3) = S^2(L_3) - \Lambda^2(L_3)$ and $L_3^2 = L_1 + L_3 = S^2(L_3) + \Lambda^2(L_3)$. Hence there must exist $\epsilon_1, \epsilon_3 \in \{\pm 1\}$ such that $\psi_2(L_3) = \epsilon_1 L_1 + \epsilon_3 L_3$. Now,

$$L_1 + 2\epsilon_1\epsilon_3 L_3 + L_3^2 = \psi_2(L_3)^2 = \psi_2(L_3^2) = (1+\epsilon_1)L_1 + \epsilon_3 L_3,$$

and so $2 = 1 + \epsilon_1$, which implies $\epsilon_1 = 1$. It follows that $\epsilon_3 = -1$.

Proposition 3.2.3. *The second Adams operation $\psi_2 : \mathcal{K}(\mathbf{Ver}_p) \to \mathcal{K}(\mathbf{Ver}_p)$ is given by*

$$\psi_2(L_t) = \sum_{s=1}^{\min(t,p-t)} (-1)^{s+1} L_{2s-1} \textit{ for } t \textit{ odd, } 1 \leq t \leq p-1, \textit{ and}$$
$$\psi_2(L_t) = \sum_{s=1}^{\min(t,p-t)} (-1)^{s} L_{2s-1} \textit{ for } t \textit{ even, } 1 \leq t \leq p-1.$$

Proof. Note that the second formula follows from the first by the equality $\psi_2(L_r) = -\psi_2(L_{p-r})$. We will use the isomorphism of $\mathbb{Q}$-algebras

$$\mathcal{K}(\mathbf{Ver}_p^+)_{\mathbb{Q}} \cong \mathbb{Q}(z + z^{-1}),$$

for z a primitive p-th root of unity, see [4, Theorem 4.5 (iv)] and Section 3.1. Consider the basis $\{z^{2i} + z^{-2i}\}_{i=1,\ldots,\frac{p-1}{2}}$ of $\mathbb{Q}(z + z^{-1})$. Via this isomorphism, we have identifications

$$L_{2j+1} = \sum_{l=1}^{j} (z^{2l} + z^{-2l}) + 1, \quad \text{for } j = 0, \ldots, (p-3)/2,$$

from which we compute

$$\begin{aligned} z^t + z^{-t} &= L_{t+1} - L_{t-1} \text{ for } t \text{ even, } 2 \leq t < p-1, \text{ and} \\ z^{p-1} + z^{-(p-1)} &= -L_{p-2}, \end{aligned} \tag{3.2.1}$$

see Section 3.1 and the proof of Theorem 3.1.1 for details.

Note that $(\psi_2)_{\mathbb{Q}} : \mathbb{Q}(z + z^{-1}) \to \mathbb{Q}(z + z^{-1})$ maps $z + z^{-1} = z^{p-1} + z^{-(p-1)}$ to $z^2 + z^{-2}$. In fact, we compute

$$\dim(S^2(L_{p-2})) = \frac{(p-2)(p-1)}{2} = 1 \mod p, \text{ and}$$
$$\dim(\Lambda^2(L_{p-2})) = \frac{(p-2)(p-3)}{2} = 3 \mod p.$$

Since $S^2(L_{p-2}) + \Lambda^2(L_{p-2}) = L_{p-2}^2 = L_1 + L_3$, and L_1 can appear in either $S^2(L_{p-2})$ or $\Lambda^2(L_{p-2})$ but not both, then it must be the case that $S^2(L_{p-2}) = L_1$ and $\Lambda^2(L_{p-2}) = L_3$.

Thus using identification (3.2.1) we get

$$\psi_2(z^{p-1}+z^{-(p-1)}) = -\psi_2(L_{p-2}) = -S^2(L_{p-2}) + \Lambda^2(L_{p-2}) = L_3 - L_1 = z^2 + z^{-2}, \quad (3.2.2)$$

as desired. In particular, this implies

$$\psi_2(z^m + z^{-m}) = z^{2m} + z^{-2m}, \quad \text{for all } 1 \leq m \leq p-1.$$

We prove now our formulas for $\psi_2(L_t)$, t odd, by induction. We do the case $1 \leq t \leq \frac{p-1}{2}$ first. We know $\psi_2(L_1) = L_1$, so the formula works for $t = 1$. Fix $1 < t \leq \frac{p-1}{2}$ odd, and suppose the formula is true for all odd $1 \leq r < t$. Then

$$\psi_2(z^{t-1} + z^{-(t-1)}) = z^{2(t-1)} + z^{-2(t-1)} = L_{2(t-1)+1} - L_{2(t-1)-1} = L_{2t-1} - L_{2t-3},$$

where in the second equality we are using the identification (3.2.1). Since $z^{t-1} + z^{-(t-1)} = L_t - L_{t-2}$, we compute using induction

$$\begin{aligned}
\psi_2(L_t) &= \psi_2(L_{t-2}) + L_{2t-1} - L_{2t-3} \\
&= \sum_{s=1}^{t-2} (-1)^{s+1} L_{2s-1} + L_{2t-1} - L_{2t-3} \\
&= \sum_{s=1}^{t} (-1)^{s+1} L_{2s-1} = \sum_{s=1}^{\min(t,p-t)} (-1)^{s+1} L_{2s-1},
\end{aligned}$$

and so the formula holds for t.

It remains to show that the formula holds for the case of odd $\frac{p-1}{2} < t < p-1$. We already computed $\psi_2(L_{p-2}) = L_1 - L_3$, so it works for $t = p-2$. Fix odd $\frac{p-1}{2} < t = p-l < p-1$ and assume the formula holds for all odd $t < r < p-1$. Since $z^{p-l+1} + z^{-(p-l+1)} = L_{p-(l-2)} - L_{p-l}$, we compute same as before

$$\begin{aligned}\psi_2(L_{p-l}) &= \psi_2(L_{p-(l-2)}) - (L_{2l-1} - L_{2l-3})\\ &= \sum_{s=1}^{l-2}(-1)^{s+1}L_{2s-1} - L_{2l-3} + L_{2l-1}\\ &= \sum_{s=1}^{l}(-1)^{s+1}L_{2s-1} = \sum_{s=1}^{\min(t,p-t)}(-1)^{s+1}L_{2s-1},\end{aligned}$$

as desired. □

We now study objects in $\mathbf{Ver}_p$ that are fixed by the second Adams operation. For a simple object X in a symmetric fusion category $\mathcal{C}$, we denote by $[Y : X]$ the multiplicity of X in Y for all $Y \in \mathcal{C}$.

Corollary 3.2.4. *An object $X \in \mathbf{Ver}_p$ is fixed by ψ_2 if and only if $X \in \mathbf{Vec}$.*

Proof. Let $X \in \mathbf{Ver}_p$ such that $\psi_2(X) = X$ and let $a_1, \dots, a_{p-1}$ be non-negative integers such that $X = \sum_{j=1}^{p-1} a_j L_j$. Since the image of ψ_2 is contained in $\mathcal{K}(\mathbf{Ver}_p^+)$ (see Remark 3.2.1) then $a_2 = a_4 = \cdots = a_{p-1} = 0$.

Using the formulas from Proposition 3.2.3 we compute

$$a_1 = [X : L_1] = [\psi_2(X) : L_1] = \sum_{j=1}^{\frac{p-1}{2}} a_{2j-1}.$$

Since a_i is non-negative for all i this implies $a_3 = a_5 = \cdots = a_{p-2} = 0$, as desired. □

Remark 3.2.5. The statement of Corollary 3.2.4 only works for actual objects in $\mathbf{Ver}_p$, that is, objects that can be written as $\mathbb{Z}_{\geq 0}$ linear combinations of $L_1, \dots, L_{p-1}$. There can exist objects in $\mathcal{K}(\mathbf{Ver}_p)$ that are fixed by the second Adams operation but are not multiples of

L_1. For example, consider $p = 17$ and $L_5 - L_7 + L_9 - L_{15} \in \mathcal{K}(\mathbf{Ver}_p)$. Then

$$\begin{aligned}\psi_2(L_5 - L_7 + L_9 - L_{15}) &= \psi_2(L_5) - \psi_2(L_7) + \psi_2(L_9) - \psi_2(L_{15}) \\ &= (L_1 - L_3 + L_5 - L_7 + L_9) - \\ &\quad - (L_1 - L_3 + L_5 - L_7 + L_9 - L_{11} + L_{13}) + \\ &\quad + (L_1 - L_3 + L_5 - L_7 + L_9 - L_{11} + L_{13} - L_{15}) - (L_1 - L_3) \\ &= L_5 - L_7 + L_9 - L_{15}\end{aligned}$$

and so $L_5 - L_7 + L_9 - L_{15}$ is fixed by ψ_2 but is not in **Vec**.

3.2.2 Powers of the Adams operation

Recall that in this section we assume $p > 2$. Here we classify symmetric fusion categories $\mathcal{C}$ such that $\psi_2^a = \psi_2^{a-1}$ for some $a \geq 1$. Namely, we show that such categories are super-Tannakian and thus classified by group data, see [15, 13, 16]. Moreover, we show that the case $a = 1$ is only possible for the trivial category. That is, we prove that if $\psi_2 = \mathrm{Id}$ in $\mathcal{K}(\mathcal{C})$ then $\mathcal{C} = \mathbf{Vec}$.

Theorem 3.2.6. *Let $p > 2$ and let $\mathcal{C}$ be a non-super-Tannakian symmetric fusion category. If the Adams operation $\psi_2 : \mathcal{K}(\mathcal{C}) \to \mathcal{K}(\mathcal{C})$ satisfies $\psi_2^a = \psi_2^b$ for some $a, b \in \mathbb{Z}_{\geq 0}$, then $2^a \equiv \pm 2^b \mod p$.*

Proof. Consider the fiber functor $F : \mathcal{C} \to \mathbf{Ver}_p$; we denote also by F the induced ring homomorphism $\mathcal{K}(\mathcal{C}) \to \mathcal{K}(\mathbf{Ver}_p)$. Suppose now that $a > 1$. Since F preserves the symmetric structure, we have that

$$\psi_2^a(F(X)) = F(\psi_2^a(X)) = F(\psi_2^b(X)) = \psi_2^b(F(X)), \text{ for all } X \in \mathcal{K}(\mathcal{C}).$$

That is, $\psi_2^a = \psi_2^b$ on the image of $\mathcal{K}(\mathcal{C})$ under F. Suppose $\mathcal{C}$ is not super Tannakian. In particular, this implies $p > 3$, since for $p = 3$ all symmetric fusion categories are super Tannakian. By Corollary 3.1.5, we know that $F(\mathcal{K}(\mathcal{C})_{\mathbb{Q}})$ is isomorphic as a $\mathbb{Q}$-algebra to $\mathbb{Q}(z+z^{-1})$ or $\mathbb{Q}(z+z^{-1})^{\oplus 2}$. Recall that $\psi_2(z+z^{-1}) = z^2+z^{-2}$, see (3.2.2). Then $(\psi_2^a)_{\mathbb{Q}} = (\psi_2^b)_{\mathbb{Q}}$ in $F(\mathcal{K}(\mathcal{C})_{\mathbb{Q}})$ would imply that $z^{2^a} + z^{-2^a} = z^{2^b} + z^{-2^b}$, and so $2^a \equiv \pm 2^b \mod p$. □

Corollary 3.2.7. *Let $p > 2$ and let $\mathcal{C}$ be a symmetric fusion category. If $\psi_2^a = \psi_2^{a-1}$ for some $a \in \mathbb{Z}_{\geq 1}$, then $\mathcal{C}$ is super-Tannakian.*

Proof. By Theorem 3.2.6, if $\mathcal{C}$ is not super Tannakian then $2^a \equiv \pm 2^{a-1} \mod p$, which implies $2 \equiv \pm 1 \mod p$, a contradiction. □

Remark 3.2.8. Let $p > 2$. If $\psi_2 : \mathcal{K}(C) \to \mathcal{K}(\mathcal{C})$ satisfies $\psi_2 = \mathrm{id}$, then $\mathcal{C}$ is actually Tannakian. In fact, since F preserves the symmetric structure, we have that

$$\psi_2(F(X)) = F(\psi_2(X)) = F(X),$$

for all $X \in \mathcal{C}$. Since $F(X) \in \mathbf{Ver}_p$ and ψ_2 fixes $F(X)$, then by Corollary 3.2.4 we have that $F(X) \in \mathrm{Vec}$ for all $X \in \mathcal{C}$, and so $\mathcal{C}$ is Tannakian, as desired.

However, we show next that $\psi_2 = \mathrm{id}$ is only possible for $\mathcal{C} = \mathbf{Vec}$.

Theorem 3.2.9. *Let $p \neq 2$. If $\mathcal{C}$ is a non-trivial symmetric fusion category then ψ_2 is not the identity.*

Proof. Let $p > 2$. Suppose ψ_2 is the identity in $\mathcal{C}$. In Remark 3.2.8 we showed that $\mathcal{C}$ is Tannakian, and thus equivalent to $\mathbf{Rep}_{\mathbf{k}}(G)$ for a finite group scheme G. A classification of finite group schemes G such that $\mathbf{Rep}_{\mathbf{k}}(G)$ is semisimple is given by Nagata's theorem

[16, IV, 3.6]; thus Remark 3.2.8 yields a classification of symmetric fusion categories such that $\psi_2 = \mathrm{Id}$. Namely, any such category is an equivariantization (see [17, Section 4]) of a pointed category such that the group of simples is an abelian p-group (see e.g. [22, 8.4]), by the action of a group H of order relatively prime to p. Suppose H is non-trivial and consider the subcategory $\mathbf{Rep}_{\mathbf{k}}(H)$ of $\mathcal{C}$.

The Adams operation acts on $\mathbf{Rep}_{\mathbf{k}}(H)$ by mapping a character $\chi(g)$ to $\chi(g^2)$ for all $g \in H$. Thus if $\psi_2 = \mathrm{Id}$ then

$$\chi(g^2) = \chi(g) \text{ for all } g \in H.$$

Hence for all g in H, g is conjugate to g^2. So if $|H|$ is even, there is an element $h \in H$ of order 2, which is conjugated to $h^2 = 1$, a contradiction.

Suppose then $|H|$ is odd. We have that for all g in H there exists some $h \in H$ such that $g^2 = hgh^{-1}$ and so

$$g = hgh^{-1}g^{-1}.$$

Thus g is in the commutator subgroup of H, and so $H \subseteq [H, H]$. This contradicts the Feit-Thompson theorem, which states that every finite group of odd order is solvable. So H must be trivial.

We thus have that $\mathcal{C}$ is a pointed category associated with an abelian p-group P. Hence ψ_2 maps $g \mapsto g^2$ for all $g \in P$ and so $g = g^2$ for all $g \in P$. Then P is trivial and $\mathcal{C}$ is equivalent to $\mathbf{Vec}$.

The result also holds in characteristic 0, since the Adams operation acts on $\mathbf{Rep}_{\mathbf{k}}(G)$ by mapping a character $\chi(g)$ to $\chi(g^2)$ for all $g \in G$. □

Remark 3.2.10. The hypothesis of $\mathcal{C}$ being finite is neccesary. Indeed, in [24] it is shown that in any characteristic there is a semisimple, but not finite, symmetric category, known as the Delannoy category, for which all Adam operations are the identity.

3.2.3 Symmetric fusion categories with two self-dual simple objects

In this subsection we prove some general properties of the Adams operation $\psi_2 : \mathcal{K}(\mathcal{C}) \to \mathcal{K}(\mathcal{C})$ for the case when $\mathcal{C}$ is a symmetric fusion category with exactly two self-dual simple objects. These results will be useful for the classification of symmetric fusion categories of ranks 3 and 4 in Sections 4.1 and 4.2. In particular, we show that if ψ_2 is an automorphism then it has even order, see Theorem 3.2.12.

Recall that throughout this section we assume $p > 2$.

Lemma 3.2.11. *Let $\mathcal{C}$ be a symmetric fusion category with exactly two self-dual simple objects $\mathbf{1}$ and Y. Then*

$$[\psi_2^{2k+1}(Y) : \mathbf{1}] \equiv 1 \mod 2$$

for all $k \geq 0$.

Proof. We proceed by induction on k. Let $\mathbf{1}, Y, X_1, X_1^*, \ldots, X_n, X_n^*$ denote the simple objects in $\mathcal{C}$. Since $\psi_2(Y) \equiv Y^2 \mod 2$ and $[Y^2 : \mathbf{1}] = 1$ then $[\psi_2(Y) : \mathbf{1}] \equiv 1 \mod 2$, which proves the base case.

Fix $k > 1$ and suppose that

$$[\psi_2^{2l+1}(Y) : \mathbf{1}] \equiv 1 \mod 2, \text{ for all } l < k.$$

We want to show this also holds for $l = k$. Since Y is self-dual then $[Y^2 : X_i] = [Y^2 : X_i^*]$ for all $i = 1, \dots, n$, and so

$$\psi_2(Y) \equiv Y^2 \equiv \mathbf{1} + \sum_{i=1}^{n} [Y^2 : X_i]\ (X_i + X_i^*) + [Y^2 : Y]\ Y \mod 2,$$

for all $i = 1, \dots, n$. Applying ψ_2^{2k} on both sides of the previous equation we get

$$\psi_2^{2k+1}(Y) \equiv \mathbf{1} + \sum_{i=1}^{n} [Y^2 : X_i] \left(\psi_2^{2k}(X_i) + \psi_2^{2k}(X_i^*) \right) + [Y^2 : Y] \psi_2^{2k}(Y) \mod 2. \tag{3.2.3}$$

Recall that ψ_2 commutes with duality. That is, $\psi_2(X_i)^* = \psi_2(X_i^*)$ and so

$$[\psi_2^l(X_i) : \mathbf{1}] = [\psi_2^l(X_i)^* : \mathbf{1}] \equiv [\psi_2^l(X_i^*) : \mathbf{1}] \mod 2, \tag{3.2.4}$$

for all $l \geq 1$. From Equations (3.2.3) and (3.2.4) we get

$$\begin{aligned} [\psi_2^{2k+1}(Y) : \mathbf{1}] &\equiv 1 + 2 \sum_{i=1}^{n} [Y^2 : X_i]\ [\psi_2^{2k}(X_i) : \mathbf{1}] + [Y^2 : Y]\ [\psi_2^{2k}(Y) : \mathbf{1}] \mod 2 \\ &\equiv 1 + [Y^2 : Y]\ [\psi_2^{2k}(Y) : \mathbf{1}] \mod 2. \end{aligned}$$

Analogously,

$$\begin{aligned} [\psi_2^{2k}(Y) : \mathbf{1}] &\equiv 1 + [Y^2 : Y][\psi_2^{2k-1}(Y) : \mathbf{1}] \mod 2 \\ &\equiv 1 + [Y^2 : Y] \mod 2, \end{aligned}$$

since we assumed $[\psi_2^{2k-1}(Y) : \mathbf{1}] \equiv 1 \mod 2$. Hence

$$\begin{aligned} [\psi_2^{2k+1}(Y) : \mathbf{1}] &\equiv 1 + [Y^2 : Y] \left(1 + [Y^2 : Y] \right) \mod 2 \\ &\equiv 1 + [Y^2 : Y] + [Y^2 : Y]^2 \mod 2 \\ &\equiv 1 \mod 2, \end{aligned}$$

as desired. □

The following is a direct application of Lemma 3.2.11.

Theorem 3.2.12. *Let $\mathcal{C}$ be a symmetric fusion category with exactly two self-dual simple objects. If ψ_2 is an automorphism of $\mathcal{K}(\mathcal{C})$ then it has even order.*

Proof. Let $k \geq 0$ such that $\psi_2^k = \mathrm{Id}$, and let $\mathbf{1}, Y$ denote the self-dual objects. By Theorem 3.2.11 the multiplicity of $\mathbf{1}$ in $\psi_2^k(Y) = Y$ is positive whenever k is odd, and thus k must be even. □

In the following proposition we restrict to the case when $\psi_2 : \mathcal{K}(\mathcal{C}) \to \mathcal{K}(\mathcal{C})$ has trivial image.

Proposition 3.2.13. *Let $\mathcal{C}$ be a symmetric fusion category with exactly two self-dual simple objects $\mathbf{1}$ and Y. If $Im(\psi_2) \cong \mathbb{Z}$ then $Y^2 = \mathbf{1}$ and $\psi_2(Y) = -\mathbf{1}$. Moreover, $[XX^* : Y] = 1$ and $[XY : Y] = 0$ for all non-self-dual simple X.*

Proof. Let X be a non-self-dual simple object. Since $\mathbf{1}$ is not a summand of X^2 then $\mathbf{1}$ has multiplicity zero in $\psi_2(X)$. On the other hand, the multiplicity of $\mathbf{1}$ in Y^2 is 1 and so its coefficient in $\psi_2(Y)$ is ± 1. Thus if ψ_2 has trivial image we get that $\psi_2(X) = 0$ for all non-self-dual simple X and $\psi_2(Y) = \epsilon$, where $\epsilon = \pm 1$. Hence

$$\mathbf{1} = \psi_2(Y)^2 = \psi_2(Y^2) = \sum_{\text{simple } Z} [Y^2 : Z]\ \psi_2(Z) = \big(1 + \epsilon \cdot [Y^2 : Y]\big) \cdot \mathbf{1},$$

which implies

$$[Y^2 : Y] = 0. \tag{3.2.5}$$

Similarly,

$$0 = \psi_2(XY) = \sum_{\text{simple } Z} [XY : Z]\psi_2(Z) = \epsilon \cdot [XY : Y] \cdot \mathbf{1}, \tag{3.2.6}$$

and thus

$$[XY : Y] = 0, \text{ for all non-self-dual simple } X.$$

From the fusion rule $[XY : Y] = [Y^2 : X]$ and Equations (3.2.5) and (3.2.6) we conclude $Y^2 = \mathbf{1}$. Lastly, note that

$$0 = \psi_2(XX^*) = \mathbf{1} + \epsilon \cdot [XX^* : Y] \cdot \mathbf{1},$$

and thus we must have $\epsilon = -1$ and $[XX^* : Y] = 1$. $\square$

Chapter 4

Classification results

This chapter contains previously published material, which appeared in [11].

4.1 Rank 3 symmetric fusion categories

In this section we classify symmetric fusion categories $\mathcal{C}$ of rank 3, making use of the properties of the Adams operation. Namely, we show that $\mathcal{C}$ is equivalent to one of the following:

- If $p = 2$, $\mathcal{C} \cong \mathbf{Rep}_{\mathbf{k}}(\mathbb{Z}_3)$.
- If $p = 3$, $\mathcal{C} \cong \mathbf{Vec}_{\mathbb{Z}_3}^{\mathbb{Z}_2}$ or $\mathcal{C} \cong \mathbf{Vec}_{\mathbb{Z}_3}$.
- If $p = 7$, $\mathcal{C} \cong \mathbf{Rep}_{\mathbf{k}}(S_3)$, $\mathcal{C} \cong \mathbf{Rep}_{\mathbf{k}}(\mathbb{Z}_3)$ or $\mathcal{C} \cong \mathbf{Ver}_7^+$.
- If $p = 5$ or $p > 7$, $\mathcal{C} \cong \mathbf{Rep}_{\mathbf{k}}(S_3)$ or $\mathcal{C} \cong \mathbf{Rep}_{\mathbf{k}}(\mathbb{Z}_3)$.

We note that if $\mathcal{C}$ is non-super-Tannakian, then by Theorem 3.1.1 we know that $p \leq 7$. Hence the only possibilities are $p = 5$ or 7, since in the cases $p = 2$ or $p = 3$ the category would be super-Tannakian.

We make use of the parametrization of self-dual based rings of rank 3 as given in [35]. Let k, l, m, n be non negative integers satisfying

$$k^2 + l^2 = kn + lm + 1, \tag{4.1.1}$$

and consider the ring $K(k, l, m, n)$ with basis $1, X, Y$ and multiplication rules

$$X^2 = 1 + mX + kY, \qquad Y^2 = 1 + lX + nY, \qquad XY = YX = kX + lY. \tag{4.1.2}$$

Note that we have a based ring isomorphism $K(k, l, m, n) \cong K(l, k, n, m)$ obtained by the interchange $X \leftrightarrow Y$. Hence we will assume $l \geq k$. By [35, Proposition 3.1] any unital based ring of rank 3 is isomorphic to either $K(k, l, m, n)$ or $K(\mathbb{Z}_3)$, where $\mathcal{K}(\mathbb{Z}_3)$ denotes the group algebra of the group $\mathbb{Z}/3\mathbb{Z}$, which has rank 3 with basis given by the group elements.

Recall that a fusion category is *integral* if the Frobenius Perron dimension of every simple object X is an integer. We have the following result.

Theorem 4.1.1. *There is an integral symmetric fusion category $\mathcal{C}$ with Grothendieck ring $K(k, l, m, n)$ with $l > k$ if and only if $(k, l, m, n) = (0, 1, 0, 1)$ and $p \geq 3$. Moreover, in such case*

1. *$\mathcal{C} \cong \mathbf{Rep}(S_3)$ if $p > 3$, or*

2. *$\mathcal{C} \cong \boldsymbol{Vec}_{\mathbb{Z}_3}^{\mathbb{Z}_2}$ if $p = 3$.*

Proof. Suppose $\mathcal{C}$ is an integral fusion category with Grothendick ring given by $K(k, l, m, n)$ with $l \geq k$. Taking Frobenius-Perron dimension on the multiplication rules for X^2 and XY (Equation (4.1.2)), we get that

$$\mathrm{FPdim}(X)^2 = 1 + m\,\mathrm{FPdim}(X) + k\,\mathrm{FPdim}(Y) \quad \text{and} \quad 1 = \frac{k}{\mathrm{FPdim}(Y)} + \frac{l}{\mathrm{FPdim}(X)},$$

respectively. From the first equality we deduce that $\mathrm{FPdim}(X)$ and $\mathrm{FPdim}(Y)$ are coprime. Hence the second equality is only possible if $k = 0$ and $l = \mathrm{FPdim}(X)$. So the multiplication rules are

$$X^2 = 1 + mX, \qquad Y^2 = 1 + \mathrm{FPdim}(X)X + nY, \qquad XY = YX = \mathrm{FPdim}(X)Y.$$

Taking Frobenius-Perron dimension on both sides of the equation for X^2 we get that $\mathrm{FPdim}(X)^2 - m\,\mathrm{FPdim}(X) - 1 = 0$, which implies

$$\mathrm{FPdim}(X) = \frac{m + \sqrt{m^2 + 4}}{2}.$$

Since $\mathrm{FPdim}(X)$ is an integer, we need for $m^2 + 4$ to be a square. It is easy to check this is only possible for $m = 0$, and so the multiplication rules are

$$X^2 = 1, \qquad Y^2 = 1 + X + nY, \qquad XY = YX = Y.$$

Taking Frobenius-Perron dimension on both sides of the equation for Y^2 we get that

$$\mathrm{FPdim}(Y) = \frac{n + \sqrt{n^2 + 8}}{2}.$$

We thus need $n^2 + 8$ to be a square, which is only possible for $n = 1$. Hence

$$X^2 = 1, \qquad Y^2 = 1 + X + Y, \qquad XY = YX = Y. \tag{4.1.3}$$

So far we have showed that if $\mathcal{C}$ is integral, then it must have fusion rules as above. We have not used the assumption that $\mathcal{C}$ is symmetric yet. We will use it in what follows to complete the proof.

We look at the case $p > 3$ first. From Equations (4.1.3) we get that $\dim(X) = 1$ and $\dim(Y) = 1$ or 2, and thus $\dim(\mathcal{C}) = 3$ or 6. Since $p > 3$, then $\dim(\mathcal{C}) \neq 0$ and thus $\mathcal{C}$

is non-degenerate. Hence we can lift $\mathcal{C}$ to a symmetric fusion category $\widetilde{\mathcal{C}}$ over a field $\mathbf{f}$ in characteristic zero, which has the same Grothendick ring as $\mathcal{C}$, see [22, Subsection 9.16] and [18]. Thus $\widetilde{\mathcal{C}}$ is equivalent to $\mathbf{Rep_f}(S_3)$ (see [13, Section 8.19]), and so by uniqueness of the lifting we get that $\mathcal{C}$ is equivalent to $\mathbf{Rep_k}(S_3)$.

For the case $p = 3$ we have that $\mathcal{C}$ contains a copy of $\mathbf{Rep}(\mathbb{Z}_2)$. Doing de-equivariantization by $\mathbb{Z}_2$ we obtain a symmetric fusion category $\mathcal{C}^{\mathbb{Z}_2}$ of dimension 3 [22, Section 2.7]. Hence $\mathcal{C}^{\mathbb{Z}_2}$ is a symmetric pointed category and must be equivalent to $\mathbf{Vec}_{\mathbb{Z}_3}$. There is only one action of $\mathbb{Z}_2$ on $\mathbb{Z}_3$, and so doing equivariantization by $\mathbb{Z}_2$ gives us back $\mathcal{C}$. Hence $\mathcal{C} \cong \mathrm{Vec}_{\mathbb{Z}_3}^{\mathbb{Z}_2}$.

On the other hand, for $p = 2$ there is no category realizing these fusion rules. In fact, in such case we would have that the category is Tannakian, and so we have a fiber functor $F : \mathcal{C} \to \mathbf{Vec}$. From Equation (4.1.3) we know that X is invertible and Y is not, and thus the same is true for $F(X)$ and $F(Y)$, respectively. Thus $d := \dim(F(Y))$ is not 1 and must satisfy $d^2 = 2 + d = d$. The only other possible solution is thus $d = 0$, and since F preserves dimensions we obtain $\dim(Y) = 0$, which is not possible. □

Theorem 4.1.2. *If $\mathcal{C}$ is a non-integral symmetric fusion category with Grothendieck ring $K(k, l, m, n)$ then $p = 7$ and $\mathcal{C} = \mathbf{Ver}_7^+$.*

Proof. Note that $p \neq 2, 3$ since in that case $\mathcal{C}$ is super-Tannakian and thus integral. Moreover, we also know that $p \leq 7$ by Theorem 3.1.1. However, our proof does not require use of this fact.

Consider the Adams operation $\psi_2 : K(k, l, m, n) \to K(k, l, m, n)$. Since X is self-dual, either $S^2(X)$ or $\Lambda^2(X)$ contains a copy of the unit object (but not both). The same is true for Y. Hence the multiplicity of $\mathbf{1}$ in both $\psi_2(X)$ and $\psi_2(Y)$ is 1 or -1, and so

$$\psi_2(X) = \epsilon_1\mathbf{1} + \alpha X + \beta Y, \qquad \psi_2(Y) = \epsilon_2\mathbf{1} + \gamma X + \delta Y,$$

for some $\epsilon_1, \epsilon_2 \in \{1, -1\}$ and $\alpha, \beta, \gamma, \delta \in \mathbb{Z}$.

Since $\mathcal{K}(\mathcal{C})$ is a based ring of rank 3 then $\mathcal{K}(\mathcal{C})_{\mathbb{Q}} := \mathcal{K}(\mathcal{C}) \otimes \mathbb{Q}$ is a semisimple commutative $\mathbb{Q}$-algebra of dimension 3, see [22, Corollary 3.7.7]. Hence we have three distinct possibilities for $\mathcal{K}(\mathcal{C})_{\mathbb{Q}}$ as a $\mathbb{Q}$-algebra, and we proceed by looking at them separetly.

▼ Case 1: $\mathcal{K}(\mathcal{C})_{\mathbb{Q}} \cong \mathbb{Q} \oplus \mathbb{Q} \oplus \mathbb{Q}$. In this case the homomorphism $\mathrm{FPdim}_{\mathbb{Q}} : \mathcal{K}(\mathcal{C})_{\mathbb{Q}} \to \mathbb{R}$ can only have rational image. Since $\mathrm{FPdim}(X)$ is an algebraic integer for all $X \in \mathcal{C}$, this implies that $\mathrm{FPdim}(X)$ is an integer for all $X \in \mathcal{C}$, and thus $\mathcal{C}$ is integral.

▼ Case 2: $\mathcal{K}(\mathcal{C})_{\mathbb{Q}} \cong \mathbb{Q} \oplus \mathbb{Q}(\sqrt{m})$ for some $m \in \mathbb{Z}$. Since $(\psi_2)_{\mathbb{Q}}$ is an endomorphism of $\mathcal{K}(\mathcal{C})_{\mathbb{Q}} \cong \mathbb{Q} \oplus \mathbb{Q}(\sqrt{m})$ mapping $(1, 1) \mapsto (1, 1)$, then it is either an automorphism of order 1 or 2, or has image the diagonal copy of $\mathbb{Q}$ inside $\mathbb{Q} \oplus \mathbb{Q}(\sqrt{m})$. Hence we have three possibilities for $\psi_2 : \mathcal{K}(\mathcal{C}) \to \mathcal{K}(\mathcal{C})$: it satisfies $\psi_2 = \mathrm{Id}$, $\psi_2^2 = \mathrm{Id}$ or $\mathrm{Im}(\psi_2) = \mathbb{Z}$. We show none of these are possible.

We assume first that $\mathrm{Im}(\psi_2) = \mathbb{Z}$. So we have $\psi_2(X) = \epsilon_1\mathbf{1}$ and $\psi_2(Y) = \epsilon_2\mathbf{1}$. The equalities

$$1 = [\psi_2(X)^2 : \mathbf{1}] = [\psi_2(X^2) : \mathbf{1}] = [\mathbf{1} + m\psi_2(X) + k\psi_2(Y) : \mathbf{1}] = 1 + m\epsilon_1 + k\epsilon_2,$$

imply that ϵ_1 and ϵ_2 must have opposite signs and that $m = k$. The analogous computation with Y shows that $l = n$. Moreover, since

$$\epsilon_1\epsilon_2 = [\psi_2(X)\psi_2(Y) : \mathbf{1}] = [\psi_2(XY) : \mathbf{1}] = [k\psi_2(X) + l\psi_2(Y) : \mathbf{1}] = k\epsilon_1 + l\epsilon_2,$$

we have that $l = k \pm 1$. Recall that we are assuming $l \geq k$ and thus $l = k + 1$. Then

$$X^2 = 1 + kX + (k+1)Y,$$

and so

$$X^2 \equiv 1 + X \mod 2 \qquad \text{or} \qquad X^2 \equiv 1 + Y \mod 2,$$

which contradicts $X^2 \equiv \psi_2(X) \equiv 1 \mod 2$. So the case $\mathrm{Im}(\psi_2) = \mathbb{Z}$ is not possible.

On the other hand, $\psi_2 = \mathrm{Id}$ is not possible by Theorem 3.2.9. Thus it remains to show that we cannot have $\psi_2^2 = \mathrm{Id}$ either. Suppose $\psi_2^2 = \mathrm{Id}$. Then

$$0 = [\psi_2^2(X) : \mathbf{1}] = \epsilon_1 + \alpha\epsilon_1 + \beta\epsilon_2 \qquad \text{and} \qquad 0 = [\psi_2^2(Y) : \mathbf{1}] = \epsilon_2 + \gamma\epsilon_1 + \delta\epsilon_2,$$

which imply

$$\beta = -(\alpha+1)\epsilon_1\epsilon_2 \qquad \text{and} \qquad \gamma = -(\delta+1)\epsilon_1\epsilon_2. \tag{4.1.4}$$

Also

$$0 = [\psi_2^2(X) : Y] = \beta(\alpha+\delta) \qquad \text{and} \qquad 0 = [\psi_2^2(Y) : X] = \gamma(\alpha+\delta).$$

If $\alpha + \delta \neq 0$ then $\beta = \gamma = 0$, which implies $\alpha = \delta = -1$. That is,

$$\psi_2(X) = \epsilon_1\mathbf{1} - X \qquad \text{and} \qquad \psi_2(Y) = \epsilon_2\mathbf{1} - Y.$$

We compute

$$\psi_2(X)\psi_2(Y) = \epsilon_1\epsilon_2\mathbf{1} + (k-\epsilon_2)X + (l-\epsilon_1)Y,$$

$$\psi_2(XY) = k\psi_2(X) + l\psi_2(Y) = (k\epsilon_1 + l\epsilon_2)\mathbf{1} - kX - lY.$$

Since the right-hand sides of the previous equations must be equal, we have that $k - \epsilon_2 = -k$ and so $2k = \pm 1$, which is not possible.

Hence we should have $\alpha + \delta = 0$, and using this together with (4.1.4) we get

$$\alpha \equiv \delta \mod 2, \qquad \beta = \gamma \mod 2 \qquad \text{and} \qquad \alpha \equiv \beta + 1 \mod 2. \tag{4.1.5}$$

On the other hand, from the equation

$$m\alpha + k\gamma = [\psi_2(X^2) : X] = [\psi_2(X)^2 : X] = 2\epsilon_1\alpha + \alpha^2 m + 2\alpha\beta k + \beta^2 l,$$

we obtain

$$m\alpha + k\gamma \equiv m\alpha + l\beta \mod 2.$$

This together with congruences (4.1.5) gives $k\beta \equiv l\beta \mod 2$. Lastly,

$$[\psi_2(XY) : X] = k\alpha + l\gamma,$$
$$[\psi_2(X)\psi_2(Y) : X] = \epsilon_1\gamma + \epsilon_2\alpha + m\gamma\alpha + k\alpha\delta + k\gamma\beta + l\beta\delta,$$

and so

$$\gamma + \alpha + m\gamma\alpha + k\alpha\delta + k\gamma\beta + l\beta\delta \equiv k\alpha + l\gamma \mod 2.$$

Note that $\alpha\gamma \equiv 0 \equiv \beta\delta \mod 2$, see (4.1.5). Hence the equation above is

$$\gamma + \alpha + k\alpha\delta + k\gamma\beta \equiv k\alpha + l\gamma \mod 2.$$

Now, using the congruences in (4.1.5) and $k\beta \equiv l\beta \mod 2$ in the equation above, we obtain $1 \equiv 0 \mod 2$. Thus $(\psi_2)_{\mathbb{Q}}$ cannot be an automorphism of order 2 and this case is not possible.

▼ Case 3: $\mathcal{K}(\mathcal{C})$ is a field extension of degree 3 over $\mathbb{Q}$. In this case $(\psi_2)_{\mathbb{Q}}$ must be an automorphism of order 3, since it cannot be the identity by Theorem 3.2.9. Note that Theorem 3.2.6 implies $p = 3$ or 7. We show $p = 7$ and $\mathcal{C} \cong \mathbf{Ver}_7^+$. Since ψ_2 has order 3, then $X, \psi_2(X)$ and $\psi_2^2(X)$ are distinct roots of the minimal polynomial of X, given by

$$m_X(t) = t^3 - (m+l)t^2 - (1 + k^2 - ml)t + l.$$

By the Vieta formulas we have that

$$m + l = [X + \psi_2(X) + \psi_2^2(X) : \mathbf{1}] = 2\epsilon_1 + \alpha\epsilon_1 + \beta\epsilon_2. \tag{4.1.6}$$

Repeating this for Y we get

$$k + n = [Y + \psi_2(Y) + \psi_2^2(Y) : \mathbf{1}] = 2\epsilon_2 + \gamma\epsilon_1 + \delta\epsilon_2. \tag{4.1.7}$$

On the other hand,

$$1 + m\epsilon_1 + k\epsilon_2 = [\psi_2(X^2) : \mathbf{1}] = [\psi_2(X)^2 : \mathbf{1}] = 1 + \alpha^2 + \beta^2, \quad \text{and}$$

$$1 + l\epsilon_1 + n\epsilon_2 = [\psi_2(Y^2) : \mathbf{1}] = [\psi_2(Y)^2 : \mathbf{1}] = 1 + \gamma^2 + \delta^2,$$

and thus

$$\alpha^2 + \beta^2 + \gamma^2 + \delta^2 = (m+l)\epsilon_1 + (k+n)\epsilon_2.$$

Combining this together with Equations (4.1.6) and (4.1.7) we get

$$\begin{aligned}\alpha^2 + \beta^2 + \gamma^2 + \delta^2 &= (2\epsilon_1 + \alpha\epsilon_1 + \beta\epsilon_2)\epsilon_1 + (2\epsilon_2 + \gamma\epsilon_1 + \delta\epsilon_2)\epsilon_2 \\ &= 2 + \alpha + \beta\epsilon_1\epsilon_2 + 2 + \delta + \gamma\epsilon_1\epsilon_2,\end{aligned}$$

and so

$$\alpha^2 - \alpha + \beta^2 - \beta\epsilon_1\epsilon_2 + \gamma^2 - \gamma\epsilon_1\epsilon_2 + \delta^2 - \delta = 4. \tag{4.1.8}$$

Hence $|\alpha|, |\beta|, |\gamma|$ and $|\delta|$ are at most 2, which implies by (4.1.6) and (4.1.7) that $m+l, k+n \leq 6$.

From each of the equalities $\psi_2(X^2) = \psi_2(X)^2$, $\psi_2(Y^2) = \psi_2(Y)^2$ and $\psi_2(XY) = \psi_2(X)\psi_2(Y)$ we get three equations on the parameters $\alpha, \beta, \gamma, \delta, k, l, m, n, \epsilon_1, \epsilon_2$. The bound $k, n \leq 6$ allows us to verify in Sage that the only solutions to said equations fulfilling (4.1.1) and (4.1.8) are

$$k = 1, l = 1, m = 1, n = 0, \alpha = -1, \beta = 1, \gamma = -1, \delta = 0, \text{ and}$$

$$k = 1, l = 1, m = 0, n = 1, \alpha = 0, \beta = -1, \gamma = 1, \delta = -1.$$

By symmetry, it is enough to consider the second case. We have multiplication rules

$$X^2 = \mathbf{1} + Y \qquad Y^2 = \mathbf{1} + X + Y, \qquad XY = YX = X + Y.$$

Note that these are the same fusion rules as $\mathbf{Ver}_7^+$. Taking dimension on the equalities above we arrive at the equation

$$\dim(Y)^3 - 2\dim(Y)^2 - \dim(Y) + 1 = 0. \tag{4.1.9}$$

On the other hand, note that $\mathcal{C}$ must be degenerate. In fact, if it was non-degenerate (see Section 2.2.11) it would lift to a symmetric category over a field of characteristic zero [18] and thus would have integer Frobenius-Perron dimensions [22, Theorem 9.9.26]. We compute

$$\begin{aligned} 0 = \dim(\mathcal{C}) &= 1 + \dim(X)^2 + \dim(Y)^2 \\ &= 1 + 1 + \dim(Y) + \dim(Y)^2, \end{aligned}$$

and so

$$\dim(Y)^2 = -2 - \dim(Y).$$

Replacing this into (4.1.9) we get

$$
\begin{aligned}
0 &= \dim(Y)^3 - 2\dim(Y)^2 - \dim(Y) + 1 \\
&= \dim(Y)(-2 - \dim(Y)) - 2(-2 - \dim(Y)) - \dim(Y) + 1 \\
&= -2\dim(Y) + 2 + \dim(Y) + 4 + 2\dim(Y) - \dim(Y) + 1 \\
&= 7.
\end{aligned}
$$

Thus we must have $p = 7$. Since $\mathcal{K}(\mathcal{C}) \cong \mathcal{K}(\mathbf{Ver}_7^+)$ we see that the fiber functor $F : \mathcal{C} \to \mathbf{Ver}_7$ gives an equivalence onto $\mathbf{Ver}_7^+$. □

Remark 4.1.3. Theorem 4.1.2 gives a positive answer for Question 1.2.1 in the case $p = 7$.

4.2 Rank 4 symmetric fusion categories

4.2.1 Exactly two self-dual simple objects

In this section we classify symmetric fusion categories $\mathcal{C}$ of rank 4 with exactly two self-dual simple objects. Namely, we show that $\mathcal{C}$ is equivalent to one of the following:

- If $p = 2$, $\mathcal{C} \cong \mathrm{Vec}_{\mathbb{Z}_4}^{\mathbb{Z}_3}$ or $\mathcal{C} \cong \mathcal{C}(\mathbb{Z}_4, q)$, where $q : \mathbb{Z}_4 \to \mathbf{k}^\times$ is one of the two group maps satisfying $q(g)^2 = 1$ for all $g \in \mathbb{Z}_4$, see [22, Section 8.4].
- If $p = 3$, $\mathcal{C} \cong \mathcal{C}(\mathbb{Z}_4, q)$.
- If $p > 3$, $\mathcal{C} \cong \mathbf{Rep}(A_4)$ or $\mathcal{C} \cong \mathcal{C}(\mathbb{Z}_4, q)$.

We use the parametrization of based rings of rank 4 with exactly two self-dual basis elements as given in [31]. Let c, e, k, l, p, q be non-negative integers satisfying the following equations:

$$kl + lc = lp + kq, \tag{4.2.1}$$

$$kp + le + kc = 2lq + k^2, \tag{4.2.2}$$

$$l^2 + c^2 = 1 + q^2 + p^2, \tag{4.2.3}$$

$$l^2 + k^2 + q^2 = 1 + 2pk + qe. \tag{4.2.4}$$

We denote by $K(c, e, k, l, p, q)$ the based ring with basis $1, X, Y, Z$ and multiplication given by

$$\begin{aligned} X^2 &= pX + lY + cZ & XY = YX &= qX + kY + lZ \\ Y^2 &= \mathbf{1} + kX + eY + kZ & YZ = ZY &= lX + kY + qZ \\ Z^2 &= cX + lY + pZ & XZ = ZX &= \mathbf{1} + pX + qY + pZ. \end{aligned} \tag{4.2.5}$$

Any based ring of rank 4 with exactly two self-dual basis elements is of the form $K(c, e, k, l, p, q)$ for some non-negative integers c, e, k, l, p, q satisfying (4.2.1)-(4.2.4), see [31].

When $p > 2$, recall that for the second Adams operation $\psi_2 : K(c, e, k, l, p, q) \to K(c, e, k, l, p, q)$ we have that $\psi_2(W) \equiv W^2 \mod 2$ for any $W \in K(c, e, k, l, p, q)$. Hence

$$\begin{aligned} \psi_2(X) &= \alpha_1 X + \alpha_2 Y + \alpha_3 Z, \\ \psi_2(Y) &= \epsilon\mathbf{1} + \beta_1 X + \beta_2 Y + \beta_3 Z, \text{ and} \\ \psi_2(Z) &= \gamma_1 X + \gamma_2 Y + \gamma_3 Z, \end{aligned} \tag{4.2.6}$$

for some $\epsilon = \pm 1$ and $\alpha_i, \beta_i, \gamma_i \in \mathbb{Z}$, $i = 1, 2, 3$. Recall that $\psi_2(W) \equiv W^2 \mod 2$ and thus

$$\begin{aligned} &\alpha_1 \equiv p \equiv \gamma_3 \mod 2 && \alpha_2 \equiv l \equiv \gamma_2 \mod 2 && \alpha_3 \equiv c \equiv \gamma_1 \mod 2 \\ &\beta_1 \equiv k \equiv \beta_3 \mod 2 && \beta_2 \equiv e \mod 2. \end{aligned}$$

We will use the previous congruences repeatedly throughout this section.

We start by discarding some possibilities for the Adams operation in the lemmas below.

Lemma 4.2.1. *Let $\mathcal{C}$ be a symmetric fusion category of rank 4 with exactly two self-dual simple objects. Then* $\mathrm{Im}(\psi_2) \neq \mathbb{Z}$.

Proof. Suppose that $\mathrm{Im}(\psi_2) = \mathbb{Z}$. Then by Proposition 3.2.13 we have that $Y^2 = \mathbf{1}$ and $[XZ : Y] = 1$, which imply $k = e = 0$ and $q = 1$. But then from Equation (4.2.4) we get that $l^2 + 1 = 1$, thus $l = 0$. Thus Equation (4.2.3) is $c^2 = 2 + p^2$, which has no integer solutions. □

Lemma 4.2.2. *Let $\mathcal{C}$ be a symmetric fusion category of rank 4 with exactly two self-dual simple objects. Then ψ_2 does not have order 2.*

Proof. Suppose for the sake of contradiction that $\psi_2^2 = \mathrm{Id}$. Then from $\psi_2^2(X) = X$ and Equation (4.2.6), we get the equations

$$0 = \alpha_2\epsilon, \qquad 1 = \alpha_1^2 + \alpha_2\beta_1 + \alpha_3\gamma_1, \qquad 0 = \alpha_1\alpha_3 + \alpha_2\beta_3 + \alpha_3\gamma_3.$$

Thus $\alpha_2 = 0$ and

$$1 = \alpha_1^2 + \alpha_3\gamma_1, \qquad 0 = \alpha_3(\alpha_1 + \gamma_3). \tag{4.2.7}$$

Similarly, from $\psi_2^2(Z) = Z$ we get that $\gamma_2 = 0$ and

$$1 = \gamma_1\alpha_3 + \gamma_3^2, \qquad 0 = \gamma_1(\alpha_1 + \gamma_3). \tag{4.2.8}$$

We divide the rest of the proof in two cases.

▼ Case 1: $\alpha_1 \neq -\gamma_3$. Then by Equations (4.2.7) and (4.2.8) we must have $\alpha_3 = 0 = \gamma_1$ and $\alpha_1^2 = 1 = \gamma_3^2$, so $\gamma_3 = \alpha_1 = \pm 1$. Then

$$\epsilon l = [\psi_2(X^2) : \mathbf{1}] = [\psi_2(X)^2 : \mathbf{1}] = \alpha_2^2 + 2\alpha_1\alpha_3 = 0,$$

which implies $l = 0$. On the other hand,

$$1 + q\epsilon = [\psi_2(XZ) : \mathbf{1}] = [\psi_2(X)\psi_2(Z) : \mathbf{1}] = \alpha_1\gamma_3 = 1,$$

which implies $q = 0$. But then by Equation (4.2.3) we have $p = 0$ which contradicts $p \equiv \alpha_1 \mod 2$.

▼ Case 2: $\alpha_1 = -\gamma_3$. Then

$$\epsilon l = [\psi_2(X^2) : \mathbf{1}] = [\psi_2(X)^2 : \mathbf{1}] = 2\alpha_1\alpha_3 = -2\gamma_3\alpha_3,$$

$$\epsilon l = [\psi_2(Z^2) : \mathbf{1}] = [\psi_2(Z)^2 : \mathbf{1}] = 2\gamma_1\gamma_3,$$

and so $\gamma_1\gamma_3 = -\gamma_3\alpha_3$. Suppose first that $\gamma_3 \neq 0$. Then $\gamma_1 = -\alpha_3$ and so by Equation (4.2.7)

$$1 = \alpha_1^2 - \alpha_3^2,$$

which implies $\alpha_1 = \pm 1$ and $\alpha_3 = 0 = \gamma_1$. But then

$$\epsilon l = [\psi_2(X^2) : \mathbf{1}] = [\psi_2(X)^2 : \mathbf{1}] = 0,$$

so $l = 0$ and

$$p\alpha_1 = [\psi_2(X^2) : X] = [\psi_2(X)^2 : X] = \alpha_1^2 p = p.$$

Since $p \equiv \alpha_1 \equiv 1 \mod 2$ this implies $\alpha_1 = 1$, and thus $\gamma_3 = -1$. But then

$$[\psi_2(Z^2) : Z] = [\psi_2(cX + pZ) : Z] = c\alpha_3 + p\gamma_3 = -p,$$

$$[\psi_2(Z)^2 : Z] = [(\gamma_3 Z)^2 : Z] = \gamma_3^2 p = p,$$

since l, α_3, γ_1 and γ_2 are all 0 and $\gamma_3 = -1$. Since the two equations above should be equal, then $p = 0$ which contradicts $p \equiv 1 \mod 2$.

The contradiction came from assuming $\gamma_3 \neq 0$, and thus we should have $\gamma_3 = 0 = \alpha_1$. Hence from the equations

$$[\psi_2(X^2) : \mathbf{1}] = [\psi_2(pX + lY + cZ) : \mathbf{1}] = \epsilon l,$$

$$[\psi_2(X)^2 : \mathbf{1}] = [(\alpha_3 Z)^2 : \mathbf{1}] = 0$$

we get that $l = 0$. On the other hand, by Equation (4.2.7) we have that $\alpha_3\gamma_1 = 1$, and so

$$[\psi_2(XZ) : \mathbf{1}] = [\psi_2(\mathbf{1} + pX + qY + pZ) : \mathbf{1}] = 1 + q\epsilon,$$

$$[\psi_2(X)\psi_2(Z) : \mathbf{1}] = [(\alpha_3 Z)(\gamma_1 X) : \mathbf{1}] = \alpha_3\gamma_1 = 1,$$

hence $q = 0$. Then from Equations (4.2.3) and (4.2.4) we conclude $k = 1 = c$ and $p = 0$. Moreover, $\psi_2^2(Y) = Y$ implies $0 = \epsilon(\beta_2 + 1)$ and so $\beta_2 = -1$. Using also that $\psi_2(XY) = \psi_2(X)\psi_2(Y)$ and $\psi_2(X) = \alpha_3(Z)$, we obtain

$$\epsilon = [\psi_2(XY) : \mathbf{1}] = [\psi_2(X)\psi_2(Y) : \mathbf{1}] = \alpha_3\beta_1,$$

$$\beta_1 = [\psi_2(XY) : X] = [\psi_2(X)\psi_2(Y) : X] = \alpha_3\beta_3,$$

$$-1 = [\psi_2(XY) : Y] = [\psi_2(X)\psi_2(Y) : Y] = -\alpha_3$$

$$\beta_3 = [\psi_2(XY) : Z] = [\psi_2(X)\psi_2(Y) : Z] = \epsilon\alpha_3,$$

which imply $1 = \alpha_3 = \gamma_1$ and $\epsilon = \beta_1 = \beta_3$. Lastly

$$e\epsilon = [\psi_2(Y^2) : 1] = [\psi_2(Y)^2 : 1] = \beta_2^2 + 2\beta_1\beta_3 = 3.$$

Thus the fusion rules for $\mathcal{C}$ would be

$$\begin{aligned} X^2 &= Z & XY &= Y, \\ Y^2 &= 1 + X + 3Y + Z & ZY &= Y, \\ Z^2 &= X & ZX &= 1. \end{aligned} \tag{4.2.9}$$

These are the fusion rules of the Izumi-Xu category (see [6]). The Frobenius Perron dimension of $\mathcal{C}$ is $\frac{21+2\sqrt{21}}{2}$, which is not possible in positive characteristic. In fact, dimensions in $\mathbf{Ver}_p$ are in the field $\mathbb{Q}(z + z^{-1})$, see [4, Theorem 4.5 (iv)]. Since we have a symmetric fiber functor $F : \mathcal{C} \to \mathbf{Ver}_p$, the same should be true for $\mathcal{C}$, which makes the obtained dimension impossible. □

Remark 4.2.3. The Adams operation is not enough on its own to classify symmetric fusion categories in positive characteristic. In fact, in the proof of the previous Lemma we found a possible based ring (4.2.9) and a suitable Adams operation, given by

$$\psi_2(X) = Z, \qquad \psi_2(Y) = \mathbf{1} + X - Y + Z, \qquad \psi_2(Z) = X.$$

However, as stated, there is no fusion category over a field of positive characteristic with (4.2.9) as its Grothendieck ring.

Lemma 4.2.4. *Let $\mathcal{C}$ be a symmetric fusion category of rank 4 with exactly two self-dual simple objects. Then $\psi_2^2 \neq \psi_2$.*

Proof. Suppose that $\psi_2^2 = \psi_2$. From the equality $\psi_2^2(X) = \psi_2(X)$ we get the equations

$$0 = \alpha_2\epsilon, \qquad \alpha_1 = \alpha_1^2 + \alpha_2\beta_1 + \alpha_3\gamma_1, \qquad \alpha_3 = \alpha_1\alpha_3 + \alpha_2\beta_3 + \alpha_3\gamma_3.$$

Thus $\alpha_2 = 0$ and $\alpha_3 = \alpha_3(\alpha_1+\gamma_3)$. If $\alpha_3 \neq 0$ then $\alpha_1+\gamma_3 = 1$ and so $p+p \equiv 1 \mod 2$ which is a contradiction. Thus we have $\alpha_3 = 0$ and $\alpha_1 = 1$ or 0. Analogously, from $\psi_2^2(Z) = \psi_2(Z)$ we get $\gamma_2 = 0 = \gamma_1$ and $\gamma_3 = 1$ or 0. Lastly, from $\psi_2^2(Y) = \psi_2(Y)$ we have that $\beta_2 = 0$.

On the other hand,

$$\epsilon l = [\psi_2(X^2) : \mathbf{1}] = [\psi_2(X)^2 : \mathbf{1}] = \alpha_2^2 + 2\alpha_1\alpha_3 = 0,$$

which implies $l = 0$, and so

$$c\gamma_3 = [\psi_2(X^2) : Z] = [\psi_2(X)^2 : Z] = c\alpha_1^2.$$

If $c = 0$ then $0 = 1 + q^2 + p^2$ by Equation (4.2.3), which is not possible. Thus $\gamma_3 = \alpha_1^2 = \alpha_1$. We divide the rest of the proof in two cases:

▼ Case 1: $\alpha_1 = \gamma_3 = 0$. Then

$$\epsilon k = [\psi_2(XY) : 1] = [\psi_2(X)\psi_2(Y) : 1] = \alpha_1\beta_3 + \alpha_3\beta_1 + \alpha_2\beta_2 = 0,$$

and so $k = 0$. Then by Equation (4.2.4) we have $q^2 = 1 + qe$, thus $q = 1$ and so $c^2 = 2 + p^2$ by Equation (4.2.3), which has no integer solutions.

▼ Case 2: $\alpha_1 = \gamma_3 = 1$. Then

$$0 = [\psi_2(XY) : Y] = [\psi_2(X)\psi_2(Y) : Y] = q\beta_3,$$

so $q = 0$ or $0 = \beta_3 = \beta_1$. If $q = 0$ then $c^2 = 1 + p^2$ by Equation (4.2.3) and so $c = 1$ and $p = 0$. Then $1 \equiv c \equiv \alpha_3 \equiv 0 \mod 2$, a contradiction. Thus we must have $0 = \beta_3$ and so

$$\epsilon k = [\psi_2(XY) : \mathbf{1}] = [\psi_2(X)\psi_2(Y) : \mathbf{1}] = \beta_3 = 0,$$

which implies $k = 0$. Since $q^2 = 1+qe$ by Equation (4.2.4) we get $q = 1$. But then $c^2 = 2+p^2$ by Equation (4.2.3), which has no integer solutions. □

We will need the following auxiliary lemma.

Lemma 4.2.5. *If $\mathcal{C}$ is a fusion category with commutative $\mathcal{K}(\mathcal{C})$ and a non-self-dual object, then there exists a ring homomorphism $\mathcal{K}(\mathcal{C}) \to \mathbb{C}$ whose image is not contained in $\mathbb{R}$.*

Proof. Let $X \in \mathcal{C}$ be a non-self-dual object, and consider the map of multiplication by $X - X^*$ in $\mathcal{K}(\mathcal{C})_{\mathbb{C}}$. In the basis given by simple objects, we can represent this map by a non-trivial skew symmetric matrix. Thus its eigenvalues are zero or non-real. Since $X \ncong X^*$ there must exist at least one non-real eigenvalue $\lambda \neq 0$. Hence $\mathcal{K}(\mathcal{C})_{\mathbb{C}}$ has a 1 dimensional representation where X acts as multiplication by λ. □

Theorem 4.2.6. *Let $\mathcal{C}$ be a symmetric fusion category of rank 4 with exactly 2 self-dual simple objects. Then $\mathcal{C}$ is integral.*

Proof. We proceed by looking at the different possibilities for $\mathcal{K}(\mathcal{C})_{\mathbb{Q}}$. Since $\mathcal{K}(\mathcal{C})_{\mathbb{Q}}$ is a semisimple commutative $\mathbb{Q}$-algebra of dimension 4, we have five cases:

▼ Case 1: $\mathcal{K}(\mathcal{C})_{\mathbb{Q}} \cong \mathbb{Q} \oplus \mathbb{Q} \oplus \mathbb{Q} \oplus \mathbb{Q}$. Note that $\mathbb{Q}$-algebra maps $\mathcal{K}(\mathcal{C})_{\mathbb{Q}} \to \mathbb{C}$ are projections to $\mathbb{Q}$, and so this case is not possible by Lemma 4.2.5.

▼ Case 2: $\mathcal{K}(\mathcal{C})_{\mathbb{Q}} \cong \mathbb{Q}(\sqrt{n}) \oplus \mathbb{Q} \oplus \mathbb{Q}$. By Lemma 4.2.5 we must have $n < 0$. But then $\mathrm{FPdim}_{\mathbb{Q}} : \mathcal{K}(\mathcal{C})_{\mathbb{Q}} \to \mathbb{R}$ can only have rational image and so $\mathcal{C}$ is integral.

▼ Case 3: $\mathcal{K}(\mathcal{C})_{\mathbb{Q}} \cong \mathbb{Q}(\sqrt{n}) \oplus \mathbb{Q}(\sqrt{m})$ with $\mathbb{Q}(\sqrt{n}) \ncong \mathbb{Q}(\sqrt{m})$. Endomorphisms of $\mathbb{Q}(\sqrt{n}) \oplus \mathbb{Q}(\sqrt{m})$ are given by

$$(1,0) \mapsto (1,0) \qquad (0,1) \mapsto (0,1) \qquad (\sqrt{n},0) \mapsto (\pm\sqrt{n},0) \qquad (0,\sqrt{m}) \mapsto (0,\pm\sqrt{m}).$$

These are all automorphisms of order 1 or 2, which is not possible for $(\psi_2)_{\mathbb{Q}}$ by Lemma 4.2.2. Hence this case is discarded.

▼ Case 3: $\mathcal{K}(\mathcal{C})_{\mathbb{Q}} \cong \mathbb{Q}(\sqrt{n}) \oplus \mathbb{Q}(\sqrt{n})$. By Lemma 4.2.5 we have $n < 0$. But $\mathbb{Q}$-algebra morphisms from $\mathbb{Q}(\sqrt{n}) \oplus \mathbb{Q}(\sqrt{n})$ to $\mathbb{C}$ are embeddings onto $\mathbb{Q}(\sqrt{n})$. This contradicts the fact that $\mathrm{FPdim} : \mathbb{Q}(\sqrt{n}) \oplus \mathbb{Q}(\sqrt{n}) \to \mathbb{C}$ should have real image, and so this case is discarded.

▼ Case 4: $\mathcal{K}(\mathcal{C})_{\mathbb{Q}} \cong \mathbf{F} \oplus \mathbb{Q}$, where $\mathbf{F}$ is a field extension of degree 3 over $\mathbb{Q}$. The only endomorphism of $\mathbf{F} \oplus \mathbb{Q}$ with non-trivial kernel is given by $(a, b) \mapsto (b, b)$ for all $a \in \mathbf{F}$ and $b \in \mathbb{Q}$. By Lemma 4.2.1 we know that $(\psi_2)_{\mathbb{Q}}$ is not of this form.

On the other hand, non-trivial automorphisms of $\mathbf{F} \oplus \mathbb{Q}$ have order 3 which is not possible for $(\psi_2)_{\mathbb{Q}}$ by Theorem 3.2.12. Hence this case is also discarded.

▼ Case 5: $\mathcal{K}(\mathcal{C})_{\mathbb{Q}}$ is a field extension of degree 4 over $\mathbb{Q}$. Since $(\psi_2)_{\mathbb{Q}} \neq \mathrm{Id}$ by Theorem 3.2.9, then it must be an automorphisms of order 2 or 4. Using Lemma 4.2.2 we can discard the former possibility. If $(\psi_2)_{\mathbb{Q}}$ has order 4 then $Y, \psi_2(Y), \psi_2^2(Y)$ and $\psi_2^3(Y)$ are distinct roots of the minimal polynomial of Y. Since the minimal polynomial of Y is given by

$$m_Y(t) = t^4 + (-2q - e)t^3 + (2qe + q^2 - k^2 - l^2 - 1)t^2 + (-q^2 e + qk^2 - lk^2 + l^2 e + 2q)t + l^2 - q^2,$$

then by the Vieta formulas the sum of the roots equals $2q + e$. Hence

$$[Y + \psi_2(Y) + \psi_2^2(Y) + \psi_2^3(Y) : \mathbf{1}] = 2q + e.$$

We compute

$$[\psi_2(Y) : \mathbf{1}] = \epsilon, \quad [\psi_2^2(Y) : \mathbf{1}] = \epsilon + \beta_2 \epsilon, \quad [\psi_2^3(Y) : \mathbf{1}] = \epsilon + \beta_2 \epsilon + \epsilon(\beta_1 \alpha_2 + \beta_2^2 + \beta_3 \gamma_2),$$

and thus

$$\epsilon + \epsilon + \beta_2 \epsilon + \epsilon + \beta_2 \epsilon + \epsilon(\beta_1 \alpha_2 + \beta_2^2 + \beta_3 \gamma_2) = 2q + e.$$

Taking congruence mod 2 on both sides we get

$$e \equiv 1 + 1 + e + 1 + e + kl + e + kl \equiv 1 + e \mod 2,$$

which is not possible, so this case is also discarded. □

Theorem 4.2.7. *Let $\mathcal{C}$ be an integral symmetric fusion category of rank 4 with exactly 2 self-dual simple objects. Then either*

- $\mathcal{C} \cong \mathcal{C}(\mathbb{Z}_4, q)$ *and* $p \geq 2$*, or*
- $\mathcal{C} \cong \mathbf{Rep}(A_4)$ *and* $p > 3$*, or*
- $\mathcal{C} \cong \mathrm{Vec}_{\mathbb{Z}_4}^{\mathbb{Z}_3}$ *and* $p = 2$*.*

Proof. Let $\mathcal{C}$ be as in the statement with Grothendieck ring $K(c, e, k, l, p, q)$, see Equation 4.2.5. Since $X^* = Z$ we have that $\mathrm{FPdim}(X) = \mathrm{FPdim}(Z)$. Thus taking Frobenius-Perron dimensions on both sides of the fusion rule $Y^2 = \mathbf{1} + kX + eY + kZ$ we get

$$\mathrm{FPdim}(Y)(\mathrm{FPdim}(Y) - e) = 1 + 2k\,\mathrm{FPdim}(X),$$

and so $\gcd(\mathrm{FPdim}(X), \mathrm{FPdim}(Y)) = 1$. From the fusion rule $XY = qX + kY + lZ$ we get $1 = \frac{q+l}{\mathrm{FPdim}(Y)} + \frac{k}{\mathrm{FPdim}(X)}$ and so since the denominators are coprime either $\mathrm{FPdim}(Y) = q + l$ and $k = 0$ or $q + l = 0$ and $\mathrm{FPdim}(X) = k$. We split the rest of the proof in two cases.

▼ Case 1: $\mathrm{FPdim}(Y) = q + l$ and $k = 0$. Since $Y^2 = \mathbf{1} + eY$ then

$$\mathrm{FPdim}(Y) = \frac{e + \sqrt{e^2 + 4}}{2}.$$

But $\mathrm{FPdim}(Y)$ is an integer and then $e^2 + 4$ must be a square, so $e = 0$. Hence $Y^2 = \mathbf{1}$ which implies $q + l = \mathrm{FPdim}(Y) = 1$. Recall that q and l are non-negative integers, and so there are only two options: either $q = 1$ and $l = 0$, or $q = 0$ and $l = 1$. The former is not possible, since in that case $c^2 = 2 + p^2$ by Equation (4.2.3), which has no integer solutions.

Hence we have that $q = 0$ and $l = 1$. Moreover, the fusion rule $X^2 = pX + Y + pZ$ implies that

$$\mathrm{FPdim}(X)^2 = 2p\,\mathrm{FPdim}(X) + 1.$$

This has integer solutions only for $p = 0$, in which case $\mathrm{FPdim}(X) = 1$. Lastly by Equation (4.2.3) we have $c = 0$. Thus the fusion rules are

$$X^2 = Y \qquad XY = YX = Z$$
$$Y^2 = \mathbf{1} \qquad YZ = ZY = X$$
$$Z^2 = Y \qquad XZ = ZX = \mathbf{1}.$$

Hence the category is pointed and $\mathcal{C} \cong \mathcal{C}(\mathbb{Z}_4, q)$, where $q : \mathbb{Z}_4 \to \mathbf{k}^\times$ is a quadratic form satisfying

$$q(gh) = q(g)q(h)b(g,h),$$

where $b(g,h) = c_{Y,X}c_{X,Y} \in \mathrm{Aut}_{\mathcal{C}}(X,Y) \cong \mathbf{k}^\times$ for X and Y simple objects representing g and h, respectively, see [22, Lemma 8.4.2]. Since $\mathcal{C}$ is symmetric, it follows that q is a group homomorphism. Finally, from the fusion rules above we get $q(g)^2 = 1$ for all $g \in \mathbb{Z}_4$.

▼ Case 2: $\mathrm{FPdim}(X) = k$ and $q + l = 0$. Since q and l are non-negative integers this implies $q = 0 = l$. The fusion rule $XZ = \mathbf{1} + pX + pZ$ implies that $\mathrm{FPdim}(X)^2 = 1 + 2p\,\mathrm{FPdim}(X)$ and so $\mathrm{FPdim}(X) = 1$ and $p = 0$. On the other hand, from Equations (4.2.3) and (4.2.4) we get that $k = 1 = c$. Lastly, since $Y^2 = \mathbf{1} + X + eY + Z$ we have that

$$\mathrm{FPdim}(Y) = \frac{e + \sqrt{e^2 + 12}}{2}.$$

Thus for $\mathrm{FPdim}(Y)$ to be an integer we need $e = 2$. Consequently. the fusion rules are

$$X^2 = Z \qquad XY = YX = Y$$

$$Y^2 = \mathbf{1} + X + 2Y + Z \qquad YZ = ZY = Y$$

$$Z^2 = X \qquad XZ = ZX = \mathbf{1}.$$

Suppose $p > 3$. The equations above imply that $\dim(X) = 1 = \dim(Z)$ and $\dim(Y) = -1$ or 3. Hence $\dim(\mathcal{C}) \neq 0$ and thus we can lift $\mathcal{C}$ to a symmetric fusion category $\widetilde{\mathcal{C}}$ over a field $\mathbf{f}$ in characteristic zero [18, Section 4.1], which has the same Grothendick ring as $\mathcal{C}$. Thus $\widetilde{\mathcal{C}}$ is equivalent to $\mathbf{Rep}_{\mathbf{f}}(A_4)$ [13, Section 8.19], and so by uniqueness of the lifting we get that $\mathcal{C}$ is equivalent to $\mathbf{Rep}_{\mathbf{k}}(A_4)$.

For the case $p = 2$ we have that the objects $\mathbf{1}, X$ and Z in $\mathcal{C}$ generate a copy of $\mathbf{Rep}(\mathbb{Z}_3)$. Doing de-equivariantization by $\mathbb{Z}_3$ we obtain a symmetric fusion category $\mathcal{C}^{\mathbb{Z}_3}$ of dimension 4 [22, Section 2.7]. Hence $\mathcal{C}^{\mathbb{Z}_3}$ is pointed and thus equivalent to $\mathrm{Vec}_{\mathbb{Z}_4}$. There is only one action of $\mathbb{Z}_3$ on $\mathbb{Z}_4$, and so doing equivariantization by $\mathbb{Z}_3$ gives us back $\mathcal{C}$. Hence $\mathcal{C} \cong \mathrm{Vec}_{\mathbb{Z}_4}^{\mathbb{Z}_3}$.

Lastly, for $p = 3$ there is no category realizing these fusion rules. In fact, we know all symmetric fusion categories in characteristic 3. Any such category is an equivariantization of a pointed category associated with a 3-group by the action of a group G of order relatively prime to 3, see [21, Section 8]. Note that the group G is non-trivial, since the category with the fusion rules above is not pointed as $\mathrm{FPdim}(Y) = 3$. Thus the category should contain a non-trivial Tannakian subcategory $\mathbf{Rep}(G)$ of rank prime to 3, which is not possible with the fusion rules above. □

Remark 4.2.8. In the proof of Theorem 4.2.6 we showed that, when $\mathcal{C}$ is a symmetric fusion category of rank 4 and exactly 2 self-dual simple objects, then the only possibilities for $\mathcal{K}(\mathcal{C})_{\mathbb{Q}}$

are $\mathbb{Q}^{\oplus 4}$ and $\mathbb{Q}(\sqrt{n}) \oplus \mathbb{Q}^{\oplus 2}$, for n a negative square-free integer. Moreover, Theorem 4.2.7 shows that such a category is equivalent to either $\mathcal{C}(\mathbb{Z}_4, q)$, $\mathbf{Rep}(A_4)$ or $\mathrm{Vec}_{\mathbb{Z}_4}^{\mathbb{Z}_3}$, the first of which satisfies $\mathcal{K}(\mathcal{C})_{\mathbb{Q}} \simeq \mathbb{Q}(\sqrt{-1}) \oplus \mathbb{Q}^{\oplus 2}$, and the other two $\mathcal{K}(\mathcal{C})_{\mathbb{Q}} \simeq \mathbb{Q}(\sqrt{-3}) \oplus \mathbb{Q}^{\oplus 2}$.

4.2.2 All simple objects are self-dual

We are not able to provide a classification of symmetric fusion categories of rank 4, but here are some comments for the remaining case, in which all simple objects are self-dual. It follows from Theorem 3.1.1 that we can have examples of such categories that are non super-Tannakian only in characteristics $p = 5$ or 7.

We take a look first at the case $p = 5$.

Proposition 4.2.9. *Let $\mathcal{C}$ be a non-super-Tannakian symmetric fusion category of rank 4 in characteristic $p = 5$. Then $\mathcal{K}(\mathcal{C})_{\mathbb{Q}} \cong \mathbb{Q}(\sqrt{5}) \oplus \mathbb{Q}^{\oplus 2}$ or $\mathcal{K}(\mathcal{C})_{\mathbb{Q}} \cong \mathbb{Q}(\sqrt{5}) \oplus \mathbb{Q}(\sqrt{m})$ for some $m \in \mathbb{Z}$.*

Proof. Since $\mathcal{K}(\mathcal{C})_{\mathbb{Q}}$ is a semisimple commutative $\mathbb{Q}$-algebra of dimension 4, it can either be $\mathbb{Q}^{\oplus 4}$, $\mathbb{Q}^{\oplus 2} \oplus \mathbb{Q}(\sqrt{n})$, $\mathbb{Q}(\sqrt{n}) \oplus \mathbb{Q}(\sqrt{m})$, $\mathbb{Q}(a) \oplus \mathbb{Q}$ or $\mathbb{Q}(b)$, for $n, m \in \mathbb{Z}$, and $a, b \in \mathbb{Q}$ such that $[\mathbb{Q}(a) : \mathbb{Q}] = 3$ and $[\mathbb{Q}(b) : \mathbb{Q}] = 4$.

Consider the Verlinde fiber functor $F : \mathcal{C} \to \mathbf{Ver}_5$, and let $\tilde{F} : \mathcal{C} \to \mathbf{Ver}_p^+$ be as in (3.1.1). We denote also by $\tilde{F}$ the induced $\mathbb{Q}$-algebra homomorphism $\mathcal{K}(\mathcal{C})_{\mathbb{Q}} \to \mathcal{K}(\mathbf{Ver}_p^+)_{\mathbb{Q}}$. By the proof of Theorem 3.1.1, since $\mathcal{C}$ is not super-Tannakian then this map is surjective and so

$$\tilde{F} : \mathcal{K}(\mathcal{C})_{\mathbb{Q}} \twoheadrightarrow \mathcal{K}(\mathbf{Ver}_5^+)_{\mathbb{Q}} \cong \mathbb{Q}(\xi_5 + \xi_5^{-1}) = \mathbb{Q}(\sqrt{5}).$$

That is, the image of $\mathcal{K}(\mathcal{C})_{\mathbb{Q}}$ under $\tilde{F}$ is $\mathbb{Q}(\sqrt{5})$. Then the only remaining possibilities for $\mathcal{K}(\mathcal{C})_{\mathbb{Q}}$ are $\mathbb{Q}^{\oplus 2} \oplus \mathbb{Q}(\sqrt{5})$ or $\mathbb{Q}(\sqrt{5}) \oplus \mathbb{Q}(\sqrt{m})$ for some $m \in \mathbb{Z}$. ∎

For $p = 7$, we have the following result.

Proposition 4.2.10. *Let $\mathcal{C}$ be a non-super-Tannakian symmetric fusion category of rank 4 in characteristic $p = 7$. Then $\mathcal{K}(\mathcal{C})_{\mathbb{Q}} \cong \mathbb{Q} \oplus \mathbb{Q}(a)$ for some a such that $[\mathbb{Q}(a) : \mathbb{Q}] = 3$. Moreover, $\psi_2^3 = Id$.*

Proof. Since $\mathcal{K}(\mathcal{C})_{\mathbb{Q}}$ is a semisimple commutative algebra of dimension 4, it can either be $\mathbb{Q} \oplus \mathbb{Q} \oplus \mathbb{Q} \oplus \mathbb{Q}$, $\mathbb{Q}(\sqrt{n}) \oplus \mathbb{Q} \oplus \mathbb{Q}$, $\mathbb{Q}(\sqrt{n}) \oplus \mathbb{Q}(\sqrt{m})$, $\mathbb{Q}(a) \oplus \mathbb{Q}$ or $\mathbb{Q}(b)$, for $a, b \in \mathbb{C}$ such that $[\mathbb{Q}(a) : \mathbb{Q}] = 3$ and $[\mathbb{Q}(b) : \mathbb{Q}] = 4$.

Thus we have that if $f \in \operatorname{End}(\mathcal{K}(\mathcal{C})_{\mathbb{Q}})$ then either $f^n = \operatorname{Id}$ for $n = 1, 2, 3, 4$, $f^k = f$ for $k = 2, 3$ or $f^3 = f^2$. By Theorem 3.2.6, the only possibility for $(\psi_2)_{\mathbb{Q}} \in \operatorname{End}(\mathcal{K}(\mathcal{C})_{\mathbb{Q}})$ is that $(\psi_2)^3_{\mathbb{Q}} = \operatorname{Id}$, which can only happen if $\mathcal{K}(\mathcal{C}) \cong \mathbb{Q} \oplus \mathbb{Q}(a)$, as desired. □

Chapter 5

Constructions from unoriented 2-dimensional cobordisms

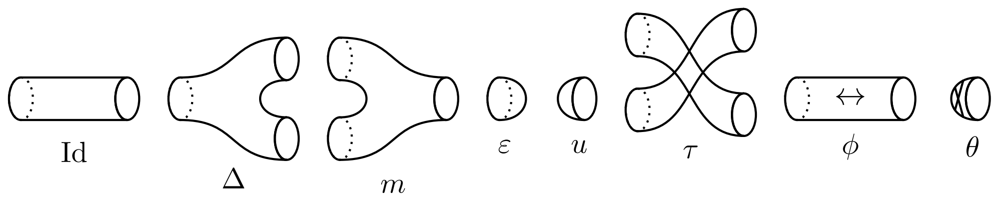
This chapter contains previously published material, which appeared in [12].

In this chapter, when we say $\mathcal{C}$ is a symmetric $\mathbf{k}$-linear monoidal category we are also assuming the bifunctor $\otimes$ is $\mathbf{k}$-bilinear.

5.1 Generators and relations for UCob_2

Consider the category UCob_2 of unoriented 2-dimensional cobordisms, as defined in Definition 2.4.1. In [40, Section 2.2], Tubbenhauer defines a category $\mathrm{uCob}^2{}_R(\emptyset)^*$ by generators and relations, which has an obvious functor to UCob_2. Arguments in [41, Section 2.2] essentially prove that this functor is an equivalence, giving a description by generators and relations of the category of 2-dimensional unoriented cobordisms.

Specifically, every morphism can be obtained by composition (from left to right) and disjoint union (vertical stacking) of the following 8 cobordisms:

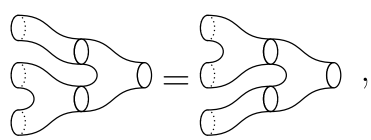


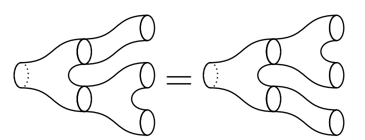
(5.1.1)

The first 6 cobordisms are the usual generators for Cob_2, the category of oriented 2-dimensional cobordisms. As is traditional, we will refer to these morphisms, from left to right, in the following way: identity, pair of pants, reverse pair of pants, cap, cup and twist. The last 2 cobordisms are new: $\phi : 1 \to 1$ denotes the orientation reversing diffeomorphism of the circle, see Example 2.4.2, and $\theta : 0 \to 1$ the once punctured projective plane, also called the Möbius or *crosscap* cobordism, which is non-orientable.

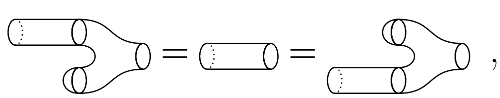
We include below the relations, as described in [40, 41].

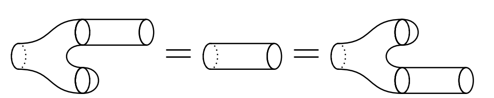
♢ Associativity and coassociativity:



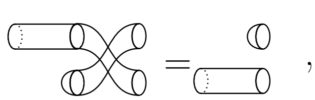


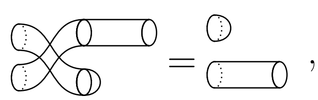
♢ Unit and counit:





(5.1.2)

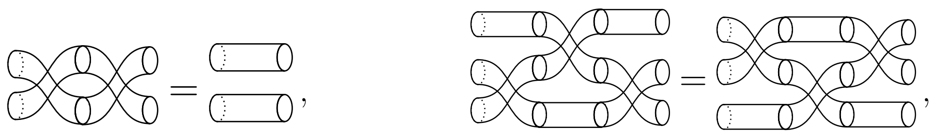




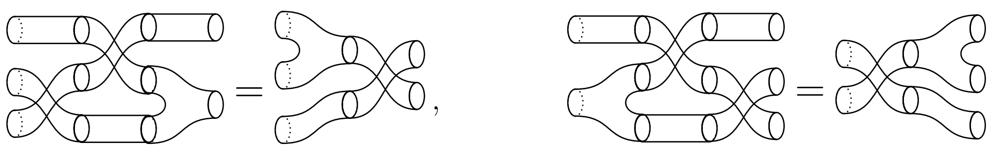
(5.1.3)

♢ Commutativity and cocommutativity:

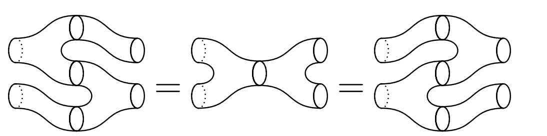
♢ First and second permutation relations:



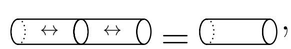
♢ Third permutation relations:



♢ Frobenius relation:



♢ ϕ is an involution:



♢ ϕ is multiplicative and comultiplicative:

(5.1.4)

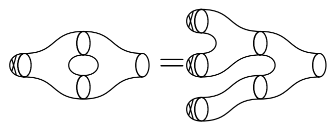
♢ ϕ is unital and counital:

(5.1.5)

◊ Extended Frobenius relations:

(5.1.6)

Using the previous relations we can deduce the following one,



(5.1.7)

which allows us to identify a triple crosscap with a handle with a unique crosscap.

Next we work towards a description of $\mathrm{Hom}_{\mathrm{UCob}_2}(m,n)$ for $m,n \geq 0$. A morphism from m to n has a finite number of closed connected components, and a finite number of connected components with boundary. Each of these can be orientable or unorientable. We consider orientable and unorientable connected components separately in what follows.

Remark 5.1.1. Note that $\mathrm{Hom}_{\mathrm{UCob}_2}(m,n) \cong \mathrm{Hom}_{\mathrm{UCob}_2}(0,n+m)$. This isomorphism is given by bending the left n circles to the rigth, see Figure 5.1.

5.1.1 Orientable connected cobordisms with boundary

The goal of this section is to give a graphical description of orientable connected morphisms, which will be useful later on. We note that orientable morphisms in UCob_2 do not contain any crosscaps.

What we will be doing in the following Proposition is similar to what is done in [30, Section 1.4.16].

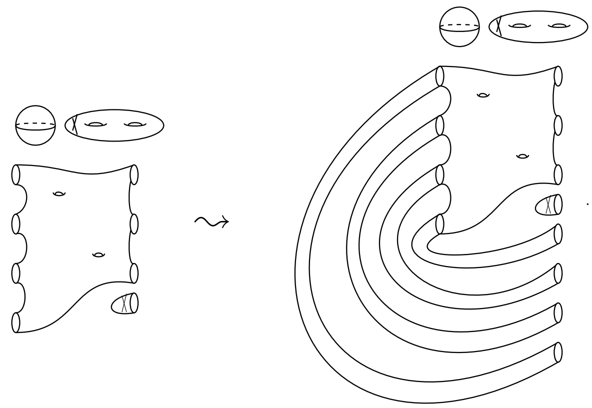
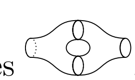


Figure 5.1: Illustration of the isomorphism $\mathrm{Hom}_{\mathrm{UCob}_2}(4,4) \cong \mathrm{Hom}_{\mathrm{UCob}_2}(0,8)$

Proposition 5.1.2. *Any orientable connected cobordism with boundary $m \to n$ in $UCob_2$ can be decomposed into three parts, which we will call the* in, mid *and* out *parts:*

♢ *The* in *part consists of:*

- *If $m = 0$, a cup.*
- *If $m > 0$, a composition of cylinders and reverse pairs of pants. The cobordism starts with m stacked cylinders, each of which is either the identity or the involution ϕ. Following the cylinders we have a composition of multiplication maps $m \to 1$.*

♢ *The* mid *part consists of a composition of* handles *. The number of handles is unique and gives the genus of the cobordism. When the genus is zero, the* mid *part is empty.*

♢ *The* out *part consists of:*

- *If $n = 0$, a cap.*

- *If $n > 0$, a composition of cylinders and pairs of pants. The cobordisms ends with a composition of pairs of pants $1 \to n$, followed by n stacked cylinders, each of which can be either the identity or ϕ.*

Example 5.1.3. The following is an orientable cobordism $4 \to 3$ of genus 3,

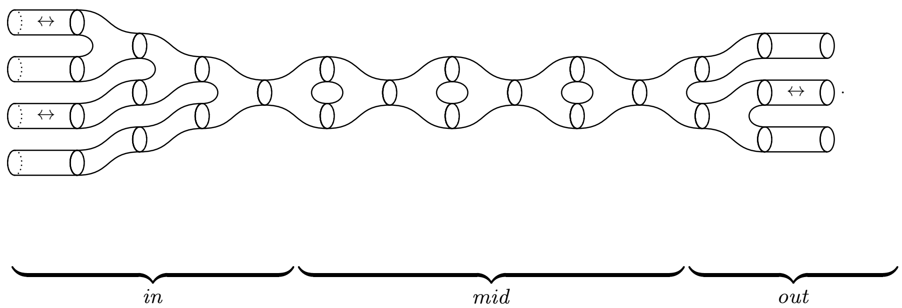


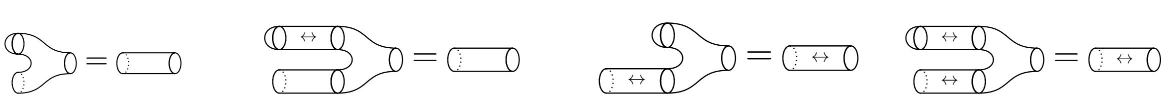
Figure 5.2: Example of an orientable connected cobordism $4 \to 3$.

Proof. Start with any connected orientable cobordism $m \to n$. Our first step will be to move all shapes of the form

(5.1.8)

if any, to the left of the cobordism, so that they consitute the *in* part. In order to do so, we must compute the composition of these shapes with the ones we may have to their left, that is, a cup, a pair of pants, or an orientation reversing cylinder (recall that this cobordism has no crosscaps, as it is orientable). We list below all possible cases.

⋆ Precomposition with a cup:



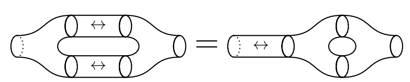
We get cylinders in every case, which we will move to the outmost left in the last step.

⋆ Precomposition with a pair of pants, forming a handle:

If the shape (5.1.8) has one orientation reversing cylinder, the composition would result in a non-orientable surface:

So this case is not possible.

If the shape (5.1.8) has two orientation reversing cylinders, we can use comultiplicativity of ϕ to get an involution composed with a handle,



see relations (5.1.4). The handle will become part of the *mid* part of the cobordism, and we will move the orientation reversing cylinder to the outmost left in the last step.

If the shape (5.1.8) has no orientation reversing cylinders, similarly to the previous case, precomposition with a pair of pants forms a handle, which stays in the *mid* part of the cobordism.

⋆ Precomposition with a pair of pants, without forming a handle: it will look like one of the following,

The picture above shows how we can move shape (5.1.8) to the left in each case.

⋆ Precomposition with an orientation reversing cylinder: this just changes the shape from (5.1.8) into one of the other possible ones shown in (5.1.8).

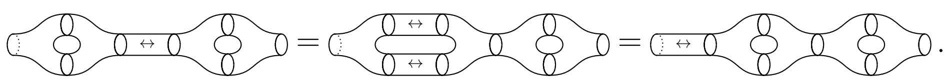
We have covered all the possible cases for precomposition, and so we are done with moving the shapes (5.1.8) to the left of the cobordism, which completes the first step.That is, all reverse pair of pants are now to the left of the cobordism.

Our second step is to move all figures of the form

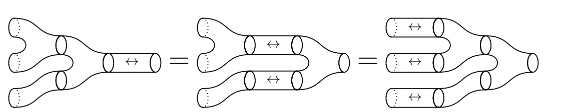
to the right. This can be achieved analogously to the previous step.

Finally, we move any orientation reversing cylinder to either the outmost left or right of the cobordism, as follows.

⋆ We move any involution precomposed with a handle to its left as in the figure below,

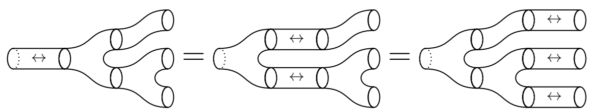


⋆ Then, we move any involution precomposed by a reverse pair of pants (but not a handle) to its left as in the figure below,



(5.1.9)

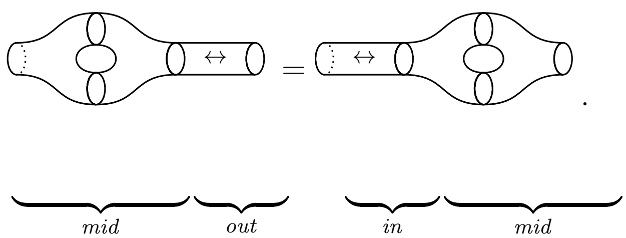
⋆ Lastly, we move any remaining involution composed with a pair of pants to its right as in the figure below,



(5.1.10)

If $m, n \neq 0$, we are done. If $m = 0$, then every orientation reversing cylinder disappears using relation (5.1.5). Moreover, by the relation (5.1.2), which concerns the cup and reverse pair of pants, we are left with just one cup on the *in* part. Analogously, if $n = 0$, using relations (5.1.5) and (5.1.2) concerning the cap, we are left with just one cap on the *out* part. □

Remark 5.1.4. The decomposition described in Proposition 5.1.2 is not unique. For example, the following two morphisms are equal in UCob_2,



Remark 5.1.5. In the case $m = 0$ (respectively, $n = 0$), the *in* part (respectively, the *out* part) consists of just a cap (respectively, just a cup).

Example 5.1.6. Pictured below is a connected orientable cobordism $0 \to 4$,

Definition 5.1.7. Let $\xi^m_{\{i_1,\dots,i_l\}}$ denote the connected cobordism in $\mathrm{Hom}_{\mathrm{UCob}_2}(0, m)$ that has genus zero, and orientation reversing cylinders in its *out* part exactly in the positions $1 \leq i_1 < \cdots < i_l \leq m$, where $1 \leq l \leq m$. That is, if we denote by Δ^m the composition

$$\Delta^{m-1} := (\mathrm{id}^{\otimes(m-2)} \otimes \Delta) \dots (\mathrm{id} \otimes \Delta)\Delta : 1 \to m,$$

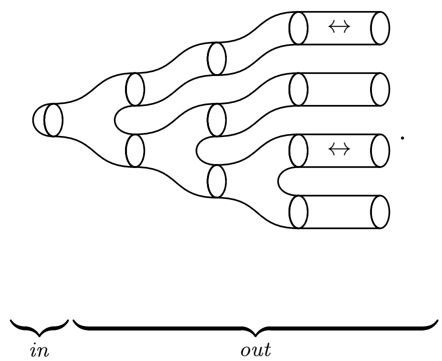


Figure 5.3: Example of a connected orientable cobordism $0 \to 4$.

and by $\phi_{i_1,\dots,i_l}$ the cobordism $m \to m$ given by

$$\phi_{i_1,\dots,i_l} = c_1 \otimes \cdots \otimes c_m, \quad \text{where } c_j = \begin{cases} \mathrm{id} & \text{if } j \notin \{i_1,\dots,i_l\} \\ \\ \phi & \text{if } j \in \{i_1,\dots,i_l\}, \end{cases}$$

then

$$\xi^m_{\{i_1,\dots,i_l\}} := \phi_{i_1,\dots,i_l} \Delta^{m-1} u.$$

For instance, the cobordism in the previous example is $\xi^4_{\{1,3\}}$.

Lemma 5.1.8. *If we have a partition* $\{1,\dots,m\} = \{i_1,\dots,i_l\} \sqcup \{j_1,\dots,j_s\}$, *then*

$$\xi^m_{\{i_1,\dots,i_l\}} = \xi^m_{\{j_1,\dots,j_s\}}.$$

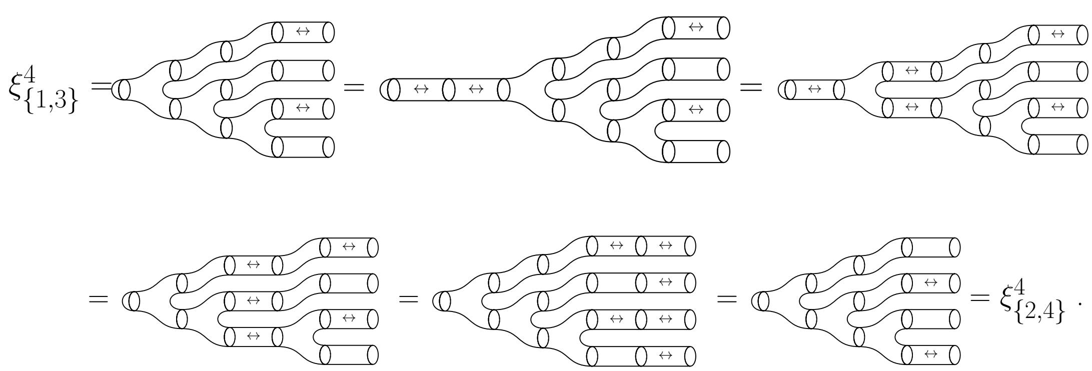
Proof. Let $\{1,\dots,m\} = \{i_1,\dots,i_l\} \sqcup \{j_1,\dots,j_s\}$. Recall that by relation (5.1.4), $\Delta\phi =$

$(\phi \otimes \phi)\Delta$. In general, this implies that $\Delta^{m-1}\phi = \phi^{\otimes m}\Delta^{m-1}$. We compute

$$\begin{aligned}
\xi^m_{i_1,\ldots,i_l} &= \phi_{i_1,\ldots,i_l}\Delta^{m-1}u \\
&= \phi_{i_1,\ldots,i_l}\Delta^{m-1}\phi^2 u \\
&= \phi_{i_1,\ldots,i_l}\phi^{\otimes m}\Delta^{m-1}\phi u \\
&= (c_1\phi \otimes \ldots c_m\phi)\Delta^{m-1}u, \\
&= \xi^m_{j_1,\ldots,j_s},
\end{aligned}$$

where we are also using the relations $\phi^2 = \mathrm{id}$, $\phi u = u$, and that $c_k\phi = \phi$ if $k \notin \{i_1,\ldots,i_l\}$ and $c_k\phi = \mathrm{id}$ if $k \in \{i_1,\ldots,i_l\}$. □

Example 5.1.9. To illustrate the previous Lemma, we show that $\xi^4_{\{1,3\}} = \xi^4_{\{2,4\}}$ using graphical calculus:



Remark 5.1.10. We note that Lemma 5.1.8 generalizes to any genus. That is, if $\xi^m_{\{i_1,\ldots,i_l\},g}$ denotes the connected cobordism in $\mathrm{Hom}_{\mathrm{UCob}_2}(0,m)$ that has genus $g \geq 0$, and orientation reversing cylinders in its *out* part exactly in the positions $1 \leq i_1 < \cdots < i_l \leq m$, then

$$\xi^m_{\{i_1,\ldots,i_l\},g} = \xi^m_{\{j_1,\ldots,j_s\},g},$$

where $\{1,\ldots,m\} = \{i_1,\ldots,i_l\} \sqcup \{j_1,\ldots,j_s\}$. We leave the proof to the reader.

5.1.2 Unorientable connected cobordisms with boundary

We describe now connected cobordisms with boundary in UCob_2 that have at least one crosscap, i.e., unorientable ones.

Proposition 5.1.11. *Any unorientable connected cobordism with bounday $m \to n$ in $UCob_2$ can be decomposed in three parts, as follows:*

◇ *The* in *part, which consists of one or two crosscaps, followed by a composition of reverse pairs of pants.*

◇ *The* mid *part, given by a composition of handles (this part is empty when the genus is zero).*

◇ *The* out *part, which is either:*

- *a cap, when $n = 0$, or*
- *a composition of pairs of pants $1 \to n$.*

Example 5.1.12. Shown in Figure 5.1.2 is an example of a connected unorientable cobordism $2 \to 3$ of genus 3 and two crosscaps.

Proof. Unorientable cobordisms have at least one crosscap. By relation (5.1.7), if we have three crosscaps we can replace them by a handle with a crosscap. Hence, we can always reduce to having either one or two crosscaps. On the other hand, relations (5.1.6) can be used to get rid of orientation reversing cylinders, as they either disappear or get replaced by two crosscaps. Lastly, analogously to the orientable case in Proposition 5.1.2, we can move pairs of pants (respectively, reverse pairs of pants) to the *out* part (respectively, to the *in* part) forming handles in the *mid* part. □

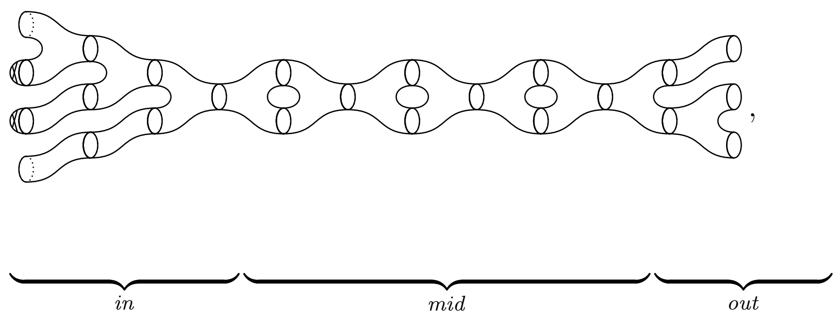


Figure 5.4: Example of a connected unorientable cobordism $2 \to 3$.

5.1.3 Closed connected components.

It will be convenient to divide closed connected surfaces in three types, according to their number of crosscaps.

♢ Type 1: orientable surfaces of genus g, denoted M_g, for all $g \geq 0$. See the figure below illustrations for $g = 1, 2$ and 3, respectively:

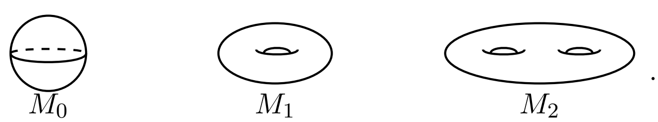


Figure 5.5: Orientable closed surfaces with no crosscaps

♢ Type 2: unorientable surfaces with one crosscap and genus g, denoted M_g^1, for all $g \geq 0$. See below illustrations for $g = 1, 2$ and 3, respectively:

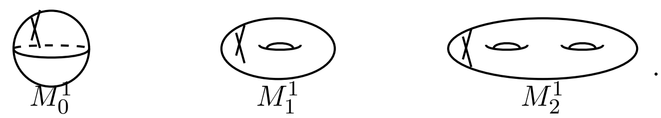


Figure 5.6: Unorientable closed surfaces with one crosscap

♢ Type 3: unorientable surfaces with two crosscaps and genus g, denoted M_g^2, for all $g \geq 0$. See below illustrations for $g = 1, 2$ and 3, respectively:

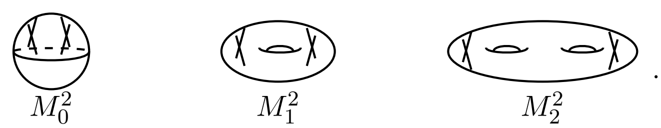


Figure 5.7: Unorientable closed surfaces with two crosscaps

5.2 The category $\mathbf{VUCob}_{\alpha,\beta,\gamma}$

Given three sequences $\alpha = (\alpha_0, \alpha_2, \dots)$, $\beta = (\beta_0, \beta_1, \dots)$ and $\gamma = (\gamma_0, \gamma_1, \dots)$ with $\alpha_i, \beta_i, \gamma_i \in \mathbf{k}$, we define a linearization of the category UCob_2, denoted by $\mathrm{VUCob}_{\alpha,\beta,\gamma}$. This is the analogue of the linearization of Cob_2 by a sequence α denoted VCob_α in [27] and Cob'_α in [28].

Definition 5.2.1. We define $\mathrm{VUCob}_{\alpha,\beta,\gamma}$ as the category with:

- Objects: Same as in UCob_2, objects are non-negative integers.

- Morphisms: Morphisms $m \to n$ are $\mathbf{k}$-linear combinations of unoriented 2-cobordisms $m \to n$, modulo the following relations. For the connected closed cobordisms M_g, M_g^1 and M_g^2 as in Subsection 5.1.3, we set

$$M_g = \alpha_g, \qquad M_g^1 = \beta_g \qquad \text{and} \qquad M_g^2 = \gamma_g.$$

- Composition: Given by glueing as induced from UCob_2, followed by evaluating closed components.

By our definition, a closed surface M in UCob_2 is evaluated to

$$M \mapsto \prod_{g \geq 0} \alpha_g^{a_g} \beta_g^{b_g} \gamma_g^{c_g}, \tag{5.2.1}$$

in $\mathrm{VUCob}_{\alpha,\beta,\gamma}$, where $a_g, b_g, c_g \in \mathbb{Z}_{\geq 0}$ denote the number of connected components of type M_g, M_g^1 and M_g^2 of M, respectively.

The category $\mathrm{VUCob}_{\alpha,\beta,\gamma}$ is a rigid symmetric $\mathbf{k}$-linear monoidal category, with tensor product and braiding induced from those on UCob_2. We call a 2-dimensional cobordism *viewable* if it has no closed components. Thus morphisms $m \to n$ in $\mathrm{VUCob}_{\alpha,\beta,\gamma}$ are linear combinations of unoriented viewable cobordisms $m \to n$. Since a cobordism can have any number of handles, Hom spaces in this category are infinite-dimensional.

Definition 5.2.2. Let $n \geq 0$. We define a trace

$$\mathrm{tr} = \mathrm{tr}_{\alpha,\beta,\gamma} : \mathrm{Hom}_{\mathrm{VUCob}_{\alpha,\beta,\gamma}}(n, n) \to \mathbf{k},$$

as follows. Closing an unoriented cobordism M from $n \to n$ by connecting its n source circles with its n target circles via n annuli results in an unoriented closed connected surface M'. Then $\mathrm{tr}_{\alpha,\beta,\gamma}(M)$ is the evaluation of M' as in equation (5.2.1).

Example 5.2.3. Consider the map $M' \in \mathrm{Hom}(2,2)$ given in the following picture,

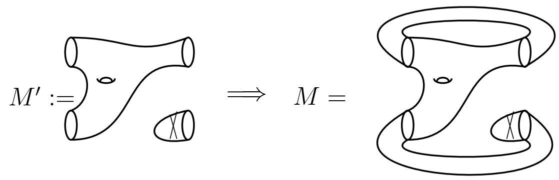


Figure 5.8: Closing a cobordism by attaching annuli

In this case, $\mathrm{tr}(M') = \beta_2$.

Definition 5.2.4. The sequences (α, β, γ) determine a $\mathbf{k}$-bilinear symmetric form

$$(\cdot,\cdot)_{\alpha,\beta,\gamma} : \mathrm{Hom}_{\mathrm{VUCob}_{\alpha,\beta,\gamma}}(0,m) \times \mathrm{Hom}_{\mathrm{VUCob}_{\alpha,\beta,\gamma}}(0,m) \to \mathbf{k},$$

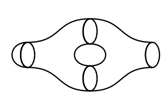
as follows. Given S_1, S_2 unoriented cobordisms $0 \to m$, define

$$(S_1, S_2)_{\alpha,\beta,\gamma} := (S_1, S_2) = \mathrm{tr}_{\alpha,\beta,\gamma}((-S_1) \sqcup S_2),$$

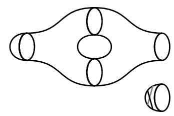
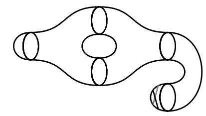
where $(-S_1) \sqcup S_2$ is the closed surface obtained by gluing the m target circles of S_1 with the respective m target circles of S_2, and $\mathrm{tr}_{\alpha,\beta,\gamma}$ is as in Definition 5.2.2. Then extend linearly to all of $\mathrm{Hom}_{\mathrm{VUCob}_{\alpha,\beta,\gamma}}(0,m)$.

The notion of trace above is the analogue of the one defined for oriented cobordisms in [27, Section 2.1].

Example 5.2.5. Consider the cobordisms

$M :=$  and $N :=$  .

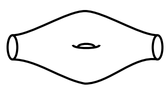
Then

$$M \sqcup N =$$  $\rightsquigarrow$  ,

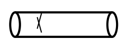
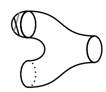
and the last cobordism has trace β_1. Hence

$$(M, N) = \beta_1.$$

We distinguish the following morphisms in $\mathrm{Hom}_{\mathrm{VUCob}_{\alpha,\beta,\gamma}}(1, 1)$:

<u>Handle</u>  ,

$x : 1 \xrightarrow{\Delta} 1 \otimes 1 \xrightarrow{m} 1$

<u>Cross</u>  $:=$  ,

$y : 1 \xrightarrow{\sim} 0 \otimes 1 \xrightarrow{\theta \otimes 1} 1 \otimes 1 \xrightarrow{m} 1$

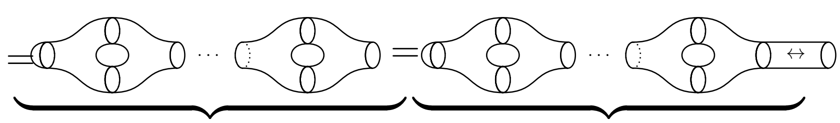
<u>Cup-Cap</u>  . (5.2.2)

$c : 1 \xrightarrow{\epsilon} 0 \xrightarrow{u} 1$

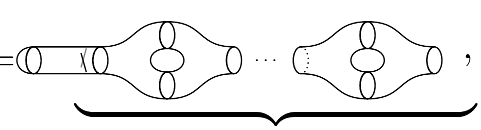
Note that $x^n, x^n y$ and $x^n y^2$ are connected cobordisms $1 \to 1$ of genus n and $0, 1$ and 2 crosscaps, respectively. To take trace, we close these cobordisms by an annulus connecting their in and out boundaries, obtaining a connected closed surface of genus $n+1$ and $0, 1$ and 2 crosscaps, respectively, see Definition 5.2.2. Hence

$$\mathrm{tr}(x^n) = \alpha_{n+1}, \qquad \mathrm{tr}(x^n y) = \beta_{n+1} \qquad \text{and} \qquad \mathrm{tr}(x^n y^2) = \gamma_{n+1}, \qquad \text{for all } n \in \mathbb{Z}_{\geq 0}.$$

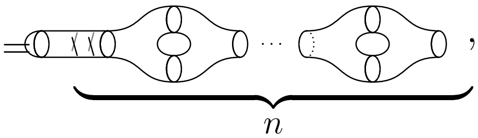
Example 5.2.6. The cobordisms

$x^n u =$  ,

n $\quad$ n

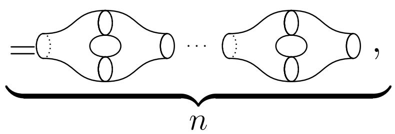
$x^n y u =$  ,

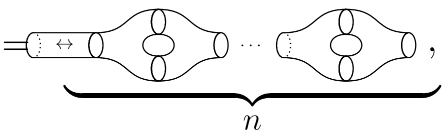
n

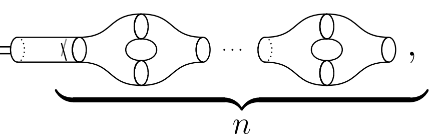
$x^n y^2 u =$ 

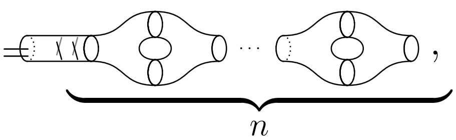
for $n \geq 0$, generate $\mathrm{Hom}_{\mathrm{VUCob}_{\alpha,\beta,\gamma}}(0,1)$ as a vector space, see Propositions 5.1.2 and 5.1.11.

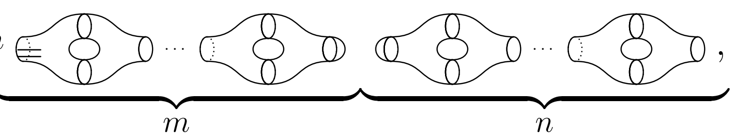
Example 5.2.7. The cobordisms

$x^n =$ 

, $x^n \phi =$ 

$x^n y =$ 

, $x^n y^2 =$ 

$y^i x^n c x^m y^j$, where $x^n c x^m =$ 

for $n \geq 0$ and $i, j = 0, 1, 2$, generate $\mathrm{End}_{\mathrm{VUCob}_{\alpha,\beta,\gamma}}(1)$ as a vector space, see Propositions 5.1.2 and 5.1.11.

5.2.1 A distinguished subset in $\mathrm{Hom}_{\mathbf{VUCob}_{\alpha,\beta,\gamma}}(0, m)$

We prove a technical result in Theorem 5.2.10, describing a linearly independent subset of orientable maps in $\mathrm{Hom}_{\mathrm{VUCob}_{\alpha,\beta,\gamma}}(0, m)$ for all $m \in \mathbb{Z}_{\geq 0}$, which will be useful later.

Let $m \in \mathbb{Z}_{\geq 0}$ and let $\mathcal{P}_m$ denote the power set of $\{1, \ldots, m\}$. We define an equivalence relation $\sim$ in $\mathcal{P}_m$ by

$$J \sim I \text{ iff } J = I \text{ or } J = I^c, \qquad \text{for all } J \in \mathcal{P}_m.$$

Let $\mathcal{R}_m$ denote the set of equivalence classes given by this relation. Note that the size of this set is $|\mathcal{R}_m| = 2^{m-1}$ if $m > 0$.

Definition 5.2.8. Let $\xi^m_{\{i_1,\dots,i_l\}} \in \mathrm{Hom}_{\mathrm{VUCob}_{\alpha,\beta,\gamma}}(0,m)$ denote the connected cobordism that has genus zero and orientation reversing cylinders exactly in the positions $1 \leq i_1, \dots, i_l \leq m$, for $1 \leq l \leq m$, see Definition 5.1.7. By Lemma 5.1.8, for each $\overline{J} \in \mathcal{R}_m$ we have a well-defined connected cobordism

$$\xi^m_{\overline{J}} : 0 \to m,$$

where $\overline{J}$ denotes the class of $J \in \mathcal{P}_m$ in $\mathcal{R}_m$.

We will need the following auxiliary Lemma.

Lemma 5.2.9. *Let $a, b \in \mathbf{k}^\times$, and let A be the $n \times n$ matrix given by*

$$A := \begin{bmatrix} a & b & \dots & b \\ b & a & \ddots & b \\ \vdots & \ddots & \ddots & \vdots \\ b & \dots & b & a \end{bmatrix}.$$

That is, A has a's on the diagonal and b's everywhere else. Then

$$\det(A) = (a-b)^{n-1} \cdot (a + (n-1)b).$$

Proof. We start by adding columns 2 to n to the first column in the matrix, obtaining

$$\begin{bmatrix} a+(n-1)b & b & \dots & b \\ a+(n-1)b & a & \ddots & b \\ \vdots & & \ddots & \vdots \\ a+(n-1)b & b & \dots & a \end{bmatrix}.$$

Now, substracting from rows 2 to n the first row, we get

$$\begin{bmatrix} a+(n-1)b & b & \dots & & b \\ 0 & a-b & 0 & \dots & 0 \\ \vdots & 0 & & \ddots & \vdots \\ & & \ddots & \ddots & 0 \\ 0 & 0 & \dots & 0 & a-b \end{bmatrix},$$

which has determinant $(a-b)^{n-1} \cdot (a+(n-1)b)$, as desired. □

Theorem 5.2.10. *Let α, β and γ be sequences in $\mathbf{k}$, and let $m \geq 2$. Suppose that*

$$\alpha_{m-1} \neq \gamma_{m-2}, (1-2^{m-1})\gamma_{m-2}. \tag{5.2.3}$$

Then the set $\{\xi^m_{\overline{J}} : \overline{J} \in \mathcal{R}_m\}$, as given in Definition 5.2.8, is a linearly independent subset of $\mathrm{Hom}_{\mathrm{VUCob}_{\alpha,\beta,\gamma}}(0, m)$.

Proof. Fix $m \geq 2$. We want to compute the matrix of inner products (see Definition 5.2.4) for cobordisms in the set $\{\xi^m_{\overline{J}} : \overline{J} \in \mathcal{R}_m\}$.

Let $\overline{J}, \overline{L} \in \mathcal{R}_m$, with $\overline{J} \neq \overline{L}$. Since ϕ is an involution then the inner product of $\xi^m_{\overline{J}}$ with itself results in the surface of type M_{m-1}, see Subsection 5.1.3, which evaluates to α_{m-1}. On the other hand, relations (5.1.6) imply that the inner product of $\xi^m_{\overline{J}}$ and $\xi^m_{\overline{L}}$ results in the surface of type M^2_{m-2}, which evaluates to γ_{m-2}. Hence, the resulting matrix of inner products is the $2^{m-1} \times 2^{m-1}$ matrix given by

$$M := \begin{bmatrix} \alpha_{m-1} & \gamma_{m-2} & \dots & \gamma_{m-2} \\ \gamma_{m-2} & \alpha_{m-1} & \ddots & \gamma_{m-2} \\ \vdots & \ddots & \ddots & \vdots \\ \gamma_{m-2} & \dots & \gamma_{m-2} & \alpha_{m-1} \end{bmatrix}.$$

By Lemma 5.2.9, this matrix has determinant

$$\det(M) = (\alpha_{m-1} - \gamma_{m-2})^{2^{m-1}-1} \cdot (\alpha_{m-1} + (2^{m-1} - 1)\gamma_{m-2}),$$

and thus cobordisms in the set $\{\xi^m_{\overline{J}} : \overline{J} \in \mathcal{R}_m\}$ are linearly independent, since by assumption

$$\alpha_{m-1} \neq \gamma_{m-2} \qquad \text{and} \qquad \alpha_{m-1} \neq (1 - 2^{m-1})\gamma_{m-2}.$$

□

We generalize the previous result to any genus. For $g \in \mathbb{N}$. Consider the maps $\xi^m_{\overline{\{i_1,\ldots,i_l\}},g} \in \mathrm{Hom}_{\mathrm{VUCob}_{\alpha,\beta,\gamma}}(0,m)$, where $\xi^m_{\overline{\{i_1,\ldots,i_l\}},g}$ represents the connected cobordism of genus g that has orientation reversing cylinders exactly in the positions $1 \leq i_1 < \cdots < i_l \leq m$, where $1 \leq l \leq m$. Note that this is well defined by Remark 5.1.10.

Theorem 5.2.11. *Let α, β and γ be sequences in $\mathbf{k}$, and let $m \geq 2$, $g \geq 1$. Suppose that*

$$\alpha_{2g+m-1} \neq \gamma_{m-2}, (1 - 2^{m-1})\gamma_{2g+m-2}. \tag{5.2.4}$$

Then the set $\{\xi^m_{\overline{J},g} : \overline{J} \in \mathcal{R}_m\}$ is a linearly independent subset of $\mathrm{Hom}_{\mathrm{VUCob}_{\alpha,\beta,\gamma}}(0,m)$.

Proof. Fix $m \geq 2, g \geq 1$. We want to compute the matrix of inner products of the set $\{\xi^m_{\overline{J},g} : \overline{J} \in \mathcal{R}_m\}$. Same as in the proof of Proposition 5.2.10, this matrix has size $2^{m-1} \times 2^{m-1}$ and is given by

$$M := \begin{bmatrix} \alpha_{2g+m-1} & \gamma_{2g+m-2} & \cdots & \gamma_{2g+m-2} \\ \gamma_{2g+m-2} & \alpha_{2g+m-1} & \ddots & \gamma_{2g+m-2} \\ \vdots & \ddots & \ddots & \vdots \\ \gamma_{2g+m-2} & \cdots & \gamma_{2g+m-2} & \alpha_{2g+m-1} \end{bmatrix}.$$

Thus by Lemma 5.2.9 it has determinant

$$\det(M) = (\alpha_{2g+m-1} - \gamma_{2g+m-2})^{2^{m-1}-1} \cdot (\alpha_{2g+m-1} + (2^{m-1}-1)\gamma_{2g+m-2}).$$

The result follows. □

5.3 The skein category $\mathbf{SUCob}_{\alpha,\beta,\gamma}$

Given three sequences $\alpha = (\alpha_0, \alpha_2, \dots)$, $\beta = (\beta_0, \beta_1, \dots)$, and $\gamma = (\gamma_0, \gamma_1, \dots)$, with $\alpha_i, \beta_i, \gamma_i \in \mathbf{k}$, we define a rigid symmetric $\mathbf{k}$-linear monoidal category with finite dimensional Hom spaces, denoted by $\mathrm{SUCob}_{\alpha,\beta,\gamma}$. This is the analogue of the category SCob_α in [27], and Cob_α in [28].

Definition 5.3.1. We say that a sequence $\eta = (\eta_0, \eta_1, \dots)$ in $\mathbf{k}$ satisfies a *linear recurrence* (or is *linearly recurrent*) if there exist fixed $K \geq r$ and $a_1, \dots, a_r \in \mathbf{k}$, such that

$$\eta_l = a_1\eta_{l-1} + \cdots + a_r\eta_{l-r}, \quad \text{for all } l \geq K. \tag{5.3.1}$$

A sequence η is linearly recurrent if and only if it has a rational generating function

$$Z_\eta(T) = \frac{p_\eta(T)}{q(T)} = \sum_{k\geq 0} \eta_k T^k,$$

where $p_\eta(T), q(T) \in \mathbf{k}[T]$ are relatively prime, see for example [25]. Normalizing so that $q(0) = 1$, the polynomials $p_\eta(T)$ and $q(T)$ are uniquely determined by the sequence η. Assume from now on that

$$Z_\eta(T) = \frac{p_\eta(T)}{q(T)} = \sum_{k\geq 0} \eta_k T^k, \quad (p_\eta(T), q(T)) = 1, \quad q(0) = 1, \tag{5.3.2}$$

and let

$$N = \deg(p_\eta(T)), \qquad M = \deg(q(T)), \qquad \text{and} \qquad K = \max(N+1, M). \tag{5.3.3}$$

Then, if

$$q(T) = 1 - a_1 T + a_2 T^2 + \cdots + (-1)^M a_M T^M, \tag{5.3.4}$$

we have that

$$\eta_l = a_1 \eta_{l-1} - a_2 \eta_{l-2} + \cdots + (-1)^{M-1} a_M \eta_{l-M}, \quad \text{for all } l \geq K. \tag{5.3.5}$$

Now we apply this to our sequences (α, β, γ). We want to take the category $\mathrm{VUCob}_{\alpha,\beta,\gamma}$ and quotient by relations defined by the sequence α in order to get finite dimensional Hom spaces.

Assume from now on that α is linearly recurrent, with generating function satisfying (5.3.2) and (5.3.4). Recall that in the category $\mathrm{VUCob}_{\alpha,\beta,\gamma}$ we have a trace, see Definition 5.2.2. Let x denote the handle cobordism, as defined on (5.2.2). It follows from the linear recurrence on α, see equation (5.3.5), that the trace of the map σ given by

$$\sigma := x^K + \sum_{i=1}^{M} (-1)^i a_i x^{K-i}, \tag{5.3.6}$$

where K is as in (5.3.3), is zero in $\mathrm{VUCob}_{\alpha,\beta,\gamma}$, see also [26, Section 2.4]. We call the equation $\sigma = 0$ the *handle relation*.

We want to show that by imposing conditions on β and γ, the quotient of $\mathrm{VUCob}_{\alpha,\beta,\gamma}$ by the tensor ideal generated by σ will be non-trivial. Recall that the set of all negligible morphisms in a spherical category $\mathcal{C}$ is a proper tensor ideal, see Section 2.2.12. Hence, if σ is negligible, then the quotient of $\mathrm{VUCob}_{\alpha,\beta,\gamma}$ by the tensor ideal generated by σ is non-trivial.

We prove in what follows that, under certain conditions on α, β and γ, the handle morphism σ is negligible. We will need the following auxiliary Lemma.

Lemma 5.3.2. *Fix an integer $B \geq 0$. Let η be a linearly recurrent sequence in $\mathbf{k}$, with generating function*

$$Z_\eta(T) = \frac{p_\eta(T)}{q(T)}, \quad \textit{where} \;\; p_\eta(T), q(T) \in \mathbf{k}[T].$$

Let $q(T) = q_0 + q_1 T + \cdots + q_L T^L$, for $q_0, \ldots, q_{L-1} \in \mathbf{k}$ and $q_L \in \mathbf{k}^\times$. A sequence μ in $\mathbf{k}$ satisfies

$$\sum_{j=0}^{l} q_{l-j}\mu_j = 0, \quad \textit{for all } l \geq B,$$

for some $B \geq 0$, if and only if it has a generating function

$$Z_\mu(T) = \frac{p_\mu(T)}{q(T)},$$

for some $p_\mu(T) \in \mathbf{k}[T]$ with $\deg(p_\mu(T)) < B$.

Proof. Define

$$p_\mu(T) := \left(\sum_{j\geq 0} \mu_j T^j\right) q_\eta(T).$$

Then

$$\sum_{j=0}^{l} q_{l-j}\mu_j = 0, \quad \text{for all } l \geq B,$$

is equivalent to $p_\mu(T)$ being a polynomial of degree at most $B-1$, as desired. ☐

As a consequence, we get the following.

Lemma 5.3.3. *Consider a linearly recurrent sequence α in $\mathbf{k}$, with generating function satisfying* (5.3.2), (5.3.3) *and* (5.3.4). *Let β and γ be sequences in $\mathbf{k}$ with generating functions*

$$Z_\beta(T) = \frac{p_\beta(T)}{q(T)} = \sum_{k\geq 0} \beta_k T^k \qquad \text{and} \qquad Z_\gamma(T) = \frac{p_\gamma(T)}{q(T)} = \sum_{k\geq 0} \gamma_k T^k,$$

respectively, where $p_\beta(T), p_\gamma(T) \in \mathbf{k}[T]$ satisfy

$$\deg(p_\beta(T)), \deg(p_\gamma(T)) < K = \max(N+1, M).$$

Then the handle morphism σ from (5.2.2) *is negligible in* $\mathrm{VUCob}_{\alpha,\beta,\gamma}$.

Proof. We want to show that the trace of $\sigma \circ z$ is zero for every $z \in \mathrm{End}_{\mathrm{VUCob}_{\alpha,\beta,\gamma}}(1,1)$. By Example 5.2.7, it is enough to show that the traces of $\sigma x^n y^i$, $\sigma x^n \phi$ and $\sigma y^i x^m u x^n y^j$ are zero for all $m, n \geq 0$ and $i, j = 0, 1, 2$.

As in Equation (5.3.5), we have that

$$\alpha_l + \sum_{s=1}^{M} (-1)^s a_s \alpha_{l-s} = 0, \quad \text{for all } l \geq K.$$

On the other hand, since $\deg(p_\beta(T)), \deg(p_\gamma(T)) \leq K$, taking $B = K$ in Lemma 5.3.2 we get

$$\beta_l + \sum_{s=1}^{M} (-1)^s a_s \beta_{l-s} = 0 \qquad \text{and} \qquad \gamma_l + \sum_{s=1}^{M} (-1)^s a_s \gamma_{l-s} = 0, \quad \text{for all } l \geq K.$$

Using this, we compute

$$tr(\sigma y^i x^n) = \begin{cases} \alpha_{K+n+1} - \sum\limits_{s=1}^{M} a_s \alpha_{K+n+1-s} = 0 & \text{if } i = 0, \\ \beta_{K+n+1} - \sum\limits_{s=1}^{M} a_s \beta_{K+n+1-s} = 0 & \text{if } i = 1, \\ \gamma_{K+n+1} - \sum\limits_{s=1}^{M} a_s \gamma_{K+n+1-s} = 0 & \text{if } i = 2. \end{cases}$$

On the other hand

$$tr(\sigma y^i x^m u x^n y^j) = \begin{cases} \alpha_{K+n+m} - \sum_{s=1}^{M} a_s \alpha_{K+n+m-s} = 0 & \text{if } i=j=0, \\ \beta_{K+n+m} - \sum_{s=1}^{M} a_s \beta_{K+n+m-s} = 0 & \text{if } i=0, j=1, \text{ or } i=1, j=0, \\ \gamma_{K+n+m} - \sum_{s=1}^{M} a_s \gamma_{K+n+m-s} = 0 & \text{if } i=2, j=0, \text{ or } i=0, j=2, \\ & \text{or } i=j=1, \\ \beta_{K+n+m+1} - \sum_{s=1}^{M} a_s \beta_{K+n+m+1-s} = 0 & \text{if } i=2, j=1, \text{ or } i=1, j=2, \\ \gamma_{K+n+m+1} - \sum_{s=1}^{M} a_s \gamma_{K+n+m+1-s} = 0 & \text{if } i=2, j=2. \end{cases}$$

Lastly, since closing the cobordism $x^n\phi$ via an annulus between its boundary circles results in the surface $ux^{n+1}\epsilon$, then $\mathrm{tr}(\sigma x^n \phi) = \mathrm{tr}(\sigma x^n) = 0$ by the equations above, for all $n \geq 0$.

Thus σ is negligible, as desired. □

Definition 5.3.4. Let (α, β, γ) be as in Lemma 5.3.3. We define the unoriented *skein* category $\mathrm{SUCob}_{\alpha,\beta,\gamma}$ as the quotient of $\mathrm{VUCob}_{\alpha,\beta,\gamma}$ by the tensor ideal generated by σ.

The category $\mathrm{SUCob}_{\alpha,\beta,\gamma}$ is a non-trivial rigid symmetric **k**-linear monoidal category, with tensor product and braiding induced from $\mathrm{VUCob}_{\alpha,\beta,\gamma}$. Hom spaces in $\mathrm{SUCob}_{\alpha,\beta,\gamma}$ consist of linear combinations of unoriented cobordisms whose connected components have genus strictly less than K. Thus they are finite dimensional.

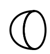
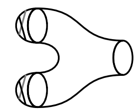
Example 5.3.5. Consider the sequences α, β, γ with generating functions

$$Z_\alpha(t) = \frac{\alpha_0}{1-\lambda t}, \qquad Z_\beta(t) = \frac{\beta_0}{1-\lambda t}, \qquad \text{and} \qquad Z_\gamma(t) = \frac{\gamma_0}{1-\lambda t},$$

respectively, for $\alpha_0, \beta_0, \gamma_0, \lambda \in \mathbf{k}^\times$. That is,

$$\alpha = (\alpha_0, \lambda\alpha_0, \lambda^2\alpha_0, \dots), \quad \beta = (\beta_0, \lambda\beta_0, \lambda^2\beta_0, \dots), \quad \text{and} \quad \gamma = (\gamma_0, \lambda\gamma_0, \lambda^2\gamma_0, \dots).$$

Suppose that $\lambda\alpha_0 \neq \gamma_0$ and $\gamma_0 \neq \pm\sqrt{\lambda}\beta_0$. Then $\mathrm{Hom}_{\mathrm{SUCob}_{\alpha,\beta,\gamma}}(0,1)$ has dimension 3, with basis

$u =$  $\quad \theta =$ , $\quad \theta[2] =$  $:=$ .

In fact, in this case $K = 1$ and the handle relation is

$$x - \lambda \mathrm{id} = 0,$$

see Equation (5.3.6). Hence, by definition, morphisms in $\mathrm{SUCob}_{\alpha,\beta,\gamma}$ are linear combinations of cobordisms whose connected components have no handles. Hence, by Propositions 5.1.2 and 5.1.11, the set $\{u, \theta, \theta[2]\}$ generates $\mathrm{Hom}_{\mathrm{SUCob}_{\alpha,\beta,\gamma}}(0,1)$. Moreover, the matrix of inner product (see Definition 5.2.4) of this set is given by

$$A := \begin{bmatrix} \alpha_0 & \beta_0 & \gamma_0 \\ \beta_0 & \gamma_0 & \lambda\beta_0 \\ \gamma_0 & \lambda\beta_0 & \lambda\gamma_0 \end{bmatrix},$$

with determinant

$$\det(A) = \alpha_0(\lambda\gamma_0^2 - \lambda^2\beta_0^2) + \gamma_0(\lambda\beta_0^2 - \gamma_0^2) = (\alpha_0\lambda - \gamma_0)(\gamma_0^2 - \lambda\beta_0^2),$$

which is nonzero by our assumptions on α_0, β_0 and γ_0. Hence $\{u, \theta, \theta[2]\}$ is a basis for $\mathrm{Hom}_{\mathrm{SUCob}_{\alpha,\beta,\gamma}}(0,1)$.

Definition 5.3.6. Let (α, β, γ) be as in Lemma 5.3.3. We denote by $\mathrm{UCob}_{\alpha,\beta,\gamma}$ the pseudo-abelian envelope of $\mathrm{SUCob}_{\alpha,\beta,\gamma}$, i.e.,

$$\mathrm{UCob}_{\alpha,\beta,\gamma} = \mathcal{K}(\mathcal{A}(\mathrm{SUCob}_{\alpha,\beta,\gamma})),$$

see Definition 2.2.10.

Thus, $\mathrm{UCob}_{\alpha,\beta,\gamma}$ is a pseudo-abelian (that is, additive and idempotent-complete) rigid symmetric $\mathbf{k}$-linear monoidal category, with finite dimensional Hom spaces. Our main results, Theorems I and II, describe the relationship between these categories and $\mathbf{Rep}(S_t \wr \mathbb{Z}_2)$.

5.4 Universal properties

In this section we state the universal properties of the categories UCob_2, $\mathrm{VUCob}_{\alpha,\beta,\gamma}$ and $\mathrm{SUCob}_{\alpha,\beta,\gamma}$. Let $\mathcal{C}$ be a symmetric monoidal category and let A be an Frobenius algebra in $\mathcal{C}$. Consider the following morphisms in $\mathrm{End}_{\mathcal{C}}(A)$,

$$\begin{aligned} x &: A \xrightarrow{\Delta_A} A \otimes A \xrightarrow{m_A} A, \quad \text{and} \\ y &: A \xrightarrow{\sim} 1 \otimes A \xrightarrow{\theta_A \otimes 1} A \otimes A \xrightarrow{m_A} A. \end{aligned} \tag{5.4.1}$$

We define

$$a_n := \epsilon_A x^n u_A, \quad b_n := \epsilon_A x^n y u_A, \quad \text{and } c_n := \epsilon_A x^n y^2 u_A \in \mathrm{End}_{\mathcal{C}}(\mathbb{1}_{\mathcal{C}}) \quad \text{for all } n \in \mathbb{Z}_{\geq 0}.$$

Since $\mathbf{k} \simeq \mathrm{End}_{\mathcal{C}}(\mathbb{1}_{\mathcal{C}})$, we have that $a_n = \alpha_n \mathrm{id}_{\mathbb{1}_{\mathcal{C}}}$, $b_n = \beta_n \mathrm{id}_{\mathbb{1}_{\mathcal{C}}}$ and $c_n = \gamma_n \mathrm{Id}_{\mathbb{1}_{\mathcal{C}}}$, for some $\alpha_n, \beta_n, \gamma_n \in \mathbf{k}$. The sequences $\alpha := (\alpha_n)_{n\geq 0}$, $\beta := (\beta_n)_{n\geq 0}$ and $\gamma := (\gamma_n)_{n\geq 0}$ will be called the *evaluation* of A, and we say that A is a *realization* of (α, β, γ) if the evaluation of A is (α, β, γ).

Proposition 2.4.8 induces an equivalence of categories

$$\left\{ \text{symmetric monoidal functors}\, \mathrm{UCob}_2 \to \mathcal{C} \right\} \leftrightarrow \left\{ \text{extended Frobenius algebras in } \mathcal{C} \right\}.$$

Proposition 5.4.1. *When $\mathcal{C}$ is a tensor category, this results in an equivalence of categories*

$$\left\{ \begin{array}{c} \textit{symmetric tensor functors} \\ \mathrm{VUCob}_{\alpha,\beta,\gamma} \to \mathcal{C} \end{array} \right\} \leftrightarrow \left\{ \begin{array}{c} \textit{extended Frobenius algebras in } \mathcal{C} \\ \textit{with evaluation } (\alpha,\beta,\gamma) \end{array} \right\}. \tag{5.4.2}$$

Proof. Let $F : \mathrm{VUCob}_{\alpha,\beta,\gamma} \to \mathcal{C}$ be a symmetric tensor functor and let $A = F(1)$ be the corresponding extended Frobenius algebra in $\mathcal{C}$. Note that

$$a_n = \epsilon_A x^n u_A = F(\varepsilon)(F(m)F(\Delta))^n F(u) = F(\varepsilon(m\Delta)^n u) = F(\alpha_n) = \alpha_n.$$

Similarly, $b_n = \beta_n$ and $c_n = \gamma_n$. Hence A has evaluation (α, β, γ).

Conversely, let A be an extended Frobenius algebra in $\mathcal{C}$ with evaluation (α, β, γ), and let $F_A : \mathrm{UCob}_2 \to \mathcal{C}$ be the symmetric tensor functor mapping $1 \to A$ and the generators of UCob_2 to the corresponding structure maps of A. Then $F_A(\varepsilon(m\Delta)^n u) = \alpha_n$, $F_A(\varepsilon(m\Delta)^n yu) = \beta_n$ and $F_A(\varepsilon(m\Delta)^n y^2 u) = \gamma_n$. Hence F_A factors through a symmetric tensor functor $F_A : \mathrm{VUCob}_{\alpha,\beta,\gamma} \to \mathcal{C}$, as desired. □

Given an extended Frobenius algebra A with extended evaluation (α, β, γ) in $\mathcal{C}$, denote by $F_A : \mathrm{VUCob}_{\alpha,\beta,\gamma} \to \mathcal{C}$ the corresponding functor mapping 1 to A. For (α, β, γ) as in Lemma 5.3.3, F_A factors through $\mathrm{SUCob}_{\alpha,\beta,\gamma}$ if and only if F_A annhiliates the handle polynomial, see Equation (5.3.6). When $\mathcal{C}$ is pseudo-abelian, there is a unique extension $F_A : \mathrm{UCob}_{\alpha,\beta,\gamma} \to \mathcal{C}$.

5.4.1 Finite realizations

Let $A \in \mathcal{C}$ be an extended Frobenius algebra with evaluation (α, β, γ). We say that the realization is *finite* if the Hom spaces $\mathrm{Hom}(\mathbb{1}, A^{\otimes n})$ in $\mathcal{C}$ are finite dimensional.

The following theorem is the unoriented analogue of [27, Theorem 3.1].

Theorem 5.4.2. *The tuple (α, β, γ) admits a finite realization if and only if the following conditions hold:*

- *α is linearly recurrent, with generating function $Z_\alpha(T)$ satisfying*

$$Z_\alpha(T) = \frac{p_\alpha(T)}{q(T)}, \qquad (p_\alpha(T), q(T)) = 1 \qquad \textit{and} \qquad q(0) = 1;$$

- *β and γ have generating functions*

$$Z_\beta(T) = \frac{p_\beta(T)}{q(T)} \qquad \textit{and} \qquad Z_\gamma(T) = \frac{p_\gamma(T)}{q(T)},$$

where $\deg(p_\beta(T)), \deg(p_\gamma(T)) \leq \max(\deg(p_\alpha(T)) + 1, \deg(q(T)))$.

Proof. Assume first that (α, β, γ) admits a finite realization A in $\mathcal{C}$. Consider the map $x \in \mathrm{End}_{\mathcal{C}}(A)$ as defined in (5.4.1). Since by assumption $\mathrm{tr}(x^n) = \alpha_n$, we aim to find a linear relation satisfied by powers of x, and thus also by α.

Since $\mathrm{End}_{\mathcal{C}}(A) = \mathrm{Hom}_{\mathcal{C}}(1, A^{\otimes 2})$ is finite dimensional, there exists a minimal polynomial $m(T) \in \mathbf{k}[T]$ for x. As $m(x) = 0$, then $x^l m(x) = 0$ for all $l \geq 0$. Taking trace on both sides of these equations yields a linearly recurrent relation with coefficients in $\mathbf{k}$ satisfied by α_l, for all $l \geq \deg(m(T))$. That is, α is linearly recurrent, and thus has a generating function

$$Z_\alpha(T) = \frac{p_\alpha(T)}{q(T)} = \sum_{n \geq 0} \alpha_n T^n, \quad \text{where} \quad (p_\alpha(T), q(T)) = 1 \quad \text{and} \quad q(0) = 1. \tag{5.4.3}$$

Let $N = \deg(p_\alpha(T)), M = \deg(q(T))$, and $K = \max(N + 1, M)$. Since the first term at which the recurrence happens is α_K, we must have $K = \deg(m(T))$.

We now want to compute generating functions for β and γ. Let y as in (5.4.1). Recall that $\mathrm{tr}(yx^n) = \beta_n$ and $\mathrm{tr}(y^2 x^n) = \gamma_n$, for all $n \geq 0$. Since $yx^i m(x) = 0$ and $y^2 x^i m(x) = 0$, taking traces on these equations we obtain linear relations satisfied by β_l and γ_l, for $l \geq K$, equal to the ones satisfied by α. Hence by Lemma 5.3.2, β and γ have generating functions

$$Z_\beta(T) = \frac{p_\beta(T)}{q(T)} \qquad \text{and} \qquad Z_\gamma(T) = \frac{p_\gamma(T)}{q(T)},$$

with $\deg(p_\beta(T)), \deg(p_\gamma(T)) < K$, as desired.

Conversely, assume α, β and γ are as in the statement. Under these assumptions, Lemma 5.3.3 implies that the cobordism σ is negligible, and thus we can quotient $\mathrm{VUCob}_{\alpha,\beta,\gamma}$ by the handle relation. The resulting category $\mathrm{SUCob}_{\alpha,\beta,\gamma}$ (see Section 5.3), with extended Frobenius algebra given by the circle S^1, equipped with the orientation reversing diffeomorphism $\phi : S^1 \to S^1$ and the Möebius band $\theta : \emptyset \to S^1$, gives a finite extended realization of (α, β, γ). □

5.5 Exterior product decompositions

This section is the analogue of [27, Section 2.4] for the unoriented case. We want to show a decomposition of $\mathrm{UCob}_{\alpha,\beta,\gamma}$ as an exterior product of categories.

Let $\mathcal{C}$ be a symmetric $\mathbf{k}$-linear monoidal category and consider an extended Frobenius algebra $A \in \mathcal{C}$ with evaluation (α, β, γ). Suppose A has finite dimensional Hom spaces and let

$$Z_\alpha(T) = \frac{p_\alpha(T)}{q(T)}, \qquad Z_\beta(T) = \frac{p_\beta(T)}{q(T)} \qquad \text{and} \qquad Z_\gamma(T) = \frac{p_\gamma(T)}{q(T)}, \tag{5.5.1}$$

be the generating functions of α, β and γ, respectively, satisfying the conditions of Theorem 5.4.2. That is, $\deg(p_\beta(T)), \deg(p_\gamma(T)) \le \max(\deg(p_\alpha(T)), \deg(q(T)) + 1)$.

Consider the algebra homomorphism

$$\Psi : \mathrm{Hom}_{\mathcal{C}}(\mathbb{1}, A) \to \mathrm{End}_{\mathcal{C}}(A),$$

induced from the action of $\mathrm{Hom}_{\mathcal{C}}(\mathbb{1}, A)$ on A. Let $m_A, \Delta_A, u_A, \theta_A$ denote the multiplication, comultiplication, unit and crosscap morphisms of A, respectively. Define the handle $x_0 := m_A \Delta_A u_A$ and cross $y_0 := m_A \Delta_A \theta_A$ morphisms of A, which are in $\mathrm{Hom}_{\mathcal{C}}(\mathbb{1}, A)$. We denote by A_0 the subalgebra of $\mathrm{Hom}_{\mathcal{C}}(\mathbb{1}, A)$ generated by x_0. Since A_0 is finite dimensional, x_0 has a minimal (monic) polynomial $m(T)$, where

$$q(T) = T^d m(T^{-1}), \quad d = \deg(m(T)). \tag{5.5.2}$$

Let $e \in A_0$ be an idempotent. Then $\Psi(e)A$ is an extended Frobenius subalgebra of A with cross morphism $\Psi(e)(\theta_A)$ and unit $\Psi(e)(u_A)$, and we have a decomposition

$$A = \Psi(e)A \oplus \Psi(1-e)A,$$

as a direct sum of extended Frobenius subalgebras. We denote their handle endomorphisms by x_0' and x_0'', and their evaluations by $(\alpha', \beta', \gamma')$ and $(\alpha'', \beta'', \gamma'')$, respectively.

Lemma 5.5.1. *If $Z_\alpha(T), Z_\beta(T), Z_\gamma(T)$ are the generating functions of (α, β, γ), then the generating functions $Z'_\alpha, Z'_\beta, Z'_\gamma$ and $Z''_\alpha, Z''_\beta, Z''_\gamma$ of $\Psi(e)A$ and $\Psi(1-e)(A)$, respectively, satisfy*

$$Z_\alpha(T) = Z'_\alpha(T) + Z''_\alpha(T), \quad Z_\beta(T) = Z'_\beta(T) + Z''_\beta(T), \quad \textit{and} \quad Z_\gamma(T) = Z'_\gamma(T) + Z''_\gamma(T).$$

Proof. We compute

$$\beta''_n = \epsilon(y_0(x''_0)^n) = \epsilon(y_0(1-e)x_0^n) = \epsilon(y_0x_0^n) - \epsilon(y_0 e x_0^n) = \beta_n - \epsilon(y_0(x'_0)^n) = \beta_n - \beta'_n.$$

That is, $\beta_n = \beta'_n + \beta''_n$, and so the statement follows for β. Analogously, we have that

$$\alpha_n = \alpha'_n + \alpha''_n \qquad \text{and} \qquad \gamma_n = \gamma'_n + \gamma''_n,$$

and so it is also true for α and γ. □

Remark 5.5.2. The parametrization of idempotents given in [27, Section 2.4] is still true in this context. That is, idempotents $e \in A_0$ are labelled by factorizations

$$m(T) = m'(T)m''(T), \tag{5.5.3}$$

where $m'(T)$ and $m''(T)$ are relatively prime. This labelling goes as follows. Given said factorization, we have that $e = a(x_0)m(x_0)$ is an idempotent, where $a(T), b(T) \in \mathbf{k}[T]$ satisfy $a(T)m'(T) + b(T)m''(T) = 1$. Conversely, given an idempotent $e = s(x_0)$, setting $m'(T) = \gcd(m(T), s(T))$ and $m''(T) = \gcd(m(T), 1-s(T))$, we get the desired factorization.

Note that a factorization (5.5.3) induces a factorization

$$q(T) = q'(T)q''(T),$$

where $q'(T), q''(T)$ are determined by $m(T)$ and $m'(T)$ as in (5.5.2). By assumption, $m'(T)$ and $m''(T)$ are relatively prime, and thus we may assume that T does not divide $m'(T)$. It follows that we have partial fraction decompostitions

$$\begin{aligned} Z_\alpha(T) &= \frac{v'_\alpha(T)}{q'(T)} + \frac{v''_\alpha(T)}{q''(T)} =: Z'_\alpha(T) + Z''_\alpha(T), \\ Z_\beta(T) &= \frac{v'_\beta(T)}{q'(T)} + \frac{v''_\beta(T)}{q''(T)} =: Z'_\beta(T) + Z''_\beta(T), \quad \text{and}, \\ Z_\gamma(T) &= \frac{v'_\gamma(T)}{q'(T)} + \frac{v''_\gamma(T)}{q''(T)} =: Z'_\gamma(T) + Z''_\gamma(T), \end{aligned} \tag{5.5.4}$$

so that $Z'_\alpha(T), Z'_\beta(T)$ and $Z'_\gamma(T)$ are proper fractions, but $Z''_\alpha(T), Z''_\beta(T)$ and $Z''_\gamma(T)$ may not be proper.

Remark 5.5.3. Since $m''(x''_0) = 0$, we get equations $(x''_0)^n m''(x''_0) = 0$, $y(x''_0)^n m''(x''_0) = 0$ and $y^2(x''_0)^n m''(x''_0) = 0$, for all $n \geq 0$. From the first set of equations we get the linear recurrence relation satisfied by the sequence α'', which begins at $K = \max(\deg(v''_\alpha(T)) + 1, q''(T))$. The other two sets of equations imply that β and γ also satisfy this recurrence relation, and thus by Lemma 5.3.2 we must have

$$\deg(v''_\beta(T)), \deg(v''_\gamma(T)) \leq K.$$

The same can be done for α', β', γ'. That is, we get that

$$\deg(v'_\beta(T)), \deg(v'_\gamma(T)) \leq K.$$

This will be important to us due to Lemma 5.3.3.

The following proposition is the analogue of [27, Proposition 2.6] for the case of extended Frobenius algebras. The proof is analogous but we include it for the sake of completeness.

Proposition 5.5.4. *If the generating functions of (α, β, γ) are as in (5.5.1), then the generating functions for the evaluations $(\alpha', \beta', \gamma')$ and $(\alpha'', \beta'', \gamma'')$ of $\Psi(e)A$ and $\Psi(1-e)(A)$ are $Z'_\alpha(T), Z'_\beta(T), Z'_\gamma(T)$ and $Z''_\alpha(T), Z''_\beta(T), Z''_\gamma(T)$ as given in (5.5.4), respectively.*

Proof. Let

$$x'_0 = ex_0,\ y'_0 = ey_0 \qquad \text{and} \qquad x''_0 = (1-e)x_0,\ y''_0 = (1-e)y_0,$$

be the handle and cross endomorphisms of $\Psi(e)A$ and $\Psi(1-e)A$, respectively. Let $s(T) \in \mathbf{k}(T)$ be such that $e = s(x_0) \in A_0$. The extended evaluation of $\Psi(e)(A)$ is given by

$$\begin{aligned} \alpha'_n &= \epsilon(x'_0)^n u = \epsilon(ex_0^n)u = \epsilon(s(x_0)x_0^n)u, \\ \beta'_n &= \epsilon(y_0 x'_0)^n u = \epsilon(ey_0 x_0^n)u = \epsilon(s(x_0)y_0 x_0^n)u, \\ \gamma'_n &= \epsilon(y_0^2 x'_0)^n u = \epsilon(ey_0^2 x_0^n)u = \epsilon(s(x_0)y_0^2 x_0^n)u, \end{aligned} \tag{5.5.5}$$

and thus it is uniquely determined by e. Analogously, the extended evaluation of $\psi(1-e)A$ is uniquely determined by e. By Remark 5.5.2, this implies that the extended evaluations of $\psi(e)A$ and $\psi(1-e)A$ are uniquely determined by the factorization of $m(T)$.

Since the evaluations depend only on the factorization of $m(T)$, we may compute them in a specific setting. Consider symmetric $\mathbf{k}$-linear monoidal categories $\mathcal{C}'$ and $\mathcal{C}''$. Let $A' \in \mathcal{C}'$

and $A'' \in \mathcal{C}''$ be extended Frobenius algebras such that the generating functions of their evaluations are $Z'_\alpha(T), Z'_\beta(T), Z'_\gamma(T)$ and $Z''_\alpha(T), Z''_\beta(T), Z''_\gamma(T)$ as in (5.5.4), respectively, and their handle endomorphisms x'_0 and x''_0 have minimal polynomials $m'(T)$ and $m''(T)$ as in (5.5.3). By Remark 5.5.3 and Section 5.3, we know such categories exist, see Definition 5.3.4. Denote by y'_0 and y''_0 the cross endomorphisms of A' and A''.

Let $\mathcal{C}' \boxtimes \mathcal{C}''$ be the external tensor product of $\mathcal{C}'$ and $\mathcal{C}''$ [37, Section 2.2], and consider the object

$$A := (A' \boxtimes \mathbb{1}) \oplus (\mathbb{1} \boxtimes A'') \in \mathcal{C}' \boxtimes \mathcal{C}'',$$

with extended Frobenius algebra structure induced from that of A' and A''. Then the handle and cross endomorphisms of A are $x_0 = x'_0 \oplus x''_0$ and $y_0 = y'_0 \oplus y''_0$, respectively, and the generating functions of A are

$$Z_\alpha = Z'_\alpha + Z''_\alpha, \qquad Z_\beta = Z'_\beta + Z''_\beta, \qquad \text{and} \qquad Z_\gamma = Z'_\gamma + Z''_\gamma.$$

Since $m'(T)$ and $m''(T)$ are relatively prime, the minimal polynomial of x_0 is $m(T) = m'(T)m''(T)$. The idempotents determined by them (see Remark 5.5.2) are the units of $A' \boxtimes \mathbb{1}$ and $\mathbb{1} \boxtimes A''$, respectively. Thus, the respective evaluations can be computed as in (5.5.5), and the result follows. □

We list in what follows a series of results from [27, Section 2.6] that still hold in our context. From now on, let $(\alpha', \beta', \gamma')$ and $(\alpha'', \beta'', \gamma'')$ have generating functions as in (5.5.4) (satisfying the conditions in Remark 5.5.3). Let

$$\mathrm{SUCob}' := \mathrm{SUCob}_{\alpha',\beta',\gamma'} \qquad \text{and} \qquad \mathrm{SUCob}'' := \mathrm{SUCob}_{\alpha'',\beta'',\gamma''}.$$

Denote by A' and A'' the extended Frobenius algebras in SUCob' and SUCob'' given by the circle object with structure maps the generating cobordisms of SUCob' and SUCob'', respectively. Consider also the category $\mathrm{SUCob}' \boxtimes \mathrm{SUCob}''$ with extended Frobenius algebra

$$\mathcal{A} := A' \boxtimes \mathbb{1} \oplus \mathbb{1} \boxtimes A'', \tag{5.5.6}$$

and denote by $\mathcal{U}(T)$ the minimal polynomial for its handle endomorphism. Define

$$\alpha := \alpha' + \alpha'', \qquad \beta := \beta' + \beta'' \qquad \text{and} \qquad \gamma = \gamma' + \gamma'', \tag{5.5.7}$$

so that $\mathcal{A}$ has extended evaluation (α, β, γ).

Lemma 5.5.5. *[27, Lemma 2.8] Let $m'(T)$ and $m''(T)$ denote the handle polynomials of A' and A'', respectively. Then*

$$\mathcal{U}(T) = lcm(m'(T), m''(T)).$$

Corollary 5.5.6. *[27, Corollary 1] Consider the category $\mathrm{SUCob}_{\alpha,\beta,\gamma}$ with extended Frobenius algebra A, for α, β and γ as in (5.5.7). The minimal polynomial $m_\alpha(T)$ for its handle endomorphism is a divisor of $\mathcal{U}(T)$.*

The following Proposition is the analogue of [27, Proposition 2.10] for the unoriented case.

Proposition 5.5.7. *With the notation above, $\mathcal{U}(T) = m_\alpha(T)$ iff there exists a functor*

$$F : \mathrm{SUCob}_{\alpha,\beta,\gamma} \to \mathrm{SUCob}' \boxtimes \mathrm{SUCob}'', \tag{5.5.8}$$

mapping the extended Frobenius algebra A given by the circle object of $SUCob_{\alpha,\beta,\gamma}$ to $\mathcal{A}$, see (5.5.6), and the extended Frobenius structure maps of A to the ones of $\mathcal{A}$.

Proof. Since both A and $\mathcal{A}$ have extended evaluation (α, β, γ), by the universal property (see Section 5.4), there exists a symmetric $\mathbf{k}$-linear monoidal functor $F : \mathrm{VUCob}_{\alpha,\beta,\gamma} \to \mathrm{SUCob}' \boxtimes \mathrm{SUCob}''$ as described. Let x_0 and z_0 denote the handle endomorphisms of $\mathrm{SUCob}_{\alpha,\beta,\gamma}$ and $\mathrm{SUCob}' \boxtimes \mathrm{SUCob}$, respectively. Then F factors through $\mathrm{SUCob}_{\alpha,\beta,\gamma}$ if and only if F maps $m_\alpha(x_0) \mapsto 0$. If $\mathcal{U}(T) = m_\alpha(T)$, then $F(m_\alpha(x_0)) = m_\alpha(z_0) = \mathcal{U}(z_0) = 0$. For the converse, if the functor F factors, then $m_\alpha(z_0) = 0$, and so by Corollary 5.5.6 we must have $m_\alpha(T) = \mathcal{U}(T)$. □

Assume from now on that $m_\alpha(T) = \mathcal{U}(T)$. The following is the analogue of [27, Proposition 2.14].

Proposition 5.5.8. *With the notation above, the functor* (5.5.8) *induces an equivalence of pseudo-abelian symmetric* $\mathbf{k}$*-linear monoidal categories*

$$F : \mathrm{UCob}_{\alpha,\beta,\gamma} \xrightarrow{\sim} \mathrm{UCob}' \boxtimes \mathrm{UCob}''.$$

Proof. Consider the functor $F : \mathrm{SUCob}_{(\alpha,\beta,\gamma)} \to \mathrm{SUCob}' \boxtimes \mathrm{SUCob}''$ as in (5.5.8), which maps $A \mapsto \mathcal{A}$. Taking the pseudo-abelian envelope, we get an induced functor

$$F : \mathrm{UCob}_{\alpha,\beta,\gamma} \to \mathrm{UCob}' \boxtimes \mathrm{UCob}''.$$

On the other hand, consider the factorization $m_\alpha(T) = m'(T)m''(T)$, which induces a decomposition $Z_\alpha(T) = Z_{\alpha'}(T) + Z_{\alpha''}(T)$ as in equation (5.5.4). By Remark 5.5.2 we have a corresponding idempotent $e \in \mathrm{Hom}(\mathbb{1}, A)$, and a decomposition

$$A = \Psi(e)A \oplus \Psi(1-e)A,$$

where the generating functions of $\Psi(e)A$ and $\Psi(1-e)A$ are as in Proposition 5.5.4. Note that A' in SUCob' and $\Psi(e)A$ in $\mathrm{UCob}_{\alpha,\beta,\gamma}$ have the same extended evaluation and thus by the universal property, see Section 5.4, there exists a symmetric **k**-linear monoidal functor $\mathrm{SUCob}' \to \mathrm{UCob}_{\alpha,\beta,\gamma}$. Similarly, we have a functor $\mathrm{SUCob}'' \to \mathrm{UCob}_{\alpha,\beta,\gamma}$ mapping $A'' \mapsto \Psi(1-e)A$. On the other hand, by the universal property of the external tensor product (see [37, Section 2.2]), we have a symmetric **k**-linear monoidal functor

$$G : \mathrm{SUCob}' \boxtimes \mathrm{SUCob}'' \to \mathrm{UCob}_{\alpha,\beta,\gamma},$$

mapping $A' \boxtimes \mathbb{1}$ to $\Psi(e)A$ and $\mathbb{1} \boxtimes A'$ to $\Psi(1-e)A$. This induces a unique **k**-linear monoidal functor from the pseudo-abelian closure of the source category,

$$G : \mathrm{UCob}' \boxtimes \mathrm{UCob}'' \to \mathrm{UCob}_{\alpha,\beta,\gamma}\,.$$

Note that the compositions

$$\mathrm{UCob}_{\alpha,\beta,\gamma} \xrightarrow{F} \mathrm{UCob}' \boxtimes \mathrm{UCob}'' \xrightarrow{G} \mathrm{UCob}_{\alpha,\beta,\gamma},$$

and

$$\mathrm{UCob}' \boxtimes \mathrm{UCob}'' \xrightarrow{G} \mathrm{UCob}_{\alpha,\beta,\gamma} \xrightarrow{F} \mathrm{UCob}' \boxtimes \mathrm{UCob}'',$$

map A to itself and $A' \boxtimes \mathbb{1}$ and $\mathbb{1} \boxtimes A''$ to themselves, respectively. Hence $G \circ F$ and $F \circ G$ are isomorphic (as **k**-linear monoidal functors) to the corresponding identity functor, and the statement follows. $\square$

Chapter 6

Equivalence with the category $\mathbf{Rep}(S_t \wr \mathbb{Z}_2)$

This chapter contains previously published material, which appeared in [12].

In this chapter, when we say $\mathcal{C}$ is a symmetric $\mathbf{k}$-linear monoidal category we are also assuming the bifunctor $\otimes$ is $\mathbf{k}$-bilinear.

6.1 (Unoriented) orientable cobordisms

Let α be a linearly recurrent sequence in $\mathbf{k}$. In this section, we define the category SOCob_α, obtained by modding out the crosscap cobordism θ. We also show that when specializing to the sequence $\alpha = (\alpha_0, \lambda\alpha_0, \lambda^2\alpha_0, \dots)$, with $\alpha_0, \lambda \in \mathbf{k}^\times$ such that $\lambda\alpha_0$ is not a non-negative even integer, its pseudo-abelian envelope OCob_α is equivalent to $\mathbf{Rep}(S_t \wr \mathbb{Z}_2)$, for $t = \frac{\lambda\alpha_0}{2}$.

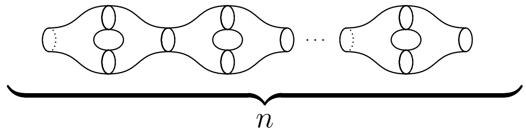
Definition 6.1.1. Let $\alpha = (\alpha_0, \alpha_1, \dots)$ be a linearly recurrent sequence in $\mathbf{k}$, and let $\beta = \gamma = (0, 0, \dots)$. We define the *orientable skein* category SOCob_α, with:

- Objects: Non-negative integers $n \in \mathbb{Z}_{\geq 0}$.
- Morphisms: $\mathrm{Hom}_{\mathrm{SOCob}_\alpha}(m, n)$ is equal to $\mathrm{Hom}_{\mathrm{SUCob}_{\alpha,\beta,\gamma}}(m, n)$ modulo the relation $\theta = 0$.
- Composition: Induced from $\mathrm{SUCob}_{\alpha,\beta,\gamma}$.

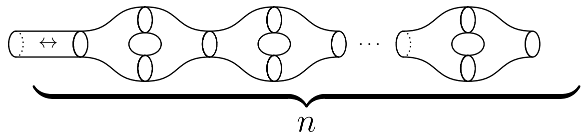
The category SOCob_α is a rigid symmetric $\mathbf{k}$-linear monoidal category. Morphisms in SOCob_α are $\mathbf{k}$-linear combinations of orientable cobordisms without closed components and genus at most K, for K as in (5.3.3). Hence, connected components of cobordisms in SOCob_α are as in Proposition 5.1.2. Same as in $\mathrm{SUCob}_{\alpha,\beta,\gamma}$, Hom spaces in SOCob_α are finite dimensional. We denote its pseudo-abelian envelope by OCob_α. We note that the category SOCob_α of orientable cobordisms is different from the category of oriented cobordisms, see Example 6.1.2.

Let $\mathcal{C}$ be a symmetric $\mathbf{k}$-linear monoidal category with an extended Frobenius algebra A, such that A has extended evaluation α and $\beta = \gamma = 0$, with α linearly recurrent. Let $F_A : \mathrm{VUCob}_{\alpha,\beta,\gamma} \to \mathcal{C}$ be the symmetric $\mathbf{k}$-linear monoidal functor given by the universal property of $\mathrm{VUCob}_{\alpha,\beta,\gamma}$, mapping 1 to A, see Section 5.4. Then F_A factors through $\mathrm{SUCob}_{\alpha,\beta,\gamma}$ if and only if F_A annhiliates the handle polynomial, and through SOCob_α if and only if it also anhiliates θ, i.e, if and only if $0 = F_A(\theta) = \theta_A$. In such case, when $\mathcal{C}$ is pseudo-abelian there is a unique extension $F_A : \mathrm{OCob}_\alpha \to \mathcal{C}$.

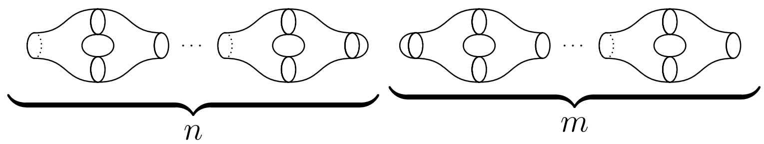
Example 6.1.2. The cobordisms below

$x^n =$ 

,

$\phi x^n =$ 

,

$x^m u \epsilon x^n =$ 

,

for $0 \leq n, m \leq K$, where K is as in (5.3.5), span $\mathrm{End}_{\mathrm{SOCob}_\alpha}(1)$, see Proposition 5.1.2.

6.2 Preliminary results

Throughout this section, let $\mathbf{k}$ be an algebraically closed field of characteristic zero.

Let

$$\alpha = (\alpha_0, \lambda\alpha_0, \lambda^2\alpha_0, \dots) \qquad \text{and} \qquad \beta = \gamma = (0, 0, \dots),$$

for some $\alpha_0, \lambda \in \mathbf{k}^\times$. Here, the generating function for α is given by

$$Z_\alpha(T) = \frac{\alpha_0}{1 - \lambda T}.$$

Hence, SOCob_α is the quotient of $\mathrm{VUCob}_{\alpha,\beta,\gamma}$ by the relations $x - \lambda\,\mathrm{Id} = 0$ and $\theta = 0$.

We are interested in finding a spanning set for $\mathrm{Hom}_{\mathrm{SOCob}_\alpha}(0, m)$ for all $m \in \mathbb{N}$. Let $\mathcal{W}_m$ denote the subspace spanned by connected cobordisms in $\mathrm{Hom}_{\mathrm{SOCob}_\alpha}(0, m)$. For $m = 1$, there is only one connected cobordism, namely u, see Section 5.1. Hence $\mathcal{W}_1$ has dimension 1.

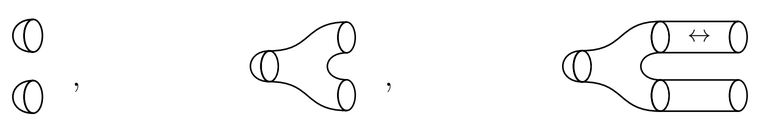
Proposition 6.2.1. *Let $m \geq 2$. The set $\{\xi^m_{\overline{J}} : \overline{J} \in \mathcal{R}_m\}$ from Definition 5.2.8 is a basis for $\mathcal{W}_m$. In particular,* $\dim(\mathcal{W}_m) = 2^{m-1}$.

Proof. We show first that $\{\xi^m_{\overline{J}} : \overline{J} \in \mathcal{R}_m\}$ spans $\mathcal{W}_m$. In fact, connected cobordisms in SOCob_α are orientable and thus they are of the form given in Proposition 5.1.2. Moreover, by the relation $x = \lambda\,\mathrm{Id}$, every handle gets replaced by a multiple of the identity and so every cobordism has genus zero. Hence every connected cobordism $0 \to m$ in SOCob_α is in the set $\{\xi^m_{\overline{J}} : \overline{J} \in \mathcal{R}_m\}$.

On the other hand, note that $\alpha_{m-1} = \lambda^{m-1}\alpha_0 \neq 0$, as $\alpha_0, \lambda \in \mathbf{k}^\times$. Since the sequence γ is constantly zero, by Theorem 5.2.10 cobordisms in the set $\{\xi^m_{\overline{J}} : \overline{J} \in \mathcal{R}_m\}$ are linearly independent. □

Remark 6.2.2. Now that we have a basis for connected cobordisms $0 \to m$, we can describe all cobordisms $0 \to m$ as follows. For every partition P of $\{1, \ldots, m\}$ and every $p \in P$, assign a connected cobordism $\xi^{|p|}_{\overline{J}}$ to p, where $\overline{J} \in \mathcal{R}_{|p|}$. Then the set of cobordisms obtained by running through all possible partitions P and all posible classes $\overline{J} \in \mathcal{R}_{|p|}$ for every $p \in P$ gives a spanning set for $\mathrm{Hom}_{\mathrm{SOCob}_\alpha}(0, m)$.

Example 6.2.3. If $\lambda\alpha_0 \neq 2$, the following is a basis for $\mathrm{Hom}_{\mathrm{SOCob}_\alpha}(0, 2)$:



In fact, we know this is a spanning set by Remark 6.2.2. On the other hand, the matrix of inner products is given by

$$A = \begin{bmatrix} \alpha_0^2 & \alpha_0 & \alpha_0 \\ \alpha_0 & \alpha_1 & 0 \\ \alpha_0 & 0 & \alpha_1 \end{bmatrix},$$

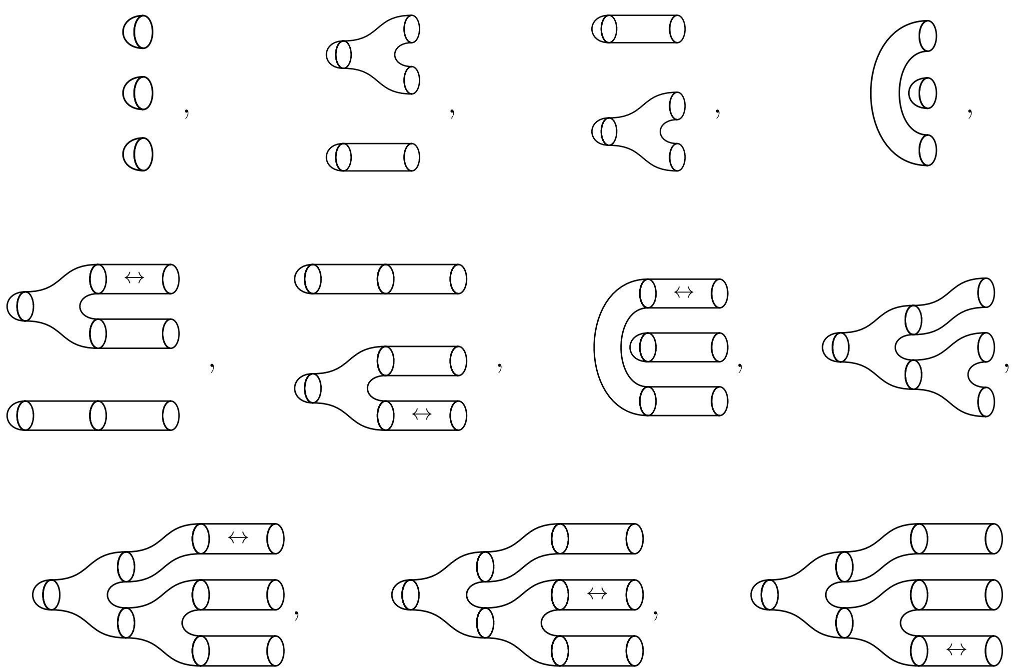
which has determinant

$$\det(A) = \alpha_0^2\alpha_1^2(\alpha_1 - 2) = \lambda\alpha_0^3(\lambda\alpha_0 - 2) \neq 0,$$

so the set is also linearly independent. Hence

$$\dim(\mathrm{Hom}_{\mathrm{SOCob}_\alpha}(0, 2)) = 3.$$

Example 6.2.4. Suppose that $\lambda\alpha_0 \neq 2, 4$. Then the following is a basis for $\mathrm{Hom}_{\mathrm{SOCob}_\alpha}(0, 3)$:



Again, this is a spanning set by Proposition 5.1.2 and Remark 6.2.2. In this case, the matrix of inner products is

$$\begin{bmatrix}
\alpha_0^3 & \alpha_0^2 & \alpha_0^2 & \alpha_0^2 & \alpha_0^2 & \alpha_0^2 & \alpha_0^2 & \alpha_0 & \alpha_0 & \alpha_0 & \alpha_0 \\
\alpha_0^2 & \lambda\alpha_0^2 & \alpha_0 & \alpha_0 & 0 & \alpha_0 & \alpha_0 & \lambda\alpha_0 & 0 & 0 & \lambda\alpha_0 \\
\alpha_0^2 & \alpha_0 & \lambda\alpha_0^2 & \alpha_0 & \alpha_0 & 0 & \alpha_0 & \lambda\alpha_0 & \lambda\alpha_0 & 0 & 0 \\
\alpha_0^2 & \alpha_0 & \alpha_0 & \lambda\alpha_0^2 & \alpha_0 & \alpha_0 & 0 & \lambda\alpha_0 & 0 & \lambda\alpha_0 & 0 \\
\alpha_0^2 & 0 & \alpha_0 & \alpha_0 & \lambda\alpha_0^2 & \alpha_0 & \alpha_0 & 0 & \lambda\alpha_0 & \lambda\alpha_0 & 0 \\
\alpha_0^2 & \alpha_0 & 0 & \alpha_0 & \alpha_0 & \lambda\alpha_0^2 & \alpha_0 & 0 & 0 & \lambda\alpha_0 & \lambda\alpha_0 \\
\alpha_0^2 & \alpha_0 & \alpha_0 & 0 & \alpha_0 & \alpha_0 & \lambda\alpha_0^2 & 0 & \lambda\alpha_0 & 0 & \lambda\alpha_0 \\
\alpha_0 & \lambda\alpha_0 & \lambda\alpha_0 & \lambda\alpha_0 & 0 & 0 & 0 & \lambda^2\alpha_0 & 0 & 0 & 0 \\
\alpha_0 & 0 & \lambda\alpha_0 & 0 & \lambda\alpha_0 & 0 & \lambda\alpha_0 & 0 & \lambda^2\alpha_0 & 0 & 0 \\
\alpha_0 & 0 & 0 & \lambda\alpha_0 & \lambda\alpha_0 & \lambda\alpha_0 & 0 & 0 & 0 & \lambda^2\alpha_0 & 0 \\
\alpha_0 & \lambda\alpha_0 & 0 & 0 & 0 & \lambda\alpha_0 & \lambda\alpha_0 & 0 & 0 & 0 & \lambda^2\alpha_0
\end{bmatrix},$$

which has determinant

$$\lambda^6\alpha_0^{11}(\lambda\alpha_0 - 2)^7(\lambda\alpha_0 - 4).$$

Hence

$$\dim(\mathrm{Hom}_{\mathrm{SOCob}_\alpha}(0,3)) = 11.$$

Let $P = \{p_1, \ldots, p_k\}$ be a partition of $\{1, \ldots, m\}$, and let $\overline{J_i} \in \mathcal{R}_{|p_i|}$ for all $1 \leq i \leq k$. Denote by $c_{P,\overline{J_1},\ldots,\overline{J_k}}$ the cobordism $0 \to m$ with k connected components, where the i-th component is of the form $\xi^{m_i}_{\overline{J_i}}$, with out-boundary given by the circles in positions $j_{i,1}, \ldots, j_{i,m_i}$, where $p_i = \{j_{i,1}, \ldots, j_{i,m_i}\}$. For instance, in Example 6.2.4 the first and seventh cobordisms are $c_{\{\{1\},\{2\},\{3\}\},\overline{\emptyset},\overline{\emptyset},\overline{\emptyset}}$ and $c_{\{\{1,3\},\{2\}\},\overline{\{1\}},\overline{\emptyset}}$, respectively. Then a spanning set for

$\mathrm{Hom}_{\mathrm{SOCob}_\alpha}(0,m)$ is given by

$$\mathcal{S}_m := \left\{ c_{P,\overline{J_1},\dots,\overline{J_k}} : P = \{p_1,\dots,p_k\} \text{ is a partition of } \{1,\dots,m\} \text{ and } \overline{J_i} \in \mathcal{R}_{|p_i|} \text{ for } 1 \le i \le k \right\}, \tag{6.2.1}$$

see Remark 6.2.2.

Lemma 6.2.5. *The determinant of the matrix of inner products of morphisms in $\mathcal{S}_m$ is a non-zero polynomial on the variables λ and α_0, for all $m \ge 1$.*

Proof. Fix $m \ge 1$. Let A be the matrix of inner products of morphisms in $\mathcal{S}_m$, and let $p(\lambda,\alpha_0)$ be its determinant as a polynomial on λ and α_0. Note that to show $p(\lambda,\alpha_0)$ is a non-zero polynomial, it is enough to show that $p(\alpha_0,\alpha_0) \neq 0$. Hence we assume from now on that $\lambda = \alpha_0$.

We will show that every row of A has the highest power of λ only in its diagonal entry. Let $c := c_{P,\overline{J_1},\dots,\overline{J_k}}$ in $\mathcal{S}_m$, and consider its corresponding row on A. Recall that, for any $d \in \mathcal{S}_m$, the inner product of c and d is given by evaluating the cobordism $c \sqcup (-d)$, see Definition 5.2.4. If $c \sqcup (-d)$ is unorientable, then $(c,d) = 0$. Otherwise,

$$(c,d) = \alpha_{l_1} \dots \alpha_{l_s} = \lambda^{l_1+\dots+l_s}\alpha_0^s = \alpha_0^{l_1+\dots+l_s+s},$$

where s is the number of connected components of $c \sqcup (-d)$, and l_i is the genus of its i-th component, for all $1 \le i \le s$. That is, the power of α_0 at the entry (c,d) will be the sum of the genuses of the connected components of $c \sqcup (-d)$ and the number of connected components of $c \sqcup (-d)$. Hence the highest it can be is

$$(|p_1| - 1) + \dots + (|p_k| - 1) + k = m - k + k = m.$$

In fact, the number s of connected components of $c \sqcup (-d)$ is at most the number of connected components k of c, and the sum of the genuses is at most $(|p_1|-1)+\cdots+(|p_k|-1)$ since that is the highest genus we can generate with c (when every space between two connected circles is closed).

We check now that this power is reached only in the diagonal entry (c,c). Note first that the sum of genuses in $c \sqcup (-d)$ can be $(|p_1|-1)+\cdots+(|p_k|-1)$ only when every pair of connected circles in c gets closed, and so d must be of the form $d : c_{Q,\overline{L_1},\ldots,\overline{L_k}}$ for some $\overline{L_i} \in \mathcal{R}_{|p_i|}$. But by relation (5.1.6) we have $(\xi^{|p_i|}_{\overline{J_i}}, \xi^{|p_i|}_{\overline{L_i}}) = 0$ if $\overline{J_i} \neq \overline{L_i}$, in which case $(c,d) = 0$. On the other hand, involutions cancel each other out in $c \sqcup (-c)$ and so $(\xi^{|p_i|}_{\overline{J_i}}, \xi^{|p_i|}_{\overline{J_i}}) = \lambda^{|p_i|-1}\alpha_0 = \alpha_0^{|p_i|}$, which implies

$$(c,c) = \Pi_{i=1}^{k} \alpha_0^{|p_i|} = \alpha_0^{m},$$

as desired.

We showed that in every row of the matrix A, the highest power of α_0 in that row happens only at the diagonal entry. Hence the term of $p(\alpha_0, \alpha_0)$ computed using the diagonal entries of A will have degree (as a polynomial on α_0) strictly greater than any other term, and thus does not cancel out. This shows the determinant of A is not the zero polynomial on λ and α_0. □

Corollary 6.2.6. *Fix $\alpha_0 \in \mathbf{k}^\times$. For all but countably many λ and α_0 in $\mathbf{k}^\times$, the dimension* $\dim_m$ *of* $\mathrm{Hom}_{\mathrm{SOCob}_\alpha}(0,m)$ *is given by*

$$\dim_m = \sum_{l=1}^{m} 2^{m-l} \begin{Bmatrix} m \\ l \end{Bmatrix}, \text{ for all } m \geq 1,$$

where $\left\{ \begin{matrix} m \\ l \end{matrix} \right\}$ *denotes the Stirling number of the second kind, which counts the number of partitions of a set with* m *elements into* l *non-empty subsets.*

Proof. As per Remark 6.2.2, the set of cobordisms $\mathcal{S}_m$, obtained by assigning to every partition P of $\{1, \ldots, m\}$ and every $p \in P$ a connected cobordism $\xi_{J}^{|p|}$ (see (7.1.3)), gives a spanning set for $\mathrm{Hom}_{\mathrm{SOCob}_\alpha}(0, m)$. Let P be a partition of $\{1, \ldots, m\}$ with parts $\{p_1, \ldots, p_l\}$, for $1 \leq l \leq m$. The number of ways of assigning connected cobordisms to P is then

$$|\mathcal{R}_{|p_1|}| \ldots |\mathcal{R}_{|p_l|}|,$$

see Section 5.2.1. Thus the number of cobordisms in the spanning set is

$$\begin{aligned} \sum_{l=1}^{m} \left(\sum_{\{p_1, \ldots, p_l\} \in \mathrm{Part(m)}} |\mathcal{R}_{|p_1|}| \ldots |\mathcal{R}_{|p_l|}| \right) &= \sum_{l=1}^{m} \left(\sum_{\{p_1, \ldots, p_l\} \in \mathrm{Part(m)}} 2^{|p_1|-1} \ldots 2^{|p_l|-1} \right) \\ &= \sum_{l=1}^{m} \left(\sum_{\{p_1, \ldots, p_l\} \in \mathrm{Part(m)}} 2^{m-l} \right) \\ &= \sum_{l=1}^{m} 2^{m-l} \left\{ \begin{matrix} m \\ l \end{matrix} \right\}. \end{aligned}$$

Lastly, note that by Lemma 6.2.5 the determinant of the matrix of inner products of cobordisms in $\mathcal{S}_m$ is a non-zero polynomial on λ and α_0, for all $m \geq 1$. Thus, for values of λ and α_0 such that these polynomials are not evaluated to zero, cobordisms in $\mathcal{S}_m$ are linearly independent, and the result follows. $\square$

Remark 6.2.7. The generating function of the sequence given by $\sum_{l=1}^{m} 2^{m-l} \left\{ \begin{matrix} m \\ l \end{matrix} \right\}$ in the Lemma above is $\exp((\exp(2x) - 1)/2)$. The first 6 terms are given by $1, 1, 3, 11, 49, 257$. See sequence A004211 at https://oeis.org/A004211 for more information.

Remark 6.2.8. By Corollary 6.2.6 and the proof of Lemma 6.2.5, the highest power of λ in the determinant of the matrix of inner products in $\mathcal{S}_m$ is given by

$$\sum_{l=1}^{m-1}(m-l)\cdot 2^{m-l}\left\{\begin{matrix} m \\ l \end{matrix}\right\}.$$

The first five terms in the sequence for $m \geq 1$ are 0, 2, 14, 92, 644.

Remark 6.2.9. It follows from Remark 6.2.8 and Corollary 6.2.6, that the highest power of α_0 in the determinant of the matrix of inner products of $\mathcal{S}_m$ is given by

$$\sum_{l=1}^{m} l\cdot 2^{m-l}\left\{\begin{matrix} m \\ l \end{matrix}\right\}.$$

The first five terms in the sequence for $m \geq 1$ are given by $1, 4, 19, 106, 641$.

Remark 6.2.10. We will see later on that the exceptional values of λ and α_0 in Corollary 6.2.6 are those such that $\lambda\alpha_0$ is a non-negative even integer. We thus predict that the determinant of the matrix of inner products of $\mathcal{S}_m$, see (7.1.3), will factor into powers of the form $(\lambda\alpha_0 - 2k)$, for $k = 0, \dots, m-1$. Note that we know this to be the case for $m = 1, 2, 3$, see Examples 6.2.3 and 6.2.4.

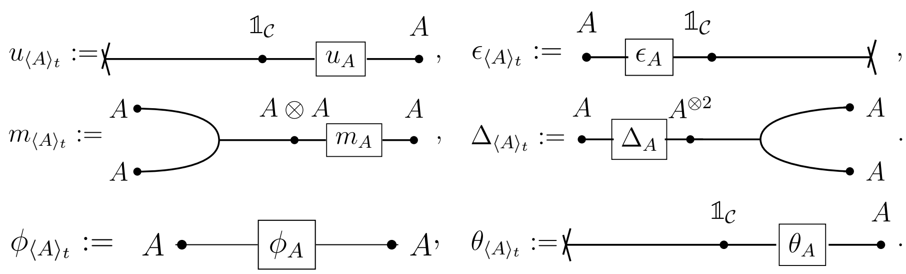
6.3 Extended Frobenius algebras in $\mathbf{Rep}(S_t \wr \mathbb{Z}_2)$

For this subsection, let $\mathcal{C} = \mathbf{Rep}(\mathbb{Z}_2)$. For $\lambda \in \mathbf{k}^\times$, let $A = A_\lambda$ be the extended Frobenius algebra given in Example 2.4.7 for $n = 1$, and consider the induced extended Frobenius algebra $\langle A\rangle_t = \langle A_\lambda\rangle_t$ in $\mathbf{Rep}(S_t \wr \mathbb{Z}_2)$, as described in Proposition 2. We remark that both of these algebras depend on λ, but we drop the subscript λ to simplify the notation.

Lemma 6.3.1. *The evaluation* (α, β, γ) *of* $\langle A \rangle_t$ *in* $\mathbf{Rep}(S_t \wr \mathbb{Z}_2)$ *is given by*

$$\alpha = (2\lambda^{-1}t, 2t, 2\lambda t, \dots) \qquad \textit{and} \qquad \beta = (0, 0, \dots) = \gamma.$$

Proof. We will use the graphical description of maps in $\mathbf{Rep}(S_t \wr \mathbb{Z}_2)$ as shown in Section 2.4.3. The structure maps of $\langle A \rangle_t$ are

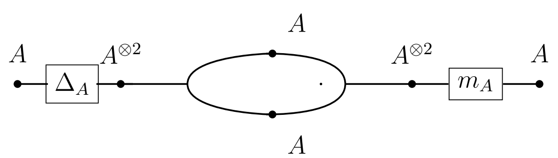


To obtain the sequences α, β and γ we need to compute

$$\alpha_n = \epsilon_{\langle A \rangle_t} x^n u_{\langle A \rangle_t}, \qquad \beta_n = \epsilon_{\langle A \rangle_t} x^n y u_{\langle A \rangle_t} \qquad \text{and} \qquad \gamma_n = \epsilon_{\langle A \rangle_t} x^n y^2 u_{\langle A \rangle_t},$$

for all $n \geq 0$, where

$$x := m_{\langle A \rangle_t} \Delta_{\langle A \rangle_t} =$$

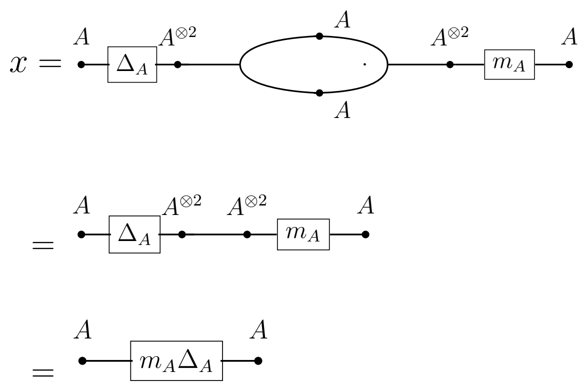


and

$$y = m_{\langle A \rangle_t}(\theta_{\langle A \rangle_t} \otimes \text{id}) = 0,$$

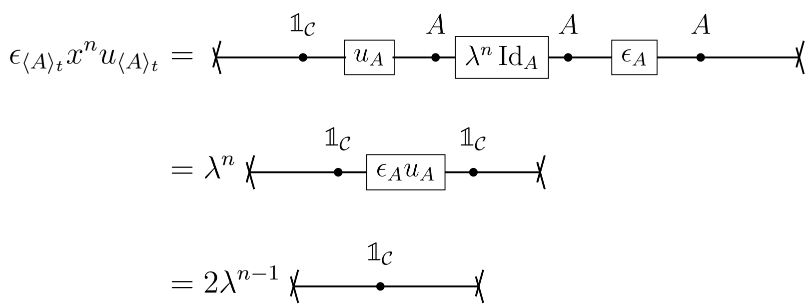
since $\theta_A = 0$. It follows trivially that $\beta = (0, \dots) = \gamma$. To compute α_n, we show first that

$x = \lambda \mathrm{Id}_A$ by graphical calculus:



$$= \lambda \mathrm{Id}_A, \tag{6.3.1}$$

since $m_A \Delta_A = \lambda \mathrm{Id}_A$ in $\mathbf{Rep}(\mathbb{Z}_2)$. Then $x^n = \lambda^n \mathrm{Id}_A$, and so



$$= 2\lambda^{n-1} t \mathrm{Id}_{\mathbb{1}}.$$

Hence

$$\alpha_n = 2\lambda^{n-1} t \quad \text{for all } n \geq 0,$$

as desired. □

Remark 6.3.2. Consider the graphical description of $\mathbf{Rep}(S_t \wr \mathbb{Z}_2)$ as given in Section 2.4.3. We will say that a map in $\mathbf{Rep}(S_t \wr \mathbb{Z}_2)$ is *connected* if its graphical representation is a connected diagram. Thus maps given by stackings of connected diagrams generate $\mathbf{Rep}(S_t \wr \mathbb{Z}_2)$ as a pseudo-abelian category.

Let $u_\mathcal{C}, \epsilon_\mathcal{C}, \mu_\mathcal{C}$ and $\Delta_\mathcal{C}$ in $\mathrm{S_t}(\mathcal{C}) = \mathbf{Rep}(S_t \wr \mathbb{Z}_2)$ as defined in Section 2.4.3.

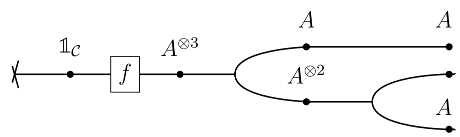
Lemma 6.3.3. *Connected maps $\mathbb{1}_{\mathbf{Rep}(S_t \wr \mathbb{Z}_2)} \to \langle A^{\otimes n} \rangle_t$ can be written as a composition*

$$\Delta_\mathcal{C}^{n-1} \circ \langle f \rangle_t \circ u_\mathcal{C},$$

where $f \in \mathrm{Hom}_\mathcal{C}(\mathbb{1}_\mathcal{C}, A^{\otimes n})$ and $\Delta_\mathcal{C}^{n-1}$ is the appropriate composite of $\Delta_\mathcal{C}$'s.

Proof. This follows from [34, Proposition 4.24]. □

Example 6.3.4. Connected maps $\mathbb{1}_{\mathbf{Rep}(S_t \wr \mathbb{Z}_2)} \to \langle A^{\otimes 3} \rangle_t$ in $\mathbf{Rep}(S_t \wr \mathbb{Z}_2)$ are given by

$\Delta_\mathcal{C}^3 \circ \langle f \rangle_t \circ u_\mathcal{C} =$ 

,

where $f : \mathbb{1} \to A^{\otimes 3}$ is a map in $\mathbf{Rep}(\mathbb{Z}_2)$.

Corollary 6.3.5. *The subspace $\mathcal{U}_n$ spanned by connected maps in $\mathrm{Hom}_{\mathbf{Rep}(S_t \wr \mathbb{Z}_2)}(\mathbb{1}, \langle A^{\otimes n} \rangle_t)$ has dimension*

$$\dim(\mathcal{U}_n) = \dim(\mathrm{Hom}_\mathcal{C}(\mathbb{1}_\mathcal{C}, A^{\otimes n})) = 2^{n-1} \quad \textit{for all } n \geq 1.$$

Proof. This follows from linearity of the functor $\mathbf{Rep}(\mathbb{Z}_2) \to \mathbf{Rep}(S_t \wr \mathbb{Z}_2)$, see [34], and the previous Lemma. □

6.4 Proof of Theorem I

We give here a proof of our first main result, stated below. For a symmetric $\mathbf{k}$-linear monoidal category $\mathcal{C}$, we denote by $\underline{\mathcal{C}}$ its quotient by the tensor ideal of negligible morpshims, see Section 5.3.

Theorem I. Let $\alpha = (\alpha_0, \lambda\alpha_0, \lambda^2\alpha_0, \dots)$ and $\beta = (0, 0, \dots) = \gamma$ be sequences in $\mathbf{k}$, for $\alpha_0, \lambda \in \mathbf{k}^\times$.We have an equivalence of symmetric $\mathbf{k}$-linear monoidal categories

$$\underline{\mathrm{OCob}_\alpha} \cong \underline{\mathbf{Rep}(\mathrm{S}_t \wr \mathbb{Z}_2)},$$

where $t = \frac{\lambda\alpha_0}{2}$.

To prove this, we will use the following result, adapted from [5, Lemma 2.6], see also [27, Proposition 2.4].

Proposition 6.4.1. *Let $\mathcal{C}$ be a semisimple Karoubian symmetric tensor category with finite dimensional* Hom *spaces. Suppose there is a symmetric tensor functor $F : \mathrm{OCob}_\alpha \to \mathcal{C}$ that is surjective on* Hom*'s. Then there is a unique fully faithful symmetric tensor functor*

$$F : \underline{\mathrm{OCob}_\alpha} \xrightarrow{\sim} \mathcal{C}.$$

For the rest of this section, fix sequences

$$\alpha = (\alpha_0, \lambda\alpha_0, \lambda^2\alpha_0, \dots) \qquad \text{and} \qquad \beta = (0, 0, \dots) = \gamma,$$

where $\alpha_0, \lambda \in \mathbf{k}^\times$, and set $t = \frac{\lambda\alpha_0}{2}$. We will show that $\underline{\mathbf{Rep}(S_t \wr \mathbb{Z}_2)}$ satisfies the conditions in the Proposition above via a series of Lemmas.

Recall we denote by $\langle A \rangle_t = \langle A_\lambda \rangle_t$ the extended Frobenius algebra in $\mathbf{Rep}(S_t \wr \mathbb{Z}_2)$, where $A = A_\lambda$ in $\mathbf{Rep}(\mathbb{Z}_2)$ is defined in Example 2.4.7 for $n = 1$, and structure maps are constructed as in Proposition 2.4.14. We recall that the structure maps of these algebras depend on λ, but we drop the subscript to simplify the notation.

Lemma 6.4.2. *There is a symmetric $\mathbf{k}$-linear monoidal functor $F_A : \mathrm{VUCob}_{\alpha,\beta,\gamma} \to \mathbf{Rep}(S_t \wr \mathbb{Z}_2)$, mapping the circle object of $\mathrm{VUCob}_{\alpha,\beta,\gamma}$ to the extended Frobenius algebra $\langle A \rangle_t$ in $\mathbf{Rep}(S_t \wr \mathbb{Z}_2)$.*

Proof. This follows from the universal property of $\mathrm{VUCob}_{\alpha,\beta,\gamma}$, see Section 5.4, and Lemma 6.3.1. □

Lemma 6.4.3. *The functor* $F_A : \mathrm{VUCob}_{\alpha,\beta,\gamma} \to \mathbf{Rep}(S_t \wr \mathbb{Z}_2)$ *factors through* SOCob_α.

Proof. We need to check that F_A annhilates the handle relation $x - \lambda\,\mathrm{Id} = 0$ and the crosscap relation $\theta = 0$. The latter is trivial since $F_A(\theta) = \theta_{\langle A\rangle_t} = 0$. On the other hand, $F_A(x - \lambda I) = m_{\langle A\rangle_t}\Delta_{\langle A\rangle_t} - \lambda\mathrm{Id}_{\langle A\rangle_t} = \lambda\mathrm{Id}_{\langle A\rangle_t} = 0$, see equation (6.3.1). □

Lemma 6.4.4. *The functor* $F_A : \mathrm{SOCob}_\alpha \to \mathbf{Rep}(S_t \wr \mathbb{Z}_2)$ *satisfies that any indecomposable object of* $\mathbf{Rep}(S_t \wr \mathbb{Z}_2)$ *is a direct summand of* $F(n)$ *for some* n *in* SOCob_α.

Proof. By definition, $F_A(1) = \langle A\rangle_t$, and so

$$F_A(n) = \langle A\rangle_t^{\otimes n}.$$

The result follows since objects of the form $\langle A\rangle_t^{\otimes n}$ generate $\mathbf{Rep}(S_t \wr \mathbb{Z}_2)$ as a pseudo-abelian category, see [34, Remark 4.25]. □

Fix $n \geq 1$. Let $\mathcal{W}_n$ denote the subspace of $\mathrm{Hom}_{\mathrm{SOCob}_\alpha}(0, n)$ spanned by connected cobordisms, and let $\mathcal{U}_n$ be the subspace of $\mathrm{Hom}_{\mathbf{Rep}(S_t \wr \mathbb{Z}_2)}(\mathbb{1}, \langle A\rangle_t^{\otimes n})$ spanned by connected diagrams, see Lemma 6.3.3.

Lemma 6.4.5. *The functor* $F_A : \mathrm{SOCob}_\alpha \to \mathbf{Rep}(S_t \wr \mathbb{Z}_2)$ *induces an isomorphism*

$$\mathcal{W}_n \xrightarrow{F_A} \mathcal{U}_n.$$

Proof. Consider the basis $\{\xi^n_{\overline{J}}\}_{\overline{J}\in\mathcal{R}_n}$ of $\mathcal{W}_n$ as in Definition 5.2.8, see also Proposition 6.2.1. That is,

$$\xi^n_{\overline{J}} = \phi_J \Delta^{n-1} u,$$

where J is the representative of $\overline{J}$ with $|J| \leq n/2$,

$$\phi_J := c_{1,J} \otimes \cdots \otimes c_{n,J}, \quad \text{with } c_{j,J} = \begin{cases} \mathrm{id} & \text{if } j \in J, \\ \\ \phi & \text{if } j \notin J, \end{cases}$$

Δ^{n-1} denotes

$$\Delta^{n-1} = (\mathrm{id}^{\otimes(n-2)} \otimes \Delta) \dots (\mathrm{id} \otimes \Delta)\Delta,$$

and u is the cap cobordism of SOCob_α.

We compute the image of ξ^n_J under F_A. By definition, $F_A(\Delta) = \Delta_{\mathcal{C}}\Delta_A$, and so

$$F_A(\Delta^{n-1}) = \left(\mathrm{id}^{\otimes(n-2)} \otimes \Delta_{\mathcal{C}}\langle\Delta_A\rangle_t\right) \dots (\mathrm{id} \otimes \Delta_{\mathcal{C}}\langle\Delta_A\rangle_t)\, \Delta_{\mathcal{C}}\langle\Delta_A\rangle_t.$$

Let

$$\Delta^k_{\mathcal{C}} = \left(\mathrm{id}^{\otimes(k-2)} \otimes \Delta_{\mathcal{C}}\right) \dots (\mathrm{id} \otimes \Delta_{\mathcal{C}})\, \Delta_{\mathcal{C}},$$

for all $k \geq 1$. Using the relation $(\langle f\rangle_t \otimes \langle g\rangle_t)\Delta_{\mathcal{C}} = \Delta_{\mathcal{C}}\langle f \otimes g\rangle_t$ in $\mathbf{Rep}(S_t \wr \mathbb{Z}_2)$, see equation (2.4.6), we have that

$$(\mathrm{id} \otimes \Delta_{\mathcal{C}}\langle\Delta_A\rangle_t)\, \Delta_{\mathcal{C}}\langle\Delta_A\rangle_t = (\mathrm{id} \otimes \Delta_{\mathcal{C}})\Delta_{\mathcal{C}}\langle(\mathrm{id} \otimes \Delta_A)\Delta_A\rangle_t = \Delta^2_{\mathcal{C}}\langle\Delta^2_A\rangle_t.$$

In general,

$$\left(\mathrm{id}^{\otimes(n-2)} \otimes \Delta_{\mathcal{C}}\langle\Delta_A\rangle_t\right) \dots (\mathbb{1} \otimes \Delta_{\mathcal{C}}\langle\Delta_A\rangle_t)\, \Delta_{\mathcal{C}}\langle\Delta_A\rangle_t = \Delta^{n-1}_{\mathcal{C}}\langle\Delta^{n-1}_A\rangle_t,$$

and thus $F_A(\Delta^{n-1}) = \Delta^{n-1}_{\mathcal{C}}\langle\Delta^{n-1}_A\rangle_t$. Thus if we define

$$\psi_{i,J} = \begin{cases} \phi_A & \text{if } i \in J, \\ \\ \mathrm{id} & \text{if } i \notin J, \end{cases}$$

then

$$
\begin{aligned}
F(\xi^n_{\overline{J}}) &= (F(c_{1,J}) \otimes \cdots \otimes F(c_{n,J}))\, F(\Delta^{n-1}_{\mathcal{C}}) F(u_A) \\
&= (\langle \psi_{1,J} \rangle_t \otimes \cdots \otimes \langle \psi_{n,J} \rangle_t)\, \Delta^{n-1}_{\mathcal{C}} \langle \Delta^{n-1}_A \rangle_t \langle u_A \rangle_t u_{\mathcal{C}} \qquad (6.4.1) \\
&= \Delta^{n-1}_{\mathcal{C}} \langle (\psi_{1,J} \otimes \cdots \otimes \psi_{n,J}) \Delta^{n-1}_A u_A \rangle_t u_{\mathcal{C}}.
\end{aligned}
$$

Hence under F_A, $\mathcal{W}_n$ is mapped to $\mathcal{U}_n$. We want to show that the set $\{F(\xi^n_{\overline{J}})\}_{\overline{J} \in \mathcal{R}_n}$ is a basis of $\mathcal{U}_n$. We prove first that it is linearly independent. By the equation above, it is enough to show that $\{f^n_{\overline{J}}\}_{\overline{J} \in \mathcal{R}_k}$ is linearly independent in $\mathbf{Rep}(\mathbb{Z}_2)$, where

$$
f^n_{\overline{J}} := (\psi_{1,J} \otimes \cdots \otimes \psi_{n,J}) \Delta^{n-1}_A u_A \in \mathrm{Hom}_{\mathbf{Rep}(\mathbb{Z}_2)}(\mathbb{1}, A^{\otimes n}).
$$

For this, we check that the value of each $f^n_{\overline{J}}$ at $1 \in \mathbb{1} \cong \mathbf{k}$ is

$$
\delta_{\overline{J}} := \delta_{1,J} \otimes \cdots \otimes \delta_{n,J} + \delta_{1,J^c} \otimes \cdots \otimes \delta_{n,J^c}, \quad \text{where } \delta_{i,J} := \begin{cases} \delta_1 & \text{if } i \in J, \\ \delta_{-1} & \text{if } i \notin J, \end{cases}
$$

for all $1 \leq i \leq n$, and $\delta_1, \delta_{-1} \in A$ are such that $\delta_1(x_l) = \delta_{l,1}$ and $\delta_{-1}(x_l) = \delta_{l,-1}$ for $l = 1, -1$, see Example 2.4.7. We compute

$$
\begin{aligned}
f^n_{\overline{J}}(1) &= (\psi_{1,J} \otimes \cdots \otimes \psi_{n,J})\, \Delta^{n-1}_A (\delta_{-1} + \delta_1) \\
&= (\psi_{1,J} \otimes \cdots \otimes \psi_{n,J})\, (\delta_{-1}^{\otimes n} + \delta_1^{\otimes n}) \\
&= \psi_{1,J}(\delta_{-1}) \otimes \cdots \otimes \psi_{n,J}(\delta_{-1}) + \psi_{1,J}(\delta_1) \otimes \cdots \otimes \psi_{n,J}(\delta_1).
\end{aligned}
$$

Recall that $\phi_A(\delta_{-1}) = \delta_1$ and $\phi_A(\delta_1) = \delta_{-1}$. Hence

$$
\psi_{i,J}(\delta_{-1}) = \begin{cases} \delta_1 & \text{if } i \in J \\ \delta_{-1} & \text{if } i \notin J \end{cases} = \delta_{i,J} \quad \text{and} \quad \psi_{i,J}(\delta_1) = \begin{cases} \delta_{-1} & \text{if } i \in J \\ \delta_1 & \text{if } i \notin J \end{cases} = \delta_{i,J^c},
$$

so we get that

$$f^n_{\overline{J}}(1) = \delta_{1,J} \otimes \cdots \otimes \delta_{n,J} + \delta_{1,J^c} \otimes \cdots \otimes \delta_{n,J^c} = \delta_{\overline{J}}.$$

So it is enough to note that $\{\delta_{\overline{J}}\}_{\overline{J}\in\mathcal{R}_n}$ is linearly independent in $A^{\otimes n}$. In fact, since $\{\delta_1, \delta_{-1}\}$ is a basis for A as a vector space, then $\{\delta_{1,J} \otimes \cdots \otimes \delta_{n,J}\}_{J\in\mathcal{P}(n)}$ is a basis for $A^{\otimes n}$, and so the set

$$\{\delta_{1,J} \otimes \cdots \otimes \delta_{n,J} + \delta_{1,J^c} \otimes \cdots \otimes \delta_{n,J^c}\}_{J\in\mathcal{P}(n)} = \{\delta_{\overline{J}}\}_{\overline{J}\in\mathcal{R}_n},$$

is linearly independent, as desired.

Lastly, by Proposition 6.2.1 and Lemma 6.3.5, we know that $\dim(\mathcal{W}_n) = 2^{n-1} = \dim(\mathcal{U}_n)$, and so the statement follows. □

Taking Karoubian envelope on the source category, $F_A : \mathrm{SOCob}_\alpha \to \mathbf{Rep}(S_t \wr \mathbb{Z}_2)$ extends uniquely to a symmetric tensor functor

$$F_A : \mathrm{OCob}_\alpha \to \mathbf{Rep}(S_t \wr \mathbb{Z}_2).$$

Moreover, by Lemma 6.4.4 F_A is essentially surjective.

Lemma 6.4.6. *The functor $F_A : \mathrm{OCob}_\alpha \to \mathbf{Rep}(S_t \wr \mathbb{Z}_2)$ is full.*

Proof. To show that F_A is surjective on morphisms, it is enough to check that the maps

$$\mathrm{Hom}_{\mathrm{OCob}_\alpha}(n, m) \xrightarrow{F_A} \mathrm{Hom}_{\mathbf{Rep}(\mathrm{S}_t\wr\mathbb{Z}_2)}(\langle A\rangle_t^{\otimes n}, \langle A\rangle_t^{\otimes m})$$

induced by F_A are surjective for all $n, m \geq 1$. Note that since $A \in \mathbf{Rep}(\mathbb{Z}_2)$ is self-dual, then by [34, Appendix A] so is $\langle A\rangle_t$ in $\mathbf{Rep}(S_t \wr \mathbb{Z}_2)$. Hence, by duality, it is enough to check surjectivity of the maps

$$\mathrm{Hom}_{\mathrm{OCob}_\alpha}(0, n) \xrightarrow{F_A} \mathrm{Hom}_{\mathbf{Rep}(\mathrm{S}_t\wr\mathbb{Z}_2)}(\mathbb{1}, \langle A\rangle_t^{\otimes n}),$$

for all $n \geq 1$. This follows from Lemma 6.4.5, since connected diagrams generate $\mathbf{Rep}(S_t \wr \mathbb{Z}_2)$ as a pseudo-abelian $\mathbf{k}$-linear monoidal category. □

We now show a proof for Theorem I.

Proof of Theorem I. Consider the symmetric tensor functor

$$\underline{F_A} := \mathrm{OCob}_\alpha \xrightarrow{F_A} \mathbf{Rep}(\mathrm{S}_t \wr \mathbb{Z}_2) \to \underline{\mathbf{Rep}(\mathrm{S}_t \wr \mathbb{Z}_2)},$$

where F_A is as defined previously, followed by the semisimplification functor. By Lemma 6.4.6, $\underline{F_A}$ satisfies the conditions of Proposition 6.4.1. Hence $\underline{F_A}$ induces a fully faithful symmetric tensor functor

$$\underline{F_A} : \underline{\mathrm{OCob}_\alpha} \to \underline{\mathbf{Rep}(\mathrm{S}_t \wr \mathbb{Z}_2)}.$$

Moreover, since F_A is essentially surjective then so is $\underline{F_A}$ and we have the desired equivalence.

□

Corollary 6.4.7. *If $\lambda\alpha_0$ is not a non-negative even integer, then*

$$\mathrm{OCob}_\alpha \cong \mathbf{Rep}(S_t \wr \mathbb{Z}_2).$$

In particular, OCob_α *is semisimple, and* $\dim_{\mathrm{OCob}_\alpha}(0, m) = \sum_{l=1}^{m} 2^{m-l} \left\{ \begin{matrix} m \\ l \end{matrix} \right\}$, *for all* $m \geq 1$.

Proof. By Corollary 6.2.6 we know that for all but countably many values of λ and α_0 in $\mathbf{k}^\times$, there are no negligible morphisms in OCob_α. From this and Theorem I, it follows that from all but countably many λ and α_0 we get an equivalence

$$\mathrm{OCob}_\alpha = \underline{\mathrm{OCob}_\alpha} \cong \underline{\mathbf{Rep}(S_t \wr \mathbb{Z}_2)},$$

where $t = \frac{\lambda\alpha_0}{2}$. Hence, when λ and α_0 are not one of the exceptional values, we have that

$$\dim \mathrm{Hom}_{\underline{\mathbf{Rep}(S_t \wr \mathbb{Z}_2)}}(\mathbb{1}, \langle A \rangle_t^{\otimes m}) = \dim \mathrm{Hom}_{\mathrm{OCob}_\alpha}(0, m) = \sum_{l=1}^{m} 2^{m-l} \begin{Bmatrix} m \\ l \end{Bmatrix},$$

see Corollary 6.2.6. But by [34], for $t \notin \mathbb{Z}_{\geq 0}$, the category $\mathbf{Rep}(S_t \wr \mathbb{Z}_2)$ is semisimple, and so $\dim_{\mathbf{Rep}(S_t \wr \mathbb{Z}_2)}(\mathbb{1}, \langle A \rangle_t^{\otimes m})$ does not depend on t. Therefore, $\dim_{\mathbf{Rep}(S_t \wr \mathbb{Z}_2)}(\mathbb{1}, \langle A \rangle_t^{\otimes m}) = \sum_{l=1}^{m} 2^{m-l} \begin{Bmatrix} m \\ l \end{Bmatrix}$ whenever $t = \frac{\lambda\alpha_0}{2} \notin \mathbb{Z}_{\geq 0}$.

Let $t = \frac{\lambda\alpha_0}{2} \notin \mathbb{Z}_{\geq 0}$, and consider again the equivalence

$$\underline{\mathrm{OCob}_\alpha} = \underline{\mathbf{Rep}(S_t \wr \mathbb{Z}_2)} = \mathbf{Rep}(S_t \wr \mathbb{Z}_2).$$

Since

$$\sum_{l=1}^{m} 2^{m-l} \begin{Bmatrix} m \\ l \end{Bmatrix} = \dim \mathrm{Hom}_{\mathbf{Rep}(S_t \wr \mathbb{Z}_2)}(\mathbb{1}, \langle A \rangle_t^{\otimes m}) = \dim \mathrm{Hom}_{\mathrm{OCob}_\alpha}(0, m) \leq \sum_{l=1}^{m} 2^{m-l} \begin{Bmatrix} m \\ l \end{Bmatrix},$$

then $\dim \mathrm{Hom}_{\mathrm{OCob}_\alpha}(0, m) = \sum_{l=1}^{m} 2^{m-l} \begin{Bmatrix} m \\ l \end{Bmatrix}$. Hence, when $\lambda\alpha_0$ is not a non-negative even integer, there are no negligible morphisms in OCob_α and thus by Theorem I we conclude

$$\mathrm{OCob}_\alpha = \underline{\mathrm{OCob}_\alpha} \cong \underline{\mathbf{Rep}(S_t \wr \mathbb{Z}_2)} = \mathbf{Rep}(S_t \wr \mathbb{Z}_2),$$

as desired. □

Chapter 7

Equivalence with the category $\mathbf{Rep}(S_t \wr \mathbb{Z}_2) \boxtimes \mathbf{Rep}(S_{t^+}) \boxtimes \mathbf{Rep}(S_{t^-})$

This chapter contains previously published material, which appeared in [12].

7.1 Preliminary results

In this section we study the category $\mathrm{SUCob}_{\alpha,\beta,\gamma}$ for the sequences

$$\alpha = (\alpha_0, \lambda\alpha_0, \lambda^2\alpha_0, \dots), \quad \beta = (\beta_0, \lambda\beta_0, \lambda^2\beta_0, \dots) \quad \text{and} \quad \gamma = (\gamma_0, \lambda\gamma_0, \lambda^2\gamma_0, \dots), \tag{7.1.1}$$

where $\alpha_0, \beta_0, \gamma_0, \lambda \in \mathbf{k}^\times$. Here, the generating functions for α, β and γ are

$$Z_\alpha(T) = \frac{\alpha_0}{1-\lambda T}, \quad Z_\beta(T) = \frac{\beta_0}{1-\lambda T}, \quad Z_\gamma(T) = \frac{\gamma_0}{1-\lambda T},$$

respectively.

Recall that for a $\mathbf{k}$-linear symmetric monoidal category $\mathcal{C}$, we denote by $\underline{\mathcal{C}}$ its quotient by the tensor ideal of negligible morpshims, see Section 2.2.12.

Theorem II. Let α, β and γ be sequences as in (7.1.1). Then we have an equivalence of symmetric $\mathbf{k}$-linear monoidal categories

$$\underline{\mathrm{UCob}_{\alpha,\beta,\gamma}} \simeq \underline{\mathbf{Rep}(\mathrm{S}_t \wr \mathbb{Z}_2)} \boxtimes \underline{\mathbf{Rep}(S_{t_+})} \boxtimes \underline{\mathbf{Rep}(S_{t_-})},$$

where $t = \frac{1}{2}(\lambda\alpha_0 - \gamma_0), t_+ = \frac{1}{2}(\gamma_0 + \sqrt{\lambda}\beta_0)$ and $t_- = \frac{1}{2}(\gamma_0 - \sqrt{\lambda}\beta_0)$.

The rest of this section is dedicated to giving a proof for Theorem II.

Recall that $\mathrm{SUCob}_{\alpha,\beta,\gamma}$ is a rigid symmetric $\mathbf{k}$-linear monoidal category with finite dimensional Hom spaces. In this case, the handle relation is $x - \lambda \mathrm{Id} = 0$, and so Hom spaces are spanned by cobordisms where all connected components have genus 0.

Proposition 7.1.1. *For $m \geq 2$, let $\{\xi^m_{\overline{J}} : \overline{J} \in \mathcal{R}_m\}$ be as in Definition 5.2.8, and let $\theta_m := \Delta^{m-1}\theta$ and $\theta_m[2] := \Delta^{m-1}m(\theta \otimes \theta)$. Then the set*

$$\boldsymbol{t}_m := \{\xi^m_{\overline{J}} : \overline{J} \in \mathcal{R}_m\} \cup \{\theta_m, \theta_m[2]\}, \tag{7.1.2}$$

is a spanning set for the subspace of $\mathrm{Hom}_{\mathrm{SUCob}_{\alpha,\beta,\gamma}}(0, m)$ *spanned by connected cobordisms.*

Proof. This follows from Propositions 5.1.2 and 5.1.11, since connected cobordisms have genus 0. □

Let $P = \{p_1, \dots, p_k\}$ be a partition of $\{1, \dots, m\}$. Let $P_1 = \{p_{l_i}\}_{i=1}^{k_1}$, P_2 and P_3 be disjoint subsets of P such that $P = P_1 \cup P_2 \cup P_3$. Choose $\overline{J_{l_i}} \in \mathcal{R}_{|p_{l_i}|}$ for all $1 \leq i \leq k_1$. Denote by $c_{P,\overline{J_{l_1}},\dots,\overline{J_{l_{k_1}}},\theta_{P_2},\theta^2_{P_3}}$ the cobordism $0 \to m$ with k connected components where:

- the l_i-th component is of the form $\xi^{|p_{l_i}|}_{\overline{J}_{l_i}}$ with out-boundary the circles in positions corresponding to the elements of p_{l_i}, for all $1 \leq i \leq k_1$,

- for $p \in P_2$, the corresponding connected component is of the form $\theta_{|p|}$ with out-boundary the circles in positions corresponding to the elements of p,

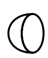
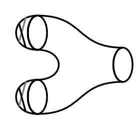
- for $p \in P_3$, the corresponding connected component is of the form $\theta_{|p|}[2]$ with out-boundary the circles in positions corresponding to the elements of p.

Then

$$\mathcal{T}_m := \left\{ c_{P,\overline{J_{l_1}},\ldots,\overline{J_{l_{k_1}}},\theta_{P_2},\theta^2_{P_3}} \right\} \tag{7.1.3}$$

moving over all possible such choices is a spanning set for $\mathrm{Hom}_{\mathrm{SOCob}_\alpha}(0,m)$.

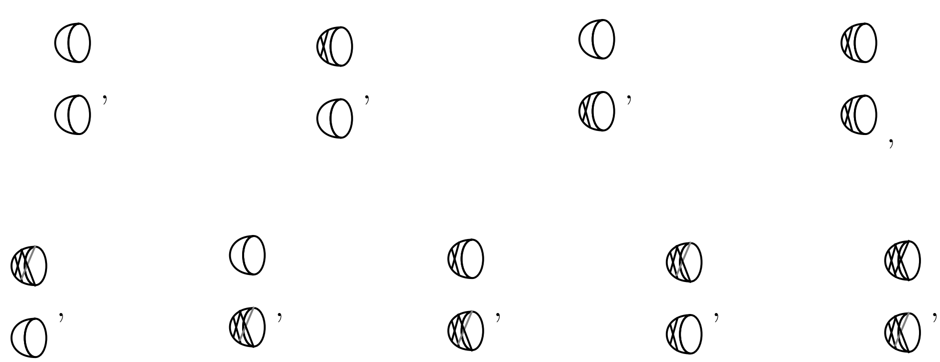
Example 7.1.2. Suppose that $\lambda\alpha_0 \neq \gamma_0$ and $\gamma_0 \neq \pm\sqrt{\lambda}\beta_0$. Then $\mathrm{Hom}_{\mathrm{SUCob}_{\alpha,\beta,\gamma}}(0,1)$ has dimension 3, with basis

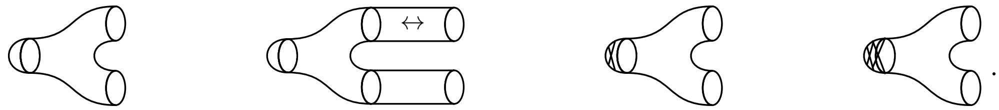
$u =$  $\theta =$ , $\theta[2] =$  $:=$ .

In fact, from Example 5.3.5 we know that the matrix of inner products has determinant

$$\alpha_0(\lambda\gamma_0^2 - \lambda^2\beta_0^2) + \gamma_0(\lambda\beta_0^2 - \gamma_0^2) = (\alpha_0\lambda - \gamma_0)(\gamma_0^2 - \lambda\beta_0^2).$$

Example 7.1.3. Suppose that $\lambda\alpha_0 - \gamma_0 \neq 0, 2$ and $\gamma_0 \pm \sqrt{\lambda}\beta_0 \neq 0, 2$. Then the following is a basis for $\mathrm{Hom}_{\mathrm{SUCob}_{\alpha,\beta,\gamma}}(0,2)$:





In fact, it is a generating set by the proposition above, and the matrix of inner products is

$$\begin{bmatrix}
\alpha_0^2 & \alpha_0\beta_0 & \alpha_0\beta_0 & \beta_0^2 & \alpha_0\gamma_0 & \alpha_0\gamma_0 & \beta_0\gamma_0 & \beta_0\gamma_0 & \gamma_0^2 & \alpha_0 & \alpha_0 & \beta_0 & \gamma_0 \\
\alpha_0\beta_0 & \alpha_0\gamma_0 & \beta_0^2 & \beta_0\gamma_0 & \alpha_0\beta_0\lambda & \beta_0\gamma_0 & \gamma_0^2 & \beta_0^2\lambda & \beta_0\gamma_0\lambda & \beta_0 & \beta_0 & \gamma_0 & \beta_0\lambda \\
\alpha_0\beta_0 & \beta_0^2 & \alpha_0\gamma_0 & \beta_0\gamma_0 & \beta_0\gamma_0 & \alpha_0\beta_0\lambda & \beta_0^2\lambda & \gamma_0^2 & \beta_0\gamma_0\lambda & \beta_0 & \beta_0 & \gamma_0 & \beta_0\lambda \\
\beta_0^2 & \beta_0\gamma_0 & \beta_0\gamma_0 & \gamma_0^2 & \beta_0^2\lambda & \beta_0^2\lambda & \beta_0\gamma_0\lambda & \beta_0\gamma_0\lambda & \beta_0^2\lambda^2 & \gamma_0 & \gamma_0 & \beta_0\lambda & \gamma_0\lambda \\
\alpha_0\gamma_0 & \alpha_0\beta_0\lambda & \beta_0\gamma_0 & \beta_0^2\lambda & \alpha_0\gamma_0\lambda & \gamma_0^2 & \beta_0\gamma_0\lambda & \beta_0\gamma_0\lambda & \gamma_0^2\lambda & \gamma_0 & \gamma_0 & \beta_0\lambda & \gamma_0\lambda \\
\alpha_0\gamma_0 & \beta_0\gamma_0 & \alpha_0\beta_0\lambda & \beta_0^2 l & \gamma_0^2 & \alpha_0\gamma_0\lambda & \beta_0\gamma_0\lambda & \beta_0\gamma_0\lambda & \gamma_0^2\lambda & \gamma_0 & \gamma_0 & \beta_0\lambda & \gamma_0\lambda \\
\beta_0\gamma_0 & \gamma_0^2 & \beta_0^2\lambda & \beta_0\gamma_0\lambda & \beta_0\gamma_0\lambda & \beta_0\gamma_0\lambda & \gamma_0^2\lambda & \beta_0^2\lambda^2 & \beta_0\gamma_0\lambda^2 & \beta_0\lambda & \beta_0\lambda & \gamma_0\lambda & \beta_0\lambda^2 \\
\beta_0\gamma_0 & \beta_0^2\lambda & \gamma_0^2 & \beta_0\gamma_0\lambda & \beta_0\gamma_0\lambda & \beta_0\gamma_0\lambda & \beta_0^2\lambda^2 & \gamma_0^2\lambda & \beta_0\gamma_0\lambda^2 & \beta_0\lambda & \beta_0\lambda & \gamma_0\lambda & \beta_0\lambda^2 \\
\gamma_0^2 & \beta_0\gamma_0\lambda & \beta_0\gamma_0\lambda & \beta_0^2\lambda^2 & \gamma_0^2\lambda & \gamma_0^2\lambda & \beta_0\gamma_0\lambda^2 & \beta_0\gamma_0\lambda^2 & \gamma_0^2\lambda^2 & \gamma_0\lambda & \gamma_0\lambda & \beta_0\lambda^2 & \gamma_0\lambda^2 \\
\alpha_0 & \beta_0 & \beta_0 & \gamma_0 & \gamma_0 & \gamma_0 & \beta_0\lambda & \beta_0\lambda & \gamma_0\lambda & \alpha_0\lambda & \gamma_0 & \beta_0\lambda & \gamma_0\lambda \\
\alpha_0 & \beta_0 & \beta_0 & \gamma_0 & \gamma_0 & \gamma_0 & \beta_0\lambda & \beta_0\lambda & \gamma_0\lambda & \gamma_0 & \alpha_0\lambda & \beta_0\lambda & \gamma_0\lambda \\
\beta_0 & \gamma_0 & \gamma_0 & \beta_0\lambda & \beta_0\lambda & \beta_0\lambda & \gamma_0\lambda & \gamma_0\lambda & \beta_0\lambda^2 & \beta_0\lambda & \beta_0\lambda & \gamma_0\lambda & \beta_0\lambda^2 \\
\gamma_0 & \beta_0\lambda & \beta_0\lambda & \gamma_0\lambda & \gamma_0\lambda & \gamma_0\lambda & \beta_0\lambda^2 & \beta_0\lambda^2 & \gamma_0\lambda^2 & \gamma_0\lambda & \gamma_0\lambda & \beta_0\lambda^2 & \gamma_0\lambda^2
\end{bmatrix},$$

which has determinant

$$\lambda^3(\gamma_0 - \sqrt{\lambda}\beta_0)^6(\gamma_0 + \sqrt{\lambda}\beta_0)^6(\gamma_0 - \sqrt{\lambda}\beta_0 - 2)(\gamma_0 + \sqrt{\lambda}\beta_0 - 2)(\lambda\alpha_0 - \gamma_0)^7(\lambda\alpha_0 - \gamma_0 - 2).$$

Hence

$$\dim(\mathrm{Hom}_{\mathrm{SUCob}_{\alpha,\beta,\gamma}}(0,2)) = 13.$$

Conjecture 7.1.4. *We conjecture that for $\lambda, \alpha_0, \beta_0, \gamma_0$ in $\mathbf{k}^\times$ such that $\lambda\alpha_0 - \gamma_0$ and $\gamma_0 \pm \sqrt{\lambda}\beta_0$ are non-negative even integers, the determinants of the matrices of inner products of morphisms in $\mathcal{T}_m$ are non-zero polynomials on $\lambda, \alpha_0, \beta_0, \gamma_0$, for all $m \geq 1$.*

We know this to be true for $m = 0, 1, 2$, see the examples above. The general computation would follow the same lines as the proof of Lemma 6.2.5. However, this case requires a more careful combinatorial analysis, since for instance, for certain $a_i, b_i \geq 0$ such that $\sum_{i=1}^{l}(a_i+b_i) = k$, we will get

$$(\theta_k, \theta_k) = \gamma_{k-1} = (\theta_k, \theta^{\otimes a_1} \otimes \theta[2]^{\otimes b_1} \otimes \theta^{\otimes a_2} \otimes \cdots \otimes \theta[2]^{\otimes b_l}).$$

That is, the diagonal entry in $\mathcal{T}_k$ corresponding to the row of θ_k will be repeated in another entry of the same row. Moreover, it will actually be repeated more than once in the same row, since

$$(\theta_k, \theta^{\otimes a_1} \otimes \theta[2]^{\otimes b_1} \otimes \theta^{\otimes a_2} \otimes \cdots \otimes \theta[2]^{\otimes b_l}) = (\theta_k, \theta^{\otimes a_1'} \otimes \theta[2]^{\otimes b_1'} \otimes \theta^{\otimes a_2'} \otimes \cdots \otimes \theta[2]^{\otimes b_j'}),$$

whenever $a_i, a_i', b_i, b_i' \geq 0$ are such that $\sum_{i=1}^{l}(a_i + b_i) = \sum_{i=1}^{j}(a_i' + b_i') = k$, and $a_1 + \cdots + a_l + 2(b_1 + \cdots + b_l) = a_1' + \cdots + a_l' + 2(b_1' + \cdots + b_l')$.

Remark 7.1.5. We conjectured that the exceptional values of $\lambda, \alpha_0, \beta_0, \gamma_0$ are those such that $\lambda\alpha_0 - \gamma_0$ and $\gamma_0 \pm \sqrt{\lambda}\beta_0$ are non-negative even integers. We thus predict that the determinant of the matrix of inner products of $\mathcal{T}_m$, see (7.1.3), will factor into powers of the form $\lambda\alpha_0 - \gamma_0 - 2k$ and $\gamma_0 \pm \sqrt{\lambda}\beta_0 - 2k$, for $k = 0, \ldots, m-1$. Note that we know this to be the case for $m = 1$ and 2, see Examples 5.3.5 and 7.1.3.

To prove Theorem II, we will need the following Proposition, adapted from [5, Lemma 2.6], see also [27, Proposition 2.4].

Proposition 7.1.6. *Let $\mathcal{C}$ be a semisimple Karoubian symmetric $\mathbf{k}$-linear monoidal category with finite dimensional Hom spaces. Suppose there is a symmetric $\mathbf{k}$-linear monoidal functor $F : \mathrm{UCob}_{\alpha,\beta,\gamma} \to \mathcal{C}$ that satisfies*

- *Any indecomposable object of $\mathcal{C}$ is a direct summand of $F(n)$ for some n in* $\mathrm{UCob}_{\alpha,\beta,\gamma}$*, and*
- *F is surjective on* Hom*'s.*

Then F induces an equivalence

$$F : \underline{\mathrm{UCob}_{\alpha,\beta,\gamma}} \xrightarrow{\sim} \mathcal{C}.$$

7.2 Extended Frobenius algebras in $\mathbf{Rep}(S_t)$

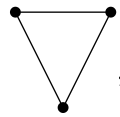
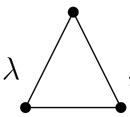
We define now Deligne's category $\mathbf{Rep}(S_t)$ for $t \in \mathbf{k}$, following [8, 14].

Definition 7.2.1. [8, Definition 2.11] Consider the category $\mathbf{Rep}_0(S_t)$ given by:

- Objects are non-negative integers. We represent $n \in \mathbb{Z}_{\geq 0}$ by n horizontal dots (zero is represented by "no dots").
- Morphisms $m \to m'$ are $\mathbf{k}$-linear combinations of partitions of the set $\{1, \dots, m, 1', \dots, m'\}$. Such a partition is represented by a diagram with m points on top labelled 1 to m and m' points on the bottom labeled $1'$ to m', such that points in the same part of the partition are connected by a path.
- Composition is as described in [8, Definition 2.11].

Definition 7.2.2. [14] Let $\mathbf{Rep}(S_t)$ be the pseudo-abelian envelope of $\mathbf{Rep}_0(S_t)$.

Remark 7.2.3. Let $t \in \mathbf{k}^\times$. Recall that we represent morphisms in $\mathbf{Rep}_0(S_t)$ as going from top to bottom. There is a canonical Frobenius algebra $A = A_\lambda$ in $\mathbf{Rep}(S_t)$, given by the object 1 (represented as 1 point) and unit, mutiplication, counit and comultiplication maps given by

$u =$, $\quad m =$  , $\quad \epsilon = \frac{1}{\lambda}$, $\quad \Delta = \lambda$  ,

respectively. Then if

$\phi =$  $\quad$ and $\quad \theta = \pm\sqrt{\lambda}u,$

we get that A with these structure maps is an extended Frobenius algebra in $\mathbf{Rep}(S_t)$, see Definition 2.4.4

From now on, let

$$t = \frac{1}{2}(\lambda\alpha_0 - \gamma_0), \qquad t_+ = \frac{1}{2}(\gamma_0 + \sqrt{\lambda}\beta_0) \qquad \text{and} \qquad t_- = \frac{1}{2}(\gamma_0 - \sqrt{\lambda}\beta_0).$$

Let $A_\pm = A_{\pm,\lambda}$ denote the extended Frobenius algebras in $\mathbf{Rep}(S_{t_\pm})$ as defined above, respectively, and let $A_t = A_{t,\lambda} := \langle A\rangle_t$ in $\mathbf{Rep}(S_t \wr \mathbb{Z}_2)$ be the extended Frobenius algebra induced from the algebra of functions introduced in Example 2.4.7 for $n = 1$. We remark that the structure maps of these algebras depend on λ, but we drop the subscript to simplify notation.

Lemma 7.2.4. *The extended Frobenius algebra*

$$\mathcal{A} := (A_t \boxtimes \mathbb{1} \boxtimes \mathbb{1}) \oplus (\mathbb{1} \boxtimes A_+ \boxtimes \mathbb{1}) \oplus (\mathbb{1} \boxtimes \mathbb{1} \boxtimes A_-)$$

in $\mathbf{Rep}(S_t \wr \mathbb{Z}_2) \boxtimes \mathbf{Rep}(S_{t_+}) \boxtimes \mathbf{Rep}(S_{t_-})$ *has evaluation* α, β, γ *as defined on Equation* (7.1.1).

Proof. The evaluation of $A_t \boxtimes \mathbb{1} \boxtimes \mathbb{1}$ was computed in Lemma 6.3.1, and is given by

$$\alpha_t = (2\lambda^{-1}t, 2t, 2\lambda t, \dots) \qquad \text{and} \qquad \beta_t = \gamma_t = (0, \dots).$$

On the other hand, it is easy to check that the evaluations of $\mathbb{1} \boxtimes A_+ \boxtimes \mathbb{1}$ and $\mathbb{1} \boxtimes \mathbb{1} \boxtimes A_-$ are

$$\alpha_\pm = (\lambda^{-1}t_\pm, t_\pm, \lambda t_\pm, \dots), \quad \beta_\pm = (\pm\lambda^{-1/2}t_\pm, \pm\lambda^{1/2}t_\pm, \pm\lambda^{3/2}t_\pm, \dots), \quad \gamma_\pm = (t_\pm, \lambda t_\pm, \lambda^2 t_\pm, \dots),$$

respectively.

Call $\tilde{\alpha}, \tilde{\beta}$ and $\tilde{\gamma}$ the evaluation sequences of $\mathcal{A}$. Note that $2t + t_+ + t_- = \lambda\alpha_0$. Hence the $\tilde{\alpha}$ sequence of the evaluation of $\mathcal{A}$ is

$$\begin{aligned}\tilde{\alpha} &= \alpha_t + \alpha_+ + \alpha_- \\ &= (\lambda^{-1}(2t + t_+ + t_-), 2t + t_+ + t_-, \lambda(2t + t_+ + t_-), \dots) \\ &= (\alpha_0, \lambda\alpha_0, \lambda^2\alpha_0, \dots),\end{aligned}$$

as desired. On the other hand, $t_+ - t_- = \sqrt{\lambda}\beta_0$ and so

$$\begin{aligned}\tilde{\beta} &= \beta_t + \beta_+ + \beta_- \\ &= (\lambda^{-1/2}(t_+ - t_-), \lambda^{1/2}(t_+ - t_-), \lambda^{3/2}(t_+ - t_-), \dots) \\ &= (\beta_0, \lambda\beta_0, \lambda^2\beta_0, \dots).\end{aligned}$$

Lastly, $t_+ + t_- = \gamma_0$, thus

$$\begin{aligned}\tilde{\gamma} &= \gamma_t + \gamma_+ + \gamma_- \\ &= (t_+ + t_-, \lambda(t_+ + t_-), \lambda^2(t_+ + t_-), \dots) \\ &= (\gamma_0, \lambda\gamma_0, \lambda^2\gamma_0, \dots).\end{aligned}$$

□

7.3 Proof of Theorem II

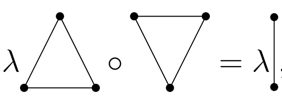
It follows from the previous section and the universal property of $\mathrm{VUCob}_{\alpha,\beta,\gamma}$, see Section 5.4, that there exists a symmetric **k**-linear monoidal functor

$$F_{\mathcal{A}} : \mathrm{VUCob}_{\alpha,\beta,\gamma} \to \mathbf{Rep}(S_t \wr \mathbb{Z}_2) \boxtimes \mathbf{Rep}(S_{t_+}) \boxtimes \mathbf{Rep}(S_{t_-}),$$

mapping the circle object of $\mathrm{VUCob}_{\alpha,\beta,\gamma}$ to $\mathcal{A}$, and the extended Frobenius algebra structure maps of the circle to those of $\mathcal{A}$.

Lemma 7.3.1. *$F_{\mathcal{A}} : \mathrm{VUCob}_{\alpha,\beta,\gamma} \to \mathbf{Rep}(S_t \wr \mathbb{Z}_2) \boxtimes \mathbf{Rep}(S_{t_+}) \boxtimes \mathbf{Rep}(S_{t_-})$ annihilates the handle relation $x - \lambda\,\mathrm{Id}$, and so it factors through $\mathrm{SUCob}_{\alpha,\beta,\gamma}$.*

Proof. We know by equation (6.3.1) that $m_{A_t}\Delta_{A_t} - \lambda \mathrm{Id}_{A_t} = 0$ in $\mathbf{Rep}(S_t \wr \mathbb{Z}_2)$. On the other hand,

$m_{A_\pm}\Delta_{A_\pm} = \lambda$  ,

in $\mathbf{Rep}(S_{t_\pm})$. It follows that $F_{\mathcal{A}}(x - \lambda \mathrm{Id}_{\mathcal{A}}) = 0$. □

Lemma 7.3.2. *The functor $F_{\mathcal{A}} : \mathrm{SUCob}_{\alpha,\beta,\gamma} \to \mathbf{Rep}(S_t \wr \mathbb{Z}_2) \boxtimes \mathbf{Rep}(S_{t_+}) \boxtimes \mathbf{Rep}(S_{t_-})$ satisfies that any indecomposable object of $\mathbf{Rep}(S_t \wr \mathbb{Z}_2) \boxtimes \mathbf{Rep}(S_{t_+}) \boxtimes \mathbf{Rep}(S_{t_-})$ is a direct sumand of $F(n)$ for some n.*

Proof. The functor $F_{\mathcal{A}}$ maps $1 \mapsto \mathcal{A}$. Any object in $\mathbf{Rep}(S_t \wr \mathbb{Z}_2)$, respectively, in $\mathbf{Rep}(S_{t_\pm})$, is a direct summand of tensor powers of A, respectively $A_\pm$, and so the statement follows. □

Lemma 7.3.3. *The unique extension*

$$F_{\mathcal{A}} : \mathrm{UCob}_{\alpha,\beta,\gamma} \to \mathbf{Rep}(S_t \wr \mathbb{Z}_2) \boxtimes \mathbf{Rep}(S_{t_+}) \boxtimes \mathbf{Rep}(S_{t_-}),$$

is surjective on Hom's.

Proof. Let $\mathcal{C} := \mathbf{Rep}(S_t \wr \mathbb{Z}_2) \boxtimes \mathbf{Rep}(S_{t_+}) \boxtimes \mathbf{Rep}(S_{t_-})$. To show that $F_{\mathcal{A}}$ is surjective on morphisms, it is enough to check that the maps

$$\mathrm{Hom}_{\mathrm{UCob}_2}(n,m) \xrightarrow{F_{\mathcal{A}}} \mathrm{Hom}_{\mathcal{C}}(\mathcal{A}^{\otimes n}, \mathcal{A}^{\otimes m}),$$

induced by $F_{\mathcal{A}}$ are surjective for all $n, m \geq 1$. Since $\mathcal{A}$ is self-dual, it is enough to check surjectivity of the maps

$$\mathrm{Hom}_{\mathrm{UCob}_2}(0,m) \xrightarrow{F_{\mathcal{A}}} \mathrm{Hom}_{\mathcal{C}}(\mathbb{1}, \mathcal{A}^{\otimes m}),$$

for all $m \geq 1$. Since $\mathcal{A} = (A_t \boxtimes \mathbb{1} \boxtimes \mathbb{1}) \oplus (\mathbb{1} \boxtimes A_+ \boxtimes \mathbb{1}) \oplus (\mathbb{1} \boxtimes \mathbb{1} \boxtimes A_-)$, it follows it is enough to check that the direct summands $\mathrm{Hom}_{\mathcal{C}}(\mathbb{1}, A_t^{\otimes m} \boxtimes \mathbb{1} \boxtimes \mathbb{1})$, $\mathrm{Hom}_{\mathcal{C}}(\mathbb{1}, \mathbb{1} \boxtimes A_+^{\otimes m} \boxtimes \mathbb{1})$ and $\mathrm{Hom}_{\mathcal{C}}(\mathbb{1}, \mathbb{1} \boxtimes \mathbb{1} \boxtimes A_-^{\otimes m})$ of $\mathrm{Hom}_{\mathcal{C}}(\mathbb{1}, \mathcal{A}^{\otimes m})$ are in the image of $F_{\mathcal{A}}$, for all $m \geq 1$. Then, all remaining summands will be in the image by induction.

We show surjectivity on $\mathrm{Hom}_{\mathcal{C}}(\mathbb{1}, \mathbb{1} \boxtimes A_+^{\otimes m} \boxtimes \mathbb{1})$ and $\mathrm{Hom}_{\mathcal{C}}(\mathbb{1}, \mathbb{1} \boxtimes \mathbb{1} \boxtimes A_-^{\otimes m})$ for all $m \geq 1$ first. Note that partitions of the form

$$\underbrace{\bullet\!-\!\bullet\!-\cdots-\!\bullet}_{k} \qquad \text{for } k \leq m,$$

generate $\mathrm{Hom}_{\mathbf{Rep}(S_{t_\pm})}(0,m)$. That is, any morphism in $\mathrm{Hom}_{\mathbf{Rep}(S_{t_\pm})}(0,m)$ is linear combination of tensor products of partitions of this form. So it is enough to show that these partitions are in the image of $F_{\mathcal{A}}$. Recall that $F_{\mathcal{A}}$ maps the structure maps of the circle object in $\mathrm{UCob}_{\alpha,\beta,\gamma}$ to those of $\mathcal{A}$. So

$$\theta_m \mapsto 0 \oplus \lambda^{m-1/2} \underbrace{\bullet\!-\!\bullet\!-\cdots-\!\bullet}_{m} \oplus (-\lambda^{m-1/2}) \underbrace{\bullet\!-\!\bullet\!-\cdots-\!\bullet}_{m}$$

and

$$\theta_m[2] \mapsto 0 \oplus \lambda^{m} \underbrace{\bullet\!-\!\bullet\!-\cdots-\!\bullet}_{m} \oplus \lambda^{m} \underbrace{\bullet\!-\!\bullet\!-\cdots-\!\bullet}_{m},$$

for all $m \geq 1$.

Hence

$$\lambda^{-(m-1/2)}\theta_m + \lambda^{-m}\theta_m[2] \mapsto 0 \oplus \underbrace{\bullet\!\!-\!\!\bullet \cdots \bullet}_{m} \oplus 0, \quad \text{and}$$

$$-\lambda^{-(m-1/2)}\theta_m + \lambda^{-m}\theta_m[2] \mapsto 0 \oplus 0 \oplus \underbrace{\bullet\!\!-\!\!\bullet \cdots \bullet}_{m}.$$

Thus we have surjectivity on $\mathrm{Hom}_{\mathcal{C}}(\mathbb{1}, \mathbb{1} \boxtimes A_+^{\otimes m} \boxtimes \mathbb{1})$ and $\mathrm{Hom}_{\mathcal{C}}(\mathbb{1}, \mathbb{1} \boxtimes \mathbb{1} \boxtimes A_-^{\otimes m})$, for all $m \geq 1$.

It remains to show that $F_{\mathcal{A}}$ is surjective on $\mathrm{Hom}_{\mathcal{C}}(\mathbb{1}, A_t^{\otimes m} \boxtimes \mathbb{1} \boxtimes \mathbb{1})$. Consider the set $\{\xi_{\overline{J}}^m\}_{\overline{J} \in \mathcal{R}_n}$ as in Definition 5.2.8. Then $F_{\mathcal{A}}$ maps

$$\xi_{\overline{J}}^m \mapsto F_{\overline{J}}^m \oplus \lambda^{m-1} \underbrace{\bullet\!\!-\!\!\bullet \cdots \bullet}_{m} \oplus \lambda^{m-1} \underbrace{\bullet\!\!-\!\!\bullet \cdots \bullet}_{m},$$

where $\{F_{\overline{J}}^m\}$ is a basis for the subspace of connected maps $\mathbb{1} \to \langle A \rangle_t^{\otimes m}$ in $\mathbf{Rep}(S_t \wr \mathbb{Z}_2)$, see Lemma 6.4.5 and Equation (6.4.1). Since we already know $F_{\mathcal{A}}$ is surjective on $\mathrm{Hom}_{\mathcal{C}}(\mathbb{1}, \mathbb{1} \boxtimes A_+ \boxtimes \mathbb{1})$ and $\mathrm{Hom}_{\mathcal{C}}(\mathbb{1}, \mathbb{1} \boxtimes \mathbb{1} \boxtimes A_+)$, this implies that $F_{\overline{J}}^n$ is in the image of $F_{\mathcal{A}}$, and it follows that $F_{\mathcal{A}}$ is surjective on $\mathrm{Hom}_{\mathcal{C}}(\mathbb{1}, A_t \boxtimes \mathbb{1} \boxtimes \mathbb{1})$, as desired. □

Proof of Theorem II. Consider the symmetric tensor functor

$$\underline{F_{\mathcal{A}}} : \mathrm{UCob}_{\alpha,\beta,\gamma} \xrightarrow{F_{\mathcal{A}}} \mathbf{Rep}(S_t \wr \mathbb{Z}_2) \boxtimes \mathbf{Rep}(S_{t_+}) \boxtimes \mathbf{Rep}(S_{t_-}) \to \underline{\mathbf{Rep}(S_t \wr \mathbb{Z}_2)} \boxtimes \underline{\mathbf{Rep}(S_{t_+})} \boxtimes \underline{\mathbf{Rep}(S_{t_-})},$$

where $F_{\mathcal{A}}$ is as defined previously, followed by the semisimplification functor. By Lemma 7.3.3, $\underline{F_{\mathcal{A}}}$ satisfies the conditions of Proposition 7.1.6. Moreover, by Lemma 7.3.2 $\underline{F_{\mathcal{A}}}$ is essentially surjective. Hence $\underline{F_{\mathcal{A}}}$ induces an equivalence

$$\underline{\mathrm{UCob}_{\alpha,\beta,\gamma}} \cong \underline{\mathbf{Rep}(S_t \wr \mathbb{Z}_2)} \boxtimes \underline{\mathbf{Rep}(S_{t_+})} \boxtimes \underline{\mathbf{Rep}(S_{t_-})},$$

as desired. □

Proposition 7.3.4. *Let α, β and γ be sequences as in (7.1.1). Suppose $\lambda\alpha_0 - \gamma_0$ and $\gamma_0 \pm \sqrt{\lambda}\beta_0$ are not non-negative even integers. If Conjecture 7.1.4 holds, we get an equivalence of* $\mathbf{k}$*-linear monoidal categories*

$$\mathrm{UCob}_{\alpha,\beta,\gamma} \cong \mathbf{Rep}(S_t \wr \mathbb{Z}_2) \boxtimes \mathbf{Rep}(S_{t_+}) \boxtimes \mathbf{Rep}(S_{t_-}).$$

Proof. If Conjecture 7.1.4 holds, then for $\lambda, \alpha_0, \beta_0$ and γ_0 as in the statement there are no negligible morphisms in $\mathrm{UCob}_{\alpha,\beta,\gamma}$. On the other hand, by [34, 13] the categories $\mathbf{Rep}(S_t \wr \mathbb{Z}_2)$ and $\mathbf{Rep}(S_{t_\pm})$ are semisimple when $t = \frac{1}{2}(\lambda\alpha_0 - \gamma_0)$ and $t_\pm = \frac{1}{2}(\gamma_0 \pm \sqrt{\lambda}\beta_0)$ are not positive integers. Hence by Theorem II we conclude

$$\mathrm{UCob}_{\alpha,\beta,\gamma} \cong \mathbf{Rep}(S_t \wr \mathbb{Z}_2) \boxtimes \mathbf{Rep}(S_{t_+}) \boxtimes \mathbf{Rep}(S_{t_-}).$$

$\square$

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