STRUCTURES AND COMPUTATIONS IN ANNULAR KHOVANOV

HOMOLOGY

by

CHAMP DAVIS

A DISSERTATION

Presented to the Department of Mathematics
and the Division of Graduate Studies of the University of Oregon
in partial fulfillment of the requirements
for the degree of
Doctor of Philosophy

June 2023

DISSERTATION APPROVAL PAGE

Student: Champ Davis

Title: Structures and Computations in Annular Khovanov Homology

This dissertation has been accepted and approved in partial fulfillment of the requirements for the Doctor of Philosophy degree in the Department of Mathematics by:

Robert Lipshitz	Chair
Nicolas Addington	Core Member
Boris Botvinnik	Core Member
Daniel Dugger	Core Member
Brittany Erickson	Institutional Representative

and

Krista Chronister	Vice Provost for Graduate Studies

Original approval signatures are on file with the University of Oregon Division of Graduate Studies.

Degree awarded June 2023

DISSERTATION ABSTRACT

Champ Davis

Doctor of Philosophy

Department of Mathematics

June 2023

Title: Structures and Computations in Annular Khovanov Homology

Let L be a link in a thickened annulus. In [GLW18], Grigsby-Licata-Wehrli showed that the annular Khovanov homology of L is equipped with an action of $\mathfrak{sl}_2(\wedge)$, the exterior current algebra of the Lie algebra $\mathfrak{sl}_2$. In this dissertation, we upgrade this result to the setting of L_∞-algebras and modules. That is, we show that $\mathfrak{sl}_2(\wedge)$ is an L_∞-algebra and that the annular Khovanov homology of L is an L_∞-module over $\mathfrak{sl}_2(\wedge)$. Up to L_∞-quasi-isomorphism, this structure is invariant under Reidemeister moves.

In proving the above result, we include explicit formulas to compute the higher L_∞-operations. Additionally, given a morphism $I : L' \to L$ of L_∞-algebras, we define a restriction of scalars operation in the setting of L_∞-modules and prove that it defines a functor $I^* : L\text{-mod} \to L'\text{-mod}$. A more abstract approach to this problem was recently given by Kraft-Schnitzer.

Finally, computer code was written to aid in the study of the above L_∞-module structure. We discuss various patterns that emerged from these computations, most notably one relating the torsion in the annular Khovanov homology groups and the location of the inner boundary of the annulus.

CURRICULUM VITAE

NAME OF AUTHOR: Champ Davis

GRADUATE AND UNDERGRADUATE SCHOOLS ATTENDED:

University of Oregon, Eugene, OR
Boston College, Chestnut Hill, MA

DEGREES AWARDED:

Doctor of Philosophy, Mathematics, 2023, University of Oregon
Bachelor of Science, Mathematics, 2017, Boston College

AREAS OF SPECIAL INTEREST:

Low-dimensional Topology

PROFESSIONAL EXPERIENCE:

Graduate Employee, University of Oregon, 2017-2023

ACKNOWLEDGEMENTS

First and foremost, I would like to express my sincere gratitude to my advisor, Robert Lipshitz, for his invaluable guidance and support throughout my academic journey. This dissertation would not have been possible without the countless conversations we had, and I am truly grateful for his mentorship.

I am also deeply grateful to John Baldwin, who introduced me to the subject of knot theory and patiently taught me while I was an undergraduate student. Additionally, I would like to extend my thanks to Dan Chambers, Maksym Fedorchuk, Eli Grigsby, and Bill Keane for all of their encouragement.

To all of my friends and to Tom Heinonen, I am forever grateful for the incredible memories we have shared over the past six years. Finally, I wish to thank my family for their unconditional support. Thank you all.

TABLE OF CONTENTS

LIST OF FIGURES

Figure   Page

CHAPTER I

INTRODUCTION

Knots are familiar objects to almost everyone, and they have been widely studied throughout history. Mathematically, they are defined as embeddings of the circle, typically into three-dimensional Euclidean space. We can represent these embeddings using two-dimensional diagrams if we keep track of the overstrand and the understrand each time the knot crosses over itself; see Figure 1.
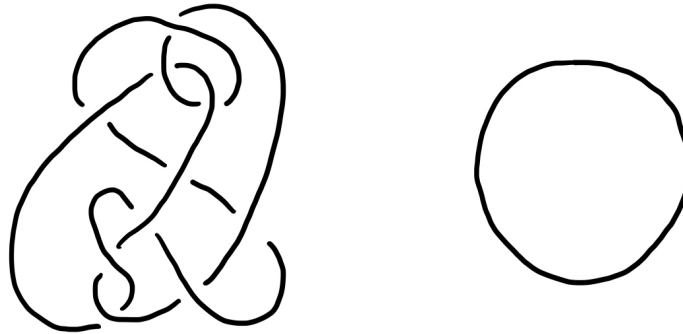
FIGURE 1. Two knot diagrams. Can the knot on the left be untangled to obtain the knot on the right?

Note that for any particular embedding of a knot, we may move the strands around to produce a different diagram for the same knot. Because one knot can have many different embeddings and associated diagrams, it makes sense to study knots up to isotopy, or, continuous deformation. Determining whether two knots belong to the same isotopy class is already a challenging question, as illustrated by Figure 1.

It is useful to associate auxiliary data to the knot that is invariant of the knot's isotopy class. In particular, if the auxiliary data is different for two knots, then the knots themselves had to be different—there is no way to deform one into

the other. This data can take many forms. It could be a number or a polynomial; it could be a group or a topological space. One of the most popular forms of data is that of a chain complex, which is what we will be studying. We will explore the Khovanov chain complex, constructed in [Kho00].

Our primary goal is to use the algebraic structure of the Khovanov complex to understand topological properties of a particular knot or link. In fact, much can be said just by examining the homology groups of the Khovanov complex. For example, a knot is the unknot if and only if the (reduced) Khovanov homology has rank one [KM11]. The Khovanov chain complex has also been used in the proofs of significant topological results. For example, Rasmussen used it to give a purely combinatorial proof of the Milnor conjecture [Ras10], which was first proved by Kronheimer-Mrowka using gauge theory [KM93]. More recently, Piccirillo used Khovanov homology to show that the Conway knot does not bound a smooth disk in the 4-ball, a longstanding open question [Pic20].

If our knot is embedded into a thickened annulus, there is a refinement of Khovanov homology, known as annular Khovanov homology. One of the main benefits of annular Khovanov homology is that there is additional structure that is not present in ordinary Khovanov homology. For example, annular Khovanov homology is an $\mathfrak{sl}_2$-representation; see [GLW18]. We will be studying the structure of annular Khovanov homology. In particular, we will be understanding it in terms of L_∞-algebras and modules, structures first appearing in rational homotopy theory, but recently seen in physics. We will review the background and theory of L_∞-algebras and L_∞-modules in Chapter II. In Chapter III, we will review the construction of Khovanov homology and annular Khovanov homology and show that both the annular Khovanov chain complex and its homology are L_∞-modules.

Finally, as the knots we study get large, the Khovanov chain complex becomes increasingly complex. Computer computation becomes increasingly necessary to guide our intuition and provide experimental data. In Chapter IV, we will discuss various patterns that have emerged from computer computation while studying the annular Khovanov chain complex.

CHAPTER II

L_∞-ALGEBRAS AND MODULES

2.1. Introduction

The study of L_∞-algebras, also known as strong homotopy Lie algebras or sh-Lie algebras, can be traced back to rational homotopy theory and the deformations of algebraic structures, where they first appeared in the form of Lie-Massey operations [All77; Ret85; SS85]. Early applications centered around the Quillen spectral sequence and rational Whitehead products, and there has been continued interest in higher order Whitehead products recently; see [Bel+17]. There has also been much interest in L_∞-algebras in physics, where Lie algebras and their representations play a major role. In particular, L_∞-algebra structures have appeared in work on higher spin particles [BBD85], as well as in closed string theory [WZ92; Zwi93]. Stasheff gives a nice overview in a recent survey article [Sta19].

Attention has also been given to modules over L_∞-algebras. The notion of an L_∞-module was introduced in [LM95], in which the correspondence between Lie algebra representations and Lie modules was generalized to the L_∞ setting. Moreover, homomorphisms between L_∞-modules were developed in [All14].

While it is possible to give a complex an L_∞ structure by writing down explicit formulas, another option is to use homological perturbation theory to transfer an existing L_∞ structure from a different complex. Information on how to do so can be found in [Hue11; HS02; GLS91], where this idea is referred to as the homological perturbation lemma, though sometimes it is referred to as the

4

homotopy transfer theorem, as in [LV12; Man10]. An approach using operads was given in [Ber14], where explicit formulas are written down for the A_∞ case. Explicit formulas for the L_∞ case can be found in [Mor22a].

Much of the literature deals with the transfer of L_∞-algebra structures; however, given a map between L_∞-algebras, it is natural to want to use this map to relate their respective categories of modules. In this chapter, we give one explicit formula to do so, giving a proof of the following:

Theorem. *Suppose L, L' are L_∞-algebras over $\mathbb{F}_2$ and $I : L' \to L$ is a map of L_∞-algebras. Then there is an induced functor $I^* : L\text{-mod} \to L'\text{-mod}$, called restriction of scalars.*

Given an L_∞-module homomorphism $f : M \to N$, our definition will satisfy $(I^* f)_1 = f_1$. It follows that I^* preserves quasi-isomorphisms; that is, if M and N are quasi-isomorphic, then so too are $I^* M$ and $I^* N$. We also observe that this generalizes the analagous result in the Lie algebra setting:

Corollary. *If L and L' are Lie algebras, and $\phi : L' \to L$ is a Lie algebra homomorphism, ϕ^* is the usual restriction of scalars for Lie algebra representations.*

Because L_∞ modules are defined in the graded setting, keeping track of signs requires a great deal of care. We will ignore signs and work over $\mathbb{F}_2$. As mentioned in [All14], A_∞-modules and maps between them can be reinterpreted in terms of differential comodules. The analagous reformulation in the L_∞ case is less-understood, but perhaps could facilitate the recording of signs. Moreover, Kraft-Schnitzer recently gave a more abstract approach to the restriction of scalars operation in [KS22]. We present an alternative interpretation, and we

emphasize that the explicit formulas developed here are of particular interest for our applications. On the other hand, [KS22] might serve as a guide for how to deal with signs in the future.

The outline of this chapter is as follows. In section 2, we review the definition of an L_∞-algebra and explain morphisms between them. We provide a similar exposition for L_∞-modules, and we describe how to compose morphisms between L_∞-modules. In section 3, we describe I^*, the restriction of scalars functor. We define I^* on objects and morphisms, and then we prove that it is functorial. In section 4, we define chain contractions to describe an additional way to transfer a existing L_∞-algebra or L_∞-module structures. The appendix includes supplementary graphics for the proofs presented in this chapter, which contain somewhat complicated formulas.

2.2. Definitions and Examples

In this section, we review L_∞-algebras and explain morphisms between them. We start by introducing some notation that we will use throughout the rest of this dissertation.

Definition 1. Let $\sigma \in S_n$ be a permutation. If X is a set, then σ induces a map $\sigma^\bullet : X^n \to X^n$, defined by $\sigma^\bullet(x_1, x_2, \ldots, x_n) = (x_{\sigma(1)}, \ldots, x_{\sigma(n)})$. If X is a vector space, σ induces a similarly-defined map on the n-fold tensor product $\sigma^\bullet : X^{\otimes n} \to X^{\otimes n}$.

Definition 2. Fix non-negative integers $i_1, i_2, \ldots, i_r$, with $i_1 + i_2 + \cdots + i_r = n$. A permutation $\sigma \in S_n$ is an $(i_1, i_2, \ldots, i_r)$-**unshuffle** if

$$\sigma(1) < \cdots < \sigma(i_1)$$

6

$$\sigma(i_1 + 1) < \cdots < \sigma(i_1 + i_2)$$

$$\vdots$$

$$\sigma(i_1 + \cdots + i_{r-1} + 1) < \cdots < \sigma(i_1 + \cdots + i_r)$$

We will denote the set of $(i_1, i_2, \ldots, i_r)$-unshuffles in S_n by $S(i_1, \ldots, i_r)$.

Definition 3. We will denote by $S'(i_1, \ldots, i_r)$ the set of $(i_1, i_2, \ldots, i_r)$-unshuffles σ in S_n satisfying $i_1 \leq i_2 \leq \cdots \leq i_r$ and $\sigma(i_1 + \cdots + i_{l-1} + 1) < \sigma(i_1 + \cdots + i_l + 1)$ if $i_l = i_{l+1}$. This second condition on σ says that the order is preserved when comparing the first elements of blocks of the same size. Indeed, if σ is a $(1, 2, 2, 3)$-unshuffle in S'_8, then $i_2 = i_3 = 2$, so the order must be preserved when comparing the first element of the i_2 block to the first element of the i_3 block.

Definition 4. We will denote by $\overline{S}(i_1, \ldots, i_r)$ the set of $(i_1, i_2, \ldots, i_r)$-unshuffles σ in $S'(i_1, \ldots, i_r)$ satisfying $\sigma(1) = 1$.

Definition 5. Let V be a graded vector space. For $\sigma \in S^n$ and $v_i \in V$, let $\epsilon(\sigma) :=$ $\epsilon(\sigma, v_1, \ldots, v_n)$ be the total Koszul sign of σ. To compute $\epsilon(\sigma)$, every time two elements of degrees x and y are transposed, we record a sign of $(-1)^{xy}$, and $\epsilon(\sigma)$ is the total product of such signs. Define $\chi(\sigma) := \epsilon(\sigma) \operatorname{sgn}(\sigma)$ to be the product of the Koszul sign and the sign of the permutation σ.

Remark. Let $f : A \to B$ and $g : C \to D$ be graded maps of graded algebras. We will also follow the Kozsul sign convention of including a sign in the evaluation of the map $f \otimes g$. That is, for an element $x \otimes y \in A \otimes C$,

$$(f \otimes g)(x \otimes y) = (-1)^{|x||g|} f(x) \otimes g(y).$$

7

Example 1. Figure 2 is an example of a $(1, 1, 2, 3)$-unshuffle in S_7. That is, $\sigma = (124653)(7)$, and we have drawn a picture describing $\sigma^\bullet$. That is, $x_{\sigma(1)} = x_2$, $x_{\sigma(2)} = x_4$, and so on. The picture describes how $\sigma^\bullet$ permutes $x_1, \ldots, x_7$.
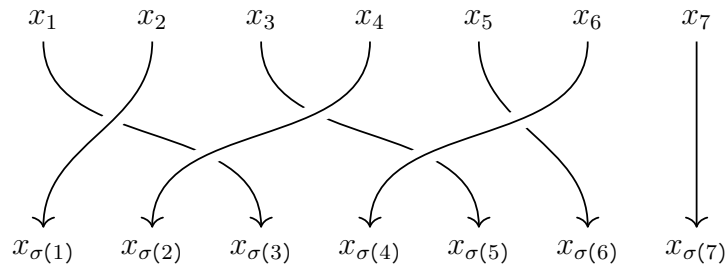
FIGURE 2. A depiction of an $(1, 1, 2, 3)$-unshuffle in S_7. Here $\sigma = (124653)(7)$, and $\sigma^\bullet(x_1, x_2, x_3, x_4, x_5, x_6, x_7) = (x_2, x_4, x_1, x_6, x_3, x_5, x_7)$.

In words, a $(1, 1, 2, 3)$-unshuffle places the numbers 1 through 7 into boxes of size 1,1,2, and 3, where the order is preserved in each box. In this example, the resulting boxes would be $(2), (4), (1, 6)$, and $(3, 5, 7)$.

Example 2. A special case of the above definition is if we only have two numbers in our partition of n. In particular, $\sigma \in S_n$ is a $(p, n - p)$-**unshuffle** if $\sigma(k) < \sigma(k + 1)$ whenever $k \neq p$. In words, this permutation will place the numbers 1 through n into two boxes, where order is preserved in each. For brevity, we will sometimes refer to a $(p, n - p)$-unshuffle as a p-unshuffle if n is clear.

Example 3. In S_4, if we use the notation $xyzw$ to denote the permutation $\left(\begin{smallmatrix} 1 & 2 & 3 & 4 \\ x & y & z & w \end{smallmatrix}\right)$, then we can write down the 1, 2, and 3-unshuffles:

> 1-unshuffles: 1234, 2134, 3124, 4123
>
> 2-unshuffles: 1234, 1324, 1423, 2314, 2413, 3412
>
> 3-unshuffles: 1234, 1243, 1342, 2341

8

We can now state the definition of an L_∞-algebra. We will include the general definition involving signs, though in the theorems we prove, we will work over $\mathbb{F}_2$.

Definition 6. Let V be a graded vector space. An L_∞**-algebra structure** on V is a collection of skew-symmetric multilinear maps $\{l_k : V^{\otimes k} \to V\}$ of degree $k - 2$. That is, each l_k is skew-symmetric in the sense that

$$l_k \circ \sigma^\bullet(x_1, x_2, \ldots, x_k) = \chi(\sigma)l_k(x_1, x_2, \ldots, x_k)$$

for all $\sigma \in S_k$ and $x_i \in V$. These maps also must satsify the generalized Jacobi identity:

$$\sum_{i+j=n+1} \sum_\sigma \chi(\sigma)(-1)^{i(j-1)} l_j \circ (l_i \otimes \mathrm{Id}) \circ \sigma^\bullet = 0$$

Here, $i \geq 1$, $j \geq 1$, $n \geq 1$, and the inner summation is taken over all $(i, n - i)$-unshuffles.

Remark. If we are working over characteristic two, then these maps are alternating as well.

Remark. We could have also written the skew-symmetry condition as $l_k \circ \sigma^\bullet = l_k$ for $\sigma \in S_k$.

Remark. Another way to write the generalized Jacobi indentity is by using the notation

$$\sum_{i+j=n+1} \sum_\sigma l_j \circ (l_i \otimes \mathrm{Id}) \circ \sigma^\bullet = 0$$

Remark. Figure 3 is a depiction of the generalized Jacobi identity.
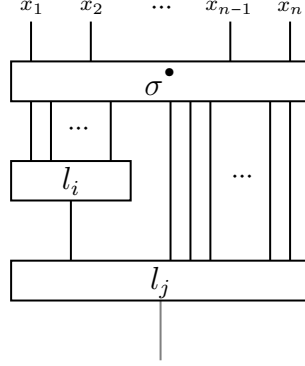
FIGURE 3. A graphical depiction of the generalized Jacobi identity. This should be interpreted as the sum of all compositions $l_j \circ (l_i \otimes \mathrm{Id}) \circ \sigma^\bullet$, applied to the input $x_1 \otimes \cdots \otimes x_n$. That is, this picture represents $\sum_{i+j=n+1} \sum_\sigma l_j \circ (l_i \otimes \mathrm{Id}) \circ \sigma^\bullet (x_1 \otimes \cdots \otimes x_n) = 0$.

Remark. This definition follows the chain complex convention. If instead our L_∞-algebra is a cochain complex, we require each l_k to have degree $2 - k$. There are similar cohain complex conventions for the following definitions.

Definition 7. Let (L, l_i) and (L', l_i') be L_∞-algebras. An L_∞**-algebra homomorphism** from L to L' is a sequence of skew-symmetric multilinear maps $\{f_n : L^{\otimes n} \to L'\}$ of degree $n - 1$ such that

$$\sum_{j+k=n+1} \sum_{\sigma \in S(k, n-k)} \epsilon_1 \cdot f_j \circ (l_k \otimes \mathrm{Id}) \circ \sigma^\bullet + \sum_{\substack{\tau \in S'(i_1, \ldots, i_r) \\ i_1 + \ldots + i_r = n}} \epsilon_2 \cdot l_r' \circ (f_{i_1} \otimes \cdots \otimes f_{i_r}) \circ \tau^\bullet = 0$$

where $\epsilon_1 = \chi(\sigma)(-1)^{k(j-1)+1}$ and $\epsilon_2 = \chi(\tau)(-1)^{\frac{r(r-1)}{2} + \sum_{s=1}^{r-1} i_s(r-s)}$.

Example 4. The $n = 2$ morphism relation says that

$$-f_1(l_2(x_1, x_2)) + f_2(l_1(x_1), x_2) - (-1)^{|x_1||x_2|} f_2(l_1(x_2), x_1)$$

$$+ l_1'(f_2(x_1, x_2)) + l_2'(f_1(x_1), f_1(x_2)) = 0$$

10

When (L, l_i) and (L', l'_i) are L_∞-algebras consisting of elements in degree 0 only, the $n = 2$ morphism relation simplifies to $f_1(l_2(x_1, x_2)) - l'_2(f_1(x_1), f_1(x_2)) = 0$, which is just a Lie algebra homomorphism: $\phi([x_1, x_2]) = [\phi(x_1), \phi(x_2)]$.

Definition 8. Let (L, l_k) be an L_∞-algebra. The data of an L_∞-**module** over L consists of a graded vector space M, together with skew-symmetric multilinear maps $\{k_n : L^{\otimes n-1} \otimes M \to M \mid 1 \le n < \infty\}$ of degree $n - 2$ satisfying:

$$\sum_{\substack{p+q=n+1 \\ p<n}} \sum_{\sigma(n)=n} \epsilon_1 \cdot k_q \circ (l_p \otimes \text{Id}) \circ \sigma^\bullet + \sum_{p+q=n+1} \sum_{\sigma(p)=n} \epsilon_2 \epsilon_3 \cdot k_q \circ \delta^\bullet \circ (k_p \otimes \text{Id}) \circ \sigma^\bullet = 0$$

where $\epsilon_1 = \epsilon_2 = \chi(\sigma)(-1)^{p(q-1)}$ and σ is a p-unshuffle in S_n. In the case of $\sigma(p) = n$, we used the skew-symmetry of k_q and introduced $\delta^\bullet$ to permute the k_p term past the remaining elements to ensure that $k_q : L^{\otimes q-1} \otimes M \to M$. Explicitly,

$$k_q\big(\underbrace{k_p(x_{\sigma(1)}, \ldots, x_{\sigma(p)})}_{\in M}, x_{\sigma(p+1)}, \ldots, x_{\sigma(n)}\big) = \epsilon_3 \cdot k_q\Big(\delta^\bullet\big(\underbrace{k_p(x_{\sigma(1)}, \ldots, x_{\sigma(p)})}_{\in M}, x_{\sigma(p+1)}, \ldots, x_{\sigma(n)}\big)\Big)$$

$$= \epsilon_3 \cdot k_q\big(x_{\sigma(p+1)}, \ldots, x_{\sigma(n)}, \underbrace{k_p(x_{\sigma(1)}, \ldots, x_{\sigma(p)})}_{\in M}\big)$$

where $\epsilon_3 = \chi(\delta) = (-1)^{q-1}(-1)^{(p+\sum_{s=1}^{p} |x_{\sigma(s)}|)(\sum_{s=p+1}^{n} |x_{\sigma(s)}|)}$.

Example 5. The $n = 1$ module relation says that M is a chain complex with differential k_1:

$$k_1(k_1(m)) = 0$$

The $n = 2$ module relation says that the action satisfies the graded Leibniz rule:

$$-k_2(l_1(x_1), x_2) - (-1)^{|x_1|} k_2(x_1, k_1(x_2)) + k_1(k_2(x_1, x_2)) = 0$$

11

Using a different notation, we could also write

$$-[\partial x_1, x_2] - (-1)^{|x_1|}[x_1, \partial x_2] + \partial[x_1, x_2] = 0$$

to remind us of differential graded Lie algebras. For reference, the $n = 3$ module relation is the following.

$$k_3(l_1(x_1), x_2, x_3)) - (-1)^{|x_1||x_2|}k_3(l_1(x_2), x_1, x_3) + (-1)^{|x_1|+|x_2|}k_3(x_1, x_2, k_1(x_3))$$

$$+k_2(l_2(x_1, x_2), x_3) + (-1)^{|x_1||x_2|}k_2(x_2, k_2(x_1, x_3)) - k_2(x_1, k_2(x_2, x_3))$$

$$+k_1(k_3(x_1, x_2, x_3)) = 0$$

Definition 9. Following [All14], let (L, l_i) be an L_∞-algebra, and let (M, k_i) and (M', k_i') be L_∞-modules over L. An L_∞**-module homomorphism** from M to M' is a collection of skew-symmetric multilinear maps $\{h_n : L^{\otimes(n-1)} \otimes M \to M'\}$ of degree $n - 1$ satisfying:

$$\sum_{\substack{i+j=n+1 \\ i<n}} \sum_{\sigma(n)=n} \epsilon_1 \cdot h_j \circ (l_i \otimes \mathrm{Id}) \circ \sigma^\bullet + \sum_{i+j=n+1} \sum_{\sigma(i)=n} \epsilon_2 \cdot h_j \circ \delta^\bullet \circ (k_i \otimes \mathrm{Id}) \circ \sigma^\bullet$$

$$+ \sum_{r+s=n+1} \sum_\tau \epsilon_3 \cdot k_r' \circ (\mathrm{Id} \otimes h_s) \circ (\tau^\bullet \otimes \mathrm{Id}) = 0$$

where $\epsilon_1 = \epsilon_2 = \chi(\sigma)(-1)^{i(j-1)+1}$ and $\epsilon_3 = \chi(\tau)(-1)^{(s-1)(\sum_{t=1}^{n-s} x_{\tau(t)})}$, σ is an i-unshuffle in S_n, and τ is an $(n - s)$-unshuffle in S_{n-1}. Similar to the definition of L_∞-module, we include the permutation δ to ensure the module element is in the correct location.

Remark. Figure 4 is a depiction of the L_∞-module homomorphism relation.
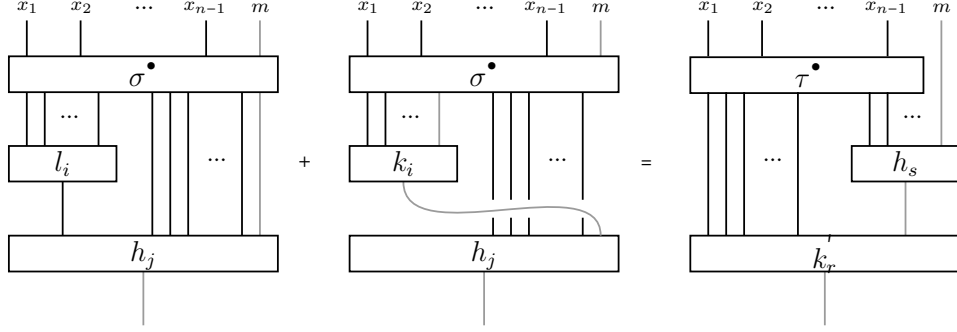
FIGURE 4. A graphical depiction of the L_∞-module homomorphism relation. This should be interpreted as $\sum h_j \circ (l_i \otimes \mathrm{Id}) \circ \sigma^\bullet + \sum h_j \circ \delta^\bullet \circ (k_i \otimes \mathrm{Id}) \circ \sigma^\bullet = \sum k'_r \circ (\mathrm{Id} \otimes h_s) \circ (\tau^\bullet \otimes \mathrm{Id})$.

Example 6. The $n = 1$ module homomorphism relation says that h_1 is a chain map: $h_1 k_1(m) = k'_1 h_1(m)$. Omitting signs, the $n = 2$ module homomorphism relation says:

$$h_2(l_1(x_1), m) + h_2(x_1, k_1(m)) + h_1(k_2(x_1, m)) = k'_2(x_1, h_1(m)) + k'_1(h_2(x_1, m))$$

Definition 10. The **identity map**, Id_M, of an L_∞ module M is defined as follows. $(\mathrm{Id}_M)_1$ is the identity map of the underlying graded vector space M, and $(\mathrm{Id}_M)_r = 0$ for $r \geq 2$. It is straightforward to check that this satisfies the definition of an L_∞-module homomorphism.

Definition 11. Let L be an L_∞-algebra, and let A, B, and C be L_∞-modules over L. Given L_∞-module homomorphisms $A \xrightarrow{f} B \xrightarrow{g} C$, we define the composition $g \circ f$ by

$$(g \circ f)_n = \sum_{i+j=n+1} \sum_{\sigma(i)=n} g_j \circ \delta^\bullet \circ (f_i \otimes \mathrm{Id}) \circ \sigma^\bullet$$

where σ is an i-unshuffle in S_n, and $\lambda^\bullet$ is the map that permutes the module element to the final input.

13
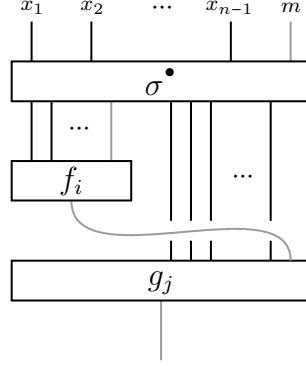
FIGURE 5. A graphical depiction of the composition of two L_∞-module homomorphisms. This should be interpreted as $(g \circ f)_n = \sum g_j \circ \delta^\bullet \circ (f_i \otimes \text{Id}) \circ \sigma^\bullet$.

The following Lemma is perhaps well-known, but we do not know a reference for it. Pictures representing each step in the proof are given in the appendix.

Lemma 1 (Composition). *Let (L, l_i) be an L_∞-algebra, and let $A, B,$ and C be L_∞-modules over L, with module operations denoted by $a_i, b_i,$ and c_i, respectively. Given L_∞-module homomorphisms $A \xrightarrow{f} B \xrightarrow{g} C$, the composition $g \circ f$ is an L_∞-module homomorphism.*

Proof. This follows from the fact that both f and g are L_∞-module homomorphisms. Below, we will apply the L_∞-module homomorphism relation for f, then we will apply the L_∞-module homomorphism relation for g, and then we will conclude the L_∞-module homomorphism relation for $g \circ f$.

Step 1. The relation that we need to show is

$$\sum_{i+j=n+1} \sum_\sigma (g \circ f)_j \circ (a_i \otimes \text{Id}) \circ \sigma^\bullet = \sum_{r+s=n+1} \sum_\tau c_r \circ (\text{Id} \otimes (g \circ f)_s) \circ \tau^\bullet$$

where σ is an $(i, n-i)$-unshuffle and τ is an $(n-s, s-1)$-unshuffle.

14

Step 2. Break the left-hand side into two parts, and replace $(g \circ f)_j$ with its definition

$$\sum_{i+j=n+1} \sum_{\sigma(i)=n} \sum_{p+q=j+1} \sum_{\theta(p)=j} g_q \circ \delta^\bullet \circ (f_p \otimes \mathrm{Id}) \circ \theta^\bullet \circ \lambda^\bullet \circ (a_i \otimes \mathrm{Id}) \circ \sigma^\bullet$$

$$+ \sum_{\substack{i+j=n+1 \\ i<n}} \sum_{\sigma(n)=n} \sum_{p+q=j+1} \sum_{\theta(p)=j} g_q \circ \delta^\bullet \circ (f_p \otimes \mathrm{Id}) \circ \theta^\bullet \circ (l_i \otimes \mathrm{Id}) \circ \sigma^\bullet$$

where $\delta^\bullet$ is the map that permutes the module element to the last input.

Step 3. In the first sum, applying $\sigma^\bullet$ and $\theta^\bullet$ results in a block of size i being inputted to a_i, a block of size $p-1$ being inputted into f_p, together with the output of a_i, and then a block of size $j-p$ remaining elements (which will be inputted into g_q). An equivalent way to achieve this is to first apply a $(p+i-1)$-unshuffle $\eta^\bullet$ and then an i-unshuffle $\psi^\bullet$. If $\eta(p+i-1) = n$ and $\psi(i) = p+i-1$, we again obtain a block of size i being inputted into a_i, then a block of size $p-1$ being inputted into f_p, together with the output of a_i, with $j-p$ elements remaining.

In the second sum, we do the same thing, except the output of l_i can either go into the first input of f_p or the first input of g_q, by the definition of unshuffle. So we decompose the second sum to reflect these two cases.

$$\sum_{i+j=n+1} \sum_{p+q=j+1} \sum_{\substack{\eta \in S(p+i-1,j-p) \\ \eta(p+i-1)=n}} \sum_{\substack{\psi \in S(i,p-1) \\ \psi(i)=p+i-1}} g_q \circ \delta^\bullet \circ (f_p \otimes \mathrm{Id}) \circ \lambda^\bullet \circ (a_i \otimes \mathrm{Id}) \circ (\psi^\bullet \otimes \mathrm{Id}) \circ \eta^\bullet$$

$$+ \sum_{\substack{i+j=n+1 \\ i<n}} \sum_{\substack{p+q=j+1 \\ p>1}} \sum_{\substack{\eta \in S(p+i-1,j-p) \\ \eta(p+i-1)=n}} \sum_{\substack{\psi \in S(i,p-1) \\ \psi(i)=i}} g_q \circ \delta^\bullet \circ (f_p \otimes \mathrm{Id}) \circ (l_i \otimes \mathrm{Id}) \circ (\psi^\bullet \otimes \mathrm{Id}) \circ \eta^\bullet$$

$$+ \sum_{\substack{i+j=n+1 \\ i<n}} \sum_{p+q=j+1} \sum_{\substack{\eta \in S(p+i,j-p-1) \\ \eta(p+i)=n}} \sum_{\substack{\psi \in S(i,p-1) \\ \psi(i)=i}} g_q \circ \delta^\bullet \circ (l_i \otimes f_p \otimes \mathrm{Id}) \circ (\psi^\bullet \otimes \mathrm{Id}) \circ \eta^\bullet$$

15

Step 4. Reindex over $\alpha = p + i$.

$$\sum_{\alpha=2}^{n+1} \sum_{p+i=\alpha} \sum_{\substack{\eta \in S(\alpha-1,n-\alpha+1) \\ \eta(\alpha-1)=n}} \sum_{\substack{\psi \in S(i,\alpha-1-i) \\ \psi(i)=\alpha-1}} g_{n+2-\alpha} \circ \delta^{\bullet} \circ (f_p \otimes \mathrm{Id}) \circ \lambda^{\bullet} \circ (a_i \otimes \mathrm{Id}) \circ (\psi^{\bullet} \otimes \mathrm{Id}) \circ \eta^{\bullet}$$

$$+ \sum_{\substack{\alpha=2 \\ }}^{n+1} \sum_{\substack{p+i=\alpha \\ 1<p,i<n}} \sum_{\substack{\eta \in S(\alpha-1,n-\alpha+1) \\ \eta(\alpha-1)=n}} \sum_{\substack{\psi \in S(i,\alpha-1-i) \\ \psi(i)=i}} g_{n+2-\alpha} \circ \delta^{\bullet} \circ (f_p \otimes \mathrm{Id}) \circ (l_i \otimes \mathrm{Id}) \circ (\psi^{\bullet} \otimes \mathrm{Id}) \circ \eta^{\bullet}$$

$$+ \sum_{\substack{\alpha=2 \\ }}^{n+1} \sum_{\substack{p+i=\alpha \\ i<n}} \sum_{\substack{\eta \in S(\alpha,n-\alpha) \\ \eta(\alpha)=n}} \sum_{\substack{\psi \in S(i,p-1) \\ \psi(i)=i}} g_{n+2-\alpha} \circ \delta^{\bullet} \circ (l_i \otimes f_p \otimes \mathrm{Id}) \circ (\psi^{\bullet} \otimes \mathrm{Id}) \circ \eta^{\bullet}$$

Step 5. Apply the module homomorphism relation for f in the first two sums. In the third sum, change notation from i to t and from p to s.

$$\sum_{\alpha=2}^{n+1} \sum_{t+s=\alpha} \sum_{\substack{\eta \in S(\alpha-1,n-\alpha+1) \\ \eta(\alpha-1)=n}} \sum_{\tau \in S(t-1,s-1)} g_{n+2-\alpha} \circ \delta^{\bullet} \circ (b_t \otimes \mathrm{Id}) \circ (\mathrm{Id} \otimes f_s \otimes \mathrm{Id}) \circ (\tau^{\bullet} \otimes \mathrm{Id}) \circ \eta^{\bullet}$$

$$+ \sum_{\substack{\alpha=2 \\ }}^{n+1} \sum_{\substack{t+s=\alpha \\ t<n}} \sum_{\substack{\eta \in S(\alpha,n-\alpha) \\ \eta(\alpha)=n}} \sum_{\substack{\psi \in S(t,s-1) \\ \psi(t)=t}} g_{n+2-\alpha} \circ \delta^{\bullet} \circ (l_t \otimes f_s \otimes \mathrm{Id}) \circ (\psi^{\bullet} \otimes \mathrm{Id}) \circ \eta^{\bullet}$$

Step 6. In the first sum, combine $\tau \in S(t-1,s-1)$ and $\eta \in S(\alpha-1, n-\alpha+1)$ into a single $(t-1, s, n-\alpha+1)$-unshuffle, denoted by π. In the second sum, combine

16

$\psi \in S(t, s-1)$ and $\eta \in S(\alpha, n-\alpha)$ into a single $(t, s, n-\alpha)$-unshuffle, denoted by π.

$$\sum_{\alpha=2}^{n+1} \sum_{t+s=\alpha} \sum_{\substack{\pi \in S(t-1,s,n-\alpha+1) \\ \pi(\alpha-1)=n}} g_{n+2-\alpha} \circ \delta^{\bullet} \circ (b_t \otimes \mathrm{Id}) \circ (\mathrm{Id} \otimes f_s \otimes \mathrm{Id}) \circ \pi^{\bullet}$$
$$+ \sum_{\alpha=2}^{n+1} \sum_{\substack{t+s=\alpha \\ t<n}} \sum_{\substack{\pi \in S(t,s,n-\alpha) \\ \pi(\alpha)=n}} g_{n+2-\alpha} \circ \delta^{\bullet} \circ (l_i \otimes f_s \otimes \mathrm{Id}) \circ \pi^{\bullet}$$

Step 7. In the first sum, π unshuffles the n elements into a block of size $t-1$, a block of size s, and a block of size $n-\alpha+1$. The block of size s is then inputted into f_s, and then the output of f_s is then inputted into b_t, as the module element, with the block of size $t-1$.

An equivalent way of achieving this is to apply an $(n-s, s-1)$-unshuffle to the $(n-1)$-algebra elements, to form blocks of size $(n-s)$ and $s-1$, and then input the $s-1$ algebra elements into f_s, with the module element. Then, apply an t-unshuffle $\sigma^{\bullet}$ to these $n-s+1$ elements. By requiring $\sigma(t) = n-s+1$, we obtain a block of size $t-1$, plus a module element, that we input into b_t. We can do an analagous reformulation of the second sum.

$$\sum_{\alpha=2}^{n+1} \sum_{t+s=\alpha} \sum_{\phi \in S(n-s,s-1)} \sum_{\substack{\sigma \in S(t,n-s+1) \\ \sigma(t)=n-s+1}} g_{n+2-\alpha} \circ \delta^{\bullet} \circ (b_t \otimes \mathrm{Id}) \circ \sigma^{\bullet} \circ (\mathrm{Id} \otimes f_s) \circ (\phi^{\bullet} \otimes \mathrm{Id})$$

17

$$+\sum_{\substack{\alpha=2 \\ }}^{n+1}\sum_{\substack{t+s=\alpha \\ t<n}}\sum_{\phi\in S(n-s,s-1)}\sum_{\substack{\sigma\in S(t,n-s+1) \\ \sigma(n-s+1)=n-s+1}} g_{n+2-\alpha}\circ(l_t\otimes\mathrm{Id})\circ\sigma^\bullet\circ(\mathrm{Id}\otimes f_s)\circ(\phi^\bullet\otimes\mathrm{Id})$$

Step 8. Reindex, noting that $\sum_{\alpha=2}^{n+1}\sum_{t+s=\alpha}=\sum_{s=1}^{n}\sum_{t=1}^{n+1-s}=\sum_{s=1}^{n}\sum_{x+y=n+2-s}$.

$$\sum_{s=1}^{n}\sum_{x+y=n+2-s}\sum_{\phi\in S(n-s,s-1)}\sum_{\substack{\sigma\in S(x,n-s+1) \\ \sigma(x)=n-s+1}} g_y\circ\delta^\bullet\circ(b_x\otimes\mathrm{Id})\circ\sigma^\bullet\circ(\mathrm{Id}\otimes f_s)\circ(\phi^\bullet\otimes\mathrm{Id})$$

$$+\sum_{s=1}^{n}\sum_{\substack{x+y=n+2-s \\ x<n}}\sum_{\phi\in S(n-s,s-1)}\sum_{\substack{\sigma\in S(x,n-s+1) \\ \sigma(n-s+1)=n-s+1}} g_y\circ(l_x\otimes\mathrm{Id})\circ\sigma^\bullet\circ(\mathrm{Id}\otimes f_s)\circ(\phi^\bullet\otimes\mathrm{Id})$$

Step 9. Apply the morphism relation for g.

$$\sum_{s=1}^{n}\sum_{r+q=n-s+2}\sum_{\phi\in S(n-s,s-1)}\sum_{\kappa\in S(r-1,q-1)} c_r\circ(\mathrm{Id}\otimes g_q)\circ(\kappa^\bullet\otimes\mathrm{Id})\circ(\mathrm{Id}\otimes f_s)\circ(\phi^\bullet\otimes\mathrm{Id})$$

Step 10. Combine κ and ϕ into a single permutation π.

$$\sum_{s=1}^{n}\sum_{r+q=n-s+2}\sum_{\pi\in S(r-1,q-1,s-1)} c_r\circ(\mathrm{Id}\otimes g_q)\circ(\mathrm{Id}\otimes f_s)\circ(\pi^\bullet\otimes\mathrm{Id})$$

Step 11. Split π into τ and ψ. The map $\lambda^\bullet$ is needed to permute the module element into the last input of g_q.

18

$$\sum_{s=1}^{n} \sum_{r+q=n-s+2} \sum_{\tau \in S(r-1,n-r)} \sum_{\psi \in S(s,q-1)} c_r \circ \big(\operatorname{Id} \otimes [g_q \circ \lambda^{\bullet} \circ (f_s \otimes \operatorname{Id}) \circ \psi^{\bullet}] \big) \circ (\tau^{\bullet} \otimes \operatorname{Id})$$

Step 12. Change how we index over s, r, q.

$$\sum_{r=1}^{n} \sum_{s+q=n+2-r} \sum_{\tau \in S(r-1,n-r)} \sum_{\psi \in S(s,q-1)} c_r \circ \big(\operatorname{Id} \otimes [g_q \circ \lambda^{\bullet} \circ (f_s \otimes \operatorname{Id}) \circ \psi^{\bullet}] \big) \circ (\tau^{\bullet} \otimes \operatorname{Id})$$

Step 13. Use the definition of $g \circ f$.

$$\sum_{r=1}^{n} \sum_{\tau \in S(r-1,n-r)} c_r \circ (\operatorname{Id} \otimes (g \circ f)_{n+1-r}) \circ (\tau^{\bullet} \otimes \operatorname{Id})$$

Step 14. This is

$$\sum_{r+s=n+1} \sum_{\tau \in S(r-1,s-1)} c_r \circ (\operatorname{Id} \otimes (g \circ f)_s) \circ (\tau^{\bullet} \otimes \operatorname{Id})$$

$\square$

2.3. Restriction of Scalars

In this section, we define the restriction of scalars functor on objects, and we prove that the result is an L_∞-module. We then define the restriction of scalars functor on morphisms, and we prove that the result is an L_∞-module homomorphism. Finally, we complete the proof of functoriality. The end of this

section contains a technical lemma that is applied several times throughout the aforementioned proofs.

Lemma 2 (Objects). *Suppose $I : (L', l') \rightarrow (L, l)$ is a map of L_∞-algebras. If (M, k) is an L-module, then $I^*M := (M, k')$ is an L'-module, where $k'_n : L^{\otimes n-1} \otimes M \rightarrow M$ is given by*

$$k'_n = \sum_{r=1}^{n-1} \sum_{\substack{\tau \in S'(i_1,\dots,i_r) \\ i_1+\dots+i_r=n-1}} k_{r+1} \circ (I_{i_1} \otimes \cdots \otimes I_{i_r} \otimes \mathrm{Id}) \circ (\tau^\bullet \otimes \mathrm{Id})$$

Proof. The idea of the proof is straightforward. We will first make a substitution using the definition of k' (steps 1-2). We will then use the L_∞-algebra homomorphism relation for I to exchange any I and l' terms (steps 3-9). The terms that remain will then cancel by applying the L_∞-module relation for k (steps 10-19). Pictures representing each step in the proof are given in the appendix.

Step 1. The L_∞ relation for k'_n that we need to show is zero is:

$$\sum_{\substack{p+q=n+1 \\ p<n}} \sum_{\sigma(n)=n} k'_q \circ (l'_p \otimes \mathrm{Id}) \circ \sigma^\bullet + \sum_{p+q=n+1} \sum_{\sigma(p)=n} k'_q \circ \delta^\bullet \circ (k'_p \otimes \mathrm{Id}) \circ \sigma^\bullet = 0$$

Step 2. Focusing only on the first double sum for now, we substitute for k'_q using its definition:

$$\sum_{\substack{p+q=n+1 \\ p<n}} \sum_{\sigma(n)=n} \sum_{\substack{\tau \in S'(i_1,\dots,i_r) \\ 1 \le r \le q-1 \\ i_1+\dots+i_r=q-1}} k_{r+1}\Big((I_{i_1} \otimes \cdots \otimes I_{i_r} \otimes \mathrm{Id}) \circ (\tau^\bullet \otimes \mathrm{Id}) \circ (l'_p \otimes \mathrm{Id}) \circ \sigma^\bullet \Big)$$

20

Step 3. The goal now is to use the morphism relation to commute the l_p' and I terms. To do so, we will break down this sum by the specific morphism relation that we will apply ($k = 1, \ldots, n-1$). In particular, this is determined by the sum of p and the size of the block to which τ sends l_p'. We will denote the block containing l_p' by i_l, and we will denote its size by s.

$$\sum_{p=1}^{n-1} \sum_{\sigma(n)=n} \sum_{\substack{\tau \in S'(i_1,\ldots,i_r) \\ 1 \leq r \leq n-p \\ i_1+\ldots+i_r=n-p}} k_{r+1}\left((I_{i_1} \otimes \cdots \otimes I_{i_r} \otimes \mathrm{Id}) \circ (\tau^\bullet \otimes \mathrm{Id}) \circ (l_p' \otimes \mathrm{Id}) \circ \sigma^\bullet \right)$$

$$= \sum_{p=1}^{n-1} \sum_{\sigma(n)=n} \sum_{s=1}^{n-p} \sum_{\substack{\tau \in S'(i_1,\ldots,i_r) \\ 1 \leq r \leq n-p \\ i_1+\ldots+i_r=n-p \\ i_l=s}} k_{r+1}\left((I_{i_1} \otimes \cdots I_{i_l} \otimes \cdots \otimes I_{i_r} \otimes \mathrm{Id}) \circ (\tau^\bullet \otimes \mathrm{Id}) \circ (l_p' \otimes \mathrm{Id}) \circ \sigma^\bullet \right)$$

We can now reindex over the sum of p and s (on the (p, s)-plane, this is summing over the diagonal) to obtain

$$\sum_{k=1}^{n-1} \sum_{p+s=k+1} \sum_{\sigma(n)=n} \sum_{\substack{\tau \in S'(i_1,\ldots,i_r) \\ 1 \leq r \leq n-p \\ i_1+\ldots+i_r=n-p \\ i_l=s}} k_{r+1}\left((I_{i_1} \otimes \cdots \otimes I_{i_l} \otimes \cdots \otimes I_{i_r} \otimes \mathrm{Id}) \circ (\tau^\bullet \otimes \mathrm{Id}) \circ (l_p' \otimes \mathrm{Id}) \circ \sigma^\bullet \right)$$

Step 4. Here, we change τ to τ' and introduce λ. Since τ is an unshuffle, we can make two observations. First, τ sends l_p' to the first input of I_{i_l}. Second, in the partition $i_1 + \ldots + i_r = n - p$, the block i_l is the first of its size (i.e. $t < l$ implies $i_t < i_l$), since the first elements of blocks of the same size are in order. This information allows us to remove l_p' as an input to τ, and then put it back in the correct spot after the remaining elements are permuted. That is, τ corresponds to an $(i_1, \ldots, i_l - 1, \ldots, i_r)$-unshuffle τ' in S_{n-p-1}, and we will send l_p' to the first input

21

of I_{i_l} via a permutation λ after we apply τ'. Special care is needed when $s = 1$, in which case $\tau' \in S(0, i_2, \ldots, i_r)$, and no element will go to the block of size 0.

Note: because $\tau \in S'(i_1, \ldots, i_r)$, we had conditions that $i_1 \leq \cdots \leq i_r$ and that the order of the first elements among these blocks is preserved. In the rest of the proof, we must remember these restrictions inherited from τ. We obtain,

$$\sum_{k=1}^{n-1} \sum_{p+s=k+1} \sum_{\substack{\sigma(n)=n}} \sum_{\substack{\tau' \in S(i_1, i_2, \ldots, i_l-1, \ldots, i_r) \\ i_1 + \ldots + i_r = n-p \\ i_l = s}}$$

$$k_{r+1}\Big((I_{i_1} \otimes \cdots \otimes I_{i_l} \otimes \cdots \otimes I_{i_r} \otimes \mathrm{Id}) \circ \lambda^{\bullet} \circ (\mathrm{Id} \otimes \tau'^{\bullet} \otimes \mathrm{Id}) \circ (l'_p \otimes \mathrm{Id}) \circ \sigma^{\bullet} \Big)$$

Step 5. Combine σ and τ' into ψ. Now we observe that applying a p-unshuffle and then τ' to the remaining inputs is equivalent to doing a $(p, i_1, \ldots, i_r)$-unshuffle to all of the inputs at once. We obtain

$$\sum_{k=1}^{n-1} \sum_{p+s=k+1} \sum_{\substack{\psi \in S(p, i_1, \ldots, i_l-1, \ldots, i_r, 1) \\ i_1 + \ldots + i_r = n-p \\ i_l = s \\ \psi(n) = n}} k_{r+1}\Big((I_{i_1} \otimes \cdots \otimes I_{i_l} \otimes \cdots \otimes I_{i_r}) \circ \lambda^{\bullet} \circ (l'_p \otimes \mathrm{Id}) \circ \psi^{\bullet} \Big)$$

Step 6. Change from ψ to μ, α, ω. Notice that a $(p, i_1, \ldots, i_l - 1, \ldots i_r)$-unshuffle is the same as first doing a $(p + i_l - 1)$-unshuffle, and then doing a $(p, i_l - 1)$-unshuffle on the $(p + i_l - 1)$-block and an $(i_1, \ldots, \hat{i_l}, \ldots, i_r)$-unshuffle on the rest. Since we are fixing $i_l = s$, note that $p + i_l - 1 = k$.

Afterwards, we need to apply a permutation ω to move the strands in the i_l block back to their original position between the i_{l-1} and i_{l+1} blocks. That is, ω is the block permuation so that applying $\omega^\bullet$ to the blocks $\{1, i_l - 1, i_1, \ldots, \widehat{i_l}, \ldots, i_r\}$ yields $\{1, i_1, \ldots, i_l - 1, \ldots i_r\}$. We apply $\lambda^\bullet$ after $\omega^\bullet$ to move the l'_p term.

$$\sum_{k=1}^{n-1} \sum_{p+s=k+1} \sum_{\substack{\mu \in S(k,i_1,\ldots,\widehat{i_l},\ldots,i_r,1) \\ i_1 + \cdots + i_r = n-p \\ i_l = s \\ \mu(n)=n}} \sum_{\alpha \in S(p,k-p)}$$

$$k_{r+1}\bigg((I_{i_1} \otimes \cdots \otimes I_{i_l} \otimes \cdots \otimes I_{i_r} \otimes \mathrm{Id}) \circ \lambda^\bullet \circ \omega^\bullet \circ (l'_p \otimes \mathrm{Id}) \circ (\alpha^\bullet \otimes \mathrm{Id}) \circ \mu^\bullet \bigg)$$

Step 7. Since k_{r+1} is skew-symmetric, we can move the I_{i_l} term to the first input.

$$\sum_{k=1}^{n-1} \sum_{p+s=k+1} \sum_{\substack{\mu \in S(k,i_1,\ldots,\widehat{i_l},\ldots,i_r,1) \\ i_1 + \cdots + i_r = n-p \\ i_l = s \\ \mu(n)=n}} \sum_{\alpha \in S(p,k-p)}$$

$$k_{r+1}\bigg((I_{i_l} \otimes I_{i_1} \otimes \cdots \otimes \widehat{I_{i_l}} \otimes \cdots \otimes I_{i_r} \otimes \mathrm{Id}) \circ (l'_p \otimes \mathrm{Id}) \circ (\alpha^\bullet \otimes \mathrm{Id}) \circ \mu^\bullet \bigg)$$

Step 8. Rewrite the maps as

$$\sum_{k=1}^{n-1} \sum_{p+s=k+1} \sum_{\substack{\mu \in S(k,i_1,\ldots,\widehat{i_l},\ldots,i_r,1) \\ i_1 + \cdots + i_r = n-p \\ i_l = s \\ \mu(n)=n}} \sum_{\alpha \in S(p,k-p)}$$

$$k_{r+1}\bigg(\big[I_{i_l} \circ (l'_p \otimes \mathrm{Id}) \circ \alpha^\bullet \big] \otimes \big[(I_{i_1} \otimes \cdots \otimes \widehat{I_{i_l}} \otimes \cdots \otimes I_{i_r} \otimes \mathrm{Id}) \big] \circ \mu^\bullet \bigg)$$

23

Step 9. Apply the L_∞-algebra homomorphism relation to the terms $I_{i_l} \circ (l'_p \otimes \mathrm{Id}) \circ \alpha^\bullet$. Since we no longer are keeping track of p, we also use the fact that $p + s = k + 1$ to rewrite the conditions for μ.

$$\sum_{\substack{k=1}}^{n-1} \sum_{\substack{1 \le t \le k \\ a_1 + \ldots + a_t = k \\ a_r \ge 1}} \sum_{\gamma \in S'(a_1, \ldots, a_t)} \sum_{\substack{\mu \in S(k, i_1, \ldots, \widehat{i_l}, \ldots, i_r, 1) \\ i_1 + \cdots \widehat{i_l} + \cdots + i_r = n-1-k \\ \mu(n) = n}}$$

$$k_{r+1}\left(\left[l_t \circ (I_{a_1} \otimes \cdots \otimes I_{a_t}) \circ \gamma^\bullet \right] \otimes \left[(I_{i_1} \otimes \cdots \otimes \widehat{I_{i_l}} \otimes \cdots \otimes I_{i_r} \otimes \mathrm{Id}) \right] \circ \mu^\bullet \right)$$

Step 10. Rewrite the maps as

$$\sum_{\substack{k=1}}^{n-1} \sum_{\substack{1 \le t \le k \\ a_1 + \ldots + a_t = k \\ a_r \ge 1}} \sum_{\gamma \in S'(a_1, \ldots, a_t)} \sum_{\substack{\mu \in S(k, i_1, \ldots, \widehat{i_l}, \ldots, i_r, 1) \\ i_1 + \cdots \widehat{i_l} + \cdots + i_r = n-1-k \\ \mu(n) = n}}$$

$$k_{r+1}\left((l_t \otimes \mathrm{Id}) \circ (I_{a_1} \otimes \cdots \otimes I_{a_t} \otimes I_{i_1} \otimes \cdots \otimes \widehat{I_{i_l}} \otimes \cdots \otimes I_{i_r}) \circ (\gamma^\bullet \otimes \mathrm{Id}) \circ \mu^\bullet \right)$$

Step 11. We can combine μ and γ into one permutation η. Indeed, applying μ and then an $(a_1, \ldots, a_t)$-unshuffle on the k-block is the same as applying an $(a_1, \ldots, a_t, i_1, \ldots, \widehat{i_l}, \ldots, i_r, 1)$-unshuffle all at once.

$$\sum_{k=1}^{n-1}\sum_{\substack{\eta\in S(a_1,\ldots,a_t,i_1,\ldots,\widehat{i_l},\ldots i_r,1)\\ 1\le t\le k\\ a_1+\ldots+a_t=k\\ i_1+\cdots+\widehat{i_l}+\cdots+i_r=n-1-k\\ \eta(n)=n}} k_{r+1}\Big((l_t\otimes\mathrm{Id})\circ(I_{a_1}\otimes\cdots\otimes I_{a_t}\otimes I_{i_1}\otimes\cdots\otimes\widehat{I_{i_l}}\otimes\cdots\otimes I_{i_r}\otimes\mathrm{Id})\circ\eta^{\bullet}\Big)$$

Step 12. Since $k = 1,\ldots,n-1$, we can drop the sum over k from the notation and just require that $a_1,\ldots,a_t,i_1,\ldots,i_r$ is a partition of $n-1$, with $a_1 \le \ldots \le a_t$, $i_1 \le \ldots \le i_r$, and $t \ge 1$ and $r \ge 1$. If we fix $\eta \in S(a_1,\ldots,a_t,i_1,\ldots,\widehat{i_l},\ldots,i_r)$, we don't have any relation between the two partitions $a_1 \le \ldots \le a_t$ and $i_1 \le \ldots \le i_r$. That is, the sizes of the blocks are in order as part of their respective partitions, but it might not be the case that $a_1,\ldots,a_t,i_1,\ldots,i_r$ is in increasing order as a whole. However, from these two partitions, we can use an unshuffle to construct a new partition where the sizes of the boxes are in order. Indeed, define σ so that $(\sigma^{-1})^{\bullet}$ arranges the $a_1,\ldots,a_t,i_1,\ldots,i_r$ in increasing order (to get a unique σ, require that the order of the a's is preserved, the order of the i's is preserved, and that, using η, the first elements of boxes of same size are in order). Then let $c_1,\ldots,c_\alpha := (\sigma^{\bullet})^{-1}(a_1,\ldots,a_t,i_1,\ldots,i_r)$. To summarize, what we have done is define a new partition $c_1,\ldots,c_\alpha$ of $n-1$ so that $c_{\sigma(1)} = a_1,\ldots,c_{\sigma(t)} = a_t, c_{\sigma(t+1)} = i_1,\ldots,c_\alpha = i_r$. Of course, since $a_1 \le \ldots \le a_t$, σ is a t-unshuffle. Moreover, we define τ by requiring that the elements that η puts into the $a_1,\ldots,a_t$ and $i_1,\ldots,i_r$-boxes are precisely those that τ puts into the $c_{\sigma(1)},\ldots,c_{\sigma(t)}$ and $c_{\sigma(t+1)},\ldots,c_\alpha$-boxes, respectively. Finally, since $\alpha = t+r-1$, we relabeled k_{r+1} as $k_{\alpha+2-t}$. Note that we can reverse this whole construction to obtain an inverse correspondence. This process is similar to Lemma 4.

$$\sum_{\substack{\tau\in S'(c_1,\dots,c_\alpha)\\ c_1+\dots+c_\alpha=n-1\\ 1\le\alpha\le n-1}} \sum_{\substack{\sigma\in S(t,\alpha+1-t)\\ \sigma(\alpha+1)=\alpha+1\\ 1\le t\le\alpha+1}} k_{\alpha+2-t}\circ(l_t\otimes\mathrm{Id})\circ\sigma^{\bullet}\circ(I_{c_1}\otimes\cdots\otimes I_{c_\alpha}\otimes\mathrm{Id})\circ(\tau^{\bullet}\otimes\mathrm{Id})$$

Step 13. On the other hand, we now examine the second term in the original sum:

$$\sum_{p+q=n+1}\sum_{\sigma(p)=n} k'_q\circ\delta^{\bullet}\circ(k'_p\otimes\mathrm{Id})\circ\sigma^{\bullet}$$

Step 14. Use the definition of k' to substitute for k'_p and k'_q. The cases $p=1$ and $q=1$ require some care; they correspond to the cases $r=0$ and $s=0$, respectively. If $r=0$, then $\phi=\mathrm{Id}$, and if $s=0$, then $\psi=\mathrm{Id}$. We also disallow r and s to be zero simultaneously.

$$\sum_{\substack{p+q=n+1\\ 1\le p\le n}}\sum_{\sigma(p)=n}\sum_{\substack{\phi\in S'(i_1,\dots,i_r)\\ 0\le r\le p-1\\ i_1+\dots+i_r=p-1}}\sum_{\substack{\psi\in S'(j_1,\dots,j_s)\\ 0\le s\le n-p\\ j_1+\dots+j_s=n-p}} k_{s+1}\circ(I_{j_1}\otimes\cdots\otimes I_{j_s}\otimes\mathrm{Id})\circ(\psi\otimes\mathrm{Id})\circ\delta^{\bullet}$$
$$\circ(k_{r+1}\otimes\mathrm{Id})\circ(I_{i_1}\otimes\cdots\otimes I_{i_r}\otimes\mathrm{Id})\circ(\phi^{\bullet}\otimes\mathrm{Id})\circ\sigma^{\bullet}$$

Step 15. Commuting composition and tensor product, and replacing δ with an analogous δ' that ensures the module element is in the correct spot, we get

$$\sum_{\substack{p+q=n+1 \\ 1\leq p\leq n}} \sum_{\sigma(p)=n} \sum_{\substack{\phi\in S'(i_1,\ldots,i_r) \\ 0\leq r\leq p-1 \\ i_1+\ldots+i_r=p-1}} \sum_{\substack{\psi\in S'(j_1,\ldots,j_s) \\ 0\leq s\leq n-p \\ j_1+\ldots+j_s=n-p}} k_{s+1}\circ\delta'^\bullet\circ(k_{r+1}\otimes\mathrm{Id})$$

$$\circ\left(I_{i_1}\otimes\cdots I_{i_r}\otimes\mathrm{Id}\otimes I_{j_1}\otimes\cdots\otimes I_{j_s}\right)\circ(\phi^\bullet\otimes\mathrm{Id}\otimes\psi^\bullet)\circ\sigma^\bullet$$

Step 16. Instead of summing over r and s separately, we can sum over the diagonal $\alpha = r + s$.

$$\sum_{p=1}^{n}\sum_{\sigma(p)=n}\sum_{\substack{1\leq\alpha\leq n-1 \\ r+s=\alpha \\ r,s\geq 0}}\sum_{\substack{\phi\in S'(i_1,\ldots,i_r) \\ i_1+\ldots+i_r=p-1}}\sum_{\substack{\psi\in S'(j_1,\ldots,j_s) \\ j_1+\ldots+j_s=n-p}}$$

$$k_{s+1}\circ\delta'^\bullet\circ(k_{r+1}\otimes\mathrm{Id})\circ\left(I_{i_1}\otimes\cdots I_{i_r}\otimes\mathrm{Id}\otimes I_{j_1}\otimes\cdots\otimes I_{j_s}\right)\circ(\phi^\bullet\otimes\mathrm{Id}\otimes\psi^\bullet)\circ\sigma^\bullet$$

Step 17. Apply Lemma 4, where $r + 1$ above corresponds to t below.

$$\sum_{\substack{\tau\in S'(c_1,\ldots,c_\alpha) \\ c_1+\ldots+c_\alpha=n-1}}\sum_{\substack{\sigma\in S(t,\alpha+1-t) \\ \sigma(t)=\alpha+1 \\ 1\leq t\leq\alpha+1}} k_{\alpha+2-t}\circ\delta'^\bullet\circ(k_t\otimes\mathrm{Id})\circ(\sigma^\bullet\otimes\mathrm{Id})\circ(I_{c_1}\otimes\cdots\otimes I_{c_\alpha}\otimes\mathrm{Id})\circ(\tau^\bullet\otimes\mathrm{Id})$$

Step 18. Summarizing what we've done so far, we've shown that the original sum

$$\sum_{p+q=n+1}\sum_{\sigma(n)=n} k'_q\circ(l'_p\otimes\mathrm{Id})\circ\sigma^\bullet$$

$$+$$

$$\sum_{p+q=n+1}\sum_{\sigma(p)=n} k'_q\circ(\mathrm{Id}\otimes k'_p)\circ\sigma^\bullet$$

27

can be rewritten as

$$\sum_{\substack{\tau \in S'(c_1,\ldots,c_\alpha) \\ c_1+\ldots+c_\alpha=n-1}} \sum_{\substack{\sigma \in S(t,\alpha+1-t) \\ \sigma(\alpha+1)=\alpha+1 \\ 1 \leq t \leq \alpha+1}} k_{\alpha+2-t} \circ (l_t \otimes \mathrm{Id}) \circ \sigma^\bullet \circ (I_{c_1} \otimes \cdots \otimes I_{c_\alpha} \otimes \mathrm{Id}) \circ (\tau^\bullet \otimes \mathrm{Id})$$

$$+ \sum_{\substack{\tau \in S'(c_1,\ldots,c_\alpha) \\ c_1+\ldots+c_\alpha=n-1}} \sum_{\substack{\sigma \in S(t,\alpha+1-t) \\ \sigma(t)=\alpha+1 \\ 1 \leq t \leq \alpha+1}} k_{\alpha+2-t} \circ \delta'^\bullet \circ (k_t \otimes \mathrm{Id}) \circ \sigma^\bullet \circ (I_{c_1} \otimes \cdots \otimes I_{c_\alpha} \otimes \mathrm{Id}) \circ (\tau^\bullet \otimes \mathrm{Id})$$

Step 19. Letting $F = (I_{c_1} \otimes \cdots \otimes I_{c_\alpha} \otimes \mathrm{Id}) \circ (\tau^\bullet \otimes \mathrm{Id})$ and setting $u = \alpha + 2 - t$, this becomes

$$\sum_{\substack{\tau \in S'(c_1,\ldots,c_\alpha) \\ c_1+\ldots+c_\alpha=n-1}} \sum_{t+u=\alpha+2} \sum_{\substack{\sigma \in S(t,\alpha+1-t) \\ \sigma(\alpha+1)=\alpha+1}} k_u \circ (l'_t \otimes \mathrm{Id}) \circ \sigma^\bullet \circ F$$

$$+ \sum_{\substack{\tau \in S'(c_1,\ldots,c_\alpha) \\ c_1+\ldots+c_\alpha=n-1}} \sum_{t+u=\alpha+2} \sum_{\substack{\sigma \in S(t,\alpha+1-t) \\ \sigma(t)=\alpha+1}} k_u \circ \delta'^\bullet \circ (k_t \otimes \mathrm{Id}) \circ \sigma^\bullet \circ F$$

which cancel by the module relation.

$$\square$$

Lemma 3 (Morphisms). *Suppose L and L' are L_∞-algebras and M and N are L-modules. Let $I : L' \to L$ be an L_∞-algebra homomorphism, and let $f : M \to N$ be an L-module homomorphism. Set $(I^* f)_1 = f_1$, and for $n \geq 2$, define*

$$(I^* f)_n : (L')^{\otimes n-1} \otimes I^* M \to I^* N$$

28

by

$$(I^* f)_n = \sum_{r=1}^{n-1} \sum_{\substack{\tau \in S'(i_1,\ldots,i_r) \\ i_1+\ldots+i_r=n-1}} f_{r+1} \circ (I_{i_1} \otimes \cdots \otimes I_{i_r} \otimes \mathrm{Id}) \circ (\tau^\bullet \otimes \mathrm{Id})$$

Then $I^* f : I^* M \to I^* N$ is a homomorphism of L'-modules.

Proof. We will start by examining the L_∞-module homomorphism relation. After replacing $I^* f$ and m_i' with their definitions on the left-hand side (steps 1-4), we will rearrange the sum (steps 5-6) and apply the L_∞-algebra relation for I (step 7). We then rewrite the terms (steps 8-9) and apply the module homomorphism relation for f (step 10). We then show that the result is equal to the right-hand side (steps 11-16).

Step 1. To start, we will denote the operations of $M, N, I^* M, I^* N$ by m, n, m', n' respectively. To show that $I^* f$ is a homomorphsism of L'-modules, we must show that it satisfies the L_∞-module homomorphism relation

$$\sum_{i+j=n+1} \sum_{\sigma} (I^* f)_j \circ (m_i' \otimes \mathrm{Id}) \circ \sigma^\bullet = \sum_{r+s=n+1} \sum_{\tau} n_r' \circ (\mathrm{Id} \otimes (I^* f)_s) \circ \tau^\bullet$$

where σ is an $(i, n-i)$-unshuffle and τ is an $(n-s, s-1)$-unshuffle.

Step 2. Focusing only on the left-hand side, we break this sum up into two parts

$$\sum_{i+j=n+1} \sum_{\sigma(i)=n} (I^* f)_j \circ \lambda^\bullet \circ (m_i' \otimes \mathrm{Id}) \circ \sigma^\bullet$$

$$+ \sum_{\substack{i+j=n+1 \\ i<n}} \sum_{\sigma(n)=n} (I^* f)_j \circ (l_i' \otimes \mathrm{Id}) \circ \sigma^\bullet$$

where we use skew-symmetry and introduce the permutation λ to insert the module element in the correct spot.

29

Step 3. Replace $I^* f$ with its definition. Note that we've allowed $r = 0$ in the first sum to include the case $j = 1$, which corresponds to $f_1 \circ m'_n \circ \sigma^\bullet$. If j is anything but 1, $r = 0$ makes no contribution to the sum.

$$\sum_{i+j=n+1} \sum_{\sigma(i)=n} \sum_{r=0}^{j-1} \sum_{\substack{\tau \in S'(i_1,\dots,i_r) \\ i_1+\dots+i_r=j-1}} f_{r+1} \circ \left(I_{i_1} \otimes \cdots \otimes I_{i_r} \otimes \mathrm{Id}\right) \circ \left(\tau^\bullet \otimes \mathrm{Id}\right) \circ \lambda^\bullet \circ \left(m'_i \otimes \mathrm{Id}\right) \circ \sigma^\bullet$$

$$+ \sum_{\substack{i+j=n+1 \\ 1\leq i<n}} \sum_{\sigma(n)=n} \sum_{r=1}^{j-1} \sum_{\substack{\tau \in S'(i_1,\dots,i_r) \\ i_1+\dots+i_r=j-1}} f_{r+1} \circ \left(I_{i_1} \otimes \cdots \otimes I_{i_r} \otimes \mathrm{Id}\right) \circ \left(\tau^\bullet \otimes \mathrm{Id}\right) \circ \left(l'_i \otimes \mathrm{Id}\right) \circ \sigma^\bullet$$

Step 4. Now focus on the first sum and replace m'_i with its definition. Similar to the above, we've allowed for the case $s = 0$ to include the case $i = 1$, which corresponds to $(I^* f)_n \circ \lambda^\bullet \circ (m'_1 \otimes \mathrm{Id}) \circ \sigma^\bullet$. If i is anything but 1, $s = 0$ makes no contribution to the sum.

$$\sum_{i+j=n+1} \sum_{\sigma(i)=n} \sum_{r=0}^{j-1} \sum_{\substack{\tau \in S'(i_1,\dots,i_r) \\ i_1+\dots+i_r=j-1}} \sum_{s=0}^{i-1} \sum_{\substack{\psi \in S'(j_1,\dots,j_a) \\ j_1+\dots+j_s=i-1}} f_{r+1} \circ \left(I_{i_1} \otimes \cdots \otimes I_{i_r} \otimes \mathrm{Id}\right) \circ \left(\tau^\bullet \otimes \mathrm{Id}\right)$$

$$\circ \lambda^\bullet \circ \left[\left(m_{s+1} \circ \left(I_{j_1} \otimes \cdots \otimes I_{j_s} \otimes \mathrm{Id}\right) \circ \psi^\bullet\right) \otimes \mathrm{Id}\right] \circ \sigma^\bullet$$

Step 5. Rewrite the sum by commuting composition and tensor product and considering the diagonal $\alpha = r + s$ instead of r and s individually. Observe that one of r and s can be 0, but not both at the same time.

$$\sum_{\substack{i+j=n+1}} \sum_{\sigma(i)=n} \sum_{\substack{1\le\alpha\le n-1 \\ r+s=\alpha \\ r,s\ge 0}} \sum_{\substack{\tau\in S'(i_1,\dots,i_r) \\ i_1+\dots+i_r=j-1}} \sum_{\substack{\psi\in S'(j_1,\dots,j_s) \\ j_1+\dots+j_s=i-1}} f_{r+1}\circ\omega^{\bullet}\circ(m_{s+1}\otimes\mathrm{Id})$$

$$\circ\,(I_{j_1}\otimes\cdots\otimes I_{j_s}\otimes\mathrm{Id}\otimes I_{i_1}\otimes\cdots\otimes I_{i_r})\circ(\psi^{\bullet}\otimes\mathrm{Id}\otimes\tau^{\bullet})\circ\sigma^{\bullet}$$

Step 6. Apply Lemma 4 to obtain

$$\sum_{\substack{\pi\in S'(c_1,\dots,c_\alpha) \\ c_1+\dots+c_\alpha=n-1}} \sum_{\substack{\theta\in S(t,\alpha+1-t) \\ \theta(t)=\alpha+1 \\ 1\le t\le\alpha+1}} f_{\alpha+2-t}\circ\omega^{\bullet}\circ(m_t\otimes\mathrm{Id})\circ\theta^{\bullet}\circ(I_{c_1}\otimes\cdots\otimes I_{c_\alpha}\otimes\mathrm{Id})\circ(\pi^{\bullet}\otimes\mathrm{Id})$$

Step 7. Now, focusing on the l terms (the second sum in Step 3), our goal is to apply the L_∞-algebra relation for I. The steps we follow here are essentially the same as in Lemma 2 (steps 3-12), and we direct the reader to them for details and for diagrams. We start with

$$\sum_{\substack{i+j=n+1 \\ 1\le i<n}} \sum_{\sigma(n)=n} \sum_{r=1}^{j-1} \sum_{\substack{\tau\in S'(i_1,\dots,i_r) \\ i_1+\dots+i_r=j-1}} f_{r+1}\circ(I_{i_1}\otimes\cdots\otimes I_{i_r}\otimes\mathrm{Id})\circ(\tau^{\bullet}\otimes\mathrm{Id})\circ(l_i'\otimes\mathrm{Id})\circ\sigma^{\bullet}$$

Denote the block where l_i' goes by I_{i_l}. Break down the sum by $i_l = s$.

$$\sum_{\substack{i+j=n+1 \\ 1\le i<n}} \sum_{\sigma(n)=n} \sum_{r=1}^{j-1}\sum_{s=1}^{j-1} \sum_{\substack{\tau\in S'(i_1,\dots,i_r) \\ i_1+\dots+i_r=j-1 \\ i_l=s}} f_{r+1}\circ(I_{i_1}\otimes\cdots\otimes I_{i_r}\otimes\mathrm{Id})\circ(\tau^{\bullet}\otimes\mathrm{Id})\circ(l_i'\otimes\mathrm{Id})\circ\sigma^{\bullet}$$

Remove j from the notation.

$$\sum_{i=1}^{n-1} \sum_{\sigma(n)=n} \sum_{r=1}^{n-i} \sum_{s=1}^{n-i} \sum_{\substack{\tau \in S'(i_1,\ldots,i_r) \\ i_1+\ldots+i_r=n-i \\ i_l=s}} f_{r+1} \circ (I_{i_1} \otimes \cdots \otimes I_{i_r} \otimes \mathrm{Id}) \circ (\tau^\bullet \otimes \mathrm{Id}) \circ (l_i' \otimes \mathrm{Id}) \circ \sigma^\bullet$$

Reindex over the sum of i and s.

$$\sum_{k=1}^{n-1} \sum_{i+s=k+1} \sum_{\sigma(n)=n} \sum_{\substack{\tau \in S'(i_1,\ldots,i_r) \\ i_1+\ldots+i_r=n-i \\ i_l=s}} f_{r+1} \circ (I_{i_1} \otimes \cdots \otimes I_{i_r} \otimes \mathrm{Id}) \circ (\tau^\bullet \otimes \mathrm{Id}) \circ (l_i' \otimes \mathrm{Id}) \circ \sigma^\bullet$$

Use the map $\lambda^\bullet$ to permute l_i' around τ and change τ to τ'.

$$\sum_{k=1}^{n-1} \sum_{i+s=k+1} \sum_{\sigma(n)=n} \sum_{\substack{\tau' \in S(i_1,\ldots,i_l-1,\ldots,i_r) \\ i_1+\ldots+i_r=n-i \\ i_l=s}}$$

$$f_{r+1} \circ (I_{i_1} \otimes \cdots \otimes I_{i_l} \otimes \cdots \otimes I_{i_r} \otimes \mathrm{Id}) \circ \lambda^\bullet \circ (\mathrm{Id} \otimes \tau'^\bullet \otimes \mathrm{Id}) \circ (l_i' \otimes \mathrm{Id}) \circ \sigma^\bullet$$

Combine τ' and σ into the permutation η.

$$\sum_{k=1}^{n-1} \sum_{i+s=k+1} \sum_{\rho \in S(i,i_l-1)} \sum_{\substack{\eta \in S(i+i_l-1,i_1,\ldots,\hat{i_l},\ldots,i_r,1) \\ i_1+\ldots+i_r=n-i \\ i_l=s \\ \eta(n)=n}}$$

$$f_{r+1} \circ (I_{i_1} \otimes \cdots \otimes I_{i_l} \otimes \cdots \otimes I_{i_r} \otimes \mathrm{Id}) \circ \lambda^\bullet \circ \omega^\bullet \circ (l_i' \otimes \mathrm{Id}) \circ (\rho^\bullet \otimes \mathrm{Id}) \circ \eta^\bullet$$

Use skew-symmetry of f_{r+1} to swap the order of the I's

$$\sum_{k=1}^{n-1} \sum_{i+s=k+1} \sum_{\rho \in S(i,i_l-1)} \sum_{\substack{\eta \in S(i+i_l-1,i_1,\ldots,\hat{i_l},\ldots,i_r,1) \\ i_1+\ldots+i_r=n-i \\ i_l=s \\ \eta(n)=n}} f_{r+1} \circ (I_{i_l} \otimes I_{i_1} \otimes \cdots \otimes I_{i_r} \otimes \mathrm{Id}) \circ (l_i' \otimes \mathrm{Id}) \circ (\rho^\bullet \otimes \mathrm{Id}) \circ \eta^\bullet$$

32

Rewrite suggestively, noting that now the I_{i_l} is omitted from $I_{i_1} \otimes \cdots \otimes I_{i_r}$.

$$\sum_{k=1}^{n-1} \sum_{\substack{i+s=k+1}} \sum_{\rho \in S(i,i_l-1)} \sum_{\substack{\eta \in S(k,i_1,\dots,\widehat{i_l},\dots,i_r,1) \\ i_1+\dots+i_r=n-i \\ i_l=s \\ \eta(n)=n}} f_{r+1}\Bigg(\Big(I_{i_l} \circ (l'_i \otimes \mathrm{Id}) \circ \rho^{\bullet}\Big) \otimes \big(I_{i_1} \otimes \cdots \otimes I_{i_r}\big) \otimes \mathrm{Id} \Bigg) \circ \eta^{\bullet}$$

Apply the morphism relations.

$$\sum_{k=1}^{n-1} \sum_{\substack{\gamma \in S'(t_1,\dots,t_z) \\ t_1+\dots+t_z=k \\ 1\leq z\leq k}} \sum_{\substack{\eta \in S(k,i_1,\dots,\widehat{i_l},\dots,i_r,1) \\ i_1+\dots+\widehat{i_l}+\dots+i_r=n-1-k \\ \eta(n)=n}} f_{r+1}\Bigg(\big(l_z \circ (I_{t_1} \otimes \cdots \otimes I_{t_z}) \circ \gamma^{\bullet}\big) \otimes \big(I_{i_1} \otimes \cdots \otimes I_{i_r}\big) \otimes \mathrm{Id} \Bigg) \circ \eta^{\bullet}$$

Combine γ and η into ψ.

$$\sum_{k=1}^{n-1} \sum_{\substack{\psi \in S(t_1,\dots,t_z,i_1,\dots,\widehat{i_l},\dots,i_r,1) \\ 1\leq z\leq k \\ t_1+\dots+t_z=k \\ i_1+\dots+\widehat{i_l}+\dots+i_r=n-1-k \\ \psi(n)=n}} f_{r+1}\Bigg((l_z \otimes \mathrm{Id}) \circ (I_{t_1} \otimes \cdots \otimes I_{t_z} \otimes I_{i_1} \otimes \cdots \otimes I_{i_r} \otimes \mathrm{Id}) \circ \psi^{\bullet} \Bigg)$$

This is equivalent to

$$\sum_{\substack{\pi \in S'(c_1,\dots,c_\alpha) \\ c_1+\dots+c_\alpha=n-1 \\ 1\leq\alpha\leq n-1}} \sum_{\substack{\theta \in S(t,\alpha+1-t) \\ \theta(\alpha+1)=\alpha+1 \\ 1\leq t<\alpha+1}} f_{\alpha+2-t} \circ (l_t \otimes \mathrm{Id}) \circ \theta^{\bullet} \circ (I_{c_1} \otimes \cdots \otimes I_{c_\alpha} \otimes \mathrm{Id}) \circ (\pi^{\bullet} \otimes \mathrm{Id})$$

Step 8. In total, combining this with Step 6, we have the sum

$$\sum_{\substack{\pi \in S'(c_1,\dots,c_\alpha) \\ c_1+\dots+c_\alpha=n-1}} \sum_{\substack{\theta \in S(t,\alpha+1-t) \\ \theta(t)=\alpha+1 \\ 1\leq t\leq\alpha+1}} f_{\alpha+2-t} \circ \omega^{\bullet} \circ (m_t \otimes \mathrm{Id}) \circ \theta^{\bullet} \circ (I_{c_1} \otimes \cdots \otimes I_{c_\alpha} \otimes \mathrm{Id}) \circ (\pi^{\bullet} \otimes \mathrm{Id})$$

33

$$+ \sum_{\substack{\pi \in S'(c_1,\ldots,c_\alpha) \\ c_1+\ldots+c_\alpha=n-1}} \sum_{\substack{\theta \in S(t,\alpha+1-t) \\ \theta(\alpha+1)=\alpha+1 \\ 1 \leq t < \alpha+1}} f_{\alpha+2-t} \circ (l_t \otimes \mathrm{Id}) \circ \theta^\bullet \circ (I_{c_1} \otimes \cdots \otimes I_{c_\alpha} \otimes \mathrm{Id}) \circ (\pi^\bullet \otimes \mathrm{Id})$$

Step 9. Change notation; change t to i and $\alpha + 2 - t$ to j.

$$\sum_{\substack{\pi \in S'(c_1,\ldots,c_\alpha) \\ c_1+\ldots+c_\alpha=n-1}} \sum_{i+j=\alpha+2} \sum_{\substack{\theta \in S(i,\alpha+1-i) \\ \theta(i)=\alpha+1}} f_j \circ \omega^\bullet \circ (m_i \otimes \mathrm{Id}) \circ \theta^\bullet \circ (I_{c_1} \otimes \cdots \otimes I_{c_\alpha} \otimes \mathrm{Id}) \circ (\pi^\bullet \otimes \mathrm{Id})$$

$$+ \sum_{\substack{\pi \in S'(c_1,\ldots,c_\alpha) \\ c_1+\ldots+c_\alpha=n-1}} \sum_{\substack{i+j=\alpha+2 \\ i<\alpha+1}} \sum_{\substack{\theta \in S(i,\alpha+1-i) \\ \theta(\alpha+1)=\alpha+1}} f_j \circ (l_i \otimes \mathrm{Id}) \circ \theta^\bullet \circ (I_{c_1} \otimes \cdots \otimes I_{c_\alpha} \otimes \mathrm{Id}) \circ (\pi^\bullet \otimes \mathrm{Id})$$

Step 10. Applying the module homomorphism relation for f, we obtain

$$\sum_{\substack{\pi \in S'(c_1,\ldots,c_\alpha) \\ c_1+\ldots+c_\alpha=n-1}} \sum_{r+s=\alpha+2} \sum_{\rho \in S(\alpha-s,s)} n_r \circ (\mathrm{Id} \otimes f_s) \circ (\rho^\bullet \otimes \mathrm{Id}) \circ (I_{c_1} \otimes \cdots \otimes I_{c_\alpha} \otimes \mathrm{Id}) \circ (\pi^\bullet \otimes \mathrm{Id})$$

Step 11. It just remains to show that the sum above is equal to

$$\sum_{r+s=n+1} \sum_\tau n_r' \circ (\mathrm{Id} \otimes (I^* f)_s) \circ (\tau^\bullet \otimes \mathrm{Id})$$

Therefore, use the definition of $I^* f$. Like usual, we start indexing at $x = 0$ to allow for the f_1 case.

$$\sum_{r+s=n+1} \sum_{\tau \in S(n-s,s)} \sum_{\substack{\phi \in S'(i_1,\ldots,i_x) \\ i_1+\ldots+i_x=s-1 \\ 0 \leq x \leq s-1}} n_r' \circ \left[\mathrm{Id} \otimes \left(f_{x+1} \circ [(I_{i_1} \otimes \cdots \otimes I_{i_x} \circ \phi^\bullet) \otimes \mathrm{Id}] \right) \right] \circ (\tau^\bullet \otimes \mathrm{Id})$$

Step 12. Now use the definition of n'. Allow for $y = 0$ to deal with the n_1 case.

$$\sum_{r+s=n+1} \sum_{\tau \in S(n-s,s)} \sum_{x=0}^{s-1} \sum_{\substack{\phi \in S(i_1,\ldots,i_x) \\ i_1+\ldots+i_x=s-1}} \sum_{y=0}^{n-s} \sum_{\substack{\gamma \in S(j_1,\ldots,j_y) \\ j_1+\ldots+j_y=n-s-1}} n_{y+1} \circ (I_{j_1} \otimes \cdots \otimes I_{j_y} \otimes \mathrm{Id}) \circ$$

$$(\gamma^\bullet \otimes \mathrm{Id}) \circ \left[\mathrm{Id} \otimes \left(f_{x+1} \circ (I_{i_1} \otimes \cdots \otimes I_{i_x} \otimes \mathrm{Id}) \circ (\phi^\bullet \otimes \mathrm{Id}) \right) \right] \circ (\tau^\bullet \otimes \mathrm{Id})$$

Step 13. Commute composition and tensor product to rewrite as

$$\sum_{r+s=n+1} \sum_{\tau \in S(n-s,s)} \sum_{x=0}^{s-1} \sum_{\substack{\phi \in S'(i_1,\ldots,i_x) \\ i_1+\ldots+i_x=s-1}} \sum_{y=0}^{n-s} \sum_{\substack{\gamma \in S'(j_1,\ldots,j_y) \\ j_1+\ldots+j_y=n-s-1}}$$

$$n_{y+1} \circ (\mathrm{Id} \otimes f_{x+1}) \circ (I_{j_1} \otimes \cdots \otimes I_{j_y} \otimes I_{i_1} \otimes \cdots \otimes I_{i_x} \otimes \mathrm{Id}) \circ (\gamma^\bullet \otimes \phi^\bullet \otimes \mathrm{Id}) \circ (\tau^\bullet \otimes \mathrm{Id})$$

Step 14. Reindex over the diagonal of $\alpha = x + y$. Observe that one of x and y can be 0, but not both at the same time.

$$\sum_{r+s=n+1} \sum_{\tau \in S(n-s,s)} \sum_{\substack{1 \le \alpha \le n-1 \\ x+y=\alpha \\ x,y \ge 0}} \sum_{\substack{\phi \in S'(i_1,\ldots,i_x) \\ i_1+\ldots+i_x=s-1}} \sum_{\substack{\gamma \in S'(j_1,\ldots,j_y) \\ j_1+\ldots+j_y=n-s-1}}$$

$$n_{y+1} \circ (\mathrm{Id} \otimes f_{x+1}) \circ (I_{j_1} \otimes \cdots \otimes I_{j_y} \otimes I_{i_1} \otimes \cdots \otimes I_{i_x} \otimes \mathrm{Id}) \circ (\gamma^\bullet \otimes \phi^\bullet \otimes \mathrm{Id}) \circ (\tau^\bullet \otimes \mathrm{Id})$$

Step 15. Apply Lemma 4 to get

35

$$\sum_{\substack{\pi \in S'(c_1,\ldots,c_\alpha) \\ c_1+\ldots+c_\alpha=n-1 \\ 1 \le \alpha \le n-1}} \sum_{\substack{\theta \in S(\alpha-s,s) \\ 1 \le s \le \alpha}} n_{\alpha+2-s} \circ (\mathrm{Id} \otimes f_s) \circ \theta^\bullet \circ (I_{c_1} \otimes \cdots \otimes I_{c_\alpha} \otimes \mathrm{Id}) \circ (\pi^\bullet \otimes \mathrm{Id})$$

Step 16. Rewriting this as

$$\sum_{\substack{\pi \in S'(c_1,\ldots,c_\alpha) \\ c_1+\ldots+c_\alpha=n-1}} \sum_{r+s=\alpha+2} \sum_{\substack{\theta \in S(\alpha-s,s) \\ 1 \le s \le \alpha}} n_r \circ (\mathrm{Id} \otimes f_s) \circ \theta^\bullet \circ (I_{c_1} \otimes \cdots \otimes I_{c_\alpha} \otimes \mathrm{Id}) \circ (\pi^\bullet \otimes \mathrm{Id})$$

shows that it is the same as the sum in Step 10, which completes the proof.

$\square$

Theorem 1 (Functoriality). *Suppose $I : (L', l') \to (L, l)$ is a map of L_∞-algebras. Then $I^* : L\text{-mod} \to L'\text{-mod}$ is a functor.*

Proof. Suppose we have L_∞-modules M, N, and Q over L and L_∞-module homomorphisms $M \xrightarrow{f} N \xrightarrow{g} Q$. We have defined I^* on objects and morphisms, so it remains to show that $I^*(\mathrm{Id}_M) = \mathrm{Id}_{I^*M}$ and that $I^*(g \circ f) = I^* g \circ I^* f$. For the former, observe that $(I^*(\mathrm{Id}_M))_1 = (\mathrm{Id}_M)_1$, and for $n \ge 2$,

$$(I^*(\mathrm{Id}_M))_n = \sum_{r=1}^{n-1} \sum_{\substack{\tau \in S'(i_1,\ldots,i_r) \\ i_1+\ldots+i_r=n-1}} (\mathrm{Id}_M)_{r+1} \circ (I_{i_1} \otimes \cdots \otimes I_{i_r} \otimes \mathrm{Id}) \circ (\tau^\bullet \otimes \mathrm{Id})$$

But $(\mathrm{Id}_M)_r = 0$ for $r > 1$, and so we conclude that $(I^*(\mathrm{Id}_M))_n = 0$ for $n \ge 2$. Hence $I^*(\mathrm{Id}_M) = \mathrm{Id}_{I^*M}$.

In remains to show that $I^*(g \circ f) = I^* g \circ I^* f$. We will follow essentially the same procedure as in Lemma 2, steps 13-17.

36

Step 1. We start with the right-hand side, and replace $[I^*g \circ I^*f]_n$ with its definition

$$\sum_{i+j=n+1} \sum_{\sigma(i)=n} (I^*g)_j \circ \lambda^\bullet \circ ((I^*f)_i \otimes \mathrm{Id}) \circ \sigma^\bullet$$

Step 2. Replace I^*g and I^*f with their definitions.

$$\sum_{i+j=n+1} \sum_{\sigma(i)=n} \sum_{r=0}^{i-1} \sum_{\substack{\phi \in S'(i_1,\ldots,i_r) \\ i_1+\ldots+i_r=i-1}} \sum_{s=0}^{j-1} \sum_{\substack{\psi \in S'(j_1,\ldots,j_s) \\ j_1+\ldots+j_s=j-1}} \left[g_{s+1} \circ (I_{j_1} \otimes \cdots \otimes I_{j_s} \otimes \mathrm{Id}) \circ (\psi^\bullet \otimes \mathrm{Id}) \right]$$

$$\circ \lambda^\bullet \circ \left(\left[f_{r+1} \circ (I_{i_1} \otimes \cdots \otimes I_{i_r} \otimes \mathrm{Id}) \circ (\phi^\bullet \otimes \mathrm{Id}) \right] \otimes \mathrm{Id} \right) \circ \sigma^\bullet$$

Note that we include the cases $r = 0$ and $s = 0$ to include the cases f_1 and g_1, respectively. In particular, $r = 0$ will contribute a nonzero term only when $i = 1$, and $s = 0$ will only contribute a nonzero term when $j = 1$.

Step 3. Commute composition and tensor product to rewrite.

$$\sum_{i+j=n+1} \sum_{\sigma(i)=n} \sum_{r=0}^{i-1} \sum_{\substack{\phi \in S'(i_1,\ldots,i_r) \\ i_1+\ldots+i_r=i-1}} \sum_{s=0}^{j-1} \sum_{\substack{\psi \in S'(j_1,\ldots,j_s) \\ j_1+\ldots+j_s=j-1}}$$

$$g_{s+1} \circ \lambda'^\bullet \circ (f_{r+1} \otimes \mathrm{Id}) \circ (I_{i_1} \otimes \cdots I_{i_r} \otimes \mathrm{Id} \otimes I_{j_1} \otimes \cdots \otimes I_{j_s}) \circ (\phi^\bullet \otimes \mathrm{Id} \otimes \psi^\bullet) \circ \sigma^\bullet$$

Here, $\lambda'^\bullet$ is the map that permutes the module element into the last input of g_{s+1}.

Step 4. By Lemma 4, we obtain

$$\sum_{\substack{\tau \in S'(c_1,\dots,c_\alpha) \\ c_1+\dots+c_\alpha=n-1 \\ 1\le\alpha\le n-1}} \sum_{\substack{\theta \in S(t+1,\alpha-t) \\ \theta(t+1)=\alpha+1 \\ 0\le t\le\alpha}} g_{\alpha+1-t} \circ \lambda'^{\bullet} \circ (f_{t+1} \otimes \mathrm{Id}) \circ \theta^{\bullet} \circ (I_{c_1} \otimes \cdots \otimes I_{c_\alpha} \otimes \mathrm{Id}) \circ (\tau^{\bullet} \otimes \mathrm{Id})$$

Step 5. Change notation; let $p = t + 1$ and $q = \alpha + 1 - t$.

$$\sum_{\substack{\tau \in S'(c_1,\dots,c_\alpha) \\ c_1+\dots+c_\alpha=n-1 \\ 1\le\alpha\le n-1}} \sum_{\substack{\theta \in S(p,\alpha+1-p) \\ \theta(p)=\alpha+1 \\ 1\le p\le\alpha+1}} g_q \circ \lambda'^{\bullet} \circ (f_p \otimes \mathrm{Id}) \circ \theta^{\bullet} \circ (I_{c_1} \otimes \cdots \otimes I_{c_\alpha} \otimes \mathrm{Id}) \circ (\tau^{\bullet} \otimes \mathrm{Id})$$

Step 6. By the definition of $g \circ f$, this is

$$\sum_{\substack{\tau \in S'(c_1,\dots,c_\alpha) \\ c_1+\dots+c_\alpha=n-1 \\ 1\le\alpha\le n-1}} (g \circ f)_{\alpha+1} \circ (I_{c_1} \otimes \cdots \otimes I_{c_\alpha} \otimes \mathrm{Id}) \circ (\tau^{\bullet} \otimes \mathrm{Id})$$

Step 7. By the definition of I^*, this is precisely $[I^*(g \circ f)]_n$, as desired.

$\square$

Corollary 1. *If L and L' are Lie algebras, and $\phi : L' \to L$ is a Lie algebra homomorphism, ϕ^* is the usual restriction of scalars for Lie algebra representations.*

Proof. Let $\rho : L \to \mathfrak{gl}(M)$ be a Lie algebra representation. For $x \in L'$ and $m \in M$, the usual restriction of scalars for Lie algebra representations is given by $x \cdot m :=$ $\phi(x) \cdot m$. Indeed, $\rho' : L' \to \mathfrak{gl}(M)$ defined by $\rho'(y) = \rho(\phi(y))$ is a homomorphism of Lie algebras. Now, regarding ϕ as an L_∞-algebra map with $\phi_i = 0$ for $i \ne 1$, because

there are also no higher operations on M as an L_∞ L-module, the formulas given in Lemma 2 for the induced operation simplify to give the usual restriction of scalars operation described above. $\square$

We now prove the technical lemma that was used in the main results above. In particular, this lemma gives two ways to interpret a particular composition of unshuffles.

Lemma 4. *For a fixed n,*

$$\sum_{p=1}^{n} \sum_{\sigma(p)=n} \sum_{\substack{1\le\alpha\le n-1 \\ r+s=\alpha \\ r,s\ge0}} \sum_{\substack{\phi\in S'(i_1,\dots,i_r) \\ i_1+\dots+i_r=p-1}} \sum_{\substack{\psi\in S'(j_1,\dots,j_s) \\ j_1+\dots+j_s=n-p}} (I_{i_1}\otimes\cdots I_{i_r}\otimes\mathrm{Id}\otimes I_{j_1}\otimes\cdots\otimes I_{j_s})\circ(\phi^\bullet\otimes\mathrm{Id}\otimes\psi^\bullet)\circ\sigma^\bullet$$

is the same as

$$\sum_{\substack{\tau\in S'(c_1,\dots,c_\alpha) \\ c_1+\dots+c_\alpha=n-1}} \sum_{\substack{\theta\in S(r+1,\alpha-r) \\ \theta(r+1)=\alpha+1 \\ 0\le r\le\alpha}} \theta^\bullet\circ(I_{c_1}\otimes\cdots\otimes I_{c_\alpha}\otimes\mathrm{Id})\circ(\tau^\bullet\otimes\mathrm{Id}).$$
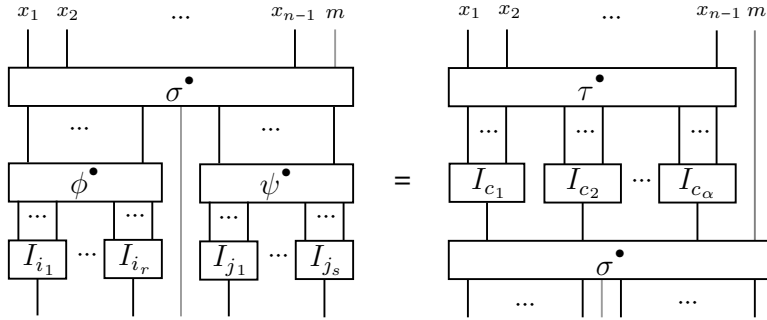
FIGURE 4. The left-hand side represents first unshuffling n elements into two boxes (with the module element by itself) via $\sigma^\bullet$ and then unshuffling these boxes further into r boxes and s boxes via $\phi^\bullet$ and $\psi^\bullet$, respectively. The right-hand side represents first unshuffling $n-1$ elements into α boxes via $\tau^\bullet$ and then unshuffling these α boxes via $\sigma^\bullet$.

39

Proof. To see this, it is helpful to examine what the first sum does for a fixed p and a fixed α. It unshuffles n elements into a box of size $p - 1$ and a box of size $n - p$, with the module element in between. It then unshuffles the box of size $p - 1$ further via ϕ into r smaller boxes and the box of size $n - p$ further via ψ into s smaller boxes.

So, if we iterate through $\alpha = r + s$, this sum describes all possible ways of unshuffling n elements into r boxes (which contain a total of $p - 1$ elements) and s boxes (which contain a total number of $n - p$ elements), with the module element in between. Then, iterating through all possible p tells us that the sum describes all ways of unshuffling n elements into $r + s$ boxes, with the module element in between. Note that the r boxes and the s boxes have to be of increasing size when considered separately, but they need not be in order when considered all together (e.g. some of the s boxes could be smaller than the last r box).

On the other hand, the second sum unshuffles the $n - 1$ algebra elements into α boxes first (here, the boxes are all of increasing size), and then it picks out r of these via an r-unshuffle θ in S_α. Since there was a module element between the r boxes and s boxes in the first sum, we can view θ as an $(r + 1)$-unshuffle in S_α where it puts the module element after the r boxes. So what we have done is the same as before: unshuffle n elements into a group of r boxes, a module element, and a group of $s = \alpha - r$ boxes, where the boxes are of increasing order when considered separately (but not necessarily when considered all together), see Figure 4. An explicit correspondence between the two sums can be written down using formulas. $\square$

2.4. Chain Contractions

It is possible to use an existing L_∞-algebra or L_∞-module to obtain a new L_∞-structure on a particular chain complex. In this section, we will use chain contractions to transfer L_∞-structures.

Definition 12. Let (A, d_A) and (B, d_B) be chain complexes. A **chain contraction** from A onto B consists of two chain maps $q : A \to B$ and $i : B \to A$ of degree 0, together with a homotopy $K : A \to A$ of degree 1. That is, we have the following diagram.

$$K \circlearrowright A \xrightarrow[\;i\;]{\;q\;} B$$

These maps q, i, and K must satisfy the following conditions:

$$q \circ i = \mathrm{Id}_B \quad \text{and} \quad \mathrm{Id}_A - i \circ q = K \circ d_A + d_A \circ K$$

$$K^2 = K \circ i = q \circ K = 0$$

We will denote a chain contraction by (A, B, i, q, K).

Remark. If (A, d_A) and (B, d_B) are cochain complexes, we require $|K| = -1$.

If L is an L_∞-algebra and L' is a chain complex, formulas exist in the literature for how to transfer the L_∞-algebra structure from L to L', given a chain contraction (L, L', i, q, K). Following [Mor22b, Theorem 1], the chain maps i and q can also be extended to L_∞-algebra homomorphisms $I : L' \to L$ and $Q : L \to L'$ such that $Q \circ I = \mathrm{Id}_{L'}$. The transferred L_∞-algebra structure on L' is unique up to quasi-isomorphism, and the formula for the transferred bracket $\{l'_k\}$ can be given

inductively as follows. Set $K\theta_1 = -i$ and define $\theta_n : (L')^{\otimes n} \to L$ for $n \geq 2$ by

$$\theta_n(x_1, \ldots, x_n) = \sum_{k=2}^{n} \sum_{\substack{\sigma \in \overline{S}(i_1, \ldots, i_k) \\ i_1 + \cdots + i_k = n \\ i_1 \leq \cdots \leq i_k}} \epsilon_1 \cdot l_k(I_{i_1} \otimes \cdots \otimes I_{i_k}) \circ \sigma^{\bullet}(x_1, \ldots, x_n)$$

where ϵ_1 is given by the Koszul sign convention. Then for all $n \geq 2$, we define $l'_n = q \circ \theta_n$ and $I_n = K \circ \theta_n$.

We can also use chain contractions to transfer an L_∞-module structure. We will make use of this technique in the proof of the invariance of the $\mathfrak{sl}_2(\wedge)_{dg}$ L_∞-module structure under Reidemeister moves in Chapter III.

Theorem 2. *Let L be an L_∞-algebra, and let M be an L_∞-module over L. Given a chain contraction*

$$T \circlearrowleft M \underset{i}{\overset{q}{\rightleftarrows}} M'$$

then M' inherits the structure of an L_∞-module over L, with transferred bracket given by

$$k'_n := \sum_{\substack{\tau \in S(i_1, \ldots, i_t) \\ i_1 + \cdots + i_t = n-1}} q \circ A_t \circ (\tau^{\bullet} \otimes i)$$

where $A_t : L^{\otimes i_1} \otimes \cdots \otimes L^{\otimes i_t} \otimes M \to M$ is defined inductively as follows. Let $A_1 = k_{i_1+1}$ and define $A_t = A_1 \circ \delta_2^{\bullet} \circ [(T \circ A_{t-1}) \otimes \mathrm{Id}] \circ \delta_1^{\bullet}$, where $i_1, \ldots, i_t$ are positive integers; see Figure 7.
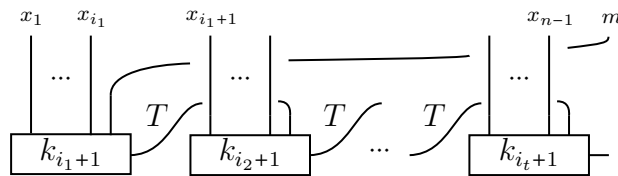
FIGURE 7. A graphical depiction of the map A_t.

Remark. The permutations δ_i in the definition of A_t above are required to ensure that the module element is the last input of each $k_{i_r+1} : L^{\otimes r} \otimes M \rightarrow M$. Explicitly, δ_i is the unique permutation so that $\delta_i^\bullet$ shifts the module element to the required position and preserves the order of the other elements. For example, in Figure 7, δ_1 is the permuation

$$
\delta_1 = \begin{pmatrix} 1 & \cdots & i_1 & i_1 + 1 & i_1 + 2 & \cdots & n \\ 1 & \cdots & i_1 & n & i_1 + 1 & \cdots & n - 1 \end{pmatrix}
$$

Throughout the proof of Theorem 2, we will make use of similar permutations λ_i to correctly place the module element while preserving the order of the remaining elements. We will not write down these permutations explicitly, but they can be readily determined by examining the figures in the appendix.

Remark. We remind the reader that we are ignoring signs in the above theorem and that the result is proved over a field of characteristic two.

Proof. We must show that the above definition for k'_n satisfies the L_∞-module relation:

$$
\sum_{\substack{p+q=n+1 \\ p<n}} \sum_{\sigma(n)=n} k'_q \circ (l_p \otimes \mathrm{Id}) \circ \sigma^\bullet = \sum_{p+q=n+1} \sum_{\sigma(p)=n} k'_q \circ \lambda^\bullet \circ (k'_p \otimes \mathrm{Id}) \circ \sigma^\bullet
$$

The idea of the proof is as follows. Start by replacing the k'_q and k'_p terms using the definition of k'_n. Next, apply the L_∞-module relation for k_n to the terms involving l_p on the left-hand side. After that, use fact that $\mathrm{Id}_M - i \circ q = k_1 T + T k_1$ to replace terms on the right-hand side. Terms will then cancel in pairs. Graphical representations of the formulas in this proof are provided in the appendix.

Step 1. Focusing on the left-hand side of the L_∞-module relation, we can replace k_n' using its definition to obtain the following sum.

$$\sum_{\substack{p+q=n+1 \\ p<n}} \sum_{\sigma(n)=n} \sum_{\substack{\tau \in S(i_1,\dots,i_t) \\ i_1+\dots+i_t=n-p}} q \circ A_t \circ (\tau^\bullet \otimes i) \circ (l_p \otimes \mathrm{Id}) \circ \sigma^\bullet$$

Step 2. We can combine σ and τ into η and ψ. Since τ is an unshuffle, the l_p term will be the first element in some block, which we denote by i_l. Defining $s = p+i_l-1$, we obtain the following sum.

$$\sum_{p=1}^{n-1} \sum_{\substack{\eta \in S(i_1,\dots,p+i_l-1,\dots,i_t) \\ i_1+\dots+i_t=n-p \\ 1\leq t\leq n-p \\ 1\leq l\leq t}} \sum_{\psi \in S(p,s-p)} q \circ A_t \circ \big[\mathrm{Id} \otimes ((l_p \otimes \mathrm{Id}) \circ \psi^\bullet) \otimes \mathrm{Id}\big] \circ (\eta^\bullet \otimes i)$$

Step 3. The goal now is to unpack the A_t terms using the definition of A_t in order to apply the L_∞-module relation. Because A_t only makes sense for $t \geq 1$, we break up the sum into several cases. In the first case, the l_p term is in the first box. In the second case, the l_p term is somewhere in the middle, in which case we need at least three boxes. In the third case, the l_p term is in the last box. Note further that the only way for there to be one box is if $p = n - 1$. We obtain the following sum.

$$\sum_{p=1}^{n-2} \sum_{\substack{\eta \in S(p+i_1-1,i_2,\dots,i_t) \\ i_1+\dots+i_t=n-p \\ 2\leq t\leq n-p \\ l=1}} \sum_{\psi \in S(p,s-p)} q \circ A_{t-1} \circ \lambda_3^\bullet \circ (T \otimes \mathrm{Id}) \circ \left[\big[k_{i_l+1} \circ (l_p \otimes \mathrm{Id}) \circ \psi^\bullet\big] \otimes \mathrm{Id}\right]$$

$$\circ \lambda_1^\bullet \circ (\eta^\bullet \otimes i)$$

44

$$+ \sum_{\substack{p=1}}^{n-3} \sum_{\substack{\eta \in S(i_1,\ldots,p+i_l-1,\ldots,i_t) \\ i_1+\cdots+i_t=n-p \\ 3 \le t \le n-p \\ 2 \le l \le t-1}} \sum_{\psi \in S(p,s-p)} q \circ A_{t-l} \circ \lambda_3^\bullet \circ (T \otimes \mathrm{Id}) \circ \left[\left[k_{i_l+1} \circ (l_p \otimes \mathrm{Id}) \circ \psi^\bullet \right] \otimes \mathrm{Id} \right]$$

$$\circ \, \lambda_2^\bullet \circ \left[(T \circ A_{l-1}) \otimes \mathrm{Id} \right] \circ \lambda_1^\bullet \circ (\eta^\bullet \otimes i)$$

$$+ \sum_{\substack{p=1}}^{n-2} \sum_{\substack{\eta \in S(i_1,\ldots,i_{t-1},p+i_t-1) \\ i_1+\cdots+i_t=n-p \\ 2 \le t \le n-p \\ l=t}} \sum_{\psi \in S(p,s-p)} q \circ \left[\left[k_{i_l+1} \circ (l_p \otimes \mathrm{Id}) \circ \psi^\bullet \right] \otimes \mathrm{Id} \right] \circ \lambda_2^\bullet$$

$$\circ \left[(T \circ A_{t-1}) \otimes \mathrm{Id} \right] \circ \lambda_1^\bullet \circ (\eta^\bullet \otimes i)$$

$$+ \sum_{s=n-1}^{s} \sum_{p=1}^{s} \sum_{\psi \in S(p,s-p)} q \circ \left[\left[k_{i_l+1} \circ (l_p \otimes \mathrm{Id}) \circ \psi^\bullet \right] \otimes \mathrm{Id} \right] \circ (\mathrm{Id} \otimes i)$$

We now reindex over the size of $s = p + i_l - 1$.

$$\sum_{s=1}^{n-2} \sum_{p=1}^{s} \sum_{\substack{\eta \in S(s,i_2,\ldots,i_t) \\ i_1+\cdots+i_t=n-p \\ 2 \le t \le n-p \\ l=1}} \sum_{\psi \in S(p,s-p)} q \circ A_{t-1} \circ \lambda_3^\bullet \circ (T \otimes \mathrm{Id}) \circ \left[\left[k_{i_l+1} \circ (l_p \otimes \mathrm{Id}) \circ \psi^\bullet \right] \otimes \mathrm{Id} \right]$$

$$\circ \, \lambda_1^\bullet \circ (\eta^\bullet \otimes i)$$

$$+ \sum_{s=1}^{n-3} \sum_{p=1}^{s} \sum_{\substack{\eta \in S(i_1,\ldots,s,\ldots,i_t) \\ i_1+\cdots+i_t=n-p \\ 3 \le t \le n-p \\ 2 \le l \le t-1}} \sum_{\psi \in S(p,s-p)} q \circ A_{t-l} \circ \lambda_3^\bullet \circ (T \otimes \mathrm{Id}) \circ \left[\left[k_{i_l+1} \circ (l_p \otimes \mathrm{Id}) \circ \psi^\bullet \right] \otimes \mathrm{Id} \right]$$

$$\circ \, \lambda_2^\bullet \circ \left[(T \circ A_{l-1}) \otimes \mathrm{Id} \right] \circ \lambda_1^\bullet \circ (\eta^\bullet \otimes i)$$

$$+ \sum_{s=1}^{n-2} \sum_{p=1}^{s} \sum_{\substack{\eta \in S(i_1,\ldots,i_{t-1},s) \\ i_1+\cdots+i_t=n-p \\ 2 \le t \le n-p \\ l=t}} \sum_{\psi \in S(p,s-p)} q \circ \left[\left[k_{i_l+1} \circ (l_p \otimes \mathrm{Id}) \circ \psi^\bullet \right] \otimes \mathrm{Id} \right]$$

$$\circ \, \lambda_2^\bullet \circ \left[(T \circ A_{t-1}) \otimes \mathrm{Id} \right] \circ \lambda_1^\bullet \circ (\eta^\bullet \otimes i)$$

$$+ \sum_{s=n-1}^{s} \sum_{p=1}^{s} \sum_{\psi \in S(p,s-p)} q \circ \left[\left[k_{i_l+1} \circ (l_p \otimes \mathrm{Id}) \circ \psi^\bullet \right] \otimes \mathrm{Id} \right] \circ (\mathrm{Id} \otimes i)$$

We can now apply the L_∞-module relation.

$$\sum_{s=1}^{n-2}\sum_{p=1}^{s+1}\sum_{\substack{\eta\in S(s,i_2,\ldots,i_t)\\ i_1+\cdots+i_t=n-p\\ 2\leq t\leq n-p\\ l=1}}\sum_{\psi\in S(p-1,s-p+1)} q\circ A_{t-l}\circ\lambda_3^\bullet\circ(T\otimes\mathrm{Id})\circ\left[\left[k_{s-p+2}\circ(k_p\otimes\mathrm{Id})\circ\psi^\bullet\right]\otimes\mathrm{Id}\right]$$

$$\circ\,\lambda_1^\bullet\circ(\eta^\bullet\otimes i)$$

$$+\sum_{s=1}^{n-3}\sum_{p=1}^{s+1}\sum_{\substack{\eta\in S(i_1,\ldots,s,\ldots,i_t)\\ i_1+\cdots+i_t=n-p\\ 3\leq t\leq n-p\\ 2\leq l\leq t-1}}\sum_{\psi\in S(p-1,s-p+1)} q\circ A_{t-l}\circ\lambda_3^\bullet\circ(T\otimes\mathrm{Id})\circ\left[\left[k_{s-p+2}\circ(k_p\otimes\mathrm{Id})\circ\psi^\bullet\right]\otimes\mathrm{Id}\right]$$

$$\circ\,\lambda_2^\bullet\circ\left[(T\circ A_{l-1})\otimes\mathrm{Id}\right]\circ\lambda_1^\bullet\circ(\eta^\bullet\otimes i)$$

$$+\sum_{s=1}^{n-2}\sum_{p=1}^{s+1}\sum_{\substack{\eta\in S(i_1,\ldots,i_{t-1},s)\\ i_1+\cdots+i_t=n-p\\ 2\leq t\leq n-p\\ l=t}}\sum_{\psi\in S(p-1,s-p+1)} q\circ\lambda_3^\bullet\circ(T\otimes\mathrm{Id})\circ\left[\left[k_{s-p+2}\circ(k_p\otimes\mathrm{Id})\circ\psi^\bullet\right]\otimes\mathrm{Id}\right]$$

$$\circ\,\lambda_2^\bullet\circ\left[(T\circ A_{t-1})\otimes\mathrm{Id}\right]\circ\lambda_1^\bullet\circ(\eta^\bullet\otimes i)$$

$$+\sum_{s=n-1}\sum_{p=1}^{s+1}\sum_{\psi\in S(p-1,s-p+1)} q\circ\left[\left[k_{s-p+2}\circ(k_p\otimes\mathrm{Id})\circ\psi^\bullet\right]\otimes\mathrm{Id}\right]\circ(\mathrm{Id}\otimes i)$$

Step 4. Combine ψ and η into κ, and reintroduce A_t into the notation, treating the cases $p=1$ and $p=s+1$ separately. Indeed, we observe that for $1<p<s+1$, we may combine both the k_p and k_{s-p+2} terms into an A_t term. Otherwise, we will have a k_1 term.

$$\sum_{s=1}^{n-2}\sum_{p=1}\sum_{\substack{\kappa\in S(s,i_2,\ldots,i_t)\\ i_1+\cdots+i_t=n-p\\ 2\leq t\leq n-p\\ l=1}} q\circ A_t\circ\lambda_3^\bullet\circ(k_1\otimes\mathrm{Id})\circ\lambda_1^\bullet\circ(\kappa^\bullet\otimes i)$$

$$+ \sum_{s=1}^{n-2} \sum_{p=2}^{s} \sum_{\substack{\kappa \in S(p-1,s-p+1,i_2,\dots,i_t) \\ i_1+\cdots+i_t=n-p \\ 2\le t\le n-p \\ l=1}} q \circ A_t \circ \lambda_3^\bullet \circ (T \otimes \mathrm{Id}) \circ (A_1 \otimes \mathrm{Id}) \circ \lambda_1^\bullet \circ (\kappa^\bullet \otimes i)$$

$$+ \sum_{s=1}^{n-2} \sum_{p=s+1} \sum_{\substack{\kappa \in S(s,i_2,\dots,i_t) \\ i_1+\cdots+i_t=n-p \\ 2\le t\le n-p \\ l=1}} q \circ A_{t-1} \circ \lambda_3^\bullet \circ ((T \circ k_1) \otimes \mathrm{Id}) \circ (A_1 \otimes \mathrm{Id}) \circ \lambda_1^\bullet \circ (\kappa^\bullet \otimes i)$$

$$+ \sum_{s=1}^{n-3} \sum_{p=1} \sum_{\substack{\kappa \in S(i_1,\dots,i_{l-1},s,i_{l+1},\dots,i_t) \\ i_1+\cdots+i_t=n-p \\ 3\le t\le n-p \\ 2\le l\le t}} q \circ A_{t-l+1} \circ \lambda_3^\bullet \circ ((k_1 \circ T) \otimes \mathrm{Id}) \circ (A_{l-1} \otimes \mathrm{Id}) \circ \lambda_1^\bullet \circ (\kappa^\bullet \otimes i)$$

$$+ \sum_{s=1}^{n-3} \sum_{p=2}^{s} \sum_{\substack{\kappa \in S(i_1,\dots,i_{l-1},p-1,s-p+1,i_{l+1},\dots,i_t) \\ i_1+\cdots+i_t=n-p \\ 3\le t\le n-p \\ 2\le l\le t}} q \circ A_{t-l+1} \circ \lambda_3^\bullet \circ A_l \otimes \mathrm{Id}) \circ \lambda_1^\bullet \circ (\kappa^\bullet \otimes i)$$

$$+ \sum_{s=1}^{n-3} \sum_{p=s+1} \sum_{\substack{\kappa \in S(i_1,\dots,i_{l-1},s,i_{l+1},\dots,i_t) \\ i_1+\cdots+i_t=n-p \\ 3\le t\le n-p \\ 2\le l\le t}} q \circ A_{t-l} \circ \lambda_3^\bullet \circ ((T \circ k_1) \otimes \mathrm{Id}) \circ (A_l \otimes \mathrm{Id}) \circ \lambda_1^\bullet \circ (\kappa^\bullet \otimes i)$$

$$+ \sum_{s=1}^{n-2} \sum_{p=1} \sum_{\substack{\kappa \in S(i_1,\dots,i_{t-1},s) \\ i_1+\cdots+i_t=n-p \\ 2\le t\le n-p \\ l=t}} q \circ A_1 \circ \lambda_3^\bullet \circ ((k_1 \circ T) \otimes \mathrm{Id}) \circ (A_{t-1} \otimes \mathrm{Id}) \circ \lambda_1^\bullet \circ (\kappa^\bullet \otimes i)$$

$$+ \sum_{s=1}^{n-2} \sum_{p=2}^{s} \sum_{\substack{\kappa \in S(i_1,\dots,i_{t-1},p-1,s-p+1) \\ i_1+\cdots+i_t=n-p \\ 2\le t\le n-p \\ l=t}} q \circ A_1 \circ \lambda_3^\bullet \circ (A_t \otimes \mathrm{Id}) \circ \lambda_1^\bullet \circ (\kappa^\bullet \otimes i)$$

$$+ \sum_{s=1}^{n-2} \sum_{p=s+1} \sum_{\substack{\kappa \in S(i_1,\dots,i_{t-1},s) \\ i_1+\cdots+i_t=n-p \\ 2\le t\le n-p \\ l=t}} q \circ k_1 \circ \lambda_3^\bullet \circ (A_t \otimes \mathrm{Id}) \circ \lambda_1^\bullet \circ (\kappa^\bullet \otimes i)$$

$$+ \sum_{s=n-1} \sum_{p=1} \sum_{\kappa=\mathrm{Id}} q \circ k_{s+1} \circ (k_1 \otimes \mathrm{Id}) \circ \lambda_1^\bullet \circ (\mathrm{Id} \otimes i)$$

$$+ \sum_{s=n-1} \sum_{p=2}^{s} \sum_{\kappa \in S(p-1,s-p+1)} q \circ k_{s-p+2} \circ (k_p \otimes \mathrm{Id}) \circ \lambda_1^\bullet \circ (\kappa^\bullet \otimes i)$$

47

$$+ \sum_{s=n-1} \sum_{p=s+1} \sum_{\kappa=\mathrm{Id}} q \circ k_1 \circ (k_{s+1} \otimes \mathrm{Id}) \circ \lambda_1^\bullet \circ (\kappa^\bullet \otimes i)$$

Step 5. We can combine the sums above. The first term is obtained by combining terms 4 and 7 above. The second term is obtained by combining terms 3 and 6 above. The third term is obtained by combining terms 2, 5, 8, and 11 above. The fourth term is obtained by combining terms 1 and 10 above. The last term is obtained by combining terms 9 and 12 above.

$$\sum_{\substack{s=1 \\ }}^{n-2} \sum_{\substack{\kappa \in S(i_1,\dots,i_{l-1},s,i_{l+1}\dots,i_t) \\ 2 \le t \le n-1 \\ 2 \le l \le t \\ i_1 + \dots + i_t = n-1 \\ i_l = s}} q \circ A_{t-l+1} \circ \lambda_2^\bullet \circ \big[(k_1 \circ T \circ A_{l-1}) \otimes \mathrm{Id} \big] \circ \lambda_1^\bullet \circ (\kappa^\bullet \otimes i)$$

$$+ \sum_{\substack{s=1 \\ }}^{n-2} \sum_{\substack{\kappa \in S(i_1,\dots,i_{l-1},s,i_{l+1}\dots,i_t) \\ 2 \le t \le n-1-s \\ 1 \le l \le t-1 \\ i_1 + \dots + i_t = n-1-s \\ i_l = 0}} q \circ A_{t-l} \circ \lambda_2^\bullet \circ \big[(T \circ k_1 \circ A_l) \otimes \mathrm{Id} \big] \circ \lambda_1^\bullet \circ (\kappa^\bullet \otimes i)$$

$$+ \sum_{s=1}^{n-1} \sum_{p=2}^{s} \sum_{\substack{\kappa \in S(i_1,\dots,i_{l-1},p-1,s-p+1,i_{l+1},\dots,i_t) \\ 1 \le t \le n-p \\ 1 \le l \le t \\ i_1 + \dots + i_t = n-p \\ i_l = s+1-p}} q \circ A_{t-l+1} \circ \lambda_2^\bullet \circ \big[(\mathrm{Id} \otimes A_l) \otimes \mathrm{Id} \big] \circ \lambda_1^\bullet \circ (\kappa^\bullet \otimes i)$$

$$+ \sum_{\substack{s=1 \\ }}^{n-1} \sum_{\substack{\kappa \in S(s,i_2,\dots,i_t) \\ 1 \le t \le n-1 \\ l=1 \\ i_1 + \dots + i_t = n-1 \\ i_l = s}} q \circ A_t \circ \lambda_3^\bullet \circ (k_1 \otimes \mathrm{Id}) \circ \lambda_1^\bullet \circ (\kappa^\bullet \otimes i)$$

$$+ \sum_{\substack{s=1 \\ }}^{n-1} \sum_{\substack{\kappa \in S(i_1,\dots,i_{t-1},s) \\ 1 \le t \le n-1-s \\ l=t \\ i_1 + \dots + i_t = n-1-s \\ i_l = 0}} q \circ k_1 \circ \lambda_3^\bullet \circ (A_t \otimes \mathrm{Id}) \circ \lambda_1^\bullet \circ (\kappa^\bullet \otimes i)$$

Step 6. Change notation. In the first sum, let $c_1, \ldots, c_w$ be $i_1, \ldots, i_{l-1}$ and $d_1, \ldots, d_x$ be $s, i_{l+1}, \ldots, i_t$. The conditions $t \geq 2$ and $2 \leq l \leq t$ imply that $w, x \geq 1$. Make similar changes to the other sums. In the second sum, let $c_1, \ldots, c_w$ be $i_1, \ldots, i_{l-1}, s$ and $d_1, \ldots, d_x$ be $i_{l+1}, \ldots, i_t$, and in the third sum let $c_1, \ldots, c_w$ be $i_1, \ldots, i_{l-1}, p-1$ and $d_1, \ldots, d_x$ be $s - p + 1, i_{l+1}, \ldots, i_t$.

$$\sum_{\substack{\kappa \in S(c_1, \ldots, c_w, d_1 \ldots, d_x) \\ c_1 + \cdots + c_w + d_1 + \cdots + d_x = n-1 \\ w, x \geq 1}} q \circ A_x \circ \lambda_2^\bullet \circ \left[(k_1 \circ T \circ A_w) \otimes \mathrm{Id} \right] \circ \lambda_1^\bullet \circ (\kappa^\bullet \otimes i)$$

$$+ \sum_{\substack{\kappa \in S(c_1, \ldots, c_w, d_1 \ldots, d_x) \\ c_1 + \cdots + c_w + d_1 + \cdots + d_x = n-1 \\ w, x \geq 1}} q \circ A_x \circ \lambda_2^\bullet \circ \left[(T \circ k_1 \circ A_w) \otimes \mathrm{Id} \right] \circ \lambda_1^\bullet \circ (\kappa^\bullet \otimes i)$$

$$+ \sum_{\substack{\kappa \in S(c_1, \ldots, c_w, d_1 \ldots, d_x) \\ c_1 + \cdots + c_w + d_1 + \cdots + d_x = n-1 \\ w, x \geq 1}} q \circ A_x \circ \lambda_2^\bullet \circ \left[(\mathrm{Id} \otimes A_w) \otimes \mathrm{Id} \right] \circ \lambda_1^\bullet \circ (\kappa^\bullet \otimes i)$$

$$+ \sum_{\substack{\kappa \in S(d_1 \ldots, d_x) \\ d_1 + \cdots + d_x = n-1 \\ x \geq 1}} q \circ A_x \circ \lambda_3^\bullet \circ (k_1 \otimes \mathrm{Id}) \circ \lambda_1^\bullet \circ (\kappa^\bullet \otimes i)$$

$$+ \sum_{\substack{\kappa \in S(c_1, \ldots, c_w) \\ c_1 + \cdots + c_w = n-1 \\ w \geq 1}} q \circ k_1 \circ \lambda_3^\bullet \circ (A_w \otimes \mathrm{Id}) \circ \lambda_1^\bullet \circ (\kappa^\bullet \otimes i)$$

Step 7. Focusing now on the right-hand side, we substitute for k_n' using its definition. We consider the cases $p = 1$ and $q = 1$ separately, and use the fact that $k_1' \circ q = q \circ k_1$ and $k_1 \circ i = i \circ k_1'$, since i and q are chain maps.

$$\sum_{p=2}^{n-1} \sum_{\sigma(p)=n} \sum_{\substack{\alpha \in S(a_1, \ldots, a_r) \\ a_1 + \cdots + a_r = p-1}} \sum_{\substack{\beta \in S(b_1, \ldots, b_s) \\ b_1 + \cdots + b_s = q-1}} q \circ A_s \circ (\beta^\bullet \otimes i) \circ \lambda^\bullet \circ \left[(q \circ A_r \circ (\alpha^\bullet \otimes i)) \otimes \mathrm{Id} \right] \circ \sigma^\bullet$$

$$+ \sum_{p=1} \sum_{\substack{\sigma(p)=n}} \sum_{\substack{\beta \in S(b_1,\dots,b_s) \\ b_1+\cdots+b_s=n-1}} q \circ A_s \circ (\beta^\bullet \otimes i) \circ \lambda^\bullet \circ [(k_1 \circ i) \otimes \mathrm{Id}] \circ \sigma^\bullet$$

$$+ \sum_{p=n} \sum_{\substack{\sigma(p)=n}} \sum_{\substack{\alpha \in S(a_1,\dots,a_r) \\ a_1+\cdots+a_r=n-1}} q \circ k_1 \circ (\beta^\bullet \otimes i) \circ \lambda^\bullet \circ [q \circ A_r \circ (\alpha^\bullet \otimes i) \otimes \mathrm{Id}] \circ \sigma^\bullet$$

Step 8. We can combine σ, α, and β into one unshuffle θ.

$$\sum_{p=2}^{n-1} \sum_{\substack{\theta \in S(a_1,\dots,a_r,b_1,\dots,b_s) \\ a_1+\cdots+a_r=p-1 \\ b_1+\cdots+b_s=q-1}} q \circ A_s \circ \lambda_2^\bullet \circ [(i \circ q \circ A_r) \otimes \mathrm{Id}] \circ \lambda_1^\bullet \circ (\theta^\bullet \otimes i)$$

$$+ \sum_{p=1} \sum_{\substack{\theta \in S(b_1,\dots,b_s) \\ b_1+\cdots+b_s=n-1}} q \circ A_s \circ \lambda_2^\bullet \circ (k_1 \otimes \mathrm{Id}) \circ \lambda_1^\bullet \circ (\theta^\bullet \otimes i)$$

$$+ \sum_{p=n} \sum_{\substack{\theta \in S(a_1,\dots,a_r) \\ a_1+\cdots+a_r=n-1}} q \circ k_1 \circ \lambda_2^\bullet \circ (A_r \otimes \mathrm{Id}) \circ \lambda_1^\bullet \circ (\theta^\bullet \otimes i)$$

Step 9. Use the fact that $\mathrm{Id}_M - i \circ q = k_1 \circ T + T \circ k_1$.

$$\sum_{p=2}^{n-1} \sum_{\substack{\theta \in S(a_1,\dots,a_r,b_1,\dots,b_s) \\ a_1+\cdots+a_r=p-1 \\ b_1+\cdots+b_s=q-1}} q \circ A_s \circ \lambda_2^\bullet \circ [(k_1 \circ T \circ A_r) \otimes \mathrm{Id}] \circ \lambda_1^\bullet \circ (\theta^\bullet \otimes i)$$

$$+ \sum_{p=2}^{n-1} \sum_{\substack{\theta \in S(a_1,\dots,a_r,b_1,\dots,b_s) \\ a_1+\cdots+a_r=p-1 \\ b_1+\cdots+b_s=q-1}} q \circ A_s \circ \lambda_2^\bullet \circ [(T \circ k_1 \circ A_r) \otimes \mathrm{Id}] \circ \lambda_1^\bullet \circ (\theta^\bullet \otimes i)$$

$$+ \sum_{p=2}^{n-1} \sum_{\substack{\theta \in S(a_1,\dots,a_r,b_1,\dots,b_s) \\ a_1+\cdots+a_r=p-1 \\ b_1+\cdots+b_s=q-1}} q \circ A_s \circ \lambda_2^\bullet \circ [(\mathrm{Id} \circ A_r) \otimes \mathrm{Id}] \circ \lambda_1^\bullet \circ (\theta^\bullet \otimes i)$$

$$+ \sum_{p=1} \sum_{\substack{\theta \in S(b_1,\dots,b_s) \\ b_1+\cdots+b_s=n-1}} q \circ A_s \circ \lambda_2^\bullet \circ (k_1 \otimes \mathrm{Id}) \circ \lambda_1^\bullet \circ (\theta^\bullet \otimes i)$$

50

$$+ \sum_{p=n} \sum_{\substack{\theta \in S(a_1,\ldots,a_r) \\ a_1 + \cdots + a_r = n-1}} q \circ k_1 \circ \lambda_2^\bullet \circ (A_r \otimes \mathrm{Id}) \circ \lambda_1^\bullet \circ (\theta^\bullet \otimes i)$$

These terms are precisely the terms in Step 6, and so the terms cancel in pairs. Hence L_∞-module relation holds for k_n'. □

Lemma 5. *In the setting of Theorem 2, if M' has the L_∞-module structure induced by a chain contraction, then the map $i : M' \to M$ can be extended to an L_∞-module homomorphism, where $I_1 = i$, and for $n \geq 2$, we define I_n by*

$$I_n := \sum_{\substack{\tau \in S(i_1,\ldots,i_t) \\ i_1 + \cdots + i_t = n-1}} T \circ A_t \circ (\tau^\bullet \otimes i)$$

The map $A_t : L^{\otimes i_1} \otimes \cdots \otimes L^{\otimes i_t} \otimes M \to M$ is defined as in the statement of Theorem 2; see Figure 7.

Proof. (Sketch). We will prove that I satisfies the L_∞-module homomorphism relation for $n = 2$. Indeed, for $x \in L$ and $m \in M'$ we must show that

$$I_2(l_1(x), m) + I_2(x, k_1'(m)) + I_1(k_2'(x, m)) = k_2(x_1, I_1(m)) + k_1(I_2(x, m)) \quad (2.4.1)$$

Working on the left-hand side and substituting in the definitions of I and k', we get

$$T \circ k_2(l_1(x), i(m)) + T \circ k_2(x_1, i \circ k_1'(m)) + i \circ q \circ k_2(x, i(m)) \quad (2.4.2)$$

Next, we use the fact that i is a chain map and that $i \circ q = T \circ k_1 + k_1 \circ T + \mathrm{Id}_M$, to

see that (2) is equal to

$$T \circ k_2(l_1(x), i(m)) + T \circ k_2(x_1, k_1(i(m))) + T \circ k_1 \circ k_2(x, i(m)) + k_1 \circ T \circ k_2(x, i(m)) + k_2(x, i(m))$$

$$(2.4.3)$$

Applying the L_∞-module relation to the first two terms in (3), we obtain

$$T \circ k_1 \circ k_2(x, i(m)) + T \circ k_1 \circ k_2(x, i(m)) + k_1 \circ T \circ k_2(x, i(m)) + k_2(x, i(m)) \quad (2.4.4)$$

Now, the first two terms cancel, and what remains is $k_2(x_1, I_1(m)) + k_1(I_2(x, m))$,

as desired. The proof of the general case is similar to the proof of Theorem 2 $\qquad \square$

CHAPTER III

ANNULAR KHOVANOV HOMOLOGY

3.1. Introduction

In [Kho00], Khovanov defined a bigraded homology group for oriented links in S^3 which is a categorification of the Jones polynomial. Following this, for a compact, oriented surface Σ, Asaeda, Przytycki, and Sikora introduced a generalization of Khovanov homology for links in $\Sigma \times [0, 1]$ that categorifies the Kauffman skein module of Σ; see [APS04]. The case where Σ is an annulus is known as annular Khovanov homology and has since garnered much attention. For example, there have been various detection results that have been obtained by exploiting the relationship of annular Khovanov homology with various Floer theories. In [XZ19], Xie-Zhang use instanton Floer homology to show that annular Khovanov homology detects both the unlink and the closure of the trivial braid. They also show that it distinguishes braid closures from other links. More recently, Binns-Martin showed that knot Floer homology detects various torus links, and they used this to show that annular Khovanov homology detects certain braid closures; see [BM20].

A key feature of annular Khovanov homology is that it is endowed with extra structure not present in ordinary Khovanov homology. In [GLW18], Grigsby-Licata-Wehrli show that the annular Khovanov homology of a link is both an $\mathfrak{sl}_2$-representation and an $\mathfrak{sl}_2(\wedge)$-representation, where $\mathfrak{sl}_2(\wedge)$ is a $\mathbb{Z}$-graded Lie superalgebra related to $\mathfrak{sl}_2$. This structure has been studied in several contexts. In one direction, Quefflec-Rose generalized this to show that annular Khovanov-

Rozansky homology carries an $\mathfrak{sl}_n$-action; see [QR18]. In another direction, Akhmechet-Krushkal-Willis have made progress towards lifting the $\mathfrak{sl}_2$-action to the stable homotopy refinement of the annular Khovanov homology; see [AKW22].

In proving that there is an $\mathfrak{sl}_2$-representation structure on the annular Khovanov complex $\mathrm{CKh}(L)$, Grigsby-Licata-Wehrli showed that the boundary maps of $\mathrm{CKh}(L)$ commute with the $\mathfrak{sl}_2$-action, which shows that the $\mathfrak{sl}_2$-action holds at the chain level. In contrast, the $\mathfrak{sl}_2(\wedge)$-action is well-defined on the annular Khovanov homology $\mathrm{AKh}(L)$, but at the chain level, it only holds up to homotopy. This observation suggests the existence of an L_∞-module structure on $\mathrm{AKh}(L)$. In this chapter, we exhibit $\mathfrak{sl}_2(\wedge)$ as an L_∞-algebra and upgrade the $\mathfrak{sl}_2(\wedge)$-representation structure to that of an L_∞-module. This module structure is an invariant of the annular link at both the chain level and on homology. In particular, we will prove the following theorem.

Theorem. *Let $L \subset A \times I$ be an annular link. There is an L_∞-module structure on both $\mathrm{CKh}(L; \mathbb{Z}/2\mathbb{Z})$ and $\mathrm{AKh}(L; \mathbb{Z}/2\mathbb{Z})$ over the L_∞-algebra $\mathfrak{sl}_2(\wedge)$. Up to L_∞-quasi-isomorphism, this module structure only depends on the isotopy class of L in $A \times I$.*

The organization of this chapter is as follows. In section 2, we recall the definitions of $\mathfrak{sl}_2$, $\mathfrak{sl}_2(\wedge)$, and $\mathfrak{sl}_2(\wedge)_{dg}$ and review some key results obtained by Grigsby-Licata-Wehrli. In section 3, we provide a more detailed background of annular Khovanov homology. In section 4, we explain how $\mathfrak{sl}_2(\wedge)_{dg}$ and $\mathfrak{sl}_2(\wedge)$ are L_∞-algebras. In sections 5 and 6, we explain how $\mathrm{CKh}(L)$ and $\mathrm{AKh}(L)$ are L_∞-modules. In section 7, we prove the invariance of these structures under Reidemeister moves. In section 8, we provide some examples showing this structure is nontrivial.

Remark. The proof of the above theorem relies on several results about L_∞-modules. In particular, the proofs of Theorem 1, Theorem 2, and Lemma 5 are given over $\mathbb{Z}/2\mathbb{Z}$. We expect these results to hold with signs, but tracking them through their respective proofs is intricate. Outside of these three proofs, we will include signs when appropriate. Working without signs affects the bracket relations in $\mathfrak{sl}_2$, $\mathfrak{sl}_2(\wedge)$, and $\mathfrak{sl}_2(\wedge)_{dg}$; see section 3.2. The absence of signs also affects the higher operations involved in the $\mathfrak{sl}_2(\wedge)$ L_∞-module structure on $\mathrm{CKh}(L)$; see Theorem 4.

3.2. The Lie algebras $\mathfrak{sl}_2, \mathfrak{sl}_2(\wedge)$, and $\mathfrak{sl}_2(\wedge)_{dg}$

In this section, we review the Lie algebras of interest. We first recall the definition of $\mathfrak{sl}_2$. Next, we define the Lie superalgebra $\mathfrak{sl}_2(\wedge)$, which will be our main L_∞-algebra of study. Finally, we define an auxiliary Lie superalgebra, $\mathfrak{sl}_2(\wedge)_{dg}$, which is closely related to $\mathfrak{sl}_2(\wedge)$ and will help us prove several key results.

3.2.1. The Lie algebra $\mathfrak{sl}_2$

To fix notation, we will denote the standard basis for the Lie algebra $\mathfrak{sl}_2$ by $\{e, f, h\}$. Over $\mathbb{Z}$, the Lie bracket relations are given by:

$$[e, f] = h, \ [e, h] = -2e, \ [f, h] = 2f.$$

3.2.2. The Lie superalgebra $\mathfrak{sl}_2(\wedge)$

In [GLW18], Grigsby-Licata-Wehrli introduce a larger Lie algebra $\mathfrak{sl}_2(\wedge)$ containing $\mathfrak{sl}_2$ as a subalgebra. In fact, $\mathfrak{sl}_2(\wedge)$ has the structure of a $\mathbb{Z}$-graded Lie superalgebra.

Definition 13. A **Lie superalgebra** $\mathfrak{g}$ is a $\mathbb{Z}/2\mathbb{Z}$-graded vector space $\mathfrak{g}_{\text{even}} \oplus \mathfrak{g}_{\text{odd}}$ equipped with a bilinear map $[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathfrak{g}$, called the super Lie bracket, satsfying the following conditions:

- $[x,y] = -(-1)^{|x||y|}[y,x]$

- $(-1)^{|x||y|}[x,[y,z]] + (-1)^{|y||x|}[y,[z,x]] + (-1)^{|z||y|}[z,[x,y]] = 0$

The first condition is known as super skew-symmetry, and the second condition is known as the super Jacobi identity. Here, x, y and z are homogeneous elements with respect to the $\mathbb{Z}/2\mathbb{Z}$-grading. The notation $|x|$ represents the degree of x, and the degree of $[x,y]$ is required to be the sum of the degrees of x and y, modulo 2. These conditions should be thought of as analogs of the usual Lie algebra axioms, but with gradings taken into consideration.

We now describe the exterior current algebra $\mathfrak{sl}_2(\wedge)$ by generators and relations, as presented in [GLW18]. As vector spaces,

$$\mathfrak{sl}_2(\wedge) \cong \mathfrak{sl}_2 \oplus \mathfrak{sl}_2,$$

where the first summand is in degree 0 and the second in degree 1 with respect to both the $\mathbb{Z}$-grading and the $\mathbb{Z}/2\mathbb{Z}$-grading. The $\mathbb{Z}/2\mathbb{Z}$-grading required for the Lie superalgebra structure is the mod 2 reduction of the $\mathbb{Z}$-grading. Denoting

the standard basis of the first $\mathfrak{sl}_2$ summand by $\{e, f, h\}$ and that of the degree 1 summand by $\{v_2, v_{-2}, v_0\}$, the bracket relations for the Lie superalgebra $\mathfrak{sl}_2(\wedge)$ are

$$[e, f] = h \qquad\qquad [h, e] = 2e \qquad\qquad [f, v_0] = 2v_{-2}$$
$$[e, v_2] = 0 \qquad\qquad [h, f] = -2f \qquad\qquad [f, v_{-2}] = 0$$
$$[e, v_0] = -2v_2 \qquad\qquad [h, v_0] = 0 \qquad\qquad [h, v_2] = 2v_2$$
$$[e, v_{-2}] = v_0 = -[f, v_2] \qquad [h, v_{-2}] = -2v_{-2} \qquad [v_i, v_j] = 0 \text{ for } i, j \in \{2, 0, -2\}.$$

3.2.3. The Lie superalgebra $\mathfrak{sl}_2(\wedge)_{dg}$

Following [GLW18], we describe the $\mathbb{Z}$-graded Lie superalgebra $\mathfrak{sl}_2(\wedge)_{dg}$. As a $\mathbb{Z}$-graded super vector space, the degree 0 generators are $\{e, f, h\}$, and the degree 1 generators are $\{v_2, v_{-2}, d, D\}$. The defining bracket relations are

$$[e, f] = h \qquad [e, v_{-2}] = -[f, v_2]; \qquad [d, y] = 0 \text{ for all } y \in \{e, f, h, v_2, v_{-2}\};$$
$$[h, e] = 2e; \qquad [f, v_{-2}] = 0; \qquad [D, y] = 0 \text{ for all } y \in \{e, f, h, v_2, v_{-2}\};$$
$$[h, f] = -2f; \qquad [h, v_2] = 2v_2; \qquad [d, d] = [D, D] = [v_2, v_2] = [v_{-2}, v_{-2}] = 0.$$
$$[e, v_2] = 0; \qquad [h, v_{-2}] = -2v_{-2}; \qquad [v_2, v_{-2}] + [d, D] = 0.$$

The structure of $\mathfrak{sl}_2(\wedge)_{dg}$ becomes more clear with the following two lemmas. The first gives us a basis for $\mathfrak{sl}_2(\wedge)_{dg}$, and the second exhibits $\mathfrak{sl}_2(\wedge)$ as a direct summand of the homology of $\mathfrak{sl}_2(\wedge)_{dg}$ by regarding $\mathfrak{sl}_2(\wedge)_{dg}$ as a chain complex with differential given by the adjoint action of d. Both lemmas are proved in [GLW18].

Lemma 6 ([GLW18]; Lemma 6). *Let* $\tilde{v}_0 = [e, v_{-2}] = -[f, v_2]$, *and let* $x = [v_2, v_{-2}] = -[d, D] = \frac{1}{2}[\tilde{v}_0, \tilde{v}_0]$. *Then the set* $\{e, f, h, v_2, v_{-2}, \tilde{v}_0, d, D, x\}$ *forms a basis of* $\mathfrak{sl}_2(\wedge)_{dg}$.

Lemma 7 ([GLW18]; Lemma 7). *The homology of the chain complex* $(\mathfrak{sl}_2(\wedge)_{dg}, [d, \cdot])$ *is isomorphic to the direct sum of* $\mathfrak{sl}_2(\wedge)$ *and the trivial Lie superalgebra. That is,* $H(\mathfrak{sl}_2(\wedge)_{dg}, [d, \cdot]) \cong \mathfrak{sl}_2(\wedge) \oplus \mathbb{Z}$.

3.3. Annular Khovanov homology

In this section, we review the construction of annular Khovanov homology and recall some of its structure. For other expositions; see [Rob13], [SZ18], and [GLW18]. To start, let $L \subset A \times I$ be a link in the thickened annulus. The link L admits a diagram $P(L) \subset A$ by considering the projection $A \times I \to A \times \{0\}$, and this diagram can be regarded as sitting inside of $S^2 - \{\mathbb{X}, \mathbb{O}\}$, where $\mathbb{X}$ is a basepoint representing the inner boundary of A, and $\mathbb{O}$ is a basepoint representing the outer boundary of A; see Figure 8.
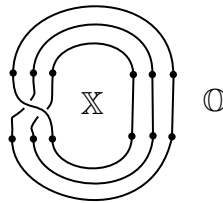
FIGURE 8. A diagram $P(L) \subset S^2 - \{\mathbb{X}, \mathbb{O}\}$ of an annular link L, where $\mathbb{X}$ and $\mathbb{O}$ represent the inner and outer boundaries of the annulus, respectively.

If we ignore the basepoint $\mathbb{X}$, we can form the ordinary Khovanov complex $\mathrm{CKh}(P(L))$. $\mathrm{CKh}(P(L))$ is generated by oriented Kauffman states, where circles are labeled either v_+ or v_-. $\mathrm{CKh}(P(L))$ is also bigraded, where an element of

$\mathrm{CKh}^{i,j}(P(L))$ is said to have homological grading i and quantum grading j. Formulas for these gradings are given in [Zha18] and [Rob13].

The addition of the basepoint $\mathbb{X}$ endows $\mathrm{CKh}(P(L))$ with a third grading k, called the k-grading or the winding-number grading. For a fixed generator, the associated Kauffman state is a collection of oriented circles, and the k-grading is defined to be the algebraic intersection number of this collection of circles with an oriented arc from $\mathbb{X}$ to $\mathbb{O}$ that misses all crossings of $P(L)$. Another way to compute the k-grading is to count the number of positively-labeled nontrivial circles and subtract the number of negatively-labeled nontrivial circles, where a nontrivial circle is a circle that separates $\mathbb{X}$ and $\mathbb{O}$. In [Rob13], it is proved that the Khovanov differential ∂ does not increase the k-grading, and so this gives rise to a filtration on $\mathrm{CKh}(P(L))$. The annular Khovanov homology $\mathrm{AKh}(P(L))$ is the homology of the associated graded object. Said differently, we can decompose the Khovanov differential as $\partial = \partial_0 + \partial_-$, where ∂_0 and ∂_- are the k-preserving and k-decreasing parts of ∂, respectively. $\mathrm{AKh}(P(L))$ is the homology of the triply-graded chain complex $(\mathrm{CKh}(P(L)), \partial_0)$. Moreover, up to isomorphism, the annular Khovanov homology does not depend on the diagram $P(L)$ representing L, so it makes sense to write $\mathrm{AKh}(L)$.

It is instructive to see how the differential ∂_0 of the annular Khovanov complex differs from the usual Khovanov differential ∂. To do so, we need to examine how the k-gradings of generators change under merge and split maps. Denoting trivial circles by T's and nontrivial circles by N's, the three possibilities are TT $\leftrightarrow$ T, NT $\leftrightarrow$ N, and NN $\leftrightarrow$ T; see Figure 9.

The formula for the differential ∂_0 depends on the types of circles involved, and we list the explicit formulas for each case below. Recall that trivial circles are labeled by either w_+ or w_- and nontrivial circles are labeled by either v_+ or v_-.

1. When two trivial circles merge into trivial circle, or when a trivial circle splits into two trivial circles:

<div align="center">

Merge   Split

</div>

$$\mathbf{w_+} \otimes \mathbf{w_+} \mapsto \mathbf{w_+} \qquad\qquad \mathbf{w_+} \mapsto \mathbf{w_+} \otimes \mathbf{w_-} + \mathbf{w_-} \otimes \mathbf{w_+}$$

$$\mathbf{w_+} \otimes \mathbf{w_-} \mapsto \mathbf{w_-} \qquad\qquad \mathbf{w_-} \mapsto \mathbf{w_-} \otimes \mathbf{w_-}$$

$$\mathbf{w_-} \otimes \mathbf{w_+} \mapsto \mathbf{w_-}$$

$$\mathbf{w_-} \otimes \mathbf{w_-} \mapsto 0$$

2. When a trivial circle and a nontrivial circle merge into a nontrivial circle, or when a nontrivial circle splits into a trivial circle and a nontrivial circle:

<div align="center">

Merge   Split

</div>

$$\mathbf{w_+} \otimes \mathbf{v_+} \mapsto \mathbf{v_+} \qquad\qquad \mathbf{v_+} \mapsto \mathbf{w_-} \otimes \mathbf{v_+}$$

$$\mathbf{w_+} \otimes \mathbf{v_-} \mapsto \mathbf{v_-} \qquad\qquad \mathbf{v_-} \mapsto \mathbf{w_-} \otimes \mathbf{v_-}$$

$$\mathbf{w_-} \otimes \mathbf{v_+} \mapsto 0$$

$$\mathbf{w_-} \otimes \mathbf{v_-} \mapsto 0$$

3. When two nontrivial circles merge into a trivial circle, or when a trivial circle splits into two nontrivial circles:

$$\underline{\text{Merge}} \qquad\qquad\qquad \underline{\text{Split}}$$

$$\mathbf{v}_+ \otimes \mathbf{v}_+ \mapsto 0 \qquad\qquad \mathbf{w}_+ \mapsto \mathbf{v}_+ \otimes \mathbf{v}_- + \mathbf{v}_- \otimes \mathbf{v}_+$$

$$\mathbf{v}_+ \otimes \mathbf{v}_- \mapsto \mathbf{w}_- \qquad\qquad \mathbf{w}_- \mapsto 0$$

$$\mathbf{v}_- \otimes \mathbf{v}_+ \mapsto \mathbf{w}_-$$

$$\mathbf{v}_- \otimes \mathbf{v}_- \mapsto 0$$

FIGURE 9. The various ways the operations of merging and splitting along a crossing (indicated by a dashed line) interact with a basepoint. The top illustrates the case of two trivial circles merging into trivial circle (or a trivial circle splitting into two trivial circles). The middle illustrates the case of a trivial circle and a nontrivial circle merging into a nontrivial circle (or a nontrivial circle splitting into a trivial circle and a nontrivial circle). The bottom illustrates the case of nontrivial circles merging into a trivial circle (or a trivial circle splitting into two nontrivial circles).

We end this section by briefly describing the $\mathfrak{sl}_2$ representation structure on AKh(L), referring the reader to [GLW18] for details. Fix a resolution of $P(L)$. Nontrivial circles, with respect to the basis $\{v_+, v_-\}$, are assigned the 2-dimensional defining representation of $\mathfrak{sl}_2$, defined by

$$h \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \; e \mapsto \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \; f \mapsto \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Trivial circles are assigned the 2-dimensional trivial representation. The resolution is then assigned the tensor product of these representations. We take the direct sum of all of these representations to obtain the structure of an $\mathfrak{sl}_2$-representation on $\mathrm{CKh}(P(L))$. This action descends to an action on the homology $\mathrm{AKh}(L)$, which Grigsby-Licata-Wehrli then upgrade to an action of $\mathfrak{sl}_2(\wedge)$. They show that the annular boundary maps commute with the $\mathfrak{sl}_2$-action, implying that the $\mathfrak{sl}_2$-action holds at the chain level. In contrast, the $\mathfrak{sl}_2(\wedge)$-action is well-defined on $\mathrm{AKh}(L)$, but at the chain level, it only holds up to homotopy [GLW18]. This observation leads us to consider this situation in terms of L_∞-algebras and modules.

3.4. The L_∞-algebra structure on $\mathfrak{sl}_2(\wedge)$

Since $\mathfrak{sl}_2(\wedge)_{dg}$ is a Lie superalgebra, it is an L_∞-algebra with no higher operations. In this section, we will use a cochain contraction to transfer this L_∞-algebra structure on $\mathfrak{sl}_2(\wedge)_{dg}$ to $\mathfrak{sl}_2(\wedge)$, and then we will show that all higher operations in the L_∞-algebra structure on $\mathfrak{sl}_2(\wedge)$ vanish.

Lemma 8. *There exist maps i and q so that the data*

$$
K \overset{\curvearrowright}{\underset{}{\bigcirc}} \ \mathfrak{sl}_2(\wedge)_{dg} \ \underset{i}{\overset{q}{\rightleftarrows}} \ H(\mathfrak{sl}_2(\wedge)_{dg}) \tag{$*$}
$$

satisfies the definition of a cochain contraction.

Proof. By Lemma 7, $H(\mathfrak{sl}_2(\wedge)_{dg}) \cong \mathfrak{sl}_2(\wedge) \oplus \mathbb{Z}$ with basis $\{v_2, v_{-2}, v_0, d, e, f, h\}$. Writing out the basis elements of $\mathfrak{sl}_2(\wedge)_{dg}$ and $H(\mathfrak{sl}_2(\wedge)_{dg})$, with their degrees above them, we have

$$
\begin{array}{cccc}
 & 2 & 1 & 0 \\
\mathfrak{sl}_2(\wedge)_{dg} & x & v_2, v_{-2}, \tilde{v}_0, d, D & e, f, h \\
H(\mathfrak{sl}_2(\wedge)_{dg}) & 0 & v_2, v_{-2}, \tilde{v}_0, d & e, f, h
\end{array}
$$

The maps i and q are easy to define. Let i lift every element to its corresponding element in $\mathfrak{sl}_2(\wedge)_{dg}$, and let q be the projection back down, sending x and D to 0. Define the chain homotopy K to be 0 for every element except for x, in which case we define $K(x) = -D$.

It is straightforward to check that i and q are chain maps. The differential in $\mathfrak{sl}_2(\wedge)$ is 0, so $i\partial = 0$. Also, $\partial i = 0$, since the elements in the image of i are in the kernel of $[d, \cdot]$, which is the differential in $\mathfrak{sl}_2(\wedge)_{dg}$. On the other hand, $\partial q = 0$. Also, $q\partial = 0$, since the only element in the image of $[d, \cdot]$ is x, which is sent to 0 by q. It is also straightforward to check that all of the chain contraction conditions are satisfied. $\qquad\square$

Lemma 9. *The Lie superalgebra $H(\mathfrak{sl}_2(\wedge)_{dg})$ inherits an L_∞-algebra structure induced by $(*)$, and this L_∞-algebra structure has no higher operations.*

Proof. Following section 2.4, the formulas for the transfered bracket tell us that

$$
I_n = \sum_{j=1}^{n-1} \sum_{\sigma \in \overline{S}(j, n-j)} \epsilon(\sigma) \cdot K \circ l_2 \circ (I_j \otimes I_{n-j}) \circ \sigma_\bullet
$$

$$
l_n' = \sum_{j=1}^{n-1} \sum_{\sigma \in \overline{S}(j, n-j)} \epsilon(\sigma) \cdot q \circ l_2 \circ (I_j \otimes I_{n-j}) \circ \sigma_\bullet
$$

Recall that $I_1 = -i$. For $n = 2$, the only unshuffle in $\overline{S}(1,1)$ is the identity. So, $I_2(x_1, x_2) = K(l_2(I_1(x_1), I_1(x_2)))$. Since $K(x) = -D$ and is 0 otherwise,

$$I_2(v_2, v_{-2}) = K(x) = -D$$

$$I_2(v_{-2}, v_2) = K(x) = -D$$

$$I_2(\tilde{v}_0, \tilde{v}_0) = K(2x) = -2D$$

$$I_2(x_1, x_2) = 0 \text{ otherwise}$$

Moreover, $l_2'(x_1, x_2) = q(l_2(I_1(x_1), I_1(x_2)))$, and so to compute the bracket of two elements in $H(\mathfrak{sl}_2(\wedge)_{dg})$, we lift them to $\mathfrak{sl}_2(\wedge)_{dg}$, take their bracket in $\mathfrak{sl}_2(\wedge)_{dg}$, and then quotient back to $H(\mathfrak{sl}_2(\wedge)_{dg})$.

Now, let $n \geq 3$. For all $m \geq 2$, I_m is in the image of K, and so $I_m(x_1, \ldots, x_m) = cD$ for some scalar c. But then for any $1 \leq j \leq n - 1$ and $\sigma \in \overline{S}(j, n-j)$, $q \circ l_2 \circ (I_j \otimes I_{n-j}) \circ \sigma_{\bullet}$ is 0, since either the $l_2 \circ (I_j \otimes I_{n-j}) \circ \sigma_{\bullet}$ term is 0, as $[D, y] = 0$ for all $y \in \{e, f, h, v_2, v_{-2}, \tilde{v}_0, D\}$, or q will send this term to 0 since the only nonzero bracket involving D is $[d, D] = -x$, and $q(x) = 0$. Hence $l_n' = 0$ for $n \geq 3$, and so the Lie superalgebra $H(\mathfrak{sl}_2(\wedge)_{dg})$ has no higher operations. $\qquad \square$

Theorem 3. *The Lie superalgebra $\mathfrak{sl}_2(\wedge)$ inherits an L_∞-algebra structure as a subalgebra of $H(\mathfrak{sl}_2(\wedge)_{dg})$, and this L_∞-algebra structure has no higher operations.*

Proof. The map $H(\mathfrak{sl}_2(\wedge)_{dg}) \rightarrow \mathfrak{sl}_2(\wedge)$ that sends $e, f, h, v_2, v_{-2}, \tilde{v}_0, d$ to $e, f, h, v_2, v_{-2}, v_0, 0$ is surjective with 1-dimensional kernel. So $H(\mathfrak{sl}_2(\wedge)_{dg}) \cong \mathfrak{sl}_2(\wedge) \oplus \mathbb{Z}$, where the $\mathbb{Z}$ summand is generated by the element d. But the bracket of d with everything in the $\mathfrak{sl}_2(\wedge)$ summand is 0, so $\mathfrak{sl}_2(\wedge)$ is a direct sum not only as a vector space, but also as an L_∞-algebra. So $\mathfrak{sl}_2(\wedge)$ is an L_∞-algebra as a subalgebra of $H(\mathfrak{sl}_2(\wedge)_{dg})$. $\qquad \square$

3.5. The L_∞-module structure on CKh(L)

Viewing $\mathfrak{sl}_2(\wedge)_{dg}$ and $\mathfrak{sl}_2(\wedge)$ as L_∞-algebras, in this section we will exhibit CKh($P(L)$) as an L_∞-module over $\mathfrak{sl}_2(\wedge)$. Also, fix a diagram $P(L)$ of the annular link L. We will simplify notation and write CKh(L) and AKh(L) instead of CKh($P(L)$) and AKh($P(L)$).

Theorem 4. *Let L be an annular link and $m \in CKh(L)$. Then $CKh(L)$ is an L_∞-module over the L_∞-algebra $\mathfrak{sl}_2(\wedge)$. One of the higher operations is given in terms of the Lee differential: $k_3(v_2, v_{-2}, m) = \partial_0^{Lee}(m)$. In particular, the L_∞-module structure is nontrivial if $\partial_0^{Lee} : CKh(L) \to CKh(L)$ is nonzero.*

Proof. To start, CKh(L) is an L_∞-module over $\mathfrak{sl}_2(\wedge)_{dg}$, where the k_2 operation is given by the usual module action, and $k_n = 0$ for $n \geq 3$. The module actions of elements of the basis $\{e, f, h, v_2, v_{-2}, \tilde{v}_0, d, D, x\}$ are as follows. The actions of e, f, h were described at the end of section 3.3, and v_2, v_{-2}, d, and D act by ∂_+^{Lee}, ∂_-, ∂_0, and ∂_0^{Lee}, respectively; see [GLW18]. The actions of $\tilde{v}_0$ and x can then be determined by the bracket relations.

Now, we have a cochain contraction from $\mathfrak{sl}_2(\wedge)_{dg}$ onto its homology, so we can transfer the $\mathfrak{sl}_2(\wedge)_{dg}$-module structure to obtain a new module structure over $H(\mathfrak{sl}_2(\wedge)_{dg})$. We can then restrict this module structure to the copy of $\mathfrak{sl}_2(\wedge)$ that sits inside of $H(\mathfrak{sl}_2(\wedge)_{dg})$.

To see that the induced module structure is nontrivial, recall the cochain contraction from Lemma 8.

$$K \,\overset{\curvearrowright}{\subset}\, \mathfrak{sl}_2(\wedge)_{dg} \;\overset{q}{\underset{i}{\rightleftarrows}}\; H(\mathfrak{sl}_2(\wedge)_{dg})$$

Examining the restriction of scalars formulas from Theorem 1, we see that

$$k_3'(x_1, x_2, m) = k_3(I_1(x_1), I_1(x_2), m) - k_2(I_2(x_1, x_2), m) = -k_2(I_2(x_1, x_2), m)$$

for $x_1, x_2 \in H(\mathfrak{sl}_2(\wedge)_{dg})$ and $m \in \mathrm{CKh}(L)$. Here, k_n is the L_∞-module operation for $\mathfrak{sl}_2(\wedge)_{dg}$, and recall that $k_n = 0$ for $n \geq 3$. Since $I_2(v_2, v_{-2}) = -D$ and $I_2(v_{-2}, v_2) = -D$, and since D acts by ∂_0^{Lee}, we conclude that $k_3'(v_2, v_{-2}, m) = \partial_0^{Lee}(m)$ and $k_3'(v_{-2}, v_2, m) = \partial_0^{Lee}(m)$, showing that we obtain higher operations. $\qquad\square$

3.6. Reidemeister Moves

3.6.1. Invariance of the $\mathfrak{sl}_2(\wedge)_{dg}$-module structure

In this section, we follow Khovanov's original proof that Khovanov homology is invariant under Reidemeister moves; see [Kho00]. There, Khovanov constructs quasi-isomorphisms between a given Khovanov complex and the complex obtained after applying a particular Reidemeister move. Here, we upgrade these quasi-isomorphisms to $\mathfrak{sl}_2(\wedge)_{dg}$ L_∞-module quasi-isomorphisms.

Theorem 5. *The L_∞-module structure on $CKh(L)$ is invariant under Reidemeister I.*

Proof. Let $[\![\curvearrowright]\!]$ and $[\![\varphi]\!]$ denote the annular chain complexes before and after applying an RI move, respectively. Our goal is to construct a quasi-isomorphism of L_∞-modules $\{h_n\} : [\![\varphi]\!] \rightarrow [\![\curvearrowright]\!]$. Because the $\mathfrak{sl}_2(\wedge)_{dg}$-module structures on these complexes have no higher operations, it suffices to give a quasi-isomorphism $h_1 : [\![\varphi]\!] \rightarrow [\![\curvearrowright]\!]$ that respects the module action, since we can then take $h_n = 0$ for

$n \geq 2$. To this end, let $\mathcal{C}$ be the complex

$$\mathcal{C} := [\![\,\unicode{x29BE}\,]\!] = [\![\,\unicode{x29BF}\,]\!] \xrightarrow{m} [\![\,\curlywedge\,]\!]\{1\}$$

and let $\mathcal{C}'$ be the subcomplex

$$\mathcal{C}' := [\![\,\unicode{x29BF}\,]\!]_{w_+} \xrightarrow{m} [\![\,\curlywedge\,]\!]\{1\},$$

where $[\![\,\unicode{x29BF}\,]\!]_{w_+}$ means that the extra circle is labeled w_+. A straightforward check of the actions of the basis elements $\{e, f, h, v_{-2}, v_2, \tilde{v}_0, d, D, x\}$ on $\mathcal{C}'$ shows that $\mathcal{C}'$ is an $sl_2(\wedge)_{dg}$-submodule. Moreover, $\mathcal{C}'$ is acyclic, since we can write $\mathcal{C}'$ as the mapping cone of the isomorphism m.

$$\mathcal{C}' = [\![\,\unicode{x29BF}\,]\!]_{w_+} \xrightarrow{m} [\![\,\curlywedge\,]\!]\{1\} = \begin{array}{c} \cdots \longrightarrow [\![\,\unicode{x29BF}\,]\!]_{w_+} \xrightarrow{d} [\![\,\unicode{x29BF}\,]\!]_{w_+} \longrightarrow \cdots \\ \oplus \quad \searrow^{m} \quad \oplus \\ \cdots \longrightarrow [\![\,\curlywedge\,]\!]\{1\} \xrightarrow{d} [\![\,\curlywedge\,]\!]\{1\} \longrightarrow \cdots \end{array}$$

Therefore, the quotient complex $\mathcal{C}/\mathcal{C}'$ is the complex $[\![\,\unicode{x29BF}\,]\!]/_{w_+=0} \rightarrow 0$, and it is isomorphic to $[\![\,\curlywedge\,]\!]$ as chain complexes via the map $z \otimes w_- \mapsto z$. To summarize, we have constructed a chain map $[\![\,\unicode{x29BE}\,]\!] \rightarrow [\![\,\curlywedge\,]\!]$ given by

$$y \otimes w_+ + z \otimes w_- + x \mapsto z \otimes w_- \mapsto z$$

for $y, z \in \mathcal{C}(*0)$ and $x \in \mathcal{C}(*1)$ (we have labeled the crossing formed by the Reidemeister I move last in the chain complex), and this map induces an

67

isomorphisms on homology

$$H([\![\mathcal{≍}]\!]) = H(\mathcal{C}) \cong H(\mathcal{C}/\mathcal{C}') \cong H([\![\cap]\!])$$

To complete the proof, we need to check that this composition respects the $\mathfrak{sl}_2(\wedge)_{dg}$ action. Certainly the first map does, as it is the quotient map of an $\mathfrak{sl}_2(\wedge)_{dg}$-submodule. For the second map, if $s \in \mathfrak{sl}_2(\wedge)_{dg}$, mapping over and then acting by s gives $z \otimes w_- \mapsto sz$. On the other hand, acting first by s and then mapping over gives $s(z \otimes w_-) = sz \otimes w_- \mapsto sz$. $\qquad\square$

Theorem 6. *The L_∞-module structure on $CKh(L)$ is invariant under Reidemeister II.*

Proof. There is a more direct way to prove RII invariance, but the method that follows will be useful in proving RIII invariance. Consider the diagrams in Figure 10.

68

$$
\begin{array}{ccc}
\beta & \xrightarrow{\quad\quad} & 0 \\
\Big\uparrow & \searrow^{\ \beta=\tau\beta} & \Big\uparrow \\
0 & \xrightarrow{\quad\quad} & \gamma \\
& (\mathcal{C}/\mathcal{C}')/\mathcal{C}''' &
\end{array}
$$

FIGURE 10. The relevant complexes in the proof of RII invariance. A similar diagram appears in [Bar02].

As complexes, the composition

$$
[\![\rightarrowtail\!\!\prec]\!] = \mathcal{C} \xrightarrow{q} \mathcal{C}/\mathcal{C}' \xrightarrow{p} (\mathcal{C}/\mathcal{C}')/\mathcal{C}''' \xrightarrow{f} [\![\succ\!\!\prec]\!]
$$

is a chain of quasi-isomorphisms; see [Bar02]. Our goal is to show that these complexes are actually quasi-isomorphic as L_∞-modules. Since $\mathcal{C}'''$ is not an L_∞-submodule, we do not immediately have an L_∞-module structure on $(\mathcal{C}/\mathcal{C}')/\mathcal{C}'''$. Our strategy then will be to give chain contractions from $\mathcal{C}$ to $\mathcal{C}/\mathcal{C}'$ and from $\mathcal{C}/\mathcal{C}'$ to $(\mathcal{C}/\mathcal{C}')/\mathcal{C}'''$ in order to equip these quotients with L_∞-module structures. Doing so will give us our desired L_∞-module quasi-isomomorphisms. To this end, define $i : \mathcal{C}/\mathcal{C}' \to \mathcal{C}$ to be the map

$$
i(z) = \begin{cases}
z - m^{-1}\partial_{\mathcal{C}}(z) & \text{if } z \text{ is in the top left} \\[2ex]
0 & \text{if } z \text{ is in the top right} \\[2ex]
z & \text{if } z \text{ is in the bottom left} \\[2ex]
z - m^{-1}\partial_{\mathcal{C}}(z) & \text{if } z \text{ is in the bottom right}
\end{cases}
$$

where the map $m^{-1} : \mathcal{C} \to \mathcal{C}$ is zero except on the top right vertex. There, it will be the inverse to the isomorphism that merges a circle with the small circle labeled w_+.

Remark. The map i above takes an element $z \in \mathcal{C}/\mathcal{C}'$ and views it as an element of $\mathcal{C}$. The complex $\mathcal{C}$ has a preferred basis of Khovanov generators, and $\mathcal{C}/\mathcal{C}'$ has a preferred basis consisting of basis elements of $\mathcal{C}$ not in $\mathcal{C}'$. So, before applying i, we should apply a map $i_0 : \mathcal{C}/\mathcal{C}' \to \mathcal{C}$ as $\mathbb{F}_2$ vector spaces, but we will suppress this for brevity.

Now, if $K : \mathcal{C} \to \mathcal{C}$ is the map

$$
K(z) = \begin{cases} 0 & \text{if } z \text{ is in the top left} \\ m^{-1}(z) & \text{if } z \text{ is in the top right} \\ 0 & \text{if } z \text{ is in the bottom left} \\ 0 & \text{if } z \text{ is in the bottom right} \end{cases}
$$

the data

$$
K \circlearrowleft \mathcal{C} \underset{i}{\overset{q}{\rightleftarrows}} \mathcal{C}/\mathcal{C}'
$$

satisfies the requirements of a chain contraction, which we can use to transfer the L_∞-module structure from $\mathcal{C}$ to $\mathcal{C}/\mathcal{C}'$. In particular, since i was a quasi-isomorphism of chain complexes, we obtain a quasi-isomorphism of L_∞-modules $I_n : \mathcal{C}/\mathcal{C}' \to \mathcal{C}$, where $\mathcal{C}/\mathcal{C}'$ has the induced L_∞-module structure. In fact, there are no higher operations on $\mathcal{C}/\mathcal{C}'$. After examining the formula for the induced operation, this follows from the fact that $\mathcal{C}$ itself has no higher operations, that the image of K is in $\mathcal{C}'$, and that $\mathcal{C}'$ is an L_∞-submodule of $\mathcal{C}$.

Next, since every nonzero element of $(\mathcal{C}/\mathcal{C}')/\mathcal{C}'''$ is equivalent to some element γ in the bottom right, we can define $j(\gamma) := \gamma$, thought of as an element of $\mathcal{C}/\mathcal{C}'$. Then if $H : \mathcal{C}/\mathcal{C}' \to \mathcal{C}/\mathcal{C}'$ is the map

$$H(z) = \begin{cases} \Delta^{-1}(z) & \text{if } z \text{ is in the top left} \\[2mm] 0 & \text{if } z \text{ is in the top right} \\[2mm] 0 & \text{if } z \text{ is in the bottom left} \\[2mm] 0 & \text{if } z \text{ is in the bottom right} \end{cases}$$

the data

$$_H \circlearrowright \mathcal{C}/\mathcal{C}' \underset{j}{\overset{p}{\rightleftarrows}} (\mathcal{C}/\mathcal{C}')/\mathcal{C}'''$$

also satisfies the requirements of a chain contraction. We obtain a quasi-isomorphism of L_∞-modules $J_n : \mathcal{C}/\mathcal{C}' \to \mathcal{C}$, where $(\mathcal{C}/\mathcal{C}')/\mathcal{C}'''$ has the induced L_∞-module structure from $\mathcal{C}/\mathcal{C}'$. There are no higher operations on $(\mathcal{C}/\mathcal{C}')/\mathcal{C}'''$ as well. To see this, note that because $\mathcal{C}/\mathcal{C}'$ has no higher operations, the induced module operation on $(\mathcal{C}/\mathcal{C}')/\mathcal{C}'''$ is of the form $k'_n = \sum_{\substack{\tau \in S(1,\dots,1) \\ i_1 = \cdots = i_{n-1} = 1}} q \circ A_{n-1} \circ (\tau^\bullet \otimes i)$; see Figure 11.
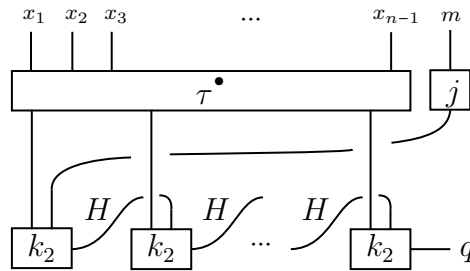
FIGURE 11. The transferred bracket on $(\mathcal{C}/\mathcal{C}')/\mathcal{C}'''$. Here, the labeled edges represent the application of that particular map. For example, $k'_3(x_1, x_2, m) = q \circ k_2(x_1, H \circ k_2(x_2, j(m))) + q \circ k_2(x_2, H \circ k_2(x_1, j(m)))$.

Since the image of j is concentrated in the bottom right corner of $\mathcal{C}/\mathcal{C}'$, and H is zero everywhere except the top-left, it follows that all higher operations vanish. As for the module operation k_2, an element $s \in \mathfrak{sl}_2(\wedge)_{dg}$ acts on $\gamma \in (\mathcal{C}/\mathcal{C}')/\mathcal{C}'''$ by

$$s \cdot \gamma = p(s \cdot j(\gamma)) = p(q(s \cdot (i \circ j(\gamma)))) = (p \circ q)(s \cdot \gamma - s \cdot m^{-1}\partial_{\mathcal{C}}(\gamma))$$

That is, we consider the difference $s \cdot \gamma - s \cdot m^{-1}\partial_{\mathcal{C}}(\gamma)$ as an element of $\mathcal{C}$, and then quotient twice. It remains to show that the degree shift map $f : (\mathcal{C}/\mathcal{C}')/\mathcal{C}''' \to [\![\bowtie]\!]$ respects this action, that is, $f(s \cdot \gamma) = s \cdot f(\gamma)$ for $s \in \mathfrak{sl}_2(\wedge)_{dg}$ and $\gamma \in (\mathcal{C}/\mathcal{C}')/\mathcal{C}'''$. We compute that

$$f(s \cdot \gamma) - s \cdot f(\gamma) = f((p \circ q)(s \cdot \gamma) - (p \circ q)(s \cdot m^{-1}\partial_{\mathcal{C}}(\gamma))) - s \cdot \gamma$$
$$= -(p \circ q)(s \cdot m^{-1}(\partial_{\mathcal{C}}\gamma))$$

Using the fact that any term $m^{-1}(\partial_{\mathcal{C}}\gamma)$ will be labeled by w_+, the action of any $s \in \mathfrak{sl}_2(\wedge)_{dg}$ on this term will quotient to 0 under $p \circ q$. In particular, we have shown that the composition

$$[\![\bowtie]\!] = \mathcal{C} \xrightarrow{q} \mathcal{C}/\mathcal{C}' \xrightarrow{p} \mathcal{C}'' \xrightarrow{f} [\![\bowtie]\!]$$

is a chain of L_∞-quasi-isomorphisms, since $[\![\bowtie]\!]$ has no higher operations. $\square$

Theorem 7. *The L_∞-module structure on $CKh(L)$ is invariant under Reidemeister III.*

Proof. **Step 1: Overview.** For RIII, the situation is summarized in Figure 12. We start by decomposing the complexes $[\![\bowtie]\!]$ and $[\![\bowtie]\!]$ into $\mathcal{C}$ and $\mathcal{D}$ (these are the

72

top left and top right cubes in Figure 12, respectively). We will then transfer the L_∞-module structures by quasi-isomorphisms $q_2 \circ q_1$ and $p_2 \circ p_1$ to the quotient complexes $(\mathcal{C}/\mathcal{C}')/C''$ and $(\mathcal{D}/\mathcal{D}')/D''$ (the bottom row) and show that these quotients are L_∞-quasi-isomorphic via an L_∞-module map f.

Step 2: The structure on $\mathcal{C}/\mathcal{C}'$ and $\mathcal{D}/\mathcal{D}'$. Analagous to the RII case, we have subcomplexes $C' \subset C$ and $D' \subset D$, which are L_∞-submodules; see Figure 13. Because C' and D' are submodules, the quotients $\mathcal{C}/\mathcal{C}'$ and $\mathcal{D}/\mathcal{D}'$ have no higher operations as L_∞-modules. Alternatively, this quotient structure agrees with the one obtained by using cochain contractions

$$H \circlearrowright \mathcal{C} \underset{i_1}{\overset{q_1}{\rightleftarrows}} \mathcal{C}/\mathcal{C}' \qquad \text{and} \qquad K \circlearrowright \mathcal{D} \underset{j_1}{\overset{p_1}{\rightleftarrows}} \mathcal{D}/\mathcal{D}'$$

to transfer the structure. Here, the maps i_1 and i_2 are

$$i_1(z) = \begin{cases} z, & \text{if } z \in 000, 001, 010, 100 \\ z - m^{-1}(\partial_{\mathcal{C}} z), & \text{if } z \in 011, 101, 110 \\ 0, & \text{if } z \in 111 \end{cases}$$

and

$$j_1(z) = \begin{cases} z, & \text{if } z \in 000, 001, 010, 100 \\ z - m^{-1}(\partial_{\mathcal{D}} z), & \text{if } z \in 011, 101, 110 \\ 0, & \text{if } z \in 111 \end{cases}$$

73
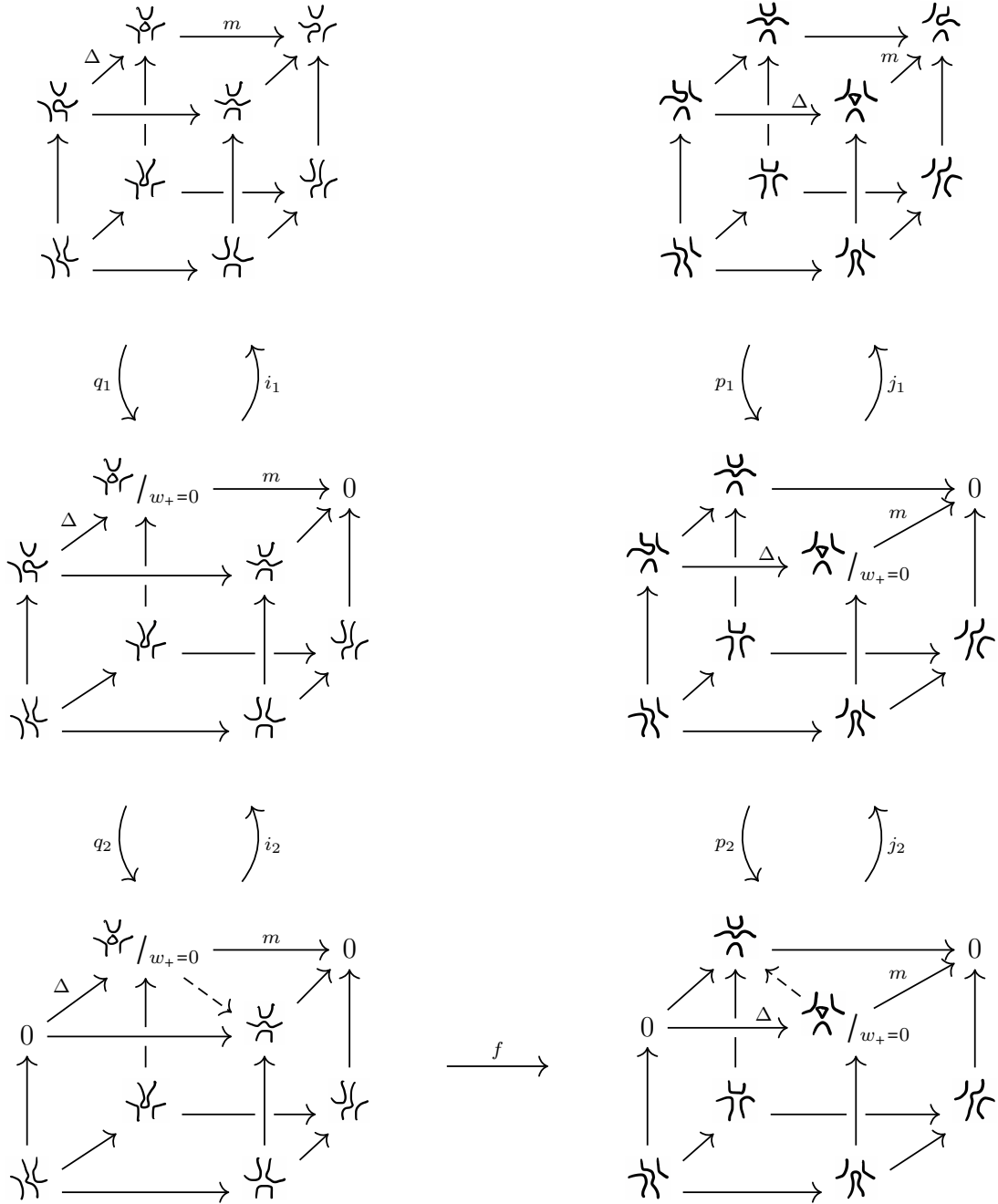
FIGURE 12. The complexes involved in RIII invariance. We have suppressed the degree shifts.

74
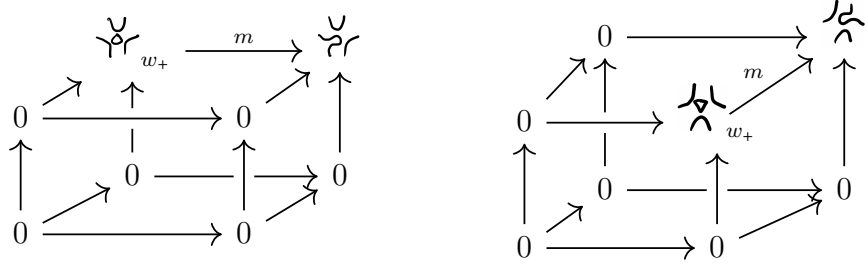
FIGURE 13. The complexes C' and D'. The w_+ means that the trivial circle is labeled w_+.

The coordinates above refer to different corners of the cubes, i.e.
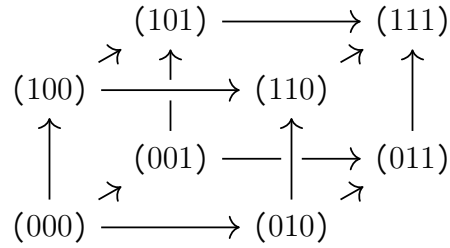
Step 3: The structure on $(C/C')/C''$ **and** $(D/D')/D''$. To go from C/C' to $(C/C')/C''$ and D/D' to $(D/D')/D''$, we identify elements in vertices 101 and 110 by imposing the relation $\beta_1 = \tau_1\beta_1$ in $(C/C')/C''$ and the relation $\beta_2 = \tau_2\beta_2$ in $(D/D')/D''$, analagous to the RII case. Similar to before, we are not quotienting by a submodule, so we need to transfer the structure from C/C' and D/D'. To this end, we define maps i_2 and j_2. Let

$$
i_2(z) = \begin{cases} z, & \text{if } z \in 000, 010, 011, 110 \\ z - \Delta^{-1}(\partial_{C/C'}z), & \text{if } z \in 001 \\ 0, & \text{if } z \in 100, 111 \end{cases}
$$

75

Note that if $z \in 101$, then z is equivalent via τ_1 to some element in 110. Also, let

$$
j_2(z) = \begin{cases} z, & \text{if } z \in 000, 001, 011, 101 \\[2mm] z - \Delta^{-1}(\partial_{\mathcal{D}/\mathcal{D}'} z), & \text{if } z \in 010 \\[2mm] 0, & \text{if } z \in 100, 111 \end{cases}
$$

where we again note that if $z \in 110$, then z is equivalent via τ_2 to some element in 101. Then, if we define $T : \mathcal{C}/\mathcal{C}' \to \mathcal{C}/\mathcal{C}'$ and $S : \mathcal{D}/\mathcal{D}' \to \mathcal{D}/\mathcal{D}'$ by

$$
T(z) = \begin{cases} \Delta^{-1}(z), & \text{if } z \in 101 \\[2mm] 0, & \text{otherwise} \end{cases} \qquad \text{and} \qquad S(z) = \begin{cases} \Delta^{-1}(z), & \text{if } z \in 110 \\[2mm] 0, & \text{otherwise} \end{cases}
$$

both

$$
{}_T \circlearrowright \mathcal{C}/\mathcal{C}' \underset{i_2}{\overset{q_2}{\rightleftarrows}} (\mathcal{C}/\mathcal{C}')/\mathcal{C}'' \qquad \text{and} \qquad {}_S \circlearrowright \mathcal{D}/\mathcal{D}' \underset{j_2}{\overset{p_2}{\rightleftarrows}} (\mathcal{D}/\mathcal{D}')/\mathcal{D}''
$$

satisfy the requirement of a cochain contraction. In particular, this allows us to transfer the L_∞-module structures from $\mathcal{C}/\mathcal{C}'$ and $\mathcal{D}/\mathcal{D}'$ to their respective quotient complexes.

Step 4: $(\mathcal{C}/\mathcal{C}')/\mathcal{C}''$ and $(\mathcal{D}/\mathcal{D}')/\mathcal{D}''$ have no higher operations. The next goal is to show that there is no higher structure on $(\mathcal{C}/\mathcal{C}')/\mathcal{C}''$ or $(\mathcal{D}/\mathcal{D}')/\mathcal{D}''$. We will explain the case of $(\mathcal{C}/\mathcal{C}')/\mathcal{C}''$. The case of $(\mathcal{D}/\mathcal{D}')/\mathcal{D}''$ is analagous. Indeed, because $\mathcal{C}/\mathcal{C}'$ has no higher L_∞-module operations, the transferred structure looks like

$$
k_n'(x_1, \ldots, x_{n-1}, m) = \sum_{\substack{\tau \in S(1, \ldots, 1) \\ i_1 = \cdots = i_{n-1} = 1}} q \circ A_{n-1} \circ (\tau^\bullet \otimes i_2)
$$

76

See Figure 11. We will show that $q \circ A_{n-1} \circ (\tau^\bullet \otimes i_2) = 0$ for any $\tau \in S_{n-1}$. That is, for $n \geq 3$, it suffices to show that $q \circ A_{n-1}(x_1, x_2, \ldots, x_{n-1}, i_2(m)) = 0$ for any choice of $x_1, x_2, \ldots, x_{n-1} \in \mathfrak{sl}_2(\wedge)_{dg}$, where $i_1, \ldots, i_{n-1} = 1$ in the definition of A_{n-1}.

Step 4.1: The case $n > 3$. We start with the case $n > 3$. Because T is only nonzero on the vertex 101, for $q \circ A_{n-1}(x_1, x_2, \ldots, x_{n-1}, i_2(m))$ to be nonzero, it must contain a nonzero composition

Here, the map x_j represents acting by the element $x_j \in \mathfrak{sl}_2(\wedge)_{dg}$. We will show that if x_j is any element of the basis $\{e, f, h, v_2, v_{-2}, \tilde{v}_0, d, D, x\}$, then this composition is zero. Indeed, x_j cannot be e, f, h, since it must change the homological degree by one to have nonzero image in vertex 101. Moreover, modulo the relation $w_+ = 0$, the actions of the elements $v_2, v_{-2}, \tilde{v}_0$, and D are all the zero map. Finally, if $x_j = -x = [d, D]$, then the component that lies in the vertex 101 is

$$D_{101}d_{10*} + D_{10*}d_{100} + d_{101}D_{10*} + d_{10*}D_{100}$$

where, for example, the notation D_{101} represents the component of D that remains in vertex 101, and d_{10*} represents the component of d obtained by acting along the edge $100 \to 101$. Now we observe that the middle terms $D_{10*}d_{100}$ and $d_{101}D_{10*}$ are both zero, because the relation $w_+ = 0$ implies that D_{10*} is the zero map. Also, the terms $D_{101}d_{10*}$ and $d_{10*}D_{100}$ cancel, because d_{10*} just appends a trivial circle labeled w_- to the resolution in vertex 100.

Therefore, we have reduced the possible nonzero $q \circ A_{n-1}(x_1, x_2, \ldots, x_{n-1}, i_2(m))$ to either the case of $q \circ A_2(x_1, x_2, i_2(m))$ or $q \circ A_{n-1}(x_1, x_2, \ldots, x_{n-1}, i_2(m))$, where $x_2 = \cdots = x_{n-2} = d$.

Step 4.2: The case $n = 3$. We now examine the case $n = 3$. From the formula for $q \circ A_2(x_1, x_2, i_2(m))$, we need $x_1 \cdot i_2(m)$ to be in vertex 101. This implies that m is either in the vertex 000 or the vertex 001. If $m \in 000$, then the only possibility for x_1 is $x_1 = x$. But then

$$ x \cdot m = -[d, D] \cdot m = \partial_0 \partial_0^{Lee} m + \partial_0^{Lee} \partial_0 m $$

Since the boundary map ∂_{*01} is a split map, and $w_+ = 0$ in vertex 101, $\partial_0^{Lee} = 0$ along this edge. So we only have a term $\partial_0 \partial_0^{Lee} m$. Therefore, we need to focus on the composition
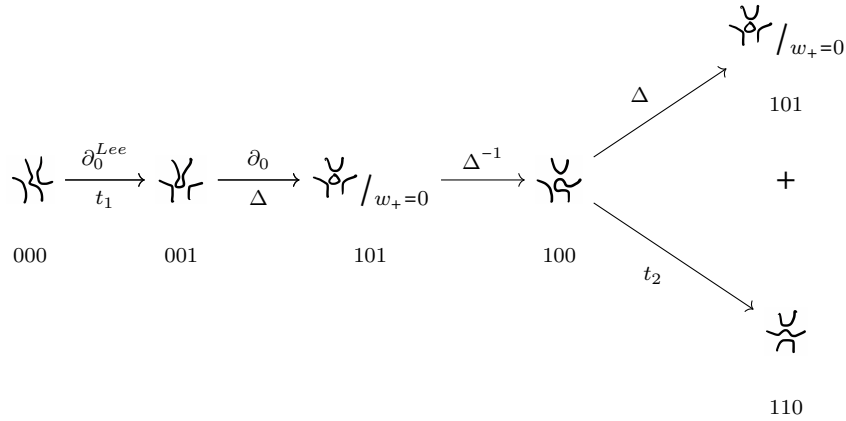
where t_1, t_2 can be either merge or split. Let a, b, c denote the circles to which the three strands in vertex 000 belong; see Figure 14. Then we have four cases: either $a = b = c$, $a = b \neq c$, $a \neq b = c$, or $a \neq b \neq c$.
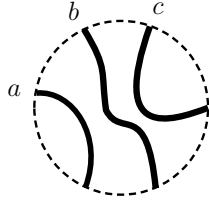
FIGURE 14. Each strand in the vertex 000 belongs to a circle. Denote these circles by a, b, and c.
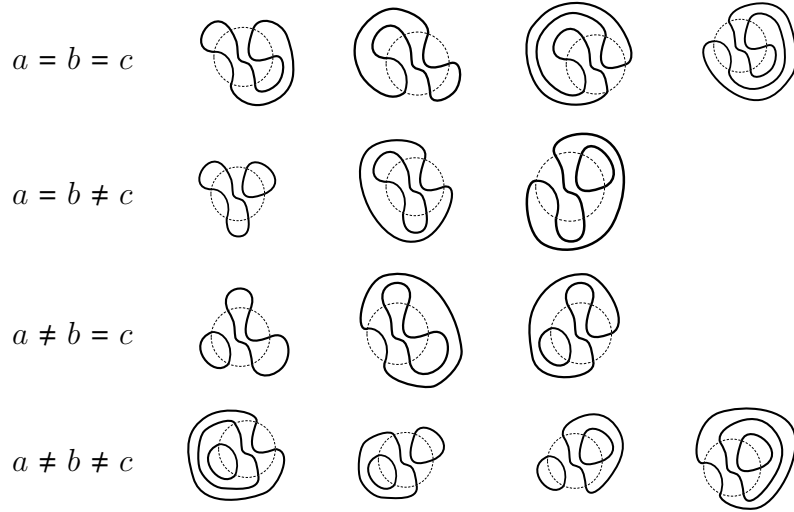
FIGURE 15. This picture shows all possible configurations of the circles a, b, and c.

We have not drawn the basepoint, which can be anywhere outside of the dashed circles. We have also not drawn the possible other circles coming from the other crossing resolutions.

Step 4.2.1: $m \in 000$ **and** $a = b = c$. If $a = b = c$, then in each case, t_1 is a split map and t_2 is a merge map. Because ∂_0^{Lee} needs to be nonzero, we must label our circle by w_-. This forces a labeling of $w_+ \otimes w_+$ in 100; see Figure 16.
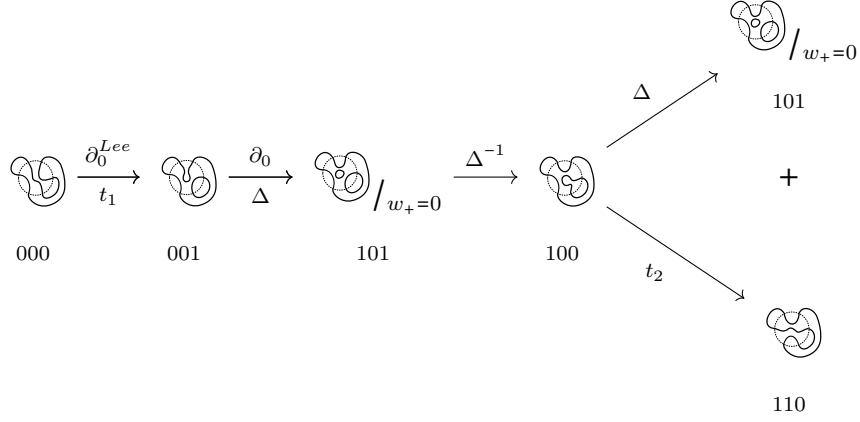
FIGURE 16. The first of four cases with $a = b = c$. In each case, the labeling of the circle in 000 must be w_-, which forces a labeling of $w_+ \otimes w_+$ in 100.

The possibilities for x_2 are $\{e, f, h, v_2, v_{-2}, \tilde{v}_0, d, D, x\}$. It cannot be e, f, h, since x_2 must change the homological be degree by one. Moreover, v_2, v_{-2} and $\tilde{v}_0$ are each the 0 map, since we are only involving trivial circles. The labeling $w_+ \otimes w_+$ implies that D is the 0 map. Finally, the terms obtained from acting by either d or x will cancel when we quotient to $(\mathcal{C}/\mathcal{C}')/\mathcal{C}''$. For example, if we act by d, then the relation $\beta_1 = \tau_1\beta_1$ identifies the terms obtained by acting by d_{10*} and d_{1*0}, and so they will cancel. On the other hand, if we act by x, the terms we obtain in vertices 101 and 110 are

$$\underbrace{d_{10*}D_{100} + d_{101}D_{10*} + D_{10*}d_{100} + D_{101}d_{10*}}_{\text{vertex } 101} + \underbrace{d_{1*0}D_{100} + d_{110}D_{1*0} + D_{1*0}d_{100} + D_{110}d_{1*0}}_{\text{vertex } 110}$$

Now, the terms involving D_{10*} and D_{1*0} are zero, because $w_+ = 0$ in vertex 101 and both circles in vertex 100 are labeled by w_+. We are left with

$$d_{10*}D_{100} + D_{101}d_{10*} + d_{1*0}D_{100} + D_{110}d_{1*0}$$

80

Because of the w_+ labelings in vertex 100, the only nonzero parts of D_{101} and D_{110} come from applying D amongst the other circles in the resolution. It follows that $D_{101}d_{10*}$ and $D_{110}d_{1*0}$ will be identified when we quotient, and so they will cancel. The $d_{10*}D_{100}$ and $d_{1*0}D_{100}$ terms will also cancel.

Step 4.2.2: $m \in 000$ **and** $a = b \neq c$. If $a = b \neq c$, then in each case, t_1 is a split map and t_2 is a split map. Again, we need to involve trivial circles for t_1, otherwise $\partial_0^{Lee} = 0$; see Figure 17.
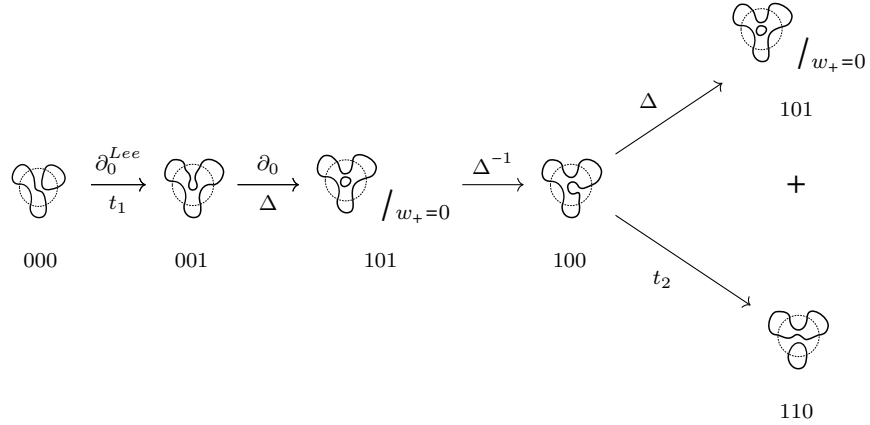
FIGURE 17. The first of three cases with $a = b \neq c$. The labeling of the circle in 000 must be $w_- \otimes w_-$, which forces a labeling of w_+ in 100.

Since the circle in vertex 100 must be labeled by w_+, by a similar argument to the case of $a = b = c$, acting by $v_2, v_{-2}, \tilde{v}_0, D$ are all 0, and the terms obtained from acting by either d or x will cancel when we quotient to $(\mathcal{C}/\mathcal{C}')/\mathcal{C}''$. Seeing that the terms will cancel in the quotient if we act by x in vertex 100 is slightly different than before. To see this explicitly, we start as in the case of $a = b = c$ by examining

the terms

$$\underbrace{d_{10*}D_{100} + d_{101}D_{10*} + D_{10*}d_{100} + D_{101}d_{10*}}_{\text{vertex } 101} + \underbrace{d_{1*0}D_{100} + d_{110}D_{1*0} + D_{1*0}d_{100} + D_{110}d_{1*0}}_{\text{vertex } 110}$$

Now, $d_{10*}D_{100}$ and $d_{1*0}D_{100}$ will cancel in the quotient. Also, D_{10*} is the zero map due to the relation $w_+ = 0$ in vertex 101. The w_+ label implies that $d_{110}D_{1*0}$ is zero. It remains to show that the terms

$$D_{101}d_{10*} + D_{1*0}d_{100} + D_{110}d_{1*0}$$

cancel. Label the circles in 100 by $c_1 \otimes \cdots \otimes c_n \otimes w_+$. The idea is to show that part of $D_{110}d_{1*0}$ will cancel with $D_{101}d_{10*}$ (the part involving the c_i themselves) and that the rest will cancel with $D_{1*0}d_{100}$ (the part involving the w_+). Indeed, we may write the $D_{110}d_{1*0}$ term as

$$
\begin{aligned}
D_{110}d_{1*0} &= D_{110}(c_1 \otimes \cdots \otimes c_n \otimes w_+ \otimes w_-) + D_{110}(c_1 \otimes \cdots \otimes c_n \otimes w_- \otimes w_+) \\
&= D_{110}^c(c_1 \otimes \cdots \otimes c_n \otimes w_+ \otimes w_-) + D_{110}^c(c_1 \otimes \cdots \otimes c_n \otimes w_- \otimes w_+) \\
&\quad + D_{110}^{w_+}(c_1 \otimes \cdots \otimes c_n \otimes w_+ \otimes w_-) + D_{110}^{w_+}(c_1 \otimes \cdots \otimes c_n \otimes w_- \otimes w_+) \\
&\quad + D_{110}^{w_-}(c_1 \otimes \cdots \otimes c_n \otimes w_+ \otimes w_-) + D_{110}^{w_-}(c_1 \otimes \cdots \otimes c_n \otimes w_- \otimes w_+)
\end{aligned}
$$

where D_{110}^c is the part of D_{110} that involves only crossings among the circles $c_1, \ldots, c_n$, D^{w_+} is the part of D_{110} that involves only crossings with the circle labeled w_+, and D^{w_-} is the part of D_{110} that involves only crossings with the circle labeled w_-. By the definition of the Lee differential, the labels imply

$$D_{110}^{w_+}(c_1 \otimes \cdots \otimes c_n \otimes w_+ \otimes w_-) = D_{110}^{w_+}(c_1 \otimes \cdots \otimes c_n \otimes w_- \otimes w_+) = 0, \text{ and so}$$

$$D_{110}d_{1*0} = D_{110}^c(c_1 \otimes \cdots \otimes c_n \otimes w_+ \otimes w_-) + D_{110}^c(c_1 \otimes \cdots \otimes c_n \otimes w_- \otimes w_+)$$

$$+ D_{110}^{w_-}(c_1 \otimes \cdots \otimes c_n \otimes w_- \otimes w_+) + D_{110}^{w_-}(c_1 \otimes \cdots \otimes c_n \otimes w_+ \otimes w_-)$$

On the other hand, $D_{101}d_{10*}$ can be written as

$$D_{101}(c_1 \otimes \cdots \otimes c_n \otimes w_+ \otimes w_-) = D_{101}^c(c_1 \otimes \cdots \otimes c_n \otimes w_+ \otimes w_-)$$

$$+ D_{101}^w(c_1 \otimes \cdots \otimes c_n \otimes w_+ \otimes w_-)$$

where D_{101}^w is the part of D_{101} involving a crossing with either the (outermost) circle labeled w_+ or the circle labeled w_-. The w_+ label together with the relation $w_+ = 0$ in vertex 101 implies that $D_{101}^w(c_1 \otimes \cdots \otimes c_n \otimes w_+ \otimes w_-) = 0$. In the quotient $(\mathcal{C}/\mathcal{C}')/\mathcal{C}''$, $D_{101}^c(c_1 \otimes \cdots \otimes c_n \otimes w_+ \otimes w_-)$ is identified with

$$D_{110}(c_1 \otimes \cdots \otimes c_n \otimes w_+ \otimes w_-) + D_{110}(c_1 \otimes \cdots \otimes c_n \otimes w_- \otimes w_+)$$

Therefore, it remains to examine the $D_{1*0}d_{100}$ term, which we may write as

$$D_{1*0}d_{100}(c_1 \otimes \cdots \otimes c_n \otimes w_+) = D_{1*0}d_{100}^c(c_1 \otimes \cdots \otimes c_n \otimes w_+)$$

$$+ D_{1*0}d_{100}^w(c_1 \otimes \cdots \otimes c_n \otimes w_+)$$

Because of the w_+ label, $D_{1*0}d_{100}^c(c_1 \otimes \cdots \otimes c_n \otimes w_+) = 0$, and so it remains to show that

$$D_{1*0}d_{100}^w(c_1 \otimes \cdots c_n \otimes w_+)$$

and

$$D_{110}^{w_+}(c_1 \otimes \cdots \otimes c_n \otimes w_- \otimes w_+) + D_{110}^{w_-}(c_1 \otimes \cdots \otimes c_n \otimes w_+ \otimes w_-)$$

cancel in $(\mathcal{C}/\mathcal{C}')/\mathcal{C}''$. This is indeed the case since to compute $D_{110}^{w_+}(c_1 \otimes \cdots \otimes c_n \otimes w_- \otimes w_+)$, we need only consider crossings where either a circle c_i labeled w_- merges with the w_- or the circle labeled w_- splits. The same is true to compute $D_{110}^{w_-}(c_1 \otimes \cdots \otimes c_n \otimes w_+ \otimes w_-)$. On the other hand, to compute $D_{1*0}d_{100}^{w}(c_1 \otimes \cdots \otimes c_n \otimes w_+)$ we again have two cases. The first case consists of crossings where a circle c_i labeled w_- merges with the w_+. These terms will cancel with those from the first case above. The second case consists of the crossings where a w_+ splits to $w_- \otimes w_+ + w_+ \otimes w_-$. These terms will cancel with the second case above.

Step 4.2.3: $m \in 000$ **and** $a \ne b = c$. We can now study the case $a \ne b = c$. In this scenario, t_1 is a split map and t_2 is a merge map; see Figure 18.
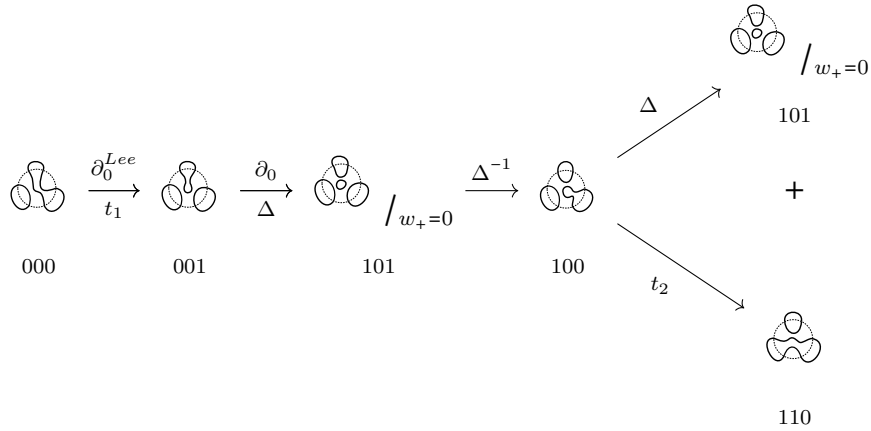
FIGURE 18. The first of three cases with $a \ne b = c$. The labeling of the circle in 000 must be $w_- \otimes w_-$, which forces a labeling of $w_- \otimes w_+ \otimes w_+$ in 100.

In each case, the labeling in 000 must be $w_- \otimes w_-$, and this forces a labeling of $w_- \otimes w_+ \otimes w_+$ in 100 in each case. Again, $v_2, v_{-2}, \tilde{v}_0, D$ are all 0, and a similar

84

argument shows that the terms obtained from acting by either d or x will cancel when we quotient to $(\mathcal{C}/\mathcal{C}')/\mathcal{C}''$.

Step 4.2.4: $m \in 000$ **and** $a \neq b \neq c$. Finally, if $a \neq b \neq c$, then t_1 is a merge map and t_2 is a merge map; see Figure 19.
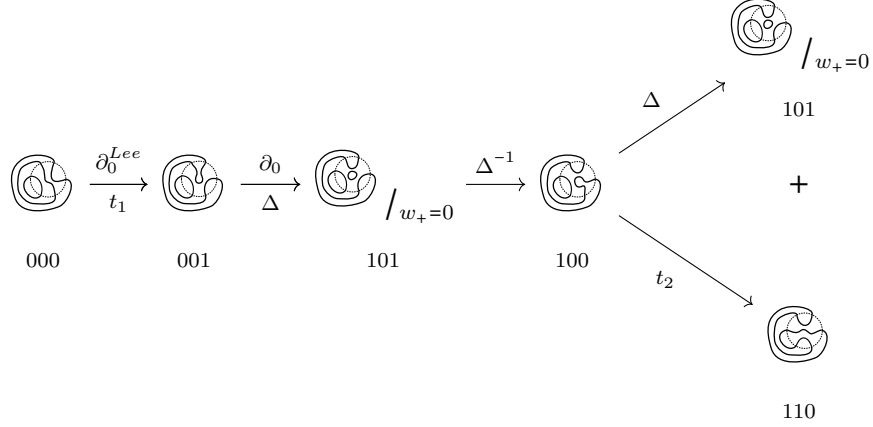
FIGURE 19. The first of four cases of $a \neq b \neq c$. The labeling of the circle in 000 must be $w_- \otimes w_- \otimes w_\bullet$, which forces a labeling of $w_+ \otimes w_\bullet$ in 100.

In each case, the labeling in 000 must be $w_- \otimes w_- \otimes w_\bullet$, where $w_\bullet$ denotes that the innermost circle can be labeled either w_+ or w_-. This forces a labeling of $w_+ \otimes w_\bullet$ in 100 in all cases. For the last time, we verify that $v_2, v_{-2}, \tilde{v}_0, D$ are all 0, and a similar argument shows that the terms obtained from acting by either d or x will cancel when we quotient to $(\mathcal{C}/\mathcal{C}')/\mathcal{C}''$. To summarize, we have thus shown that $q \circ A_2(x_1, x_2, i_2(m)) = 0$ for all $m \in 000$ and $x_1, x_2 \in \mathfrak{sl}_2(\wedge)_{dg}$, and we conclude that $k_3'(x_1, x_2, m) = 0$ for all $m \in 000$ and $x_1, x_2 \in \mathfrak{sl}_2(\wedge)_{dg}$ as well.

Step 4.2.5: $m \in 001$. We next examine $q \circ A_2(x_1, x_2, i_2(m)) = 0$ in the case $m \in 001$. The relevant composition in the RIII cube is given in Figure 20.
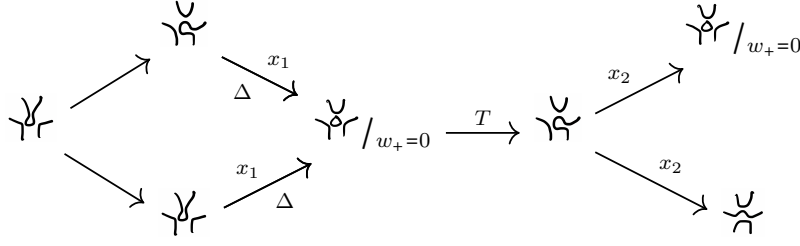
FIGURE 20. The relevant part of the RIII cube. If we start with an element in
001, $i_2 : (\mathcal{C}/\mathcal{C}')/\mathcal{C}'' \to \mathcal{C}/\mathcal{C}'$ gives a sum of elements in 001 and 100. We then act by
x_1, apply the homotopy T, act by x_2, and then quotient back to $(\mathcal{C}/\mathcal{C}')/\mathcal{C}''$.

As before, the possibilities for x_1 are $\{e, f, h, v_2, v_{-2}, \tilde{v}_0, d, D, x\}$. Because x_1
needs to increase the homological degree of m, it cannot be e, f, or h. Since we are
working modulo $w_+ = 0$, both ∂^{Lee} and ∂_- are the zero map, and so D, v_2, v_{-2}, and
$\tilde{v}_0$ are all the zero map. Moreover, x_1 cannot be d, since the resolutions in 001 and
100 have the same label, which means that they will cancel when mapped to 101.
Similarly, the fact that both resolutions have the same label also implies that the
terms in $x = -[d, D]$ will cancel. We conclude that $q \circ A_2(x_1, x_2, i_2(m)) = 0$ for all
$m \in 001$ and $x_1, x_2 \in \mathfrak{sl}_2(\wedge)_{dg}$, and so we have thus shown that $k_3'(x_1, x_2, m) = 0$ on
$(\mathcal{C}/\mathcal{C}')/\mathcal{C}''$.

Step 4.2.6: Conclusion. From the above case analysis, the only possible
higher operation is k_n' for $n > 3$, which could include a nonzero term of
$q \circ A_{n-1}(x_1, x_2, \ldots, x_{n-1}, i_2(m))$ with $x_2 = \cdots = x_{n-2} = d$. But because d is just
the inverse to the chain homotopy T, this will cycle the module element back and
forth between vertices 101 and 100. In particular, $q \circ A_{n-1}(x_1, x_2, \ldots, x_{n-1}, i_2(m)) =$
$q \circ A_2(x_1, x_{n-1}, i_2(m))$, which we have already shown is zero. We conclude that
$k_n' = 0$ on $(\mathcal{C}/\mathcal{C}')/\mathcal{C}''$ for $n > 3$, and this completes the proof that $(\mathcal{C}/\mathcal{C}')/\mathcal{C}''$ has no
higher operations. The symmetry between $(\mathcal{C}/\mathcal{C}')/\mathcal{C}''$ and $(\mathcal{D}/\mathcal{D}')/\mathcal{D}''$ implies that
$(\mathcal{D}/\mathcal{D}')/\mathcal{D}''$ also has no higher operations.

Step 5: The cubes $(\mathcal{C}/\mathcal{C}')/\mathcal{C}''$ and $(\mathcal{D}/\mathcal{D}')/\mathcal{D}''$ are quasi-isomorphic.

It remains to construct the map $f : (\mathcal{C}/\mathcal{C}')/\mathcal{C}'' \to (\mathcal{D}/\mathcal{D}')/\mathcal{D}''$ and show that it respects the (trivial) L_∞-module structures. Indeed, in $(\mathcal{C}/\mathcal{C}')/\mathcal{C}''$, each $\beta_1 \in 101$ is equivalent via τ_1 to some $\gamma_1 \in 110$. The map f will send an element in 110 to itself, but as an element of 101 in $(\mathcal{D}/\mathcal{D}')/\mathcal{D}''$, and it will keep the bottom layer of the cube fixed. This is an isomorphism on spaces, and Bar-Natan checks that this map is a chain map; see [Bar02]. So, for $s \in \mathfrak{sl}_2(\wedge)_{dg}$ and $x \in (\mathcal{C}/\mathcal{C}')/\mathcal{C}''$ we need to compare $f(s \cdot x)$ and $s \cdot f(x)$, where the module structure is $s \cdot x = q_2(s \cdot i_2(x)) = q_2(q_1(s \cdot i_1(i_2(x))))$.

Step 5.1: The case $s \in \{e, f, h\}$. Suppose that $s \in \{e, f, h\}$. First we examine the case where z is on the bottom face of the cube. If z is in 000 or 010, then

$$q_2(q_1(s \cdot i_1(i_2(z)))) = q_2(q_1(s \cdot z))$$

Note that we abuse notation and think of z as an element of $\mathcal{C}$ on the right-hand side. If z is in 001, then

$$
\begin{aligned}
q_2(q_1(s \cdot i_1(i_2(z)))) &= q_2(q_1(s \cdot i_1(z - \Delta^{-1}(\partial_{\mathcal{C}/\mathcal{C}'} z)))) \\
&= q_2(q_1(s \cdot z - s \cdot \Delta^{-1}(\partial_{\mathcal{C}/\mathcal{C}'} z))) \\
&= q_2(q_1(s \cdot z)) - q_2 q_1(s \cdot \Delta^{-1}(\partial_{\mathcal{C}/\mathcal{C}'} z)) \\
&= q_2(q_1(s \cdot z))
\end{aligned}
$$

because $s \cdot \Delta^{-1}(\partial_{\mathcal{C}/\mathcal{C}'} z)$ is in 100, which quotients to 0. If z is in 011, then

$$q_2(q_1(s \cdot i_1(i_2(z)))) = q_2(q_1(s \cdot i_1(z)))$$

$$= q_2(q_1(s \cdot (z - m^{-1}\partial_{\mathcal{C}} z)))$$

$$= q_2(q_1(s \cdot z)) - q_2 q_1(s \cdot m^{-1}(\partial_{\mathcal{C}} z))$$

$$= q_2(q_1(s \cdot z))$$

because $s \cdot m^{-1}(\partial_{\mathcal{C}} z)$ is labeled w_+, which quotients to 0. A similar argument shows that $s \cdot z = p_2(p_1(s \cdot z))$, if z is thought of as an element of $(\mathcal{D}/\mathcal{D}')/\mathcal{D}''$. Since f is the identity on the bottom face, it follows that $s \cdot f(z) = f(s \cdot z)$ for $z \in 000, 010, 001, 100$.

If z is on the top face, we need only consider the case $z \in 110$, since any element in 101 is equivalent to some $z \in 110$. Then in $(\mathcal{C}/\mathcal{C}')/\mathcal{C}''$,

$$s \cdot z = q_2(q_1(s \cdot i_1(i_2(z)))) = q_2(q_1(s \cdot i_1(z)))$$

$$= q_2(q_1(s \cdot (z - m^{-1}\partial_{\mathcal{C}} z)))$$

$$= q_2(q_1(s \cdot z)) - q_2 q_1(s \cdot m^{-1}(\partial_{\mathcal{C}} z))$$

$$= q_2(q_1(s \cdot z))$$

because $s \cdot m^{-1}(\partial_{\mathcal{C}} z)$ is labeled w_+, which quotients to 0. On the other hand, if we consider z as an element of 101 in $(\mathcal{D}/\mathcal{D}')/\mathcal{D}''$,

$$s \cdot z = p_2(p_1(s \cdot j_1(j_2(z)))) = p_2(p_1(s \cdot j_1(z)))$$

$$= p_2(p_1(s \cdot (z - m^{-1}\partial_{\mathcal{D}} z)))$$

$$= p_2(p_1(s \cdot z)) - p_2 p_1(s \cdot m^{-1}(\partial_{\mathcal{D}} z))$$

$$= p_2(p_1(s \cdot z))$$

Since f identically maps elements in 110 in $(\mathcal{C}/\mathcal{C}')/\mathcal{C}''$ to those in 101 in $(\mathcal{D}/\mathcal{D}')/\mathcal{D}''$, it follows that $s \cdot f(z) = f(s \cdot z)$ on the top face.

Step 5.2: The case $s \in \{v_2, v_{-2}, \tilde{v}_0, d, D\}$. Suppose that $s \in \{v_2, v_{-2}, \tilde{v}_0, d, D\}$. We again start with the case that z is on the bottom face of $(\mathcal{C}/\mathcal{C}')/\mathcal{C}''$. The cases $z \in 000$ and $z \in 011$ are straightforward to check, since f is the identity on the bottom face. If $z \in 001$, then in $(\mathcal{C}/\mathcal{C}')/\mathcal{C}''$,

$$
\begin{aligned}
q_2(q_1(s \cdot i_1(i_2(z)))) &= q_2(q_1(s \cdot i_1(z - \Delta^{-1}(\partial_{\mathcal{C}/\mathcal{C}'}z)))) \\
&= q_2(q_1(s \cdot z - s \cdot \Delta^{-1}(\partial_{\mathcal{C}/\mathcal{C}'}z))) \\
&= q_2(q_1(s \cdot z)) - q_2(q_1(s \cdot \Delta^{-1}(\partial_{\mathcal{C}/\mathcal{C}'}z))) \\
&= q_2(q_1(s_0 \cdot z + s_{*01} \cdot z + s_{0*1} \cdot z)) \\
&\quad - q_2(q_1(s_{10*} \cdot \Delta^{-1}\partial_{\mathcal{C}/\mathcal{C}'}z + s_{1*0} \cdot \Delta^{-1}\partial_{\mathcal{C}/\mathcal{C}'}z))
\end{aligned}
$$

and in $(\mathcal{D}/\mathcal{D}')/\mathcal{D}''$,

$$
s \cdot f(z) = p_2(p_1(s_0 \cdot f(z) + s_{*01} \cdot f(z) + s_{0*1} \cdot f(z)))
$$

and we must show that f maps the former to the latter. Indeed, in $(\mathcal{C}/\mathcal{C}')/\mathcal{C}''$, the terms $q_2q_1(s_{*01} \cdot z)$ and $q_2q_1(s_{10*} \cdot \Delta^{-1}(\partial_{\mathcal{C}/\mathcal{C}'}z))$ will cancel. This is because $\Delta^{-1}(\partial_{\mathcal{C}/\mathcal{C}'}z)$ has the same labeling as z, and both maps to 101 are split maps. Furthermore, $q_2q_1(s_{1*0} \cdot \Delta^{-1}(\partial_{\mathcal{C}/\mathcal{C}'}z))$ in $(\mathcal{C}/\mathcal{C}')/\mathcal{C}'''$ will be mapped via f to $s_{*01} \cdot f(z)$. This is because $\Delta^{-1}(\partial_{\mathcal{C}/\mathcal{C}'}z)$ has the same labeling as z and the maps ∂_{1*0} in $(\mathcal{C}/\mathcal{C}')/\mathcal{C}''$ and ∂_{*01} in $(\mathcal{D}/\mathcal{D}')/\mathcal{D}''$ are of the same type (i.e. they are either both merge or both split), meaning s will act the same across these maps. The case $z \in 010$ is analogous. Next, suppose that z is in the top face of the cube. If

89

$z \in 110$, then

$$s \cdot z = q_2(q_1(s \cdot i_1(i_2(z)))) = q_2(q_1(s \cdot i_1(z)))$$
$$= q_2(q_1(s \cdot (z - m^{-1}\partial_{\mathcal{C}} z)))$$
$$= q_2(q_1(s_{110} \cdot z)) - q_2 q_1(s_{101} \cdot m^{-1}(\partial_{\mathcal{C}} z))$$
$$= q_2(q_1(s_{110} \cdot z))$$

where $s_{110} \cdot z$ is the part of $s \cdot z$ that remains in 110 and $s_{101} \cdot m^{-1}(\partial_{\mathcal{C}} z)$ is the part of $s \cdot m^{-1}(\partial_{\mathcal{C}} z)$ that remains in 101. But the latter quotients to 0, as it is labeled by w_+. On the other hand, if we consider z as an element of 101 in $(\mathcal{D}/\mathcal{D}')/\mathcal{D}''$,

$$s \cdot z = p_2(p_1(s \cdot j_1(j_2(z)))) = p_2(p_1(s \cdot j_1(z)))$$
$$= p_2(p_1(s \cdot (z - m^{-1}\partial_{\mathcal{D}} z)))$$
$$= p_2(p_1(s_{101} \cdot z)) - p_2 p_1(s_{110} \cdot m^{-1}(\partial_{\mathcal{D}} z))$$
$$= p_2(p_1(s_{101} \cdot z))$$

where $s_{101} \cdot z$ is the part of $s \cdot z$ that remains in 101 and $s_{110} \cdot m^{-1}(\partial_{\mathcal{D}} z)$ is the part of $s \cdot m^{-1}(\partial_{\mathcal{D}} z)$ that remains in 110. Similar to before, the latter quotients to 0, as it is labeled by w_+. Since f identically maps elements in 110 in $(\mathcal{C}/\mathcal{C}')/\mathcal{C}''$ to those in 101 in $(\mathcal{D}/\mathcal{D}')/\mathcal{D}''$, we conclude that $s \cdot f(z) = f(s \cdot z)$ on the top face.

Step 5.3: The case $s = x = -[d, D]$. Finally, suppose that $s = x = -[d, D]$. For $z \in (\mathcal{C}/\mathcal{C}')/\mathcal{C}''$,

$$f(s \cdot z) = f((-dD - Dd) \cdot z) = -df(D \cdot z) - Df(d \cdot z) = (-dD - Dd) \cdot f(z) = s \cdot f(z)$$

Step 5.4: Conclusion. To summarize, we have shown that for every element s in a basis of $\mathfrak{sl}_2(\wedge)_{dg}$, $f(s \cdot z) = s \cdot f(z)$. We conclude that $f : (\mathcal{C}/\mathcal{C}')/\mathcal{C}'' \to (\mathcal{D}/\mathcal{D}')/\mathcal{D}''$ is an L_∞-module quasi-isomorphism, and so up to quasi-isomorphism, the L_∞-module structure on $\text{CKh}(L)$ is invariant under the Reidemeister III move.

$\square$

3.6.2. Invariance of the $\mathfrak{sl}_2(\wedge)$-module structure

Now that we have shown the invariance of the $\mathfrak{sl}_2(\wedge)_{dg}$ L_∞-module structure on $\text{CKh}(L)$ under Reidemeister moves, we can show that the $\mathfrak{sl}_2(\wedge)$ L_∞-module structure on $\text{CKh}(L)$ is invariant as well.

Theorem 8. *Up to L_∞-quasi-isomorphism, the $\mathfrak{sl}_2(\wedge)$ L_∞-module structure is invariant under Reidemeister moves.*

Proof. This follows from the fact that the $\mathfrak{sl}_2(\wedge)$ L_∞-module structure on $\text{CKh}(L)$ was obtained from the $\mathfrak{sl}_2(\wedge)_{dg}$ L_∞-module structure by restricting scalars through an L_∞-algebra homomorphism $I : H(\mathfrak{sl}_2(\wedge)_{dg}) \to \mathfrak{sl}_2(\wedge)_{dg}$. In particular, restricton of scalars preserves L_∞-quasi-isomorphisms (see [Dav22]), so applying the restriction of scalars functor to the quasi-isomorphisms constructed in the proof of invariance for $\mathfrak{sl}_2(\wedge)_{dg}$ yields quasi-isomorphisms of these complexes considered as L_∞-modules over $H(\mathfrak{sl}_2(\wedge))$. Finally, the $\mathfrak{sl}_2(\wedge)$ L_∞-module structure is invariant, since $\mathfrak{sl}_2(\wedge)$ is an L_∞-subalgebra of $H(\mathfrak{sl}_2(\wedge)_{dg})$. $\square$

3.7. The L_∞-module structure on $\text{AKh}(L)$

In this section, we explain how the annular Khovanov homology $\text{AKh}(L)$ has an L_∞-module structure that is invariant under Reidemeister moves.

Theorem 9. *Let L be an annular link. There is an L_∞-module structure on $AKh(L)$, invariant under Reidemeister moves. It is well-defined up to L_∞-quasi-isomorphism.*

Proof. The situation can be summarized by the following diagram.

$$
\begin{array}{ccc}
\mathfrak{sl}_2(\wedge)_{dg} & \longrightarrow & \mathrm{CKh}(L) \\
\downarrow & \nearrow & \downarrow \\
\mathfrak{sl}_2(\wedge) & \dashrightarrow & \mathrm{AKh}(L)
\end{array}
$$

Theorem 8 proved that, up to L_∞-quasi-isomorphism, the L_∞-module structure on $\mathrm{CKh}(L)$ over $\mathfrak{sl}_2(\wedge)$ is invariant under Reidemeister moves. By Theorem 2, $\mathrm{AKh}(L)$ inherits an L_∞-module structure over $\mathfrak{sl}_2(\wedge)$ via any choice of chain contraction $\mathrm{CKh}(L) \to \mathrm{AKh}(L)$. By Lemma 5, $\mathrm{AKh}(L)$ is quasi-isomorphic to $\mathrm{CKh}(L)$, so if L and L' differ by Reidemeister moves, we have the following diagram:

$$
\begin{array}{ccc}
\mathrm{CKh}(L) & \overset{\cong}{\Longrightarrow} & \mathrm{CKh}(L') \\
\downarrow{\scriptstyle\cong} & & \downarrow{\scriptstyle\cong} \\
\mathrm{AKh}(L) & \dashrightarrow & \mathrm{AKh}(L')
\end{array}
$$

This shows that $\mathrm{AKh}(L)$ and $\mathrm{AKh}(L')$ are quasi-isomorphic as L_∞-modules over $\mathfrak{sl}_2(\wedge)$, and so this L_∞-module structure is well-defined up to L_∞-quasi-isomorphism. $\square$

3.8. Examples

In this section, we explore the L_∞-module structure of several knots and links.

Example 7. Let L be any link in S^3 where ∂^{Lee} is nonzero on Khovanov homology. We may view L as an annular link by placing the basepoint away

from the link. If we denote the L_∞-module operation on $\mathrm{AKh}(L)$ by k_n, ∂_0^{Lee} will yield a corresponding nontrivial $k_3(v_2, v_{-2}, m)$ on $\mathrm{AKh}(L)$. Indeed, the L_∞-module structure on $\mathrm{AKh}(L)$ is induced from a cochain contraction $T \mathrel{\reflectbox{$\circlearrowright$}} \mathrm{CKh}(L) \underset{i}{\overset{q}{\rightleftarrows}} \mathrm{AKh}(L)$. If k_n' is the L_∞-module operation on $\mathrm{CKh}(L)$, the following equation gives a formula for $k_3(x_1, x_2, m)$.

$$k_3(x_1, x_2, m) = k_3'(x_1, x_2, m) + q \circ k_2'(x_1, T \circ k_2(x_2, i(m))) + q \circ k_2'(x_2, T \circ k_2(x_1, i(m)))$$

The k_2' operations vanish because all of the circles involved are trivial.

Example 8. In the above example, suppose we put an unknot U around the basepoint. Let $w \in \mathrm{AKh}(L)$ be a generator on which ∂_0^{Lee} acts nontrivially. After choosing a cochain contraction that respects $\mathrm{CKh}(U \sqcup L) = V \otimes \mathrm{CKh}(L)$, then in $\mathrm{AKh}(U \sqcup L)$, the generators $v_\pm \otimes w$ have both nontrivial k_2 and k_3 actions.

Example 9. The left-handed trefoil with the basepoint in the center is an example of a knot K where $\mathrm{AKh}(K)$ has both nontrivial k_2 and k_3 operations; see Figure 21.

Indeed, in resolution 000, $k_3(v_2, v_{-2}, w_- \otimes w_- \otimes w_-) = \partial_0^{Lee}(w_- \otimes w_- \otimes w_-)$ is nonzero in homology. Also, the usual module action of $\mathfrak{sl}_2(\wedge)$ acts nontrivially on the generator $v_+ \otimes v_+$ in resolution 111. Notice that the mirror (the right-handed trefoil) does not have a nontrivial k_3 operation in the lowest homological degree.

Example 10. The above example generalizes to any torus knot or link where the basepoint is in the center. If every boundary map coming from the lowest homological degree is a merge map, the resolution with each circle labeled w_- will have a nontrivial k_3 operation, and the module will act nontrivially on a generator in the highest homological degree.
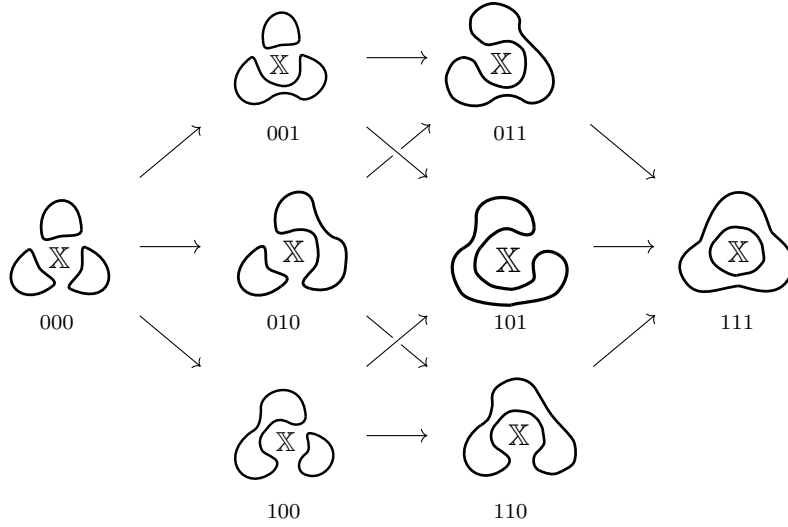
FIGURE 21. The cube of resolutions for the left-handed trefoil knot with basepoint in the center.

The examples above illustrate that for an annular link L, AKh(L) can have both nontrivial k_2 and k_3 operations. In the case where L is a split link (i.e., at least one component is disjoint), it is further possible for a specific generator to have both nontrivial k_2 and k_3 operations. On the other hand, we end this section with a question regarding non-split links.

Question 1. *Does there exist a non-split link $L \subset A \times I$ such that AKh(L) contains a homology class on which the k_2 and k_3 operations of $\mathfrak{sl}_2(\wedge)$ are nontrivial? In other words, for a non-split annular link L, can there exist $m \in$ AKh(L) and $x, y_1, y_2 \in \mathfrak{sl}_2(\wedge)$ such that $k_2(x, m) \neq 0$ and $k_3(y_1, y_2, m) \neq 0$?*

CHAPTER IV

COMPUTATIONS

4.1. Overview

In trying to further understand the L_∞-module structure on the annular Khovanov homology of a given knot or link, implementing a computer program to compute the annular Khovanov homology has been beneficial.

Given a knot or link diagram, we can record the diagram as follows. First, number both the crossings and the arcs in the diagram. Each crossing then corresponds to an array of four numbers. This array is obtained by listing the surrounding arcs in clockwise order, starting with one of the arcs that is part of the understrand. The collection of all of the crossing arrays is enough to determine the knot or link diagram. The basepoint is recorded by drawing an arc from the basepoint to infinity, documenting the arcs it passes through in a separate array. This method of inputting knots has been used by others to compute ordinary Khovanov homology; see, for example, the KnotTheory package in Mathematica [Kno11].

Having inputted the knot or link, executing the program will generate the annular Khovanov chain complex. The computation of the homology is performed by importing these complexes into the computational algebra system, Sage. The program can also compute the gradings of the generators that are outputted by Sage during the homology computation. This computation requires some additional user input, such as the number of positive and negative crossings.

This section presents a selection of data generated by the code. Data is available for all knots with up to 8 crossings and all links with up to 7 crossings. For knots with more than 11 crossings, the computational demands begin to render the program impractical.

To illustrate the patterns that emerge, we have included two examples: the 7_3 knot and the Borromean rings. These examples demonstrate general patterns that are observed in the data.

4.2.1. The knot 7_3
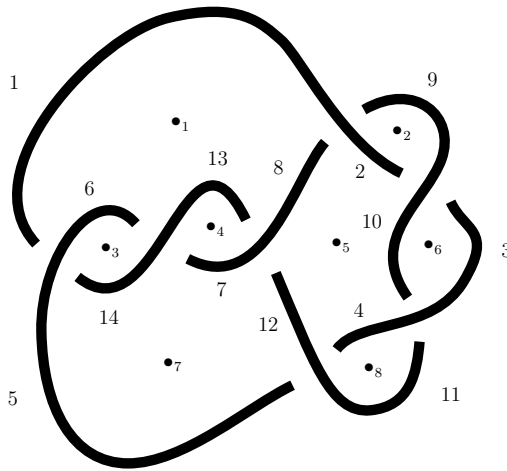
FIGURE 22. A diagram for the knot 7_3. The arcs are labeled, as well as the possible locations of the basepoint.

We can see how the code works by studying the knot 7_3, shown in Figure 22. After labeling the crossings, the crossing array might be inputted as the following code.

```
1   crossings =
    [[9,2,8,1],[14,5,1,6],[6,13,7,14],[13,8,12,7],[2,9,3,10],
    [10,3,11,4],[4,11,5,12]]
```

On the other hand, we can document the basepoint by drawing a line from the basepoint to infinity and recording the arcs that this line intersects. For example, to record basepoint 5, we might input the following code.

```
1   specialarcs = [4,11]
```

Executing the program produces the data in Figure 23.

0	1	2	3	4	5	6	7
$\mathbb{Z}^2$	$\mathbb{Z}$	$\mathbb{Z}^2 \times C_2$	$\mathbb{Z}^2 \times C_2$	$\mathbb{Z}^3 \times C_2$	$\mathbb{Z}^2 \times C_2 \times C_2$	$\mathbb{Z}$	$\mathbb{Z} \times C_2$
$\mathbb{Z}^2$	$\mathbb{Z}^2$	$\mathbb{Z}^4$	$\mathbb{Z}^4$	$\mathbb{Z}^6$	$\mathbb{Z}^4$	$\mathbb{Z}^2$	$\mathbb{Z}^2$
$\mathbb{Z}^2$	$\mathbb{Z}^2$	$\mathbb{Z}^4$	$\mathbb{Z}^4$	$\mathbb{Z}^6$	$\mathbb{Z}^4$	$\mathbb{Z}^2$	$\mathbb{Z}^2$
$\mathbb{Z}^3$	$\mathbb{Z}^4$	$\mathbb{Z}^7$	$\mathbb{Z}^8$	$\mathbb{Z}^9$	$\mathbb{Z}^6 \times C_2$	$\mathbb{Z}^2$	$\mathbb{Z} \times C_2$
$\mathbb{Z}^4$	$\mathbb{Z}^6$	$\mathbb{Z}^8$	$\mathbb{Z}^8$	$\mathbb{Z}^9$	$\mathbb{Z}^6 \times C_2$	$\mathbb{Z}^2$	$\mathbb{Z} \times C_2$
$\mathbb{Z}^3$	$\mathbb{Z}^4$	$\mathbb{Z}^5$	$\mathbb{Z}^3 \times C_2$	$\mathbb{Z}^3 \times C_2$	$\mathbb{Z}^2 \times C_2 \times C_2$	$\mathbb{Z}$	$\mathbb{Z} \times C_2$
$\mathbb{Z}^2$	$\mathbb{Z}^2$	$\mathbb{Z}^4$	$\mathbb{Z}^4$	$\mathbb{Z}^6$	$\mathbb{Z}^4$	$\mathbb{Z}^2$	$\mathbb{Z}^2$
$\mathbb{Z}^2$	$\mathbb{Z}^2$	$\mathbb{Z}^4$	$\mathbb{Z}^4$	$\mathbb{Z}^6$	$\mathbb{Z}^4$	$\mathbb{Z}^2$	$\mathbb{Z}^2$
$\mathbb{Z}^2$	$\mathbb{Z}^2$	$\mathbb{Z}^4$	$\mathbb{Z}^4$	$\mathbb{Z}^6$	$\mathbb{Z}^4$	$\mathbb{Z}^2$	$\mathbb{Z}^2$

FIGURE 23. The result of executing the program for the knot 7_3. Each row corresponds to the annular Khovanov homology of 7_3 with respect to a particular basepoint. The first row is the ordinary Khovanov homology. The second row corresponds to basepoint 1 in Figure 22. The third row corresponds to the homology computed with respect to basepoint 2 in Figure 22, and so on. The various columns represent the various homological degrees.

4.2.2. Borromean Rings

The program can also compute the annular Khovanov homology for links. One particular example is the Borromean rings; see Figure 24.
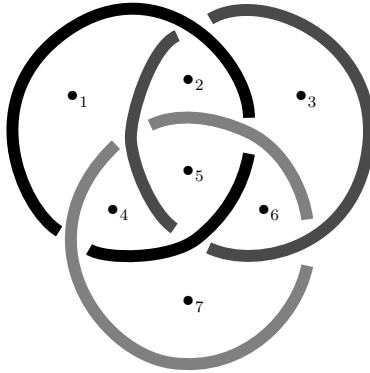
FIGURE 24. A diagram for the Borromean rings, also known as the link $L6a4$. The possible locations for the basepoint are labeled.

The result of executing the code is presented in the table in Figure 26.

-3	-2	-1	0	1	2	3
$\mathbb{Z}$	$\mathbb{Z}^3 \times C_2$	$\mathbb{Z}^2 \times C_2 \times C_2$	$\mathbb{Z}^8$	$\mathbb{Z}^2$	$\mathbb{Z}^3 \times C_2 \times C_2$	$\mathbb{Z} \times C_2$
$\mathbb{Z}^2$	$\mathbb{Z}^6$	$\mathbb{Z}^4$	$\mathbb{Z}^8$	$\mathbb{Z}^4$	$\mathbb{Z}^6$	$\mathbb{Z}^2$
$\mathbb{Z}$	$\mathbb{Z}^3 \times C_2$	$\mathbb{Z}^4 \times C_2 \times C_2$	$\mathbb{Z}^{12}$	$\mathbb{Z}^9$	$\mathbb{Z}^{10}$	$\mathbb{Z}^3$
$\mathbb{Z}^2$	$\mathbb{Z}^6$	$\mathbb{Z}^4$	$\mathbb{Z}^8$	$\mathbb{Z}^4$	$\mathbb{Z}^6$	$\mathbb{Z}^2$
$\mathbb{Z}$	$\mathbb{Z}^3 \times C_2$	$\mathbb{Z}^4 \times C_2 \times C_2$	$\mathbb{Z}^{12}$	$\mathbb{Z}^9$	$\mathbb{Z}^{10}$	$\mathbb{Z}^3$
$\mathbb{Z}^2$	$\mathbb{Z}^6$	$\mathbb{Z}^6$	$\mathbb{Z}^{12}$	$\mathbb{Z}^6$	$\mathbb{Z}^6$	$\mathbb{Z}^2$
$\mathbb{Z}$	$\mathbb{Z}^3 \times C_2$	$\mathbb{Z}^4 \times C_2 \times C_2$	$\mathbb{Z}^{12}$	$\mathbb{Z}^9$	$\mathbb{Z}^{10}$	$\mathbb{Z}^3$
$\mathbb{Z}^2$	$\mathbb{Z}^6$	$\mathbb{Z}^4$	$\mathbb{Z}^8$	$\mathbb{Z}^4$	$\mathbb{Z}^6$	$\mathbb{Z}^2$

FIGURE 25. The result of executing the program for the link $L6a4$. Each row corresponds to the annular Khovanov homology of $L6a4$ with respect to a particular basepoint. The first row is the ordinary Khovanov homology. The second row corresponds to basepoint 1 in Figure 24. The third row corresponds to the homology computed with respect to basepoint 2 in Figure 24, and so on. The various columns represent the various homological degrees.

As mentioned, it is possible to extract the gradings of the generators of the above homology groups. For example, if we select basepoint 1, we obtain data in Figure 26.

	-3	-2	-1	0	1	2	3
6							1 -1
4						1,1,1 -1,-1,-1	
2					1,1,1 -1,-1,-1		
0				3 1,1,1,1,1 -1,-1,-1,-1,-1 -3			
-2			1,1,1 -1,-1,-1				
-4		1,1,1 -1,-1,-1					
-6	1 -1						

FIGURE 26. The gradings of the generators in the annular Khovanov homology of the Borromean rings with basepoint 1, as in Figure 24. The columns represent the homological gradings and the rows represent the filtration-adjusted quantum gradings, as described in [GLW18]. Each cell contains the k-gradings of the generators in a particular homological grading and filtration-adjusted quantum grading.

99

	-3	-2	-1	0	1	2	3
7							$\mathbb{Z}$
5						$\mathbb{Z}^2$	$\mathbb{Z}_2$
3						$\mathbb{Z} \oplus \mathbb{Z}_2^2$	
1				$\mathbb{Z}^4$	$\mathbb{Z}^2$		
-1			$\mathbb{Z}^2$	$\mathbb{Z}^4$			
-3		$\mathbb{Z}$	$\mathbb{Z}_2^2$				
-5		$\mathbb{Z}^2 \oplus \mathbb{Z}_2$					
-7	$\mathbb{Z}$						

FIGURE 27. The ordinary integral Khovanov homology of the Borromean rings. The columns represent the homological gradings and the rows represent the quantum gradings. Each cell contains the homology group present in that particular homological grading and quantum grading. This data was computed with Mathematica, using the KnotTheory package [Kno11]

.

It is perhaps instructive to compare this data to the ordinary Khovanov homology of the Borromean rings; see Figure 27. For one, it gives a way to understand the spectral sequence from annular Khovanov homology to ordinary Khovanov homology.

4.3. Observations

These two examples highlight a general phenomenon. In particular, the annular Khovanov homologies with torsion correspond to basepoints with respect to which the knot or link has even winding number—that is, basepoints where we can draw an arc from the basepoint to infinity intersecting the knot or link an even number of times. Another observation that we can make is that it seems as if more torsion is occuring the more crossings the region containing the basepoint abuts.

In the case of the first observation, we would ideally like to make the claim that if a knot or link has an odd winding number around the basepoint, then this

implies that there is no torsion in the annular Khovanov homology. However, the knot 8_{19} is the only known counterexample to this claim. In particular, 8_{19} has winding number three around one of the possible basepoints, but the annular Khovanov homology has 3-torsion. It is perhaps worth noting that 8_{19} is not alternating, and it also has 3-torsion in its odd Khovanov homology. Therefore, we can amend the claim in several ways. The first way is to simply make a conjecture about alternating knots, as 8_{19} is non-alternating.

Conjecture 1. *If a non-split alternating link has an odd winding number around the basepoint, then its annular Khovanov homology has no torsion.*

Alternatively, because 8_{19} has winding number three around the basepoint in the counterexample, it is also possible to make the following conjecture.

Conjecture 2. *If a non-split link has winding number one around the basepoint, then its annular Khovanov homology has no torsion.*

Finally, because the torsion involved in the 8_{19} counterexample is 3-torsion, we could also conjecture the following.

Conjecture 3. *If a non-split link has an odd winding number around the basepoint, then its annular Khovanov homology has no 2-torsion.*

A search for counterexamples has begun with connected sums of knots, though not much is known at this point.

APPENDIX A

RESTRICTION OF SCALARS: COMPOSITION

This appendix contains graphical representations of the formulas presented in the proof of Lemma 1.

Step 1

Step 2

Step 3

Step 5

103

Step 6

Step 8

104
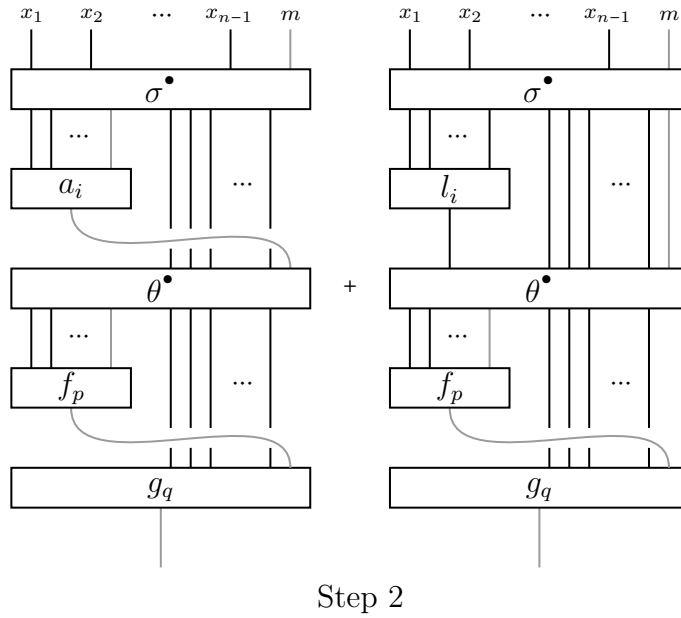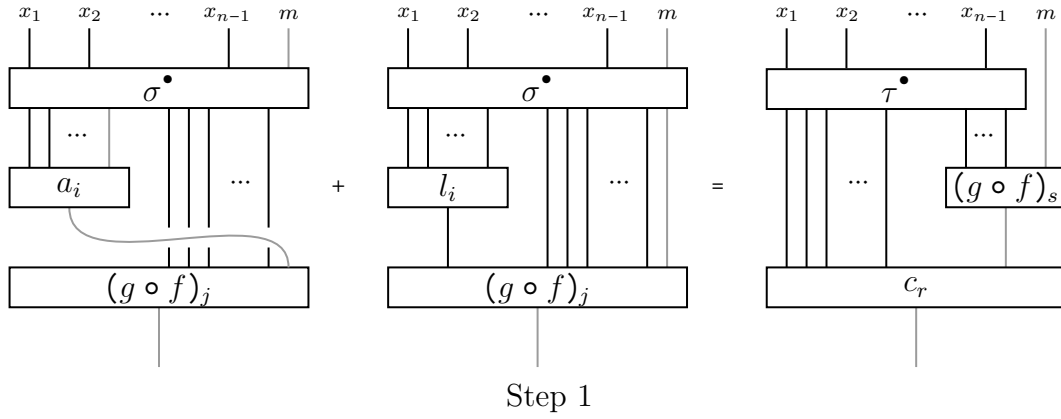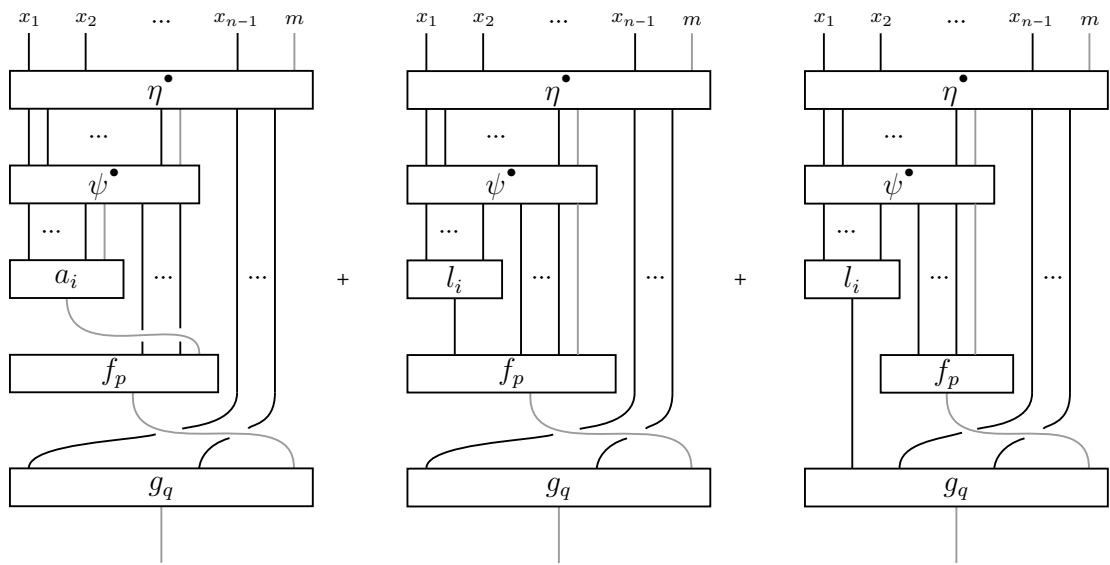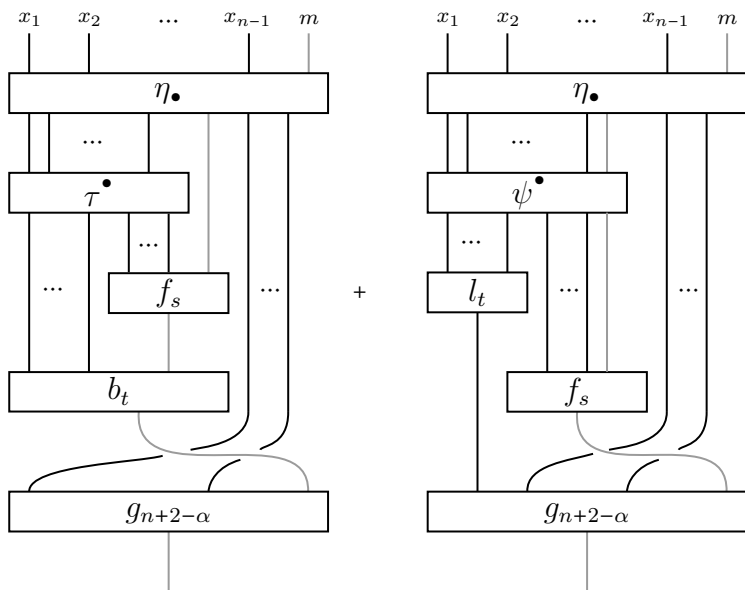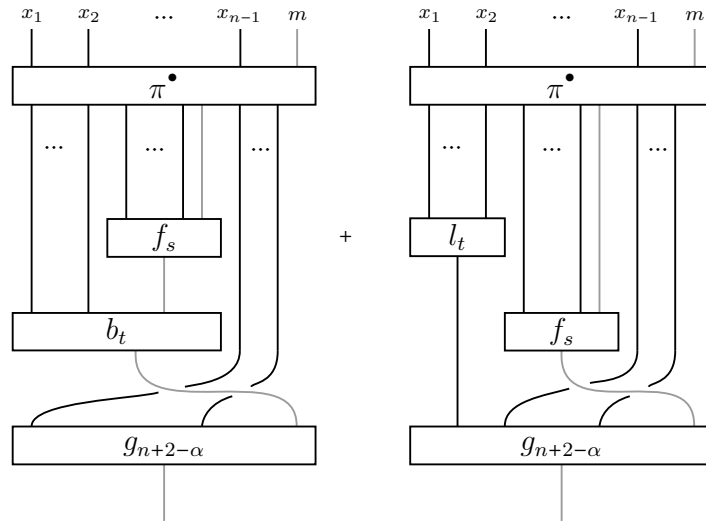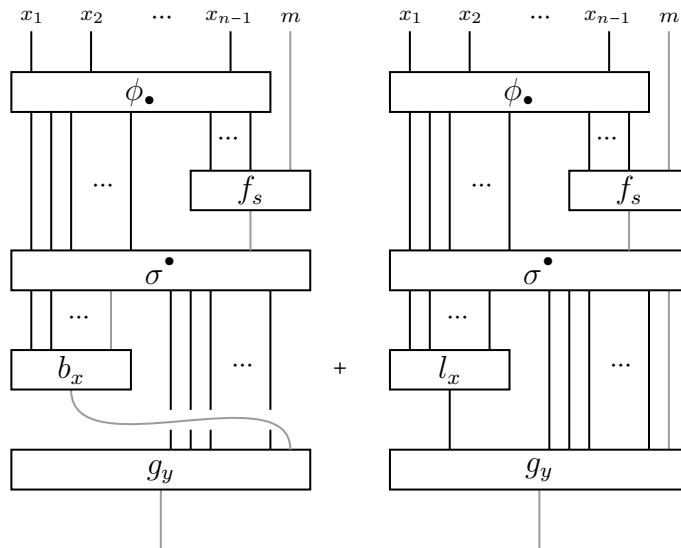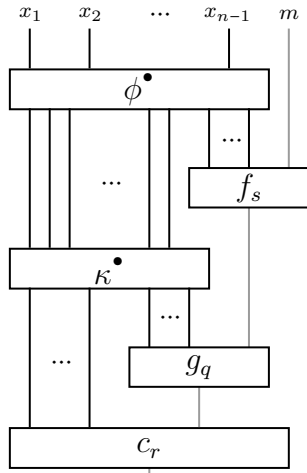
Step 9
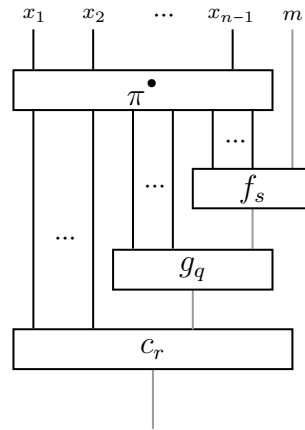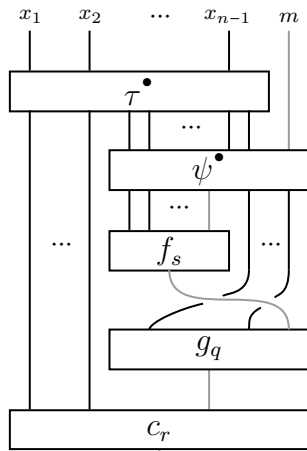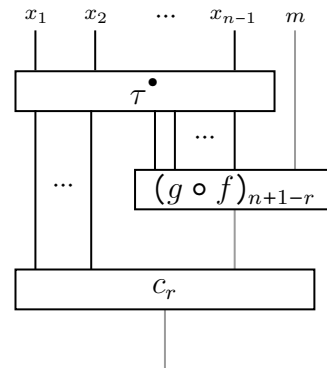
Step 10

Step 11

Step 13

APPENDIX B

RESTRICTION OF SCALARS: OBJECTS

This appendix contains graphical representations of the formulas presented in the proof of Lemma 2.

Step 1

Step 2
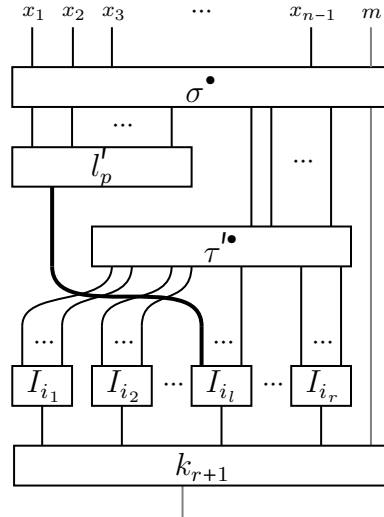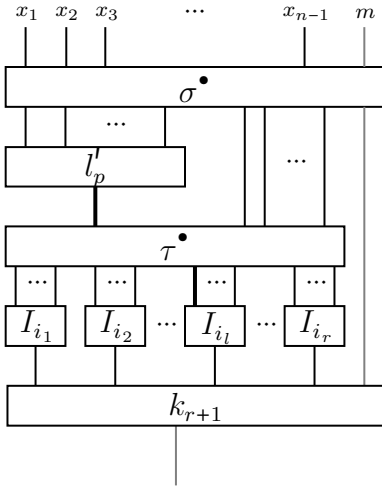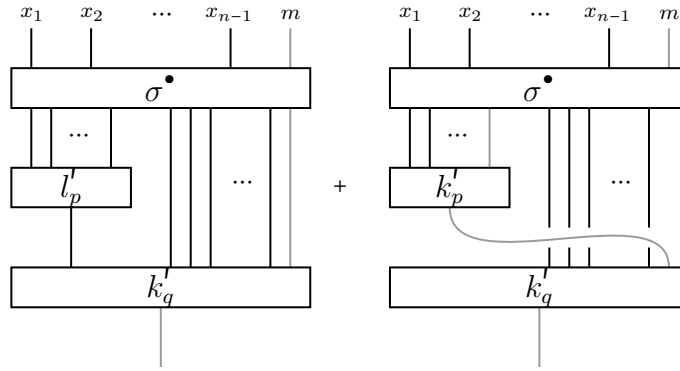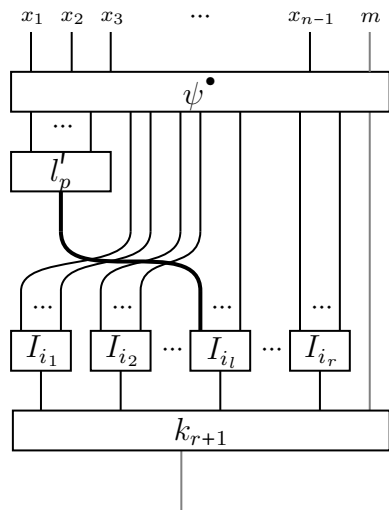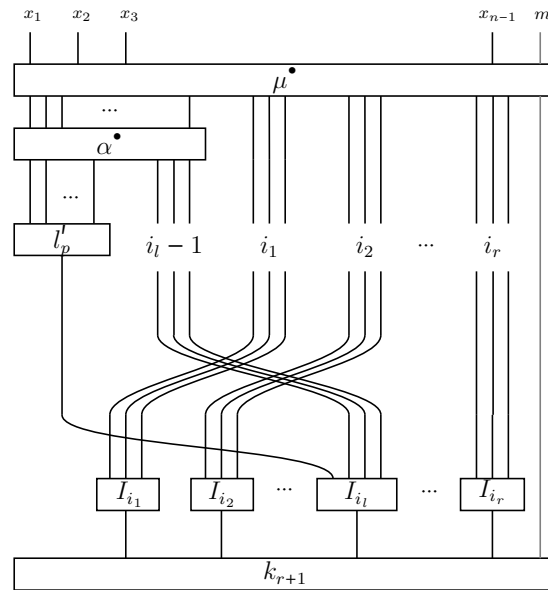
Step 4
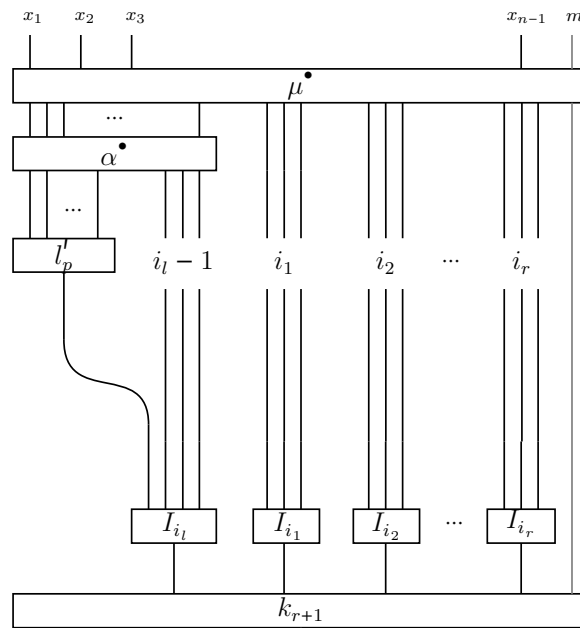
Step 5

Step 6

Step 7

107
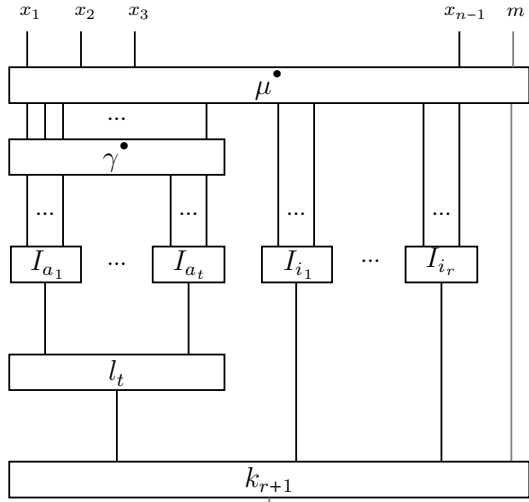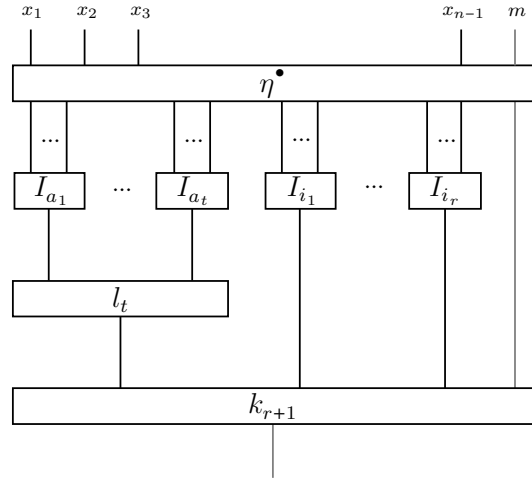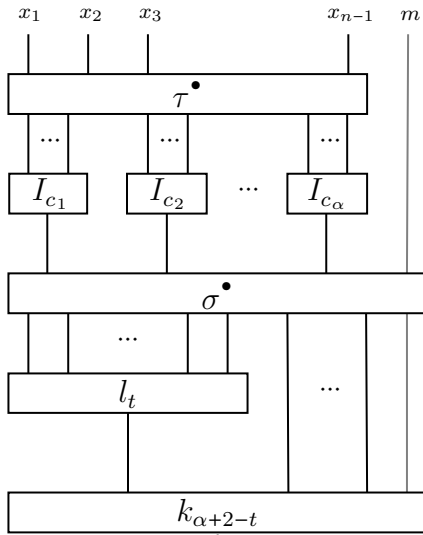
Step 10

Step 11

Step 12
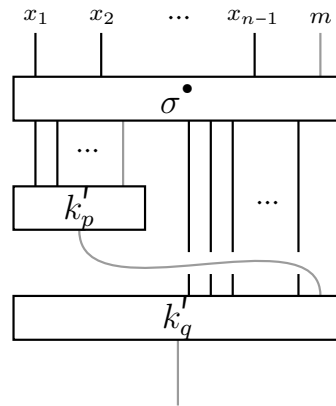
Step 13
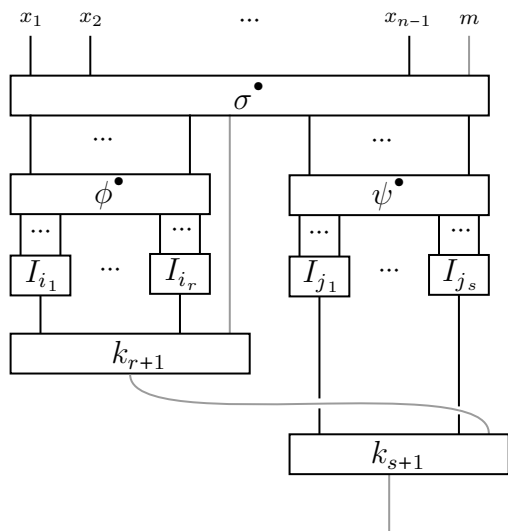
Step 15
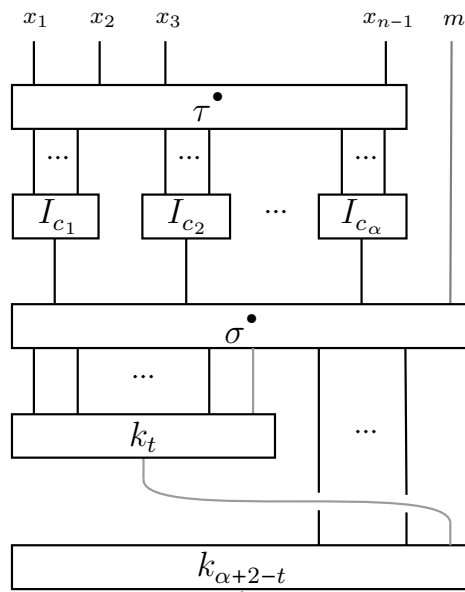
Step 17
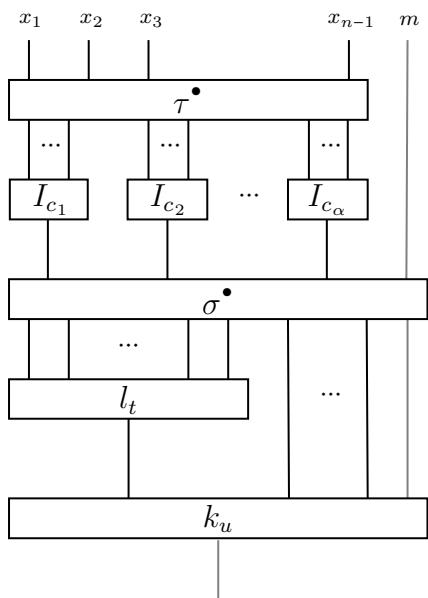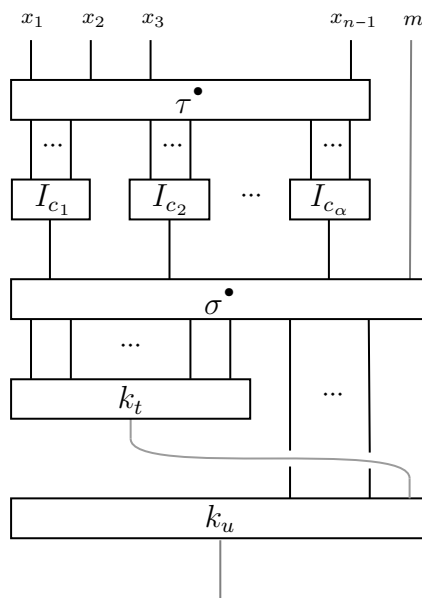
Step 19

109

APPENDIX C

RESTRICTION OF SCALARS: MORPHISMS

This appendix contains graphical representations of the formulas presented in the proof of Lemma 3.

Step 1

Step 3

Step 4

Step 5

Step 6

Step 7

111

Step 9

Step 10

Step 11

112

Step 12

Step 13

Step 15

113
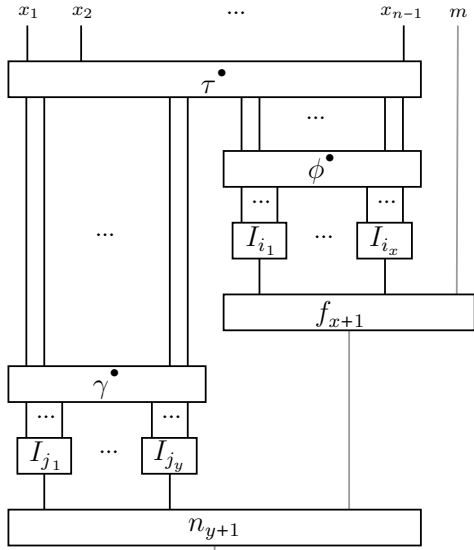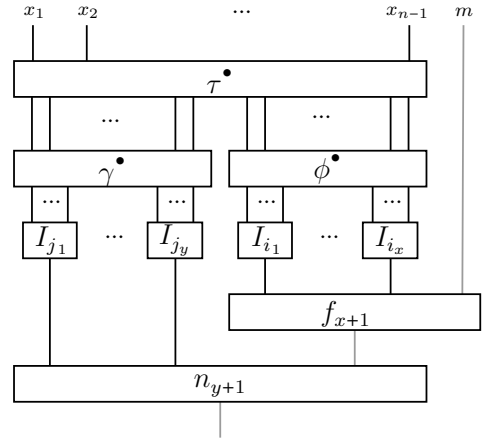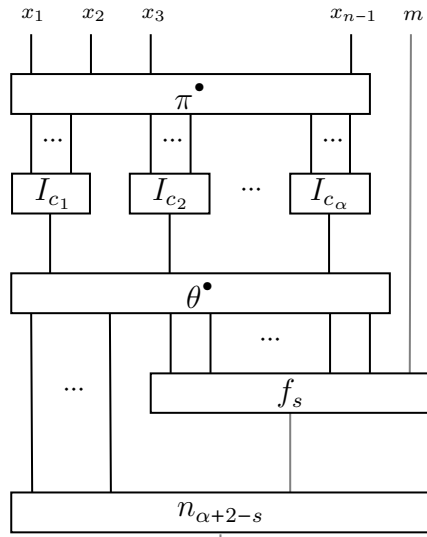
RESTRICTION OF SCALARS: FUNCTORIALITY

This appendix contains graphical representations of the formulas presented in the proof of Theorem 1.
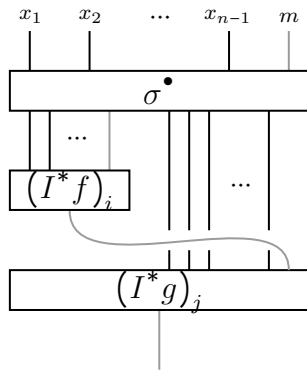
Step 1
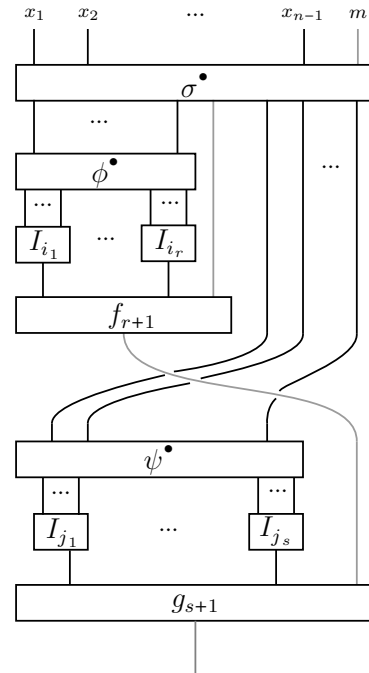
Step 2

Step 3

Step 5

Step 6

115
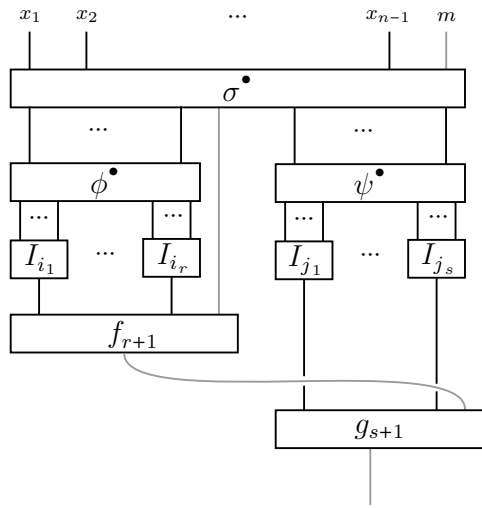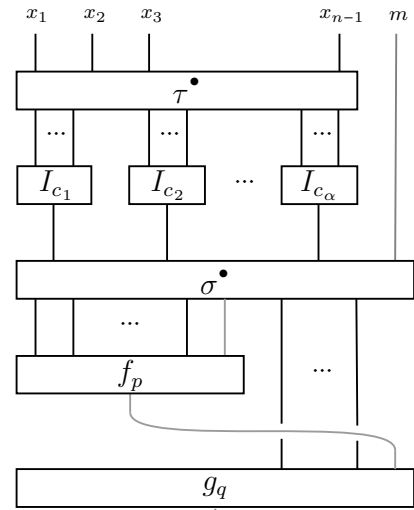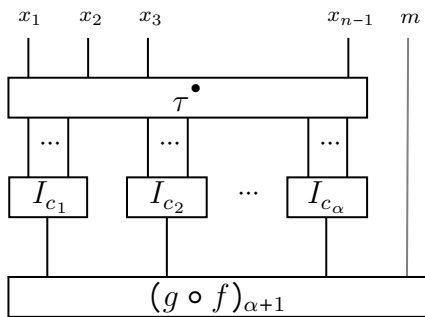
TRANSFER OF STRUCTURE VIA CHAIN CONTRACTIONS

This appendix contains graphical representations of the formulas presented in the proof of Theorem 2.
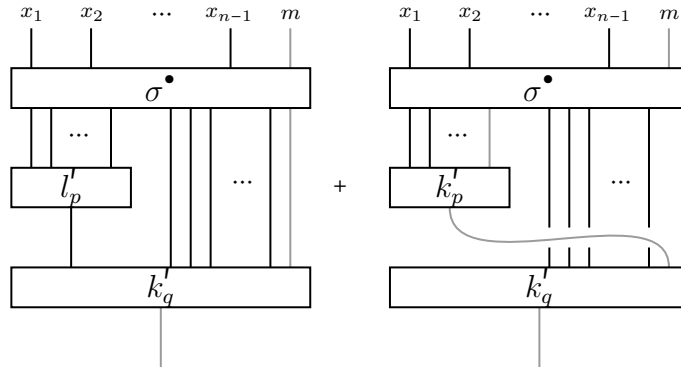
FIGURE 28. A graphical depiction of the L_∞-module relation, as in [Dav22].
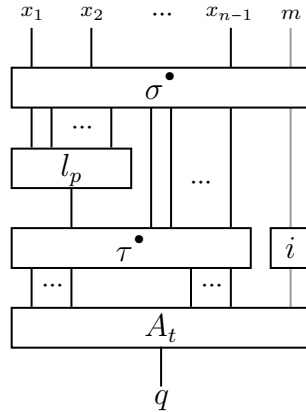
Step 1. We start with the terms on the left-hand side of the L_∞-module relation and replace k'_q with its definition.
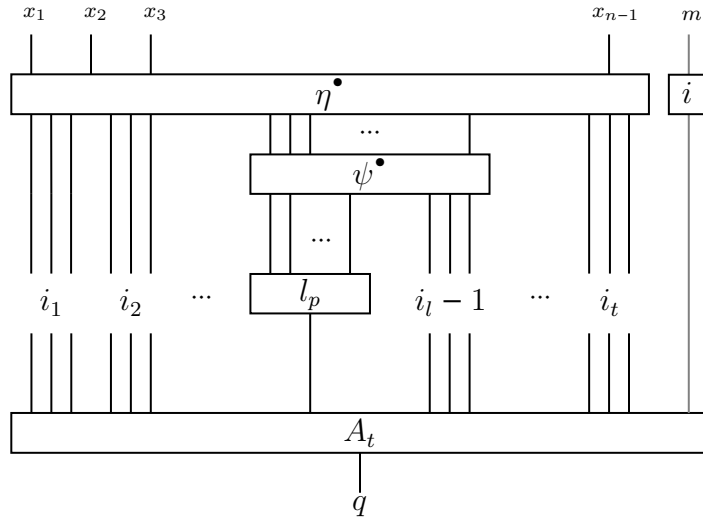
Step 2. By the definition of unshuffle, the l_p term in Step 1 goes to the first element in one of the boxes of size $i_1, \ldots, i_t$ determined by τ. This observation allows us to combine σ and τ into an unshuffle η.
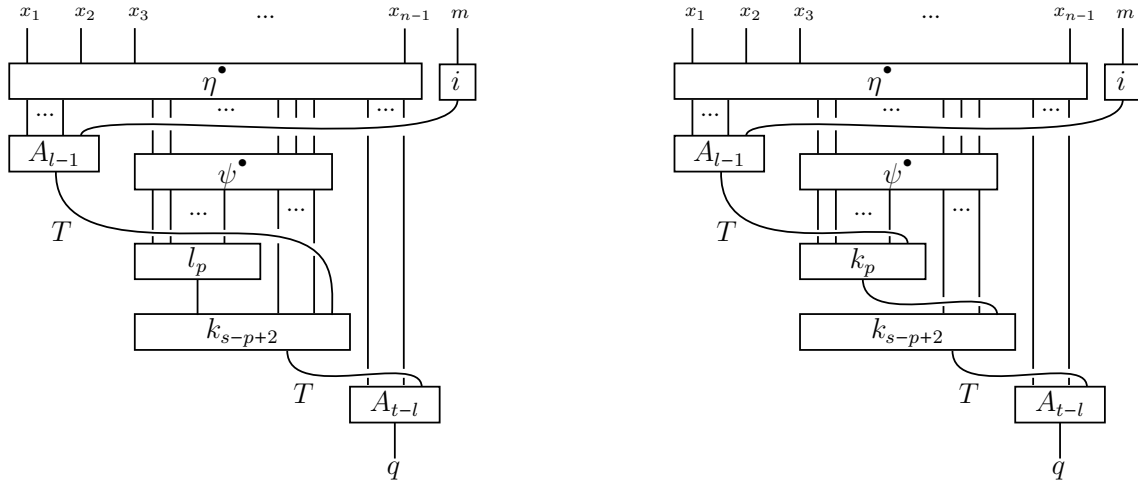
Step 3. After unpacking the definition of A_t, the left-hand side in the above figure represents the second term in the proof. The cases where $l = 1$, $l = t$, and $p = n - 1$ are not pictured here. We obtain the right-hand side after applying the L_∞-module relation.
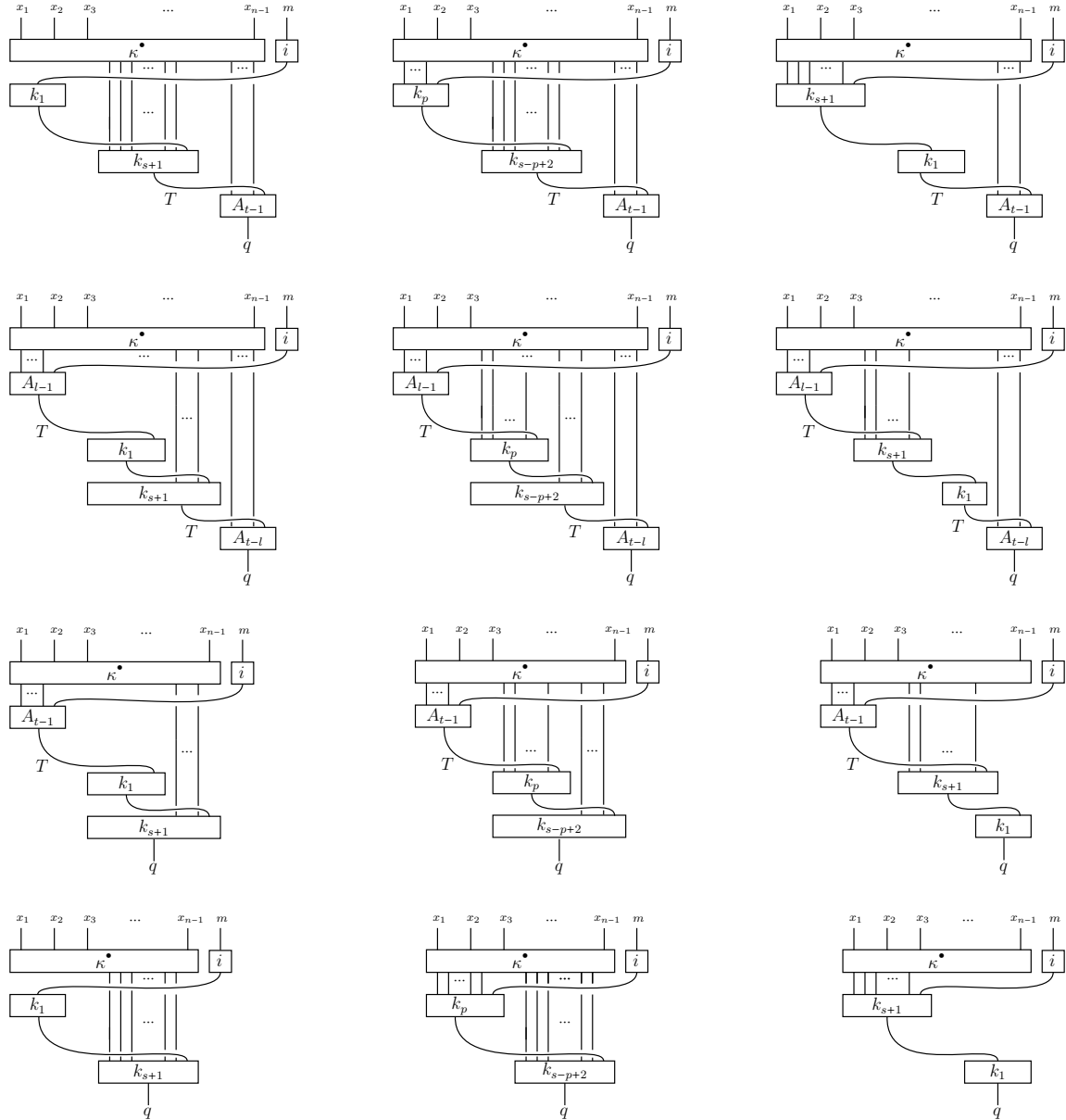
Step 4. Combine the permuations ψ and η into κ. There are four terms in step three, and each row in this figure represents one of those terms, where the cases $p = 1$, $2 \leq p \leq s$, and $p = s + 1$ are considered separately (pictured left to right). For $1 < p < s + 1$, we may combine the k_p and k_{s-p+2} operations into the A_t operation to obtain the formulas in Step 4.

118
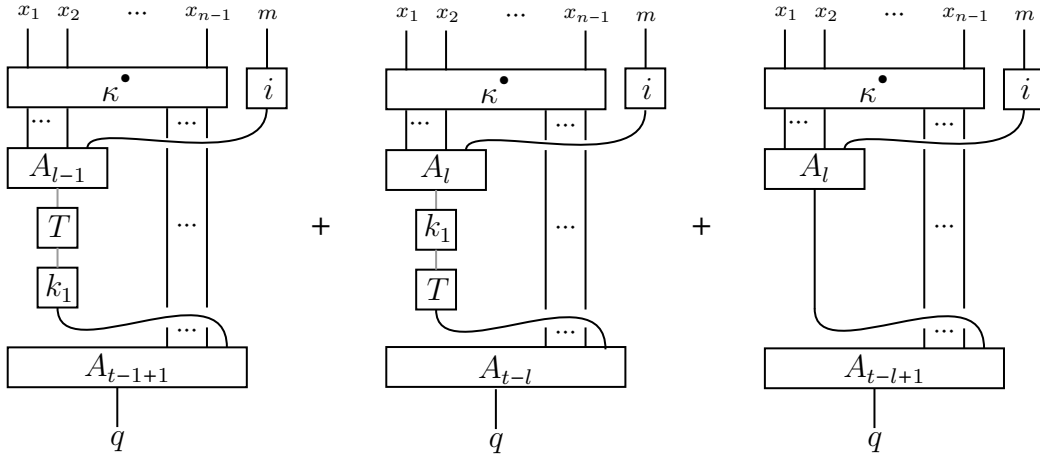
Step 5. We can combine some of the terms in Step 4. In the graphic for Step 4 above, label the terms in the first row by 1, 2, 3, the terms in the second row by 4, 5, 6, the terms in the third row by 7, 8, 9, and the terms in the last row by 10, 11, 12. Then terms 4 and 7 combine to give the first term above on the left. The middle term is obtained by combining terms 3 and 6. The last term is obtained by combining terms 2, 5, 8, and 11. Moreover, the terms 1 and 10 combine, and so too do 9 and 12, but these two cases are not pictured here.
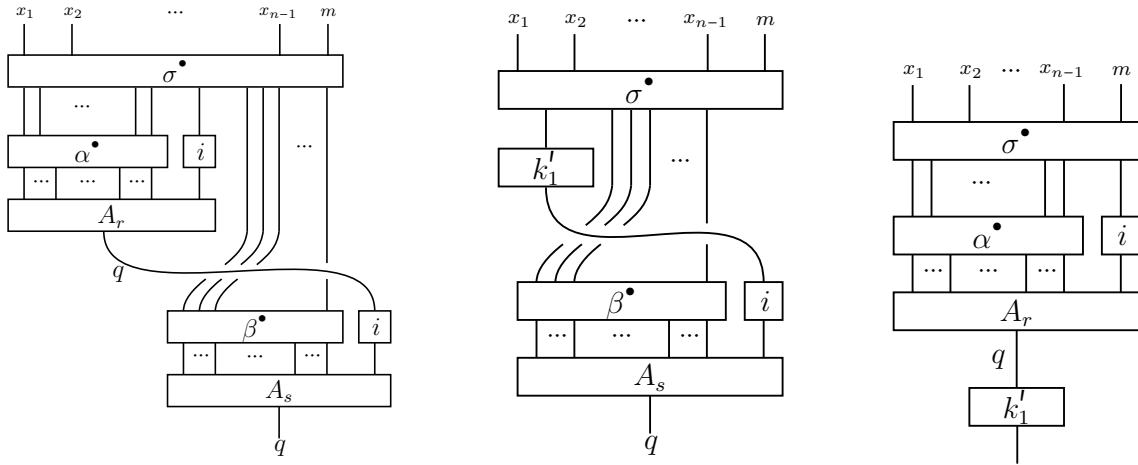
Step 7. Focusing now on the right-hand side of the original L_∞-module relation, we substitute for k'_n using its definition. On the left is the case $2 \leq p \leq n - 1$, in the center is the case $p = 1$, and on the right is the case $p = n - 1$. After using the fact that $i \circ k'_1 = k_1 \circ i$ and $k'_1 \circ q = q \circ k_1$, we obtain the formulas in Step 7.

119
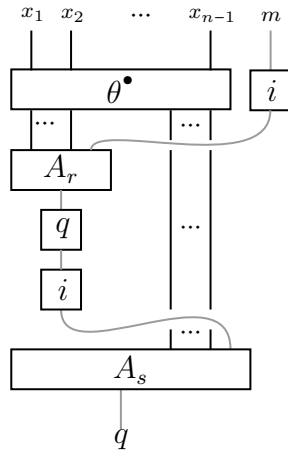
Step 8. Combine σ, α, and β into one unshuffle θ. Drawn above is the case $2 \le p \le n - 1$. The cases of $p = 1$ and $p = n - 1$ are omitted.
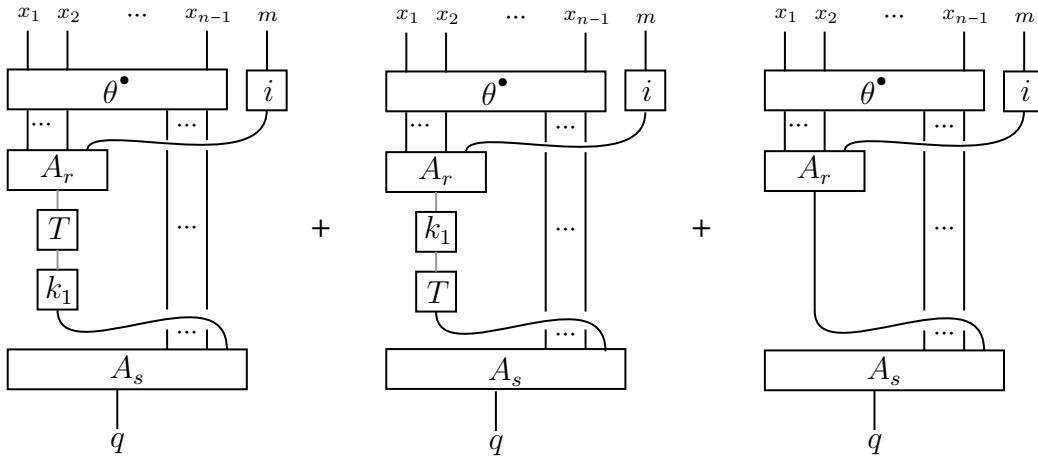
Step 9. In Step 8, we can replace $i \circ q$ with $\mathrm{Id}_M + k_1 \circ T + T \circ k_1$. The result is precisely what we had in Step 5. Again, the cases of $p = 1$ and $p = n - 1$ are not included in this picture.

APPENDIX F

ANNULAR KNOT DIAGRAMS

This appendix contains knot diagrams for all knots and links with up to seven crossings. The possible basepoints are labeled in the diagrams.
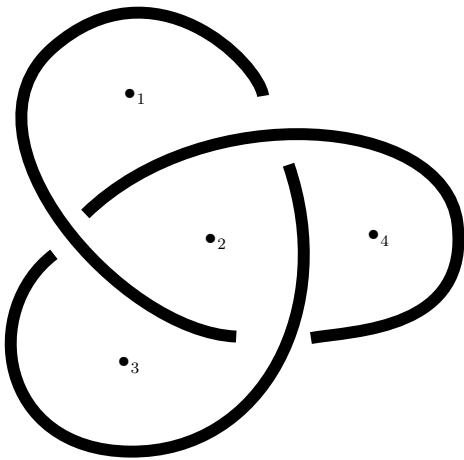
FIGURE 29. 3_1
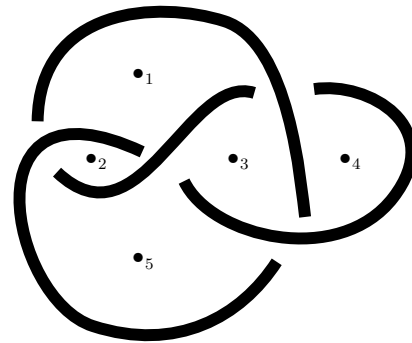
FIGURE 30. 4_1
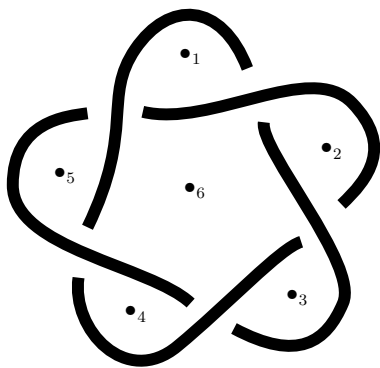
FIGURE 31. 5_1
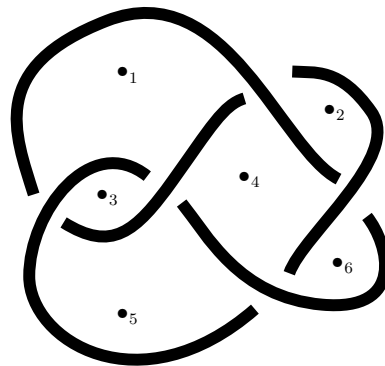
FIGURE 32. 5_2
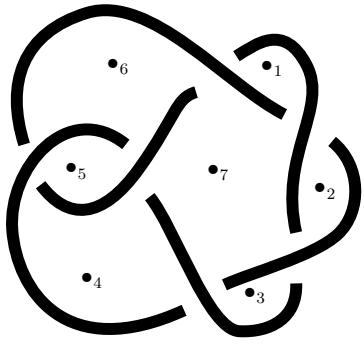
FIGURE 33. 6_1
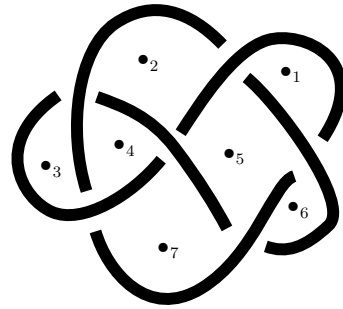
FIGURE 34. 6_2
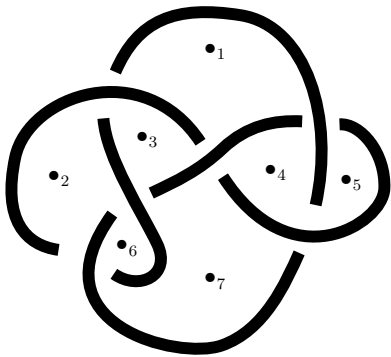
FIGURE 35. 6_3
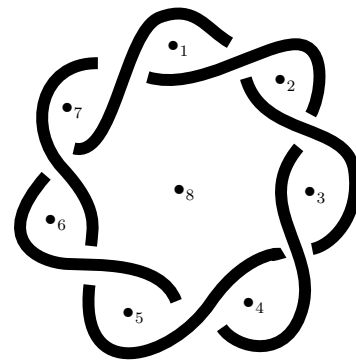
FIGURE 36. 7_1
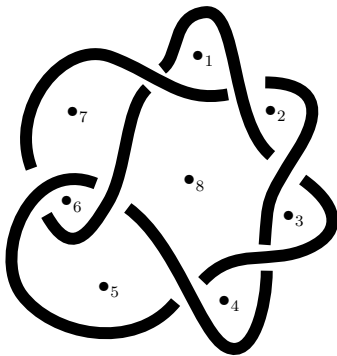
FIGURE 37. 7_2

FIGURE 38. 7_3

122
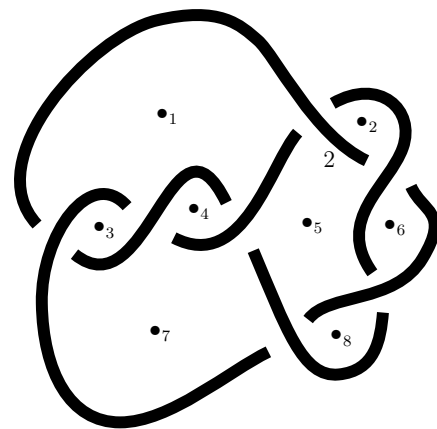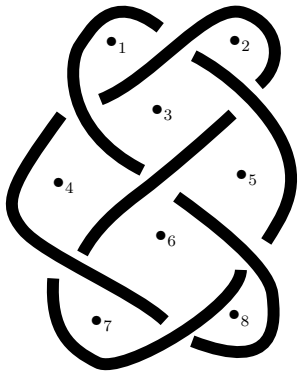
FIGURE 39. 7_4
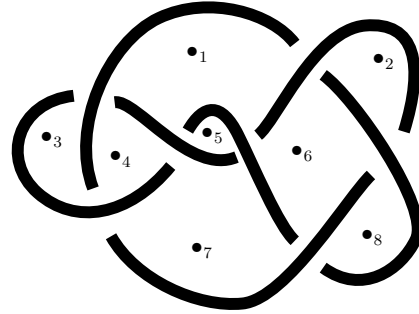
FIGURE 40. 7_5
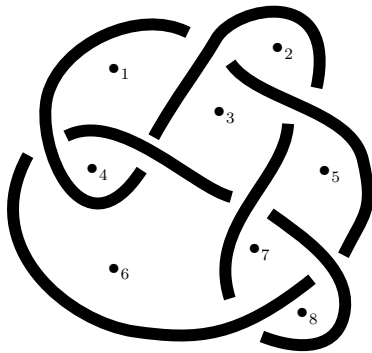
FIGURE 41. 7_6
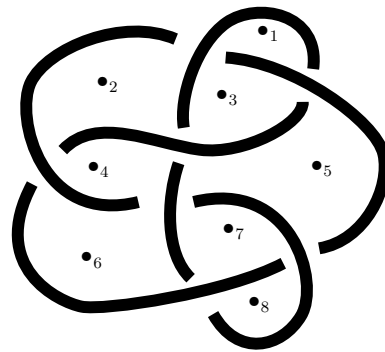
FIGURE 42. 7_7
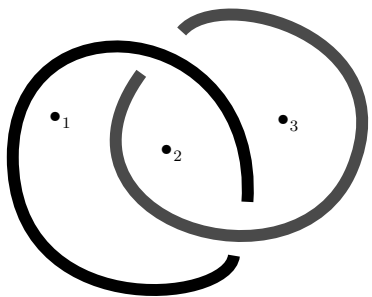
FIGURE 43. $L2a1$

FIGURE 44. $L4a1$

123
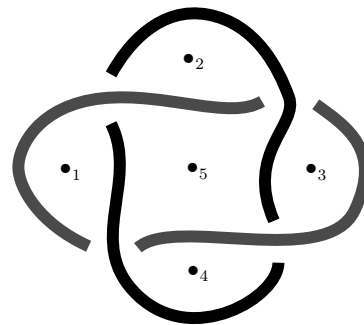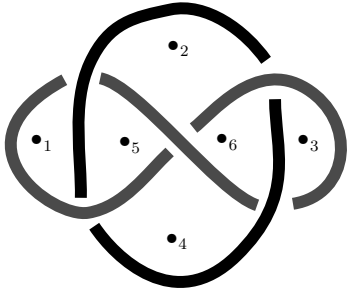
FIGURE 45. *L5a1*
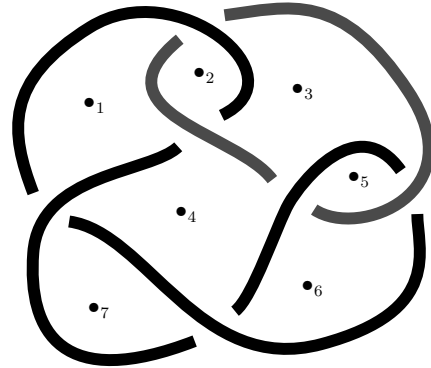
FIGURE 46. *L6a1*
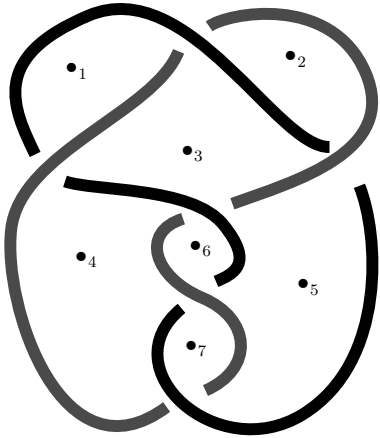
FIGURE 47. *L6a2*
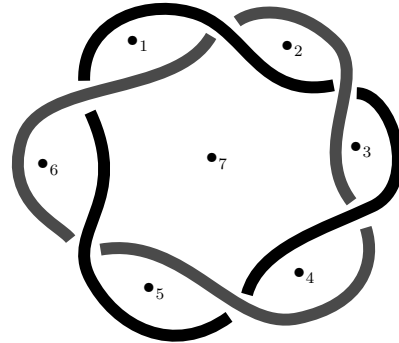
FIGURE 48. *L6a3*
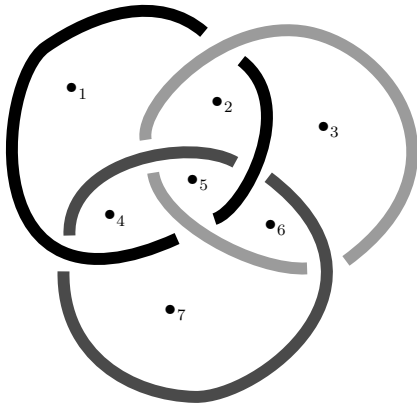
FIGURE 49. *L6a4*
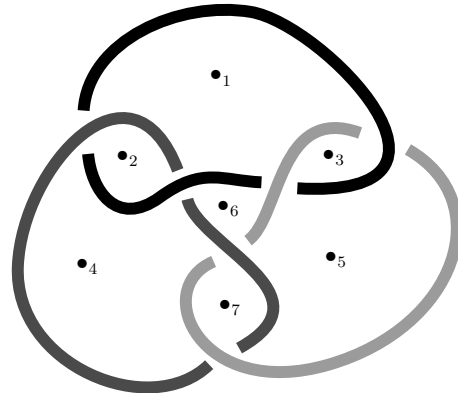
FIGURE 50. *L6a5*
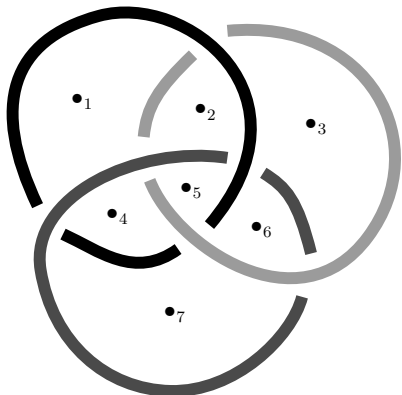
FIGURE 51. *L6n1*
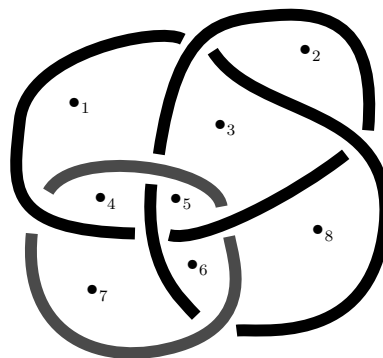
FIGURE 52. *L7a1*
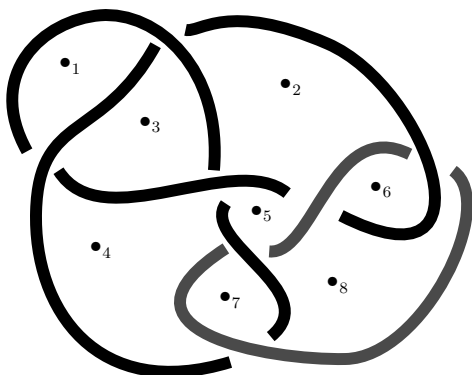
FIGURE 53. *L7a2*
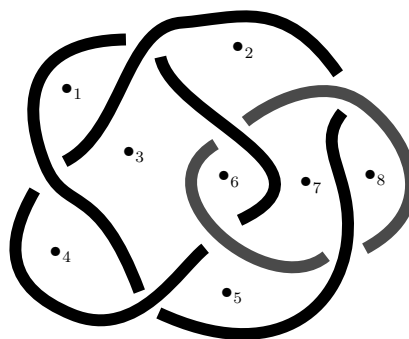
FIGURE 54. *L7a3*
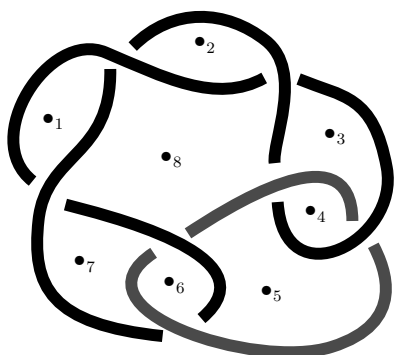
FIGURE 55. *L7a4*
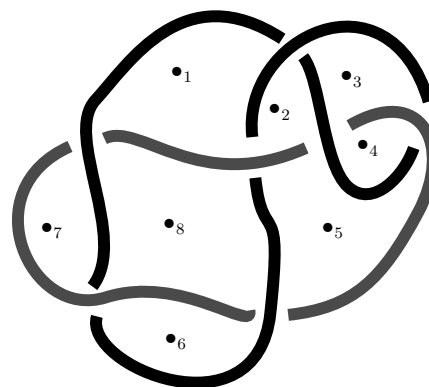
FIGURE 56. *L7a5*

125
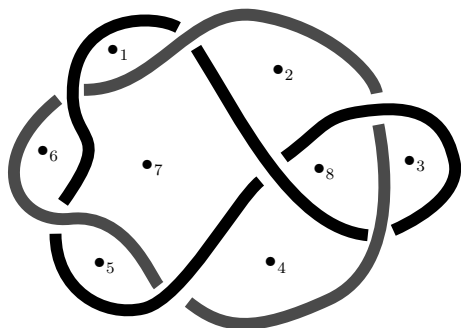
FIGURE 57. *L7a6*
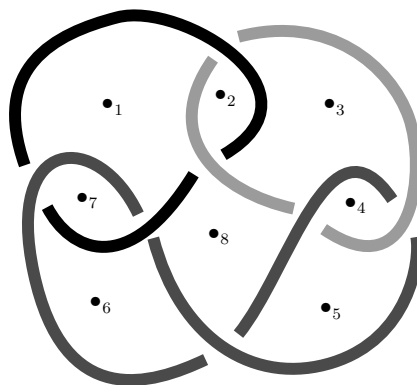
FIGURE 58. *L7a7*
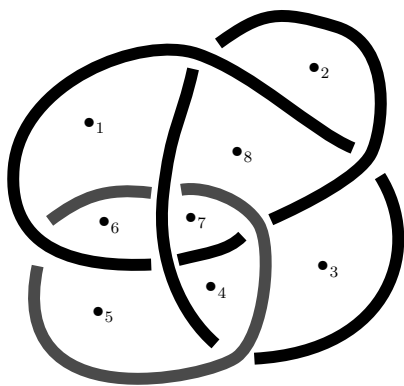
FIGURE 59. *L7n1*
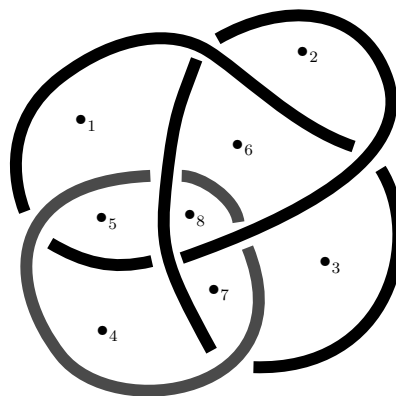
FIGURE 60. *L7n2*

APPENDIX G

HOMOLOGY CALCULATIONS

This appendix contains annular Khovanov homology data for all knots and links with up to seven crossings. These results were generated by the computer code described in Chapter IV. The first row contains the ordinary Khovanov homology. The ith row thereafter contains the annular Khovanov homology computed with respect to the ith basepoint, as labeled in Appendix F. Each column represents a different homological grading.

<u>Knots</u>

3_1

$\mathbb{Z}$	$\mathbb{Z} \times C_2$	0	$\mathbb{Z} \times \mathbb{Z}$
$\mathbb{Z} \times \mathbb{Z}$	$\mathbb{Z} \times \mathbb{Z}$	0	$\mathbb{Z} \times \mathbb{Z}$
$\mathbb{Z}$	$\mathbb{Z} \times C_2$	$\mathbb{Z}$	$\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$
$\mathbb{Z} \times \mathbb{Z}$	$\mathbb{Z} \times \mathbb{Z}$	0	$\mathbb{Z} \times \mathbb{Z}$
$\mathbb{Z} \times \mathbb{Z}$	$\mathbb{Z} \times \mathbb{Z}$	0	$\mathbb{Z} \times \mathbb{Z}$

4_1

$\mathbb{Z}$	$\mathbb{Z} \times C_2$	$\mathbb{Z} \times \mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z} \times C_2$
$\mathbb{Z} \times \mathbb{Z}$	$\mathbb{Z} \times \mathbb{Z}$	$\mathbb{Z} \times \mathbb{Z}$	$\mathbb{Z} \times \mathbb{Z}$	$\mathbb{Z} \times \mathbb{Z}$
$\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$	$\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$	$\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$	$\mathbb{Z} \times \mathbb{Z}$	$\mathbb{Z} \times C_2$
$\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$	$\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$	$\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z} \times C_2$
$\mathbb{Z} \times \mathbb{Z}$	$\mathbb{Z} \times \mathbb{Z}$	$\mathbb{Z} \times \mathbb{Z}$	$\mathbb{Z} \times \mathbb{Z}$	$\mathbb{Z} \times \mathbb{Z}$
$\mathbb{Z} \times \mathbb{Z}$	$\mathbb{Z} \times \mathbb{Z}$	$\mathbb{Z} \times \mathbb{Z}$	$\mathbb{Z} \times \mathbb{Z}$	$\mathbb{Z} \times \mathbb{Z}$

5_1

$\mathbb{Z}$	$\mathbb{Z}\times C_2$	$\mathbb{Z}$	$\mathbb{Z}\times C_2$	0	$\mathbb{Z}^2$
$\mathbb{Z}^2$	$\mathbb{Z}^2$	$\mathbb{Z}^2$	$\mathbb{Z}^2$	0	$\mathbb{Z}^2$
$\mathbb{Z}^2$	$\mathbb{Z}^2$	$\mathbb{Z}^2$	$\mathbb{Z}^2$	0	$\mathbb{Z}^2$
$\mathbb{Z}^2$	$\mathbb{Z}^2$	$\mathbb{Z}^2$	$\mathbb{Z}^2$	0	$\mathbb{Z}^2$
$\mathbb{Z}^2$	$\mathbb{Z}^2$	$\mathbb{Z}^2$	$\mathbb{Z}^2$	0	$\mathbb{Z}^2$
$\mathbb{Z}^2$	$\mathbb{Z}^2$	$\mathbb{Z}^2$	$\mathbb{Z}^2$	0	$\mathbb{Z}^2$
$\mathbb{Z}$	$\mathbb{Z}\times C_2$	$\mathbb{Z}$	$\mathbb{Z}\times C_2$	$\mathbb{Z}$	$\mathbb{Z}^3$

5_2

$\mathbb{Z}$	$\mathbb{Z}\times C_2$	$\mathbb{Z}$	$\mathbb{Z}^2\times C_2$	$\mathbb{Z}\times C_2$	$\mathbb{Z}^2$
$\mathbb{Z}^2$	$\mathbb{Z}^2$	$\mathbb{Z}^2$	$\mathbb{Z}^4$	$\mathbb{Z}^2$	$\mathbb{Z}^2$
$\mathbb{Z}^2$	$\mathbb{Z}^2$	$\mathbb{Z}^2$	$\mathbb{Z}^4$	$\mathbb{Z}^2$	$\mathbb{Z}^2$
$\mathbb{Z}^3$	$\mathbb{Z}^4$	$\mathbb{Z}^4$	$\mathbb{Z}^5$	$\mathbb{Z}^2\times C_2$	$\mathbb{Z}^2$
$\mathbb{Z}^3$	$\mathbb{Z}^4$	$\mathbb{Z}^2$	$\mathbb{Z}^2\times C_2$	$\mathbb{Z}\times C_2$	$\mathbb{Z}^2$
$\mathbb{Z}^2$	$\mathbb{Z}^2$	$\mathbb{Z}^2$	$\mathbb{Z}^4$	$\mathbb{Z}^2$	$\mathbb{Z}^2$
$\mathbb{Z}^2$	$\mathbb{Z}^2$	$\mathbb{Z}^2$	$\mathbb{Z}^4$	$\mathbb{Z}^2$	$\mathbb{Z}^2$

6_1

0	$\mathbb{Z}$	$\mathbb{Z}\times C_2$	$\mathbb{Z}^2$	$\mathbb{Z}$	$\mathbb{Z}\times C_2$	0
0	$\mathbb{Z}^2$	$\mathbb{Z}^2$	$\mathbb{Z}^2$	$\mathbb{Z}^2$	$\mathbb{Z}^2$	0
0	$\mathbb{Z}^2$	$\mathbb{Z}^2$	$\mathbb{Z}^2$	$\mathbb{Z}^2$	$\mathbb{Z}^2$	0
0	$\mathbb{Z}^2$	$\mathbb{Z}^2$	$\mathbb{Z}^2$	$\mathbb{Z}^2$	$\mathbb{Z}^2$	0
0	$\mathbb{Z}^2$	$\mathbb{Z}^2$	$\mathbb{Z}^2$	$\mathbb{Z}^2$	$\mathbb{Z}^2$	0
0	$\mathbb{Z}^3$	$\mathbb{Z}^4$	$\mathbb{Z}^4$	$\mathbb{Z}^2$	$\mathbb{Z}\times C_2$	0
0	$\mathbb{Z}^2$	$\mathbb{Z}^2$	$\mathbb{Z}^2$	$\mathbb{Z}^2$	$\mathbb{Z}^2$	0
0	$\mathbb{Z}^3$	$\mathbb{Z}^4$	$\mathbb{Z}^3$	$\mathbb{Z}$	$\mathbb{Z}\times C_2$	0

6_2

$\mathbb{Z}$	$\mathbb{Z}^2 \times C_2$	$\mathbb{Z}^2 \times C_2$	$\mathbb{Z}^2 \times C_2$	$\mathbb{Z}^3 \times C_2$	$\mathbb{Z}$	$\mathbb{Z} \times C_2$
$\mathbb{Z}^2$	$\mathbb{Z}^4$	$\mathbb{Z}^4$	$\mathbb{Z}^4$	$\mathbb{Z}^4$	$\mathbb{Z}^2$	$\mathbb{Z}^2$
$\mathbb{Z}^2$	$\mathbb{Z}^4$	$\mathbb{Z}^4$	$\mathbb{Z}^4$	$\mathbb{Z}^4$	$\mathbb{Z}^2$	$\mathbb{Z}^2$
$\mathbb{Z}^2$	$\mathbb{Z}^4$	$\mathbb{Z}^4$	$\mathbb{Z}^4$	$\mathbb{Z}^4$	$\mathbb{Z}^2$	$\mathbb{Z}^2$
$\mathbb{Z}$	$\mathbb{Z}^2 \times C_2$	$\mathbb{Z}^3 \times C_2$	$\mathbb{Z}^5 \times C_2$	$\mathbb{Z}^6$	$\mathbb{Z}^4$	$\mathbb{Z}^3$
$\mathbb{Z}$	$\mathbb{Z}^2 \times C_2$	$\mathbb{Z}^2 \times C_2$	$\mathbb{Z}^3 \times C_2$	$\mathbb{Z}^5 \times C_2$	$\mathbb{Z}^4$	$\mathbb{Z}^3$
$\mathbb{Z}^2$	$\mathbb{Z}^4$	$\mathbb{Z}^4$	$\mathbb{Z}^4$	$\mathbb{Z}^4$	$\mathbb{Z}^2$	$\mathbb{Z}^2$
$\mathbb{Z}^2$	$\mathbb{Z}^4$	$\mathbb{Z}^4$	$\mathbb{Z}^4$	$\mathbb{Z}^4$	$\mathbb{Z}^2$	$\mathbb{Z}^2$

6_3

$\mathbb{Z}$	$\mathbb{Z}^2 \times C_2$	$\mathbb{Z}^2 \times C_2$	$\mathbb{Z}^4 \times C_2$	$\mathbb{Z}^2 \times C_2$	$\mathbb{Z}^2 \times C_2$	$\mathbb{Z} \times C_2$
$\mathbb{Z}^2$	$\mathbb{Z}^4$	$\mathbb{Z}^4$	$\mathbb{Z}^6$	$\mathbb{Z}^4$	$\mathbb{Z}^4$	$\mathbb{Z}^2$
$\mathbb{Z}^2$	$\mathbb{Z}^4$	$\mathbb{Z}^4$	$\mathbb{Z}^6$	$\mathbb{Z}^4$	$\mathbb{Z}^4$	$\mathbb{Z}^2$
$\mathbb{Z}^3$	$\mathbb{Z}^7$	$\mathbb{Z}^8$	$\mathbb{Z}^8$	$\mathbb{Z}^3 \times C_2$	$\mathbb{Z}^2 \times C_2$	$\mathbb{Z} \times C_2$
$\mathbb{Z}^3$	$\mathbb{Z}^7$	$\mathbb{Z}^6$	$\mathbb{Z}^6 \times C_2$	$\mathbb{Z}^3 \times C_2$	$\mathbb{Z}^2 \times C_2$	$\mathbb{Z} \times C_2$
$\mathbb{Z}^2$	$\mathbb{Z}^4$	$\mathbb{Z}^4$	$\mathbb{Z}^6$	$\mathbb{Z}^4$	$\mathbb{Z}^4$	$\mathbb{Z}^2$
$\mathbb{Z}^3$	$\mathbb{Z}^7$	$\mathbb{Z}^8$	$\mathbb{Z}^{10}$	$\mathbb{Z}^6$	$\mathbb{Z}^3 \times C_2$	$\mathbb{Z} \times C_2$
$\mathbb{Z}^2$	$\mathbb{Z}^4$	$\mathbb{Z}^4$	$\mathbb{Z}^6$	$\mathbb{Z}^4$	$\mathbb{Z}^4$	$\mathbb{Z}^2$

7_1

$$
\begin{array}{cccccccc}
0 & \mathbb{Z} & \mathbb{Z}\times C_2 & \mathbb{Z} & \mathbb{Z}\times C_2 & 0 & \mathbb{Z}^2 & 0 \\
0 & \mathbb{Z}^2 & \mathbb{Z}^2 & \mathbb{Z}^2 & \mathbb{Z}^2 & 0 & \mathbb{Z}^2 & 0 \\
0 & \mathbb{Z}^2 & \mathbb{Z}^2 & \mathbb{Z}^2 & \mathbb{Z}^2 & 0 & \mathbb{Z}^2 & 0 \\
0 & \mathbb{Z}^2 & \mathbb{Z}^2 & \mathbb{Z}^2 & \mathbb{Z}^2 & 0 & \mathbb{Z}^2 & 0 \\
0 & \mathbb{Z}^2 & \mathbb{Z}^2 & \mathbb{Z}^2 & \mathbb{Z}^2 & 0 & \mathbb{Z}^2 & 0 \\
0 & \mathbb{Z}^2 & \mathbb{Z}^2 & \mathbb{Z}^2 & \mathbb{Z}^2 & 0 & \mathbb{Z}^2 & 0 \\
0 & \mathbb{Z}^2 & \mathbb{Z}^2 & \mathbb{Z}^2 & \mathbb{Z}^2 & 0 & \mathbb{Z}^2 & 0 \\
0 & \mathbb{Z}^2 & \mathbb{Z}^2 & \mathbb{Z}^2 & \mathbb{Z}^2 & 0 & \mathbb{Z}^2 & 0 \\
0 & \mathbb{Z} & \mathbb{Z}\times C_2 & \mathbb{Z} & \mathbb{Z}\times C_2 & \mathbb{Z} & \mathbb{Z}^3 & 0
\end{array}
$$

7_2

$$
\begin{array}{cccccccc}
\mathbb{Z} & \mathbb{Z}\times C_2 & \mathbb{Z} & \mathbb{Z}^2\times C_2 & \mathbb{Z}^2\times C_2 & \mathbb{Z}^2\times C_2 & \mathbb{Z}\times C_2 & \mathbb{Z}^2 \\
\mathbb{Z}^2 & \mathbb{Z}^2 & \mathbb{Z}^2 & \mathbb{Z}^4 & \mathbb{Z}^4 & \mathbb{Z}^4 & \mathbb{Z}^2 & \mathbb{Z}^2 \\
\mathbb{Z}^2 & \mathbb{Z}^2 & \mathbb{Z}^2 & \mathbb{Z}^4 & \mathbb{Z}^4 & \mathbb{Z}^4 & \mathbb{Z}^2 & \mathbb{Z}^2 \\
\mathbb{Z}^2 & \mathbb{Z}^2 & \mathbb{Z}^2 & \mathbb{Z}^4 & \mathbb{Z}^4 & \mathbb{Z}^4 & \mathbb{Z}^2 & \mathbb{Z}^2 \\
\mathbb{Z}^2 & \mathbb{Z}^2 & \mathbb{Z}^2 & \mathbb{Z}^4 & \mathbb{Z}^4 & \mathbb{Z}^4 & \mathbb{Z}^2 & \mathbb{Z}^2 \\
\mathbb{Z}^2 & \mathbb{Z}^2 & \mathbb{Z}^2 & \mathbb{Z}^4 & \mathbb{Z}^4 & \mathbb{Z}^4 & \mathbb{Z}^2 & \mathbb{Z}^2 \\
\mathbb{Z}^3 & \mathbb{Z}^4 & \mathbb{Z}^4 & \mathbb{Z}^5 & \mathbb{Z}^5\times C_2 & \mathbb{Z}^5 & \mathbb{Z}^2\times C_2 & \mathbb{Z}^2 \\
\mathbb{Z}^2 & \mathbb{Z}^2 & \mathbb{Z}^2 & \mathbb{Z}^4 & \mathbb{Z}^4 & \mathbb{Z}^4 & \mathbb{Z}^2 & \mathbb{Z}^2 \\
\mathbb{Z}^3 & \mathbb{Z}^4 & \mathbb{Z}^2 & \mathbb{Z}^2\times C_2 & \mathbb{Z}^2\times C_2 & \mathbb{Z}^2\times C_2 & \mathbb{Z}\times C_2 & \mathbb{Z}^2
\end{array}
$$

7_3

$$
\begin{array}{llllllll}
\mathbb{Z}^2 & \mathbb{Z} & \mathbb{Z}^2 \times C_2 & \mathbb{Z}^2 \times C_2 & \mathbb{Z}^3 \times C_2 & \mathbb{Z}^2 \times C_2 \times C_2 & \mathbb{Z} & \mathbb{Z} \times C_2 \\
\mathbb{Z}^2 & \mathbb{Z}^2 & \mathbb{Z}^4 & \mathbb{Z}^4 & \mathbb{Z}^6 & \mathbb{Z}^4 & \mathbb{Z}^2 & \mathbb{Z}^2 \\
\mathbb{Z}^2 & \mathbb{Z}^2 & \mathbb{Z}^4 & \mathbb{Z}^4 & \mathbb{Z}^6 & \mathbb{Z}^4 & \mathbb{Z}^2 & \mathbb{Z}^2 \\
\mathbb{Z}^3 & \mathbb{Z}^4 & \mathbb{Z}^7 & \mathbb{Z}^8 & \mathbb{Z}^9 & \mathbb{Z}^6 \times C_2 & \mathbb{Z}^2 & \mathbb{Z} \times C_2 \\
\mathbb{Z}^4 & \mathbb{Z}^6 & \mathbb{Z}^8 & \mathbb{Z}^8 & \mathbb{Z}^9 & \mathbb{Z}^6 \times C_2 & \mathbb{Z}^2 & \mathbb{Z} \times C_2 \\
\mathbb{Z}^3 & \mathbb{Z}^4 & \mathbb{Z}^5 & \mathbb{Z}^3 \times C_2 & \mathbb{Z}^3 \times C_2 & \mathbb{Z}^2 \times C_2 \times C_2 & \mathbb{Z} & \mathbb{Z} \times C_2 \\
\mathbb{Z}^2 & \mathbb{Z}^2 & \mathbb{Z}^4 & \mathbb{Z}^4 & \mathbb{Z}^6 & \mathbb{Z}^4 & \mathbb{Z}^2 & \mathbb{Z}^2 \\
\mathbb{Z}^2 & \mathbb{Z}^2 & \mathbb{Z}^4 & \mathbb{Z}^4 & \mathbb{Z}^6 & \mathbb{Z}^4 & \mathbb{Z}^2 & \mathbb{Z}^2 \\
\mathbb{Z}^2 & \mathbb{Z}^2 & \mathbb{Z}^4 & \mathbb{Z}^4 & \mathbb{Z}^6 & \mathbb{Z}^4 & \mathbb{Z}^2 & \mathbb{Z}^2 \\
\end{array}
$$

7_4

$$
\begin{array}{llllllll}
\mathbb{Z}^2 & \mathbb{Z}^2 & \mathbb{Z}^3 \times C_2 \times C_2 & \mathbb{Z}^2 \times C_2 & \mathbb{Z}^3 \times C_2 & \mathbb{Z}^2 \times C_2 \times C_2 & \mathbb{Z} & \mathbb{Z} \times C_2 \\
\mathbb{Z}^2 & \mathbb{Z}^4 & \mathbb{Z}^6 & \mathbb{Z}^4 & \mathbb{Z}^6 & \mathbb{Z}^4 & \mathbb{Z}^2 & \mathbb{Z}^2 \\
\mathbb{Z}^2 & \mathbb{Z}^4 & \mathbb{Z}^6 & \mathbb{Z}^4 & \mathbb{Z}^6 & \mathbb{Z}^4 & \mathbb{Z}^2 & \mathbb{Z}^2 \\
\mathbb{Z}^2 & \mathbb{Z}^2 & \mathbb{Z}^3 \times C_2 \times C_2 & \mathbb{Z}^3 \times C_2 & \mathbb{Z}^6 \times C_2 & \mathbb{Z}^5 \times C_2 & \mathbb{Z}^4 & \mathbb{Z}^3 \\
\mathbb{Z}^2 & \mathbb{Z}^4 & \mathbb{Z}^6 & \mathbb{Z}^4 & \mathbb{Z}^6 & \mathbb{Z}^4 & \mathbb{Z}^2 & \mathbb{Z}^2 \\
\mathbb{Z}^2 & \mathbb{Z}^4 & \mathbb{Z}^6 & \mathbb{Z}^4 & \mathbb{Z}^6 & \mathbb{Z}^4 & \mathbb{Z}^2 & \mathbb{Z}^2 \\
\mathbb{Z}^2 & \mathbb{Z}^2 & \mathbb{Z}^3 \times C_2 \times C_2 & \mathbb{Z}^3 \times C_2 & \mathbb{Z}^6 \times C_2 & \mathbb{Z}^5 \times C_2 & \mathbb{Z}^4 & \mathbb{Z}^3 \\
\mathbb{Z}^2 & \mathbb{Z}^4 & \mathbb{Z}^6 & \mathbb{Z}^4 & \mathbb{Z}^6 & \mathbb{Z}^4 & \mathbb{Z}^2 & \mathbb{Z}^2 \\
\mathbb{Z}^2 & \mathbb{Z}^4 & \mathbb{Z}^6 & \mathbb{Z}^4 & \mathbb{Z}^6 & \mathbb{Z}^4 & \mathbb{Z}^2 & \mathbb{Z}^2 \\
\end{array}
$$

7_5

$\mathbb{Z}$	$\mathbb{Z}^2 \times C_2$	$\mathbb{Z}^3 \times C_2$	$\mathbb{Z}^3 \times C_2 \times C_2$	$\mathbb{Z}^3 \times C_2$	$\mathbb{Z}^3 \times C_2 \times C_2$	$\mathbb{Z} \times C_2$	$\mathbb{Z}^2$
$\mathbb{Z}^2$	$\mathbb{Z}^4$	$\mathbb{Z}^6$	$\mathbb{Z}^6$	$\mathbb{Z}^6$	$\mathbb{Z}^6$	$\mathbb{Z}^2$	$\mathbb{Z}^2$
$\mathbb{Z}^2$	$\mathbb{Z}^4$	$\mathbb{Z}^6$	$\mathbb{Z}^6$	$\mathbb{Z}^6$	$\mathbb{Z}^6$	$\mathbb{Z}^2$	$\mathbb{Z}^2$
$\mathbb{Z}^2$	$\mathbb{Z}^4$	$\mathbb{Z}^6$	$\mathbb{Z}^6$	$\mathbb{Z}^6$	$\mathbb{Z}^6$	$\mathbb{Z}^2$	$\mathbb{Z}^2$
$\mathbb{Z}$	$\mathbb{Z}^2 \times C_2$	$\mathbb{Z}^4 \times C_2$	$\mathbb{Z}^7 \times C_2 \times C_2$	$\mathbb{Z}^9$	$\mathbb{Z}^8 \times C_2$	$\mathbb{Z}^4$	$\mathbb{Z}^3$
$\mathbb{Z}$	$\mathbb{Z}^3 \times C_2$	$\mathbb{Z}^7 \times C_2$	$\mathbb{Z}^{10} \times C_2$	$\mathbb{Z}^{12}$	$\mathbb{Z}^{11}$	$\mathbb{Z}^6$	$\mathbb{Z}^4$
$\mathbb{Z}$	$\mathbb{Z}^2 \times C_2$	$\mathbb{Z}^3 \times C_2$	$\mathbb{Z}^4 \times C_2 \times C_2$	$\mathbb{Z}^7 \times C_2$	$\mathbb{Z}^8 \times C_2$	$\mathbb{Z}^4$	$\mathbb{Z}^3$
$\mathbb{Z}^2$	$\mathbb{Z}^4$	$\mathbb{Z}^6$	$\mathbb{Z}^6$	$\mathbb{Z}^6$	$\mathbb{Z}^6$	$\mathbb{Z}^2$	$\mathbb{Z}^2$
$\mathbb{Z}^2$	$\mathbb{Z}^4$	$\mathbb{Z}^6$	$\mathbb{Z}^6$	$\mathbb{Z}^6$	$\mathbb{Z}^6$	$\mathbb{Z}^2$	$\mathbb{Z}^2$

7_6

$\mathbb{Z}$	$\mathbb{Z}^2 \times C_2$	$\mathbb{Z}^3 \times C_2$	$\mathbb{Z}^4 \times C_2 \times C_2$	$\mathbb{Z}^3 \times C_2 \times C_2$	$\mathbb{Z}^4 \times C_2$	$\mathbb{Z}^2 \times C_2$	$\mathbb{Z} \times C_2$
$\mathbb{Z}^2$	$\mathbb{Z}^4$	$\mathbb{Z}^6$	$\mathbb{Z}^8$	$\mathbb{Z}^6$	$\mathbb{Z}^6$	$\mathbb{Z}^4$	$\mathbb{Z}^2$
$\mathbb{Z}^2$	$\mathbb{Z}^4$	$\mathbb{Z}^6$	$\mathbb{Z}^8$	$\mathbb{Z}^6$	$\mathbb{Z}^6$	$\mathbb{Z}^4$	$\mathbb{Z}^2$
$\mathbb{Z}$	$\mathbb{Z}^2 \times C_2$	$\mathbb{Z}^3 \times C_2$	$\mathbb{Z}^5 \times C_2 \times C_2$	$\mathbb{Z}^7 \times C_2 \times C_2$	$\mathbb{Z}^{10}$	$\mathbb{Z}^7$	$\mathbb{Z}^3$
$\mathbb{Z}$	$\mathbb{Z}^3 \times C_2$	$\mathbb{Z}^7 \times C_2$	$\mathbb{Z}^{10} \times C_2$	$\mathbb{Z}^{10} \times C_2$	$\mathbb{Z}^{11}$	$\mathbb{Z}^7$	$\mathbb{Z}^3$
$\mathbb{Z}^2$	$\mathbb{Z}^4$	$\mathbb{Z}^6$	$\mathbb{Z}^8$	$\mathbb{Z}^6$	$\mathbb{Z}^6$	$\mathbb{Z}^4$	$\mathbb{Z}^2$
$\mathbb{Z}^2$	$\mathbb{Z}^4$	$\mathbb{Z}^6$	$\mathbb{Z}^8$	$\mathbb{Z}^6$	$\mathbb{Z}^6$	$\mathbb{Z}^4$	$\mathbb{Z}^2$
$\mathbb{Z}$	$\mathbb{Z}^2 \times C_2$	$\mathbb{Z}^4 \times C_2$	$\mathbb{Z}^7 \times C_2 \times C_2$	$\mathbb{Z}^7 \times C_2$	$\mathbb{Z}^9 \times C_2$	$\mathbb{Z}^7$	$\mathbb{Z}^3$
$\mathbb{Z}^2$	$\mathbb{Z}^4$	$\mathbb{Z}^6$	$\mathbb{Z}^8$	$\mathbb{Z}^6$	$\mathbb{Z}^6$	$\mathbb{Z}^4$	$\mathbb{Z}^2$

7_7

$\mathbb{Z}$	$\mathbb{Z}^3 \times C_2$	$\mathbb{Z}^3 \times C_2 \times C_2$	$\mathbb{Z}^5 \times C_2$	$\mathbb{Z}^4 \times C_2 \times C_2$	$\mathbb{Z}^3 \times C_2 \times C_2$	$\mathbb{Z}^2 \times C_2$	$\mathbb{Z} \times C_2$
$\mathbb{Z}^2$	$\mathbb{Z}^6$	$\mathbb{Z}^6$	$\mathbb{Z}^8$	$\mathbb{Z}^8$	$\mathbb{Z}^6$	$\mathbb{Z}^4$	$\mathbb{Z}^2$
$\mathbb{Z}^2$	$\mathbb{Z}^6$	$\mathbb{Z}^6$	$\mathbb{Z}^8$	$\mathbb{Z}^8$	$\mathbb{Z}^6$	$\mathbb{Z}^4$	$\mathbb{Z}^2$
$\mathbb{Z}$	$\mathbb{Z}^3 \times C_2$	$\mathbb{Z}^4 \times C_2 \times C_2$	$\mathbb{Z}^9 \times C_2$	$\mathbb{Z}^{10} \times C_2$	$\mathbb{Z}^9 \times C_2$	$\mathbb{Z}^7$	$\mathbb{Z}^3$
$\mathbb{Z}$	$\mathbb{Z}^3 \times C_2$	$\mathbb{Z}^4 \times C_2 \times C_2$	$\mathbb{Z}^8 \times C_2$	$\mathbb{Z}^{11} \times C_2 \times C_2$	$\mathbb{Z}^{11}$	$\mathbb{Z}^7$	$\mathbb{Z}^3$
$\mathbb{Z}^2$	$\mathbb{Z}^6$	$\mathbb{Z}^6$	$\mathbb{Z}^8$	$\mathbb{Z}^8$	$\mathbb{Z}^6$	$\mathbb{Z}^4$	$\mathbb{Z}^2$
$\mathbb{Z}^2$	$\mathbb{Z}^6$	$\mathbb{Z}^6$	$\mathbb{Z}^8$	$\mathbb{Z}^8$	$\mathbb{Z}^6$	$\mathbb{Z}^4$	$\mathbb{Z}^2$
$\mathbb{Z}$	$\mathbb{Z}^3 \times C_2$	$\mathbb{Z}^4 \times C_2 \times C_2$	$\mathbb{Z}^9 \times C_2$	$\mathbb{Z}^{10} \times C_2$	$\mathbb{Z}^9 \times C_2$	$\mathbb{Z}^7$	$\mathbb{Z}^3$
$\mathbb{Z}^2$	$\mathbb{Z}^6$	$\mathbb{Z}^6$	$\mathbb{Z}^8$	$\mathbb{Z}^8$	$\mathbb{Z}^6$	$\mathbb{Z}^4$	$\mathbb{Z}^2$

Links

L6a1

$\mathbb{Z}$	$\mathbb{Z}^2 \times C_2$	$\mathbb{Z}^3 \times C_2$	$\mathbb{Z}^2$	$\mathbb{Z}^3 \times C_2 \times C_2$	$\mathbb{Z} \times C_2$	$\mathbb{Z}^2$
$\mathbb{Z}^2$	$\mathbb{Z}^4$	$\mathbb{Z}^4$	$\mathbb{Z}^4$	$\mathbb{Z}^6$	$\mathbb{Z}^2$	$\mathbb{Z}^2$
$\mathbb{Z}^3$	$\mathbb{Z}^7$	$\mathbb{Z}^8$	$\mathbb{Z}^6$	$\mathbb{Z}^6 \times C_2$	$\mathbb{Z}^2 \times C_2$	$\mathbb{Z}^2$
$\mathbb{Z}^2$	$\mathbb{Z}^4$	$\mathbb{Z}^4$	$\mathbb{Z}^4$	$\mathbb{Z}^6$	$\mathbb{Z}^2$	$\mathbb{Z}^2$
$\mathbb{Z}^3$	$\mathbb{Z}^7$	$\mathbb{Z}^7$	$\mathbb{Z}^3$	$\mathbb{Z}^3 \times C_2 \times C_2$	$\mathbb{Z} \times C_2$	$\mathbb{Z}^2$
$\mathbb{Z}^3$	$\mathbb{Z}^7$	$\mathbb{Z}^8$	$\mathbb{Z}^6$	$\mathbb{Z}^6 \times C_2$	$\mathbb{Z}^2 \times C_2$	$\mathbb{Z}^2$
$\mathbb{Z}^2$	$\mathbb{Z}^4$	$\mathbb{Z}^4$	$\mathbb{Z}^4$	$\mathbb{Z}^6$	$\mathbb{Z}^2$	$\mathbb{Z}^2$
$\mathbb{Z}^2$	$\mathbb{Z}^4$	$\mathbb{Z}^4$	$\mathbb{Z}^4$	$\mathbb{Z}^6$	$\mathbb{Z}^2$	$\mathbb{Z}^2$

L6a2

$$\begin{array}{ccccccc}
\mathbb{Z}^2 & \mathbb{Z} & \mathbb{Z}^2\times C_2 & \mathbb{Z}^2\times C_2 & \mathbb{Z}^2\times C_2 & \mathbb{Z}\times C_2 & \mathbb{Z}^2 \\
\mathbb{Z}^2 & \mathbb{Z}^2 & \mathbb{Z}^4 & \mathbb{Z}^4 & \mathbb{Z}^4 & \mathbb{Z}^2 & \mathbb{Z}^2 \\
\mathbb{Z}^2 & \mathbb{Z}^2 & \mathbb{Z}^4 & \mathbb{Z}^4 & \mathbb{Z}^4 & \mathbb{Z}^2 & \mathbb{Z}^2 \\
\mathbb{Z}^3 & \mathbb{Z}^4 & \mathbb{Z}^5 & \mathbb{Z}^3\times C_2 & \mathbb{Z}^2\times C_2 & \mathbb{Z}\times C_2 & \mathbb{Z}^2 \\
\mathbb{Z}^2 & \mathbb{Z}^2 & \mathbb{Z}^4 & \mathbb{Z}^4 & \mathbb{Z}^4 & \mathbb{Z}^2 & \mathbb{Z}^2 \\
\mathbb{Z}^2 & \mathbb{Z}^2 & \mathbb{Z}^4 & \mathbb{Z}^4 & \mathbb{Z}^4 & \mathbb{Z}^2 & \mathbb{Z}^2 \\
\mathbb{Z}^4 & \mathbb{Z}^6 & \mathbb{Z}^8 & \mathbb{Z}^8 & \mathbb{Z}^6 & \mathbb{Z}^2\times C_2 & \mathbb{Z}^2 \\
\mathbb{Z}^3 & \mathbb{Z}^4 & \mathbb{Z}^7 & \mathbb{Z}^8 & \mathbb{Z}^6 & \mathbb{Z}^2\times C_2 & \mathbb{Z}^2
\end{array}$$

L6a3

$$\begin{array}{ccccccc}
\mathbb{Z}^2 & 0 & \mathbb{Z} & \mathbb{Z}\times C_2 & \mathbb{Z} & \mathbb{Z}\times C_2 & \mathbb{Z}^2 \\
\mathbb{Z}^2 & 0 & \mathbb{Z}^2 & \mathbb{Z}^2 & \mathbb{Z}^2 & \mathbb{Z}^2 & \mathbb{Z}^2 \\
\mathbb{Z}^2 & 0 & \mathbb{Z}^2 & \mathbb{Z}^2 & \mathbb{Z}^2 & \mathbb{Z}^2 & \mathbb{Z}^2 \\
\mathbb{Z}^2 & 0 & \mathbb{Z}^2 & \mathbb{Z}^2 & \mathbb{Z}^2 & \mathbb{Z}^2 & \mathbb{Z}^2 \\
\mathbb{Z}^2 & 0 & \mathbb{Z}^2 & \mathbb{Z}^2 & \mathbb{Z}^2 & \mathbb{Z}^2 & \mathbb{Z}^2 \\
\mathbb{Z}^2 & 0 & \mathbb{Z}^2 & \mathbb{Z}^2 & \mathbb{Z}^2 & \mathbb{Z}^2 & \mathbb{Z}^2 \\
\mathbb{Z}^2 & 0 & \mathbb{Z}^2 & \mathbb{Z}^2 & \mathbb{Z}^2 & \mathbb{Z}^2 & \mathbb{Z}^2 \\
\mathbb{Z}^3 & \mathbb{Z} & \mathbb{Z} & \mathbb{Z}\times C_2 & \mathbb{Z} & \mathbb{Z}\times C_2 & \mathbb{Z}^2
\end{array}$$

L6a4

$$
\begin{array}{ccccccc}
\mathbb{Z} & \mathbb{Z}^3 \times C_2 & \mathbb{Z}^2 \times C_2 \times C_2 & \mathbb{Z}^8 & \mathbb{Z}^2 & \mathbb{Z}^3 \times C_2 \times C_2 & \mathbb{Z} \times C_2 \\
\mathbb{Z}^2 & \mathbb{Z}^6 & \mathbb{Z}^4 & \mathbb{Z}^8 & \mathbb{Z}^4 & \mathbb{Z}^6 & \mathbb{Z}^2 \\
\mathbb{Z} & \mathbb{Z}^3 \times C_2 & \mathbb{Z}^4 \times C_2 \times C_2 & \mathbb{Z}^{12} & \mathbb{Z}^9 & \mathbb{Z}^{10} & \mathbb{Z}^3 \\
\mathbb{Z}^2 & \mathbb{Z}^6 & \mathbb{Z}^4 & \mathbb{Z}^8 & \mathbb{Z}^4 & \mathbb{Z}^6 & \mathbb{Z}^2 \\
\mathbb{Z} & \mathbb{Z}^3 \times C_2 & \mathbb{Z}^4 \times C_2 \times C_2 & \mathbb{Z}^{12} & \mathbb{Z}^9 & \mathbb{Z}^{10} & \mathbb{Z}^3 \\
\mathbb{Z}^2 & \mathbb{Z}^6 & \mathbb{Z}^6 & \mathbb{Z}^{12} & \mathbb{Z}^6 & \mathbb{Z}^6 & \mathbb{Z}^2 \\
\mathbb{Z} & \mathbb{Z}^3 \times C_2 & \mathbb{Z}^4 \times C_2 \times C_2 & \mathbb{Z}^{12} & \mathbb{Z}^9 & \mathbb{Z}^{10} & \mathbb{Z}^3 \\
\mathbb{Z}^2 & \mathbb{Z}^6 & \mathbb{Z}^4 & \mathbb{Z}^8 & \mathbb{Z}^4 & \mathbb{Z}^6 & \mathbb{Z}^2
\end{array}
$$

L6a5

$$
\begin{array}{ccccccc}
\mathbb{Z}^2 & \mathbb{Z}^2 & \mathbb{Z}^3 \times C_2 \times C_2 & \mathbb{Z} \times C_2 & \mathbb{Z}^6 & \mathbb{Z} & \mathbb{Z} \times C_2 \\
\mathbb{Z}^2 & \mathbb{Z}^4 & \mathbb{Z}^6 & \mathbb{Z}^2 & \mathbb{Z}^6 & \mathbb{Z}^2 & \mathbb{Z}^2 \\
\mathbb{Z}^3 & \mathbb{Z}^7 & \mathbb{Z}^{11} & \mathbb{Z}^6 & \mathbb{Z}^8 & \mathbb{Z}^2 & \mathbb{Z} \times C_2 \\
\mathbb{Z}^3 & \mathbb{Z}^7 & \mathbb{Z}^{11} & \mathbb{Z}^6 & \mathbb{Z}^8 & \mathbb{Z}^2 & \mathbb{Z} \times C_2 \\
\mathbb{Z}^2 & \mathbb{Z}^4 & \mathbb{Z}^6 & \mathbb{Z}^2 & \mathbb{Z}^6 & \mathbb{Z}^2 & \mathbb{Z}^2 \\
\mathbb{Z}^2 & \mathbb{Z}^4 & \mathbb{Z}^6 & \mathbb{Z}^2 & \mathbb{Z}^6 & \mathbb{Z}^2 & \mathbb{Z}^2 \\
\mathbb{Z}^4 & \mathbb{Z}^9 & \mathbb{Z}^{12} & \mathbb{Z}^6 & \mathbb{Z}^7 & \mathbb{Z} & \mathbb{Z} \times C_2 \\
\mathbb{Z}^3 & \mathbb{Z}^7 & \mathbb{Z}^{11} & \mathbb{Z}^6 & \mathbb{Z}^8 & \mathbb{Z}^2 & \mathbb{Z} \times C_2
\end{array}
$$

L6n1

$\mathbb{Z}^2$	0	$\mathbb{Z}$	$\mathbb{Z}\times C_2$	$\mathbb{Z}^6$	0	0
$\mathbb{Z}^2$	0	$\mathbb{Z}^2$	$\mathbb{Z}^2$	$\mathbb{Z}^6$	0	0
$\mathbb{Z}^3$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}\times C_2$	$\mathbb{Z}^6$	0	0
$\mathbb{Z}^2$	0	$\mathbb{Z}^2$	$\mathbb{Z}^2$	$\mathbb{Z}^6$	0	0
$\mathbb{Z}^3$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}\times C_2$	$\mathbb{Z}^6$	0	0
$\mathbb{Z}^4$	$\mathbb{Z}^2$	$\mathbb{Z}^2$	$\mathbb{Z}^2$	$\mathbb{Z}^6$	0	0
$\mathbb{Z}^3$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}\times C_2$	$\mathbb{Z}^6$	0	0
$\mathbb{Z}^2$	0	$\mathbb{Z}^2$	$\mathbb{Z}^2$	$\mathbb{Z}^6$	0	0

L7a1

$\mathbb{Z}$	$\mathbb{Z}^3\times C_2$	$\mathbb{Z}^4\times C_2\times C_2$	$\mathbb{Z}^4\times C_2\times C_2$	$\mathbb{Z}^7\times C_2\times C_2$	$\mathbb{Z}^3\times C_2$	$\mathbb{Z}^3\times C_2\times C_2$	$\mathbb{Z}\times C_2$
$\mathbb{Z}^2$	$\mathbb{Z}^6$	$\mathbb{Z}^8$	$\mathbb{Z}^8$	$\mathbb{Z}^{10}$	$\mathbb{Z}^6$	$\mathbb{Z}^6$	$\mathbb{Z}^2$
$\mathbb{Z}^2$	$\mathbb{Z}^6$	$\mathbb{Z}^8$	$\mathbb{Z}^8$	$\mathbb{Z}^{10}$	$\mathbb{Z}^6$	$\mathbb{Z}^6$	$\mathbb{Z}^2$
$\mathbb{Z}$	$\mathbb{Z}^3\times C_2$	$\mathbb{Z}^4\times C_2\times C_2$	$\mathbb{Z}^6\times C_2\times C_2$	$\mathbb{Z}^{11}\times C_2\times C_2$	$\mathbb{Z}^{10}\times C_2$	$\mathbb{Z}^{10}$	$\mathbb{Z}^3$
$\mathbb{Z}$	$\mathbb{Z}^3\times C_2$	$\mathbb{Z}^5\times C_2\times C_2$	$\mathbb{Z}^9\times C_2\times C_2$	$\mathbb{Z}^{15}\times C_2$	$\mathbb{Z}^{12}$	$\mathbb{Z}^{10}$	$\mathbb{Z}^3$
$\mathbb{Z}^2$	$\mathbb{Z}^6$	$\mathbb{Z}^8$	$\mathbb{Z}^{10}$	$\mathbb{Z}^{14}$	$\mathbb{Z}^8$	$\mathbb{Z}^6$	$\mathbb{Z}^2$
$\mathbb{Z}$	$\mathbb{Z}^3\times C_2$	$\mathbb{Z}^5\times C_2\times C_2$	$\mathbb{Z}^9\times C_2\times C_2$	$\mathbb{Z}^{15}\times C_2$	$\mathbb{Z}^{12}$	$\mathbb{Z}^{10}$	$\mathbb{Z}^3$
$\mathbb{Z}^2$	$\mathbb{Z}^6$	$\mathbb{Z}^8$	$\mathbb{Z}^8$	$\mathbb{Z}^{10}$	$\mathbb{Z}^6$	$\mathbb{Z}^6$	$\mathbb{Z}^2$
$\mathbb{Z}^2$	$\mathbb{Z}^6$	$\mathbb{Z}^8$	$\mathbb{Z}^8$	$\mathbb{Z}^{10}$	$\mathbb{Z}^6$	$\mathbb{Z}^6$	$\mathbb{Z}^2$

L7a2

$\mathbb{Z}^2$	$\mathbb{Z}^2$	$\mathbb{Z}^4 \times C_2 \times C_2$	$\mathbb{Z}^3 \times C_2 \times C_2$	$\mathbb{Z}^5 \times C_2$	$\mathbb{Z}^3 \times C_2 \times C_2$	$\mathbb{Z}^2 \times C_2$	$\mathbb{Z} \times C_2$
$\mathbb{Z}^2$	$\mathbb{Z}^4$	$\mathbb{Z}^8$	$\mathbb{Z}^6$	$\mathbb{Z}^8$	$\mathbb{Z}^6$	$\mathbb{Z}^4$	$\mathbb{Z}^2$
$\mathbb{Z}^2$	$\mathbb{Z}^4$	$\mathbb{Z}^8$	$\mathbb{Z}^6$	$\mathbb{Z}^8$	$\mathbb{Z}^6$	$\mathbb{Z}^4$	$\mathbb{Z}^2$
$\mathbb{Z}^3$	$\mathbb{Z}^7$	$\mathbb{Z}^{12}$	$\mathbb{Z}^8 \times C_2$	$\mathbb{Z}^7 \times C_2$	$\mathbb{Z}^4 \times C_2 \times C_2$	$\mathbb{Z}^2 \times C_2$	$\mathbb{Z} \times C_2$
$\mathbb{Z}^2$	$\mathbb{Z}^4$	$\mathbb{Z}^8$	$\mathbb{Z}^6$	$\mathbb{Z}^8$	$\mathbb{Z}^6$	$\mathbb{Z}^4$	$\mathbb{Z}^2$
$\mathbb{Z}^4$	$\mathbb{Z}^9$	$\mathbb{Z}^{15}$	$\mathbb{Z}^{13}$	$\mathbb{Z}^{10}$	$\mathbb{Z}^4 \times C_2 \times C_2$	$\mathbb{Z}^2 \times C_2$	$\mathbb{Z} \times C_2$
$\mathbb{Z}^3$	$\mathbb{Z}^7$	$\mathbb{Z}^{14}$	$\mathbb{Z}^{13}$	$\mathbb{Z}^{12}$	$\mathbb{Z}^7 \times C_2$	$\mathbb{Z}^3 \times C_2$	$\mathbb{Z} \times C_2$
$\mathbb{Z}^3$	$\mathbb{Z}^7$	$\mathbb{Z}^{14}$	$\mathbb{Z}^{13}$	$\mathbb{Z}^{12}$	$\mathbb{Z}^7 \times C_2$	$\mathbb{Z}^3 \times C_2$	$\mathbb{Z} \times C_2$
$\mathbb{Z}^2$	$\mathbb{Z}^4$	$\mathbb{Z}^8$	$\mathbb{Z}^6$	$\mathbb{Z}^8$	$\mathbb{Z}^6$	$\mathbb{Z}^4$	$\mathbb{Z}^2$

L7a3

$\mathbb{Z}$	$\mathbb{Z}^2 \times C_2$	$\mathbb{Z}^3 \times C_2$	$\mathbb{Z}^3 \times C_2 \times C_2$	$\mathbb{Z}^2 \times C_2$	$\mathbb{Z}^5 \times C_2$	$\mathbb{Z}$	$\mathbb{Z} \times C_2$
$\mathbb{Z}^2$	$\mathbb{Z}^4$	$\mathbb{Z}^6$	$\mathbb{Z}^6$	$\mathbb{Z}^4$	$\mathbb{Z}^6$	$\mathbb{Z}^2$	$\mathbb{Z}^2$
$\mathbb{Z}^2$	$\mathbb{Z}^4$	$\mathbb{Z}^6$	$\mathbb{Z}^6$	$\mathbb{Z}^4$	$\mathbb{Z}^6$	$\mathbb{Z}^2$	$\mathbb{Z}^2$
$\mathbb{Z}$	$\mathbb{Z}^2 \times C_2$	$\mathbb{Z}^3 \times C_2$	$\mathbb{Z}^3 \times C_2 \times C_2$	$\mathbb{Z}^4 \times C_2$	$\mathbb{Z}^8 \times C_2$	$\mathbb{Z}^4$	$\mathbb{Z}^3$
$\mathbb{Z}^2$	$\mathbb{Z}^4$	$\mathbb{Z}^6$	$\mathbb{Z}^6$	$\mathbb{Z}^4$	$\mathbb{Z}^6$	$\mathbb{Z}^2$	$\mathbb{Z}^2$
$\mathbb{Z}^2$	$\mathbb{Z}^4$	$\mathbb{Z}^6$	$\mathbb{Z}^6$	$\mathbb{Z}^4$	$\mathbb{Z}^6$	$\mathbb{Z}^2$	$\mathbb{Z}^2$
$\mathbb{Z}^2$	$\mathbb{Z}^4$	$\mathbb{Z}^6$	$\mathbb{Z}^6$	$\mathbb{Z}^6$	$\mathbb{Z}^{10}$	$\mathbb{Z}^4$	$\mathbb{Z}^2$
$\mathbb{Z}$	$\mathbb{Z}^2 \times C_2$	$\mathbb{Z}^3 \times C_2$	$\mathbb{Z}^4 \times C_2 \times C_2$	$\mathbb{Z}^6 \times C_2$	$\mathbb{Z}^9$	$\mathbb{Z}^4$	$\mathbb{Z}^3$
$\mathbb{Z}^2$	$\mathbb{Z}^4$	$\mathbb{Z}^6$	$\mathbb{Z}^6$	$\mathbb{Z}^4$	$\mathbb{Z}^6$	$\mathbb{Z}^2$	$\mathbb{Z}^2$

L7a4

$\mathbb{Z}$	$\mathbb{Z}^2 \times C_2$	$\mathbb{Z}^2 \times C_2$	$\mathbb{Z}^3 \times C_2$	$\mathbb{Z}^3 \times C_2 \times C_2$	$\mathbb{Z}^5 \times C_2$	$\mathbb{Z}$	$\mathbb{Z} \times C_2$
$\mathbb{Z}^2$	$\mathbb{Z}^4$	$\mathbb{Z}^4$	$\mathbb{Z}^6$	$\mathbb{Z}^6$	$\mathbb{Z}^6$	$\mathbb{Z}^2$	$\mathbb{Z}^2$
$\mathbb{Z}^2$	$\mathbb{Z}^4$	$\mathbb{Z}^4$	$\mathbb{Z}^6$	$\mathbb{Z}^6$	$\mathbb{Z}^6$	$\mathbb{Z}^2$	$\mathbb{Z}^2$
$\mathbb{Z}^2$	$\mathbb{Z}^4$	$\mathbb{Z}^4$	$\mathbb{Z}^6$	$\mathbb{Z}^6$	$\mathbb{Z}^6$	$\mathbb{Z}^2$	$\mathbb{Z}^2$
$\mathbb{Z}^3$	$\mathbb{Z}^7$	$\mathbb{Z}^8$	$\mathbb{Z}^9$	$\mathbb{Z}^7 \times C_2$	$\mathbb{Z}^7 \times C_2$	$\mathbb{Z}^2$	$\mathbb{Z} \times C_2$
$\mathbb{Z}^2$	$\mathbb{Z}^4$	$\mathbb{Z}^4$	$\mathbb{Z}^6$	$\mathbb{Z}^6$	$\mathbb{Z}^6$	$\mathbb{Z}^2$	$\mathbb{Z}^2$
$\mathbb{Z}^3$	$\mathbb{Z}^7$	$\mathbb{Z}^8$	$\mathbb{Z}^9$	$\mathbb{Z}^7 \times C_2$	$\mathbb{Z}^7 \times C_2$	$\mathbb{Z}^2$	$\mathbb{Z} \times C_2$
$\mathbb{Z}^2$	$\mathbb{Z}^4$	$\mathbb{Z}^4$	$\mathbb{Z}^6$	$\mathbb{Z}^6$	$\mathbb{Z}^6$	$\mathbb{Z}^2$	$\mathbb{Z}^2$
$\mathbb{Z}^3$	$\mathbb{Z}^7$	$\mathbb{Z}^6$	$\mathbb{Z}^4 \times C_2$	$\mathbb{Z}^3 \times C_2 \times C_2$	$\mathbb{Z}^5 \times C_2$	$\mathbb{Z}$	$\mathbb{Z} \times C_2$

L7a5

$\mathbb{Z}$	$\mathbb{Z}^2 \times C_2$	$\mathbb{Z}^4 \times C_2$	$\mathbb{Z}^3 \times C_2$	$\mathbb{Z}^5 \times C_2 \times C_2$	$\mathbb{Z}^2 \times C_2$	$\mathbb{Z}^2 \times C_2$	$\mathbb{Z} \times C_2$
$\mathbb{Z}^2$	$\mathbb{Z}^4$	$\mathbb{Z}^6$	$\mathbb{Z}^6$	$\mathbb{Z}^8$	$\mathbb{Z}^4$	$\mathbb{Z}^4$	$\mathbb{Z}^2$
$\mathbb{Z}$	$\mathbb{Z}^2 \times C_2$	$\mathbb{Z}^5 \times C_2$	$\mathbb{Z}^7 \times C_2$	$\mathbb{Z}^{11} \times C_2$	$\mathbb{Z}^8$	$\mathbb{Z}^7$	$\mathbb{Z}^3$
$\mathbb{Z}^2$	$\mathbb{Z}^4$	$\mathbb{Z}^6$	$\mathbb{Z}^6$	$\mathbb{Z}^8$	$\mathbb{Z}^4$	$\mathbb{Z}^4$	$\mathbb{Z}^2$
$\mathbb{Z}$	$\mathbb{Z}^3 \times C_2$	$\mathbb{Z}^7 \times C_2$	$\mathbb{Z}^{10} \times C_2$	$\mathbb{Z}^{13}$	$\mathbb{Z}^8$	$\mathbb{Z}^7$	$\mathbb{Z}^3$
$\mathbb{Z}^2$	$\mathbb{Z}^4$	$\mathbb{Z}^6$	$\mathbb{Z}^6$	$\mathbb{Z}^8$	$\mathbb{Z}^4$	$\mathbb{Z}^4$	$\mathbb{Z}^2$
$\mathbb{Z}^2$	$\mathbb{Z}^4$	$\mathbb{Z}^6$	$\mathbb{Z}^6$	$\mathbb{Z}^8$	$\mathbb{Z}^4$	$\mathbb{Z}^4$	$\mathbb{Z}^2$
$\mathbb{Z}^2$	$\mathbb{Z}^4$	$\mathbb{Z}^6$	$\mathbb{Z}^6$	$\mathbb{Z}^8$	$\mathbb{Z}^4$	$\mathbb{Z}^4$	$\mathbb{Z}^2$
$\mathbb{Z}$	$\mathbb{Z}^2 \times C_2$	$\mathbb{Z}^4 \times C_2$	$\mathbb{Z}^4 \times C_2$	$\mathbb{Z}^7 \times C_2 \times C_2$	$\mathbb{Z}^6 \times C_2$	$\mathbb{Z}^7$	$\mathbb{Z}^3$

L7a6

$\mathbb{Z}$	$\mathbb{Z}^2 \times C_2$	$\mathbb{Z}^2 \times C_2$	$\mathbb{Z}^4 \times C_2$	$\mathbb{Z}^2 \times C_2$	$\mathbb{Z}^3 \times C_2$	$\mathbb{Z}$	$\mathbb{Z} \times C_2$
$\mathbb{Z}^2$	$\mathbb{Z}^4$	$\mathbb{Z}^4$	$\mathbb{Z}^6$	$\mathbb{Z}^4$	$\mathbb{Z}^4$	$\mathbb{Z}^2$	$\mathbb{Z}^2$
$\mathbb{Z}^2$	$\mathbb{Z}^4$	$\mathbb{Z}^4$	$\mathbb{Z}^6$	$\mathbb{Z}^4$	$\mathbb{Z}^4$	$\mathbb{Z}^2$	$\mathbb{Z}^2$
$\mathbb{Z}^2$	$\mathbb{Z}^4$	$\mathbb{Z}^4$	$\mathbb{Z}^6$	$\mathbb{Z}^4$	$\mathbb{Z}^4$	$\mathbb{Z}^2$	$\mathbb{Z}^2$
$\mathbb{Z}^2$	$\mathbb{Z}^4$	$\mathbb{Z}^4$	$\mathbb{Z}^6$	$\mathbb{Z}^4$	$\mathbb{Z}^4$	$\mathbb{Z}^2$	$\mathbb{Z}^2$
$\mathbb{Z}^2$	$\mathbb{Z}^4$	$\mathbb{Z}^4$	$\mathbb{Z}^6$	$\mathbb{Z}^4$	$\mathbb{Z}^4$	$\mathbb{Z}^2$	$\mathbb{Z}^2$
$\mathbb{Z}^2$	$\mathbb{Z}^4$	$\mathbb{Z}^4$	$\mathbb{Z}^6$	$\mathbb{Z}^4$	$\mathbb{Z}^4$	$\mathbb{Z}^2$	$\mathbb{Z}^2$
$\mathbb{Z}$	$\mathbb{Z}^2 \times C_2$	$\mathbb{Z}^2 \times C_2$	$\mathbb{Z}^4 \times C_2$	$\mathbb{Z}^3 \times C_2$	$\mathbb{Z}^5 \times C_2$	$\mathbb{Z}^4$	$\mathbb{Z}^3$
$\mathbb{Z}$	$\mathbb{Z}^2 \times C_2$	$\mathbb{Z}^3 \times C_2$	$\mathbb{Z}^6 \times C_2$	$\mathbb{Z}^5 \times C_2$	$\mathbb{Z}^6$	$\mathbb{Z}^4$	$\mathbb{Z}^3$

L7a7

$\mathbb{Z}$	$\mathbb{Z}^3 \times C_2$	$\mathbb{Z}^3 \times C_2 \times C_2$	$\mathbb{Z}^7 \times C_2$	$\mathbb{Z}^3$	$\mathbb{Z}^4 \times C_2 \times C_2 \times C_2$	$\mathbb{Z} \times C_2$	$\mathbb{Z}^2$
$\mathbb{Z}^2$	$\mathbb{Z}^6$	$\mathbb{Z}^6$	$\mathbb{Z}^8$	$\mathbb{Z}^6$	$\mathbb{Z}^8$	$\mathbb{Z}^2$	$\mathbb{Z}^2$
$\mathbb{Z}^3$	$\mathbb{Z}^{10}$	$\mathbb{Z}^{12}$	$\mathbb{Z}^{14}$	$\mathbb{Z}^8$	$\mathbb{Z}^7 \times C_2 \times C_2$	$\mathbb{Z}^2 \times C_2$	$\mathbb{Z}^2$
$\mathbb{Z}^2$	$\mathbb{Z}^6$	$\mathbb{Z}^6$	$\mathbb{Z}^8$	$\mathbb{Z}^6$	$\mathbb{Z}^8$	$\mathbb{Z}^2$	$\mathbb{Z}^2$
$\mathbb{Z}^3$	$\mathbb{Z}^{10}$	$\mathbb{Z}^{12}$	$\mathbb{Z}^{14}$	$\mathbb{Z}^8$	$\mathbb{Z}^7 \times C_2 \times C_2$	$\mathbb{Z}^2 \times C_2$	$\mathbb{Z}^2$
$\mathbb{Z}^2$	$\mathbb{Z}^6$	$\mathbb{Z}^6$	$\mathbb{Z}^8$	$\mathbb{Z}^6$	$\mathbb{Z}^8$	$\mathbb{Z}^2$	$\mathbb{Z}^2$
$\mathbb{Z}^2$	$\mathbb{Z}^6$	$\mathbb{Z}^6$	$\mathbb{Z}^8$	$\mathbb{Z}^6$	$\mathbb{Z}^8$	$\mathbb{Z}^2$	$\mathbb{Z}^2$
$\mathbb{Z}^3$	$\mathbb{Z}^{10}$	$\mathbb{Z}^{12}$	$\mathbb{Z}^{14}$	$\mathbb{Z}^8$	$\mathbb{Z}^7 \times C_2 \times C_2$	$\mathbb{Z}^2 \times C_2$	$\mathbb{Z}^2$
$\mathbb{Z}^3$	$\mathbb{Z}^{10}$	$\mathbb{Z}^{12}$	$\mathbb{Z}^{12}$	$\mathbb{Z}^4$	$\mathbb{Z}^4 \times C_2 \times C_2 \times C_2$	$\mathbb{Z} \times C_2$	$\mathbb{Z}^2$

139

L7n1

$\mathbb{Z}^2$	0	$\mathbb{Z}$	$\mathbb{Z}\times C_2$	$\mathbb{Z}^3$	$\mathbb{Z}\times C_2$	0	0
$\mathbb{Z}^2$	0	$\mathbb{Z}^2$	$\mathbb{Z}^2$	$\mathbb{Z}^4$	$\mathbb{Z}^2$	0	0
$\mathbb{Z}^2$	0	$\mathbb{Z}^2$	$\mathbb{Z}^2$	$\mathbb{Z}^4$	$\mathbb{Z}^2$	0	0
$\mathbb{Z}^2$	0	$\mathbb{Z}^2$	$\mathbb{Z}^2$	$\mathbb{Z}^4$	$\mathbb{Z}^2$	0	0
$\mathbb{Z}^2$	$\mathbb{Z}$	$\mathbb{Z}^4$	$\mathbb{Z}^4$	$\mathbb{Z}^7$	$\mathbb{Z}^4$	0	0
$\mathbb{Z}^2$	0	$\mathbb{Z}^2$	$\mathbb{Z}^2$	$\mathbb{Z}^4$	$\mathbb{Z}^2$	0	0
$\mathbb{Z}^2$	$\mathbb{Z}$	$\mathbb{Z}^4$	$\mathbb{Z}^4$	$\mathbb{Z}^7$	$\mathbb{Z}^4$	0	0
$\mathbb{Z}^2$	0	$\mathbb{Z}^2$	$\mathbb{Z}^4$	$\mathbb{Z}^6$	$\mathbb{Z}^2\times C_2\times C_2$	0	0
$\mathbb{Z}^3$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}\times C_2$	$\mathbb{Z}^4$	$\mathbb{Z}^2\times C_2$	0	0

L7n2

0	0	$\mathbb{Z}^4$	$\mathbb{Z}$	$\mathbb{Z}^2\times C_2$	$\mathbb{Z}\times C_2$	$\mathbb{Z}$	$\mathbb{Z}\times C_2$
0	0	$\mathbb{Z}^4$	$\mathbb{Z}^2$	$\mathbb{Z}^4$	$\mathbb{Z}^2$	$\mathbb{Z}^2$	$\mathbb{Z}^2$
0	0	$\mathbb{Z}^4$	$\mathbb{Z}^2$	$\mathbb{Z}^4$	$\mathbb{Z}^2$	$\mathbb{Z}^2$	$\mathbb{Z}^2$
0	0	$\mathbb{Z}^4$	$\mathbb{Z}^2$	$\mathbb{Z}^4$	$\mathbb{Z}^2$	$\mathbb{Z}^2$	$\mathbb{Z}^2$
0	0	$\mathbb{Z}^4$	$\mathbb{Z}^2$	$\mathbb{Z}^4$	$\mathbb{Z}^2$	$\mathbb{Z}^2$	$\mathbb{Z}^2$
0	0	$\mathbb{Z}^7$	$\mathbb{Z}^7$	$\mathbb{Z}^8$	$\mathbb{Z}^5$	$\mathbb{Z}^2$	$\mathbb{Z}\times C_2$
0	0	$\mathbb{Z}^6$	$\mathbb{Z}^5$	$\mathbb{Z}^5$	$\mathbb{Z}^2\times C_2$	$\mathbb{Z}$	$\mathbb{Z}\times C_2$
0	0	$\mathbb{Z}^5$	$\mathbb{Z}^2$	$\mathbb{Z}^2\times C_2$	$\mathbb{Z}^2\times C_2$	$\mathbb{Z}^4$	$\mathbb{Z}^3$
0	0	$\mathbb{Z}^6$	$\mathbb{Z}^4$	$\mathbb{Z}^4$	$\mathbb{Z}^2$	$\mathbb{Z}^2$	$\mathbb{Z}^2$

REFERENCES CITED

[AKW22]   Rostislav Akhmechet, Vyacheslav Krushkal, and Michael Willis. "Towards an $\mathfrak{sl}_2$ action on the annular Khovanov spectrum". In: *Adv. Math.* 408.part A (2022), Paper No. 108581, 65. ISSN: 0001-8708.

[All14]   Michael P. Allocca. "Homomorphisms of L_∞ modules". In: *J. Homotopy Relat. Struct.* 9.2 (2014), pp. 285–298. ISSN: 2193-8407.

[All77]   Christopher Allday. "Rational Whitehead products and a spectral sequence of Quillen. II". In: *Houston J. Math.* 3.3 (1977), pp. 301–308. ISSN: 0362-1588.

[APS04]   Marta M. Asaeda, Józef H. Przytycki, and Adam S. Sikora. "Categorification of the Kauffman bracket skein module of I-bundles over surfaces". In: *Algebr. Geom. Topol.* 4 (2004), pp. 1177–1210. ISSN: 1472-2747.

[Bar02]   Dror Bar-Natan. "On Khovanov's categorification of the Jones polynomial". In: *Algebr. Geom. Topol.* 2 (2002), pp. 337–370. ISSN: 1472-2747.

[BBD85]   F. A. Berends, G. J. H. Burgers, and H. van Dam. "On the theoretical problems in constructing interactions involving higher-spin massless particles". In: *Nuclear Phys. B* 260.2 (1985), pp. 295–322. ISSN: 0550-3213.

[Bel+17]   Francisco Belchí et al. "Higher order Whitehead products and L_∞ structures on the homology of a DGL". In: *Linear Algebra Appl.* 520 (2017), pp. 16–31. ISSN: 0024-3795.

[Ber14]   Alexander Berglund. "Homological perturbation theory for algebras over operads". In: *Algebr. Geom. Topol.* 14.5 (2014), pp. 2511–2548. ISSN: 1472-2747.

[BM20]   Fraser Binns and Gage Martin. *Knot Floer homology, link Floer homology and link detection.* arXiv:2011.02005. 2020. URL: https://arxiv.org/abs/2011.02005.

[Dav22]   Champ Davis. *Restriction of Scalars for L_∞-Modules.* arXiv:2208.13942. 2022. URL: https://arxiv.org/abs/2208.13942.

[GLS91]   V. K. A. M. Gugenheim, L. A. Lambe, and J. D. Stasheff. "Perturbation theory in differential homological algebra. II". In: *Illinois J. Math.* 35.3 (1991), pp. 357–373. ISSN: 0019-2082.

[GLW18]   J. Elisenda Grigsby, Anthony M. Licata, and Stephan M. Wehrli. "Annular Khovanov homology and knotted Schur-Weyl representations". In: *Compos. Math.* 154.3 (2018), pp. 459–502. ISSN: 0010-437X.

[HS02]   Johannes Huebschmann and Jim Stasheff. "Formal solution of the master equation via HPT and deformation theory". In: *Forum Math.* 14.6 (2002), pp. 847–868. ISSN: 0933-7741.

[Hue11]   Johannes Huebschmann. "The sh-Lie algebra perturbation lemma". In: *Forum Math.* 23.4 (2011), pp. 669–691. ISSN: 0933-7741.

[Kho00]   Mikhail Khovanov. "A categorification of the Jones polynomial". In: *Duke Math. J.* 101.3 (2000), pp. 359–426. ISSN: 0012-7094.

[KM11]   P. B. Kronheimer and T. S. Mrowka. "Khovanov homology is an unknot-detector". In: *Publ. Math. Inst. Hautes Études Sci.* 113 (2011), pp. 97–208. ISSN: 0073-8301.

[KM93]   P. B. Kronheimer and T. S. Mrowka. "Gauge theory for embedded surfaces. I". In: *Topology* 32.4 (1993), pp. 773–826. ISSN: 0040-9383.

[Kno11]   Knot Atlas. *KnotTheory*. 2011. URL: http://katlas.org/wiki/The_Mathematica_Package_KnotTheory.

[KS22]   Andreas Kraft and Jonas Schnitzer. *An Introduction to L_∞-Algebras and their Homotopy Theory*. arXiv:2207.01861. 2022. URL: https://arxiv.org/abs/2207.01861.

[LM95]   Tom Lada and Martin Markl. "Strongly homotopy Lie algebras". In: *Comm. Algebra* 23.6 (1995), pp. 2147–2161. ISSN: 0092-7872.

[LV12]   Jean-Louis Loday and Bruno Vallette. *Algebraic operads*. Vol. 346. Grundlehren der mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]. Springer, Heidelberg, 2012, pp. xxiv+634. ISBN: 978-3-642-30361-6.

[Man10]   Marco Manetti. "A relative version of the ordinary perturbation lemma". In: *Rend. Mat. Appl. (7)* 30.2 (2010), pp. 221–238. ISSN: 1120-7183.

[Mor22a]   José Manuel Moreno Fernández. "The Milnor-Moore theorem for L_∞ algebras in rational homotopy theory". In: *Math. Z.* 300.3 (2022), pp. 2147–2165. ISSN: 0025-5874.

[Mor22b] José Manuel Moreno Fernández. "The Milnor-Moore theorem for L_∞ algebras in rational homotopy theory". In: *Math. Z.* 300.3 (2022), pp. 2147–2165. ISSN: 0025-5874.

[Pic20] Lisa Piccirillo. "The Conway knot is not slice". In: *Ann. of Math. (2)* 191.2 (2020), pp. 581–591. ISSN: 0003-486X.

[QR18] Hoel Queffelec and David E. V. Rose. "Sutured annular Khovanov-Rozansky homology". In: *Trans. Amer. Math. Soc.* 370.2 (2018), pp. 1285–1319. ISSN: 0002-9947.

[Ras10] Jacob Rasmussen. "Khovanov homology and the slice genus". In: *Invent. Math.* 182.2 (2010), pp. 419–447. ISSN: 0020-9910.

[Ret85] V. S. Retakh. "Massey operations in Lie superalgebras and differentials of the Quillen spectral sequence". In: *Colloq. Math.* 50.1 (1985), pp. 81–94. ISSN: 0010-1354.

[Rob13] Lawrence P. Roberts. "On knot Floer homology in double branched covers". In: *Geom. Topol.* 17.1 (2013), pp. 413–467. ISSN: 1465-3060.

[SS85] Michael Schlessinger and James Stasheff. "The Lie algebra structure of tangent cohomology and deformation theory". In: *J. Pure Appl. Algebra* 38.2-3 (1985), pp. 313–322. ISSN: 0022-4049.

[Sta19] Jim Stasheff. "*L*-infinity and *A*-infinity structures". In: *High. Struct.* 3.1 (2019), pp. 292–326.

[SZ18] Matthew Stoffregen and Melissa Zhang. *Localization in Khovanov homology.* arXiv:1810.04769. 2018. URL: https://arxiv.org/abs/1810.04769.

[WZ92]   Edward Witten and Barton Zwiebach. "Algebraic structures and differential geometry in two-dimensional string theory". In: *Nuclear Phys. B* 377.1-2 (1992), pp. 55–112. ISSN: 0550-3213.

[XZ19]   Yi Xie and Boyu Zhang. "Instanton Floer homology for sutured manifolds with tangles". In: (2019). URL: `https://arxiv.org/abs/1907.00547`.

[Zha18]   Melissa Zhang. "A rank inequality for the annular Khovanov homology of 2-periodic links". In: *Algebr. Geom. Topol.* 18.2 (2018), pp. 1147–1194. ISSN: 1472-2747.

[Zwi93]   Barton Zwiebach. "Closed string field theory: quantum action and the Batalin-Vilkovisky master equation". In: *Nuclear Phys. B* 390.1 (1993), pp. 33–152. ISSN: 0550-3213.