# LINES ON CUBIC THREEFOLDS AND FOURFOLDS CONTAINING A PLANE

by

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DISSERTATION ABSTRACT

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This thesis describes the Fano scheme $F(Y)$ of lines on a general cubic threefold $Y$ containing a plane over a field $k$ of characteristic different from 2. One irreducible component of $F(Y)$ is birational (over $k$) to a torsor $T$ of an abelian surface, and we apply the geometry and arithmetic of this torsor to answer two questions. First, when is a cubic threefold containing a plane rational over $k$, and second, how can one describe the rational Lagrangian fibration from the Fano variety of lines on a cubic fourfold containing a plane? To answer the first question, we apply recently developed intermediate Jacobian torsor obstructions and show that the existence over $k$ of certain classical rationality constructions completely determines whether the threefold is rational over $k$. The second question, motivated by hyperkähler geometry, we answer by giving an elementary construction that works over a broad class of base fields where hyperkähler tools are not available; moreover, we relate our construction to other descriptions of the rational Lagrangian fibration in the case $k = \mathbb{C}$.

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For my aunt Loretta Wollett and my grandfather Dr. Francis Brooke

TABLE OF CONTENTS

LIST OF FIGURES

Figure | Page

# CHAPTER I

## INTRODUCTION AND BACKGROUND

A cubic fourfold is a smooth hypersurface $X \subset \mathbb{P}^5$ of degree three. For several reasons, cubic fourfolds have attracted significant attention from algebraic geometers. Foremost, the question of whether a generic cubic fourfold is rational has proved very difficult: conjecturally, rational cubic fourfolds are rare, but not a single example has been proved irrational yet. This contrasts cubic surfaces (proved to be rational by Clebsch in 1866) and cubic threefolds (proved to be irrational by Clemens and Griffiths in 1972). Second, cubic fourfolds have rich connections to hyperkähler geometry: many families of cubic fourfolds have associated K3 surfaces, explained later, and the variety of lines on a complex cubic fourfold is a hyperkähler fourfold. Finally, largely due to work of Kuznetsov, the derived category of coherent sheaves on a cubic fourfold encodes a surprising amount of information about the geometry of that cubic fourfold.

Let us fix a cubic fourfold $X$ over a field $k$. The *Fano variety of lines on* $X$ is a variety $F \subset Gr(2,6)$ parametrizing the lines in $\mathbb{P}^5$ that are contained in $X$. The variety $F$ is smooth of dimension 4, and in [6], Beauville and Donagi show that when $k = \mathbb{C}$, $F$ is a hyperkähler manifold of $\mathrm{K3}^{[2]}$-type, i.e. deformation equivalent to the Hilbert scheme of two points on a K3 surface. This establishes a two-way flow of information between the study of complex cubic fourfolds and hyperkähler fourfolds. In this spirit, this work draws intuition from hyperkähler geometry to study the Fano varieties $F$ of lines on some special cubic fourfolds $X$ over arbitrary fields. The central theorem is the following:

**Theorem 1.0.1.** *Let $X$ be a cubic fourfold over a field $k$ of characteristic different from 2. Suppose $X$ contains a plane $P$, let $F$ be the Fano variety of lines on $X$,* *and let $M$ be the Mukai flop of $F$ along the dual plane $P^*$. Then $M$ admits a fibration $M \to \mathbb{P}^2$ whose smooth fibers are torsors of abelian surfaces.*

We describe the fibration $M \to \mathbb{P}^2$ in explicit geometric terms and again from the perspective of the derived category of coherent sheaves on $X$. The advantage of the explicit construction is that it enables an examination of the arithmetic of the fibers. This amounts to studying the Fano variety of lines on a cubic threefold $Y$ containing a plane obtained as a hyperplane section of $X$. I prove the following:

**Theorem 1.0.2.** *Let $T$ be a smooth fiber of $M \to \mathbb{P}^2$. Then there is a genus 2 curve $C$ for which $T$ is a torsor of $\mathrm{Pic}^0_C$. In the Weil-Châtelet group $H^1(k, \mathrm{Pic}^0_C)$, the relations $2[T] = [\mathrm{Pic}^1_C]$ and $4[T] = 0$ hold.*

This arithmetic calls to mind similar results from [35], where Wang proved the same relations for torsors coming from Fano varieties of linear spaces in a complete intersection of two quadrics. Wang's result has proved fruitful in studying the rationality of an intersection of two quadrics over nonclosed fields; for examples, see [7] and [16]. Similarly, we prove the following result about the rationality of cubic threefolds containing planes.

**Theorem 1.0.3.** *Let $Y$ be a cubic threefold containing a plane $P$ over a field $k$, and suppose the singular locus of $Y$ consists of four isolated nodes along $P$. Then $Y$ is rational over $k$ if and only if one of the following is true:*

1. *one of the nodes on $Y$ is defined over $k$, or*

2. *there is a line in $Y \setminus P$ defined over $k$.*

Note that in the above setup, $Y$ is rational over the separable closure $k^s$ regardless of whether it is rational over $k$. This result promises to produce interesting examples of rationality phenomena for threefolds over nonclosed fields: for example, is there a smooth, geometrically rational threefold $Y$ over a number field $K$ which is rational over $K_v$ for all places $v$ but irrational over $K$?

### 1.1 Lines on cubic hypersurfaces

Let $X \subset \mathbb{P}^{n+1}$ be an irreducible but not necessarily smooth cubic hypersurface of dimension $n > 1$. An important scheme attached to $X$ is its Fano scheme of lines $F \subset Gr(2, n+2)$, parametrizing the lines contained in $X$. For two famous examples, the Cayley-Salmon Theorem asserts that the Fano variety of lines on a smooth cubic surface consists of 27 isolated points, and Clemens and Griffiths showed that the Fano variety of lines on a smooth, complex cubic threefold is an abelian surface not isogenous to a product of curves to prove the irrationality of smooth cubic threefolds [9].

It is not difficult to make general statements about the geometry of $F$: for example, the following proposition relates the dimension and singularities of $F$ to those of $X$. For similar treatments, see [4], or [11, Ch. 6], or [21, Ch. 2].

**Proposition 1.1.1.** *If $X \subset \mathbb{P}^{n+1}$ is a cubic hypersurface over an algebraically closed field $k$ and $F$ is the Fano variety of lines on $X$, then*

*(i) $F$ is nonempty of dimension at least $2n-4$,*

*(ii) if $X$ is smooth, then $F$ is smooth of dimension $2n-4$.*

*Proof.* Let $X \subset \mathbb{P}^{n+1}$ be a smooth cubic $n$-fold containing a line $L$ (for example, the Fermat cubic), and let $F$ be its Fano variety of lines. We first prove that $F$ is $(2n-4)$-dimensional and smooth at $[L]$. The exact sequence of normal bundles

$$0 \to N_{L/X} \to N_{L/\mathbb{P}^{n+1}} \to N_{X/\mathbb{P}^{n+1}}|_L \to 0$$

is

$$0 \to N_{L/X} \to \mathcal{O}_L(1)^{\oplus n} \to \mathcal{O}_L(3) \to 0.$$

Writing

$$N_{L/X} \simeq \bigoplus_{i=1}^{n-1} \mathcal{O}_L(a_i),$$

we have

$$\sum_{i=1}^{n-1} a_i = n-3$$

and since $N_{L/X}$ embeds in $\mathcal{O}_L(1)^{\oplus n}$, $a_i \leq 1$ for all $i$. It must be the case then that $-1 \leq a_i \leq 1$ for each $i$, so $H^1(L, N_{L/X}) = 0$. The long exact sequence of cohomology associated to the sequence of normal bundles yields $H^0(L, N_{L/X}) = 2n-4$. Using the identification $T_{[L]}F \simeq H^0(L, N_{L/X})$ and the fact that $F$ is smooth at $[L]$ if $H^1(L, N_{L/X}) = 0$ (see [21, Ch. 2, Prop. 1.10]), we obtain our first claim since there is an identification $T_{[L]}F \simeq H^0(L, N_{L/X})$. Moreover, $F$ is smooth at every point $[L]$ by the same argument.

Writing $N = \binom{n+4}{3}$, let $\mathbb{P}^{N-1}$ be the space of cubic hypersurfaces in $\mathbb{P}^{n+1}$, and let

$$W = \{(L, X) \mid L \subset X\} \subset Gr(2, n+2) \times \mathbb{P}^{N-1}$$

be the universal line along with its projections $\pi_1$ and $\pi_2$. For a line $L \subset \mathbb{P}^{n+1}$, we may identify the fiber $\pi_1^{-1}(L)$ with $\mathbb{P}H^0(L, \mathcal{I}_{L/\mathbb{P}^{n+1}}(3))$. From the long exact sequence in cohomology associated to the exact sequence

$$0 \to \mathcal{I}_{L/\mathbb{P}^{n+1}}(3) \to \mathcal{O}_{\mathbb{P}^{n+1}}(3) \to \mathcal{O}_L(3) \to 0,$$

we obtain $\dim \pi_1^{-1}(L) = N-5$. As the dimension of $Gr(2, n+2)$ is $2n$,

$$\dim W = N + 2n - 5 = (2n-4) + \dim \mathbb{P}^{N-1}.$$

We proved prior that the fiber of $\pi_2$ over any point in $\mathbb{P}^{N-1}$ corresponding to a smooth cubic hypersurface containing a line is $(2n-4)$-dimensional, so by upper semi-continuity and the fact that $\pi_2$ is closed, $\pi_2$ must by surjective. Therefore every smooth cubic hypersurface contains at least one line so has a smooth, $(2n-4)$-dimensional Fano variety, proving (ii). Again using upper semi-continuity, each fiber of $\pi_2$ has dimension at least $2n-4$, proving (i). □

*Remark* 1.1.2. If $X$ is a singular cubic $n$-fold, then it is possible for the Fano variety $F$ of lines on $X$ to be of dimension greater than $2n-4$: for example, if $X \subset \mathbb{P}^3$ is the cone over a cubic plane curve, then $\dim F = 1$. On the other hand, if $X \subset \mathbb{P}^4$ is a cubic threefold containing a plane, then $X$ is singular, but the Fano variety of lines on $X$ is nevertheless a surface, as discussed in detail in Chapter II.

**Proposition 1.1.3.** *Let $X \subset \mathbb{P}^{n+1}$ be a cubic hypersurface and $F$ its Fano variety of lines. Suppose $F$ is of the expected dimension $2n-4$. Then $F$ is smooth at $[L]$ if and only if $X$ is smooth along $L$.*

*Proof.* Let $L \subset X$. Regardless of whether $X$ is smooth along $L$, there is an exact sequence of sheaves

$$0 \to N_{L/X} \to N_{L/\mathbb{P}^{n+1}} \xrightarrow{\alpha} N_{X/\mathbb{P}^{n+1}}|_L,$$

and $\alpha$ is surjective if and only if $X$ is smooth along $L$ by [11, Prop. 6.24]. We have seen already in the course of proving Proposition 1.1.1 how to use the above sequence to show $F$ is smooth at $[L]$ when $\alpha$ is surjective. Conversely, suppose $F$ is smooth at $[L]$. Then

$$\dim H^0(L, N_{L/X}) = \dim T_{[L]}F = 2n-4.$$

From the long exact sequence in cohomology associated to the short exact sequence

$$0 \to N_{L/X} \to N_{L/\mathbb{P}^{n+1}} \to \operatorname{im}(\alpha) \to 0,$$

we see $\dim H^0(L, \mathrm{im}(\alpha)) \geq 4$. Since $\mathrm{im}(\alpha) \subset N_{X/\mathbb{P}^{n+1}}|_L \simeq \mathcal{O}_L(3)$, it must be the case that $\mathrm{im}(\alpha) \simeq \mathcal{O}_L(3)$, i.e. $\alpha$ is surjective. This completes the argument. $\square$

*Example* 1.1.4. Revisit from Remark 1.1.2 the cone $S \subset \mathbb{P}^3$ over a smooth plane cubic curve $E$. The Fano variety $F$ of $S$ is isomorphic to $E$ so is smooth despite the fact that each line in $S$ passes through the cone point. This example highlights the need for the assumption in Proposition 1.1.3 that $F$ has the expected dimension.

*Example* 1.1.5. The Fano variety of lines on a smooth cubic fourfold is again a smooth fourfold, and the Fano variety of lines on a nodal cubic threefold is singular of dimension at least two. These Fano varieties are the central characters in what follows.

### 1.2 Special cubic fourfolds

We now specialize to the case of smooth cubic fourfolds over the complex numbers. The primary reference for this topic is Hassett's survey paper [14]. To begin, it is worth reiterating the major question surrounding cubic fourfolds: which complex cubic fourfolds are rational? Conjecturally, almost none are, but none have been proved irrational. In contrast, some smooth cubic fourfolds are rational:

*Example* 1.2.1. Suppose a cubic fourfold $X \subset \mathbb{P}^5$ contains two disjoint planes $P_1$, $P_2$. For example, [14] presents the cubic fourfold with defining equation

$$x_0x_3^2 + x_1x_4^2 + x_2x_5^2 = x_0^2x_3 + x_1^2x_4 + x_2^2x_5,$$

which contains the disjoint planes

$$x_0 = x_1 = x_2 = 0 \text{ and } x_3 = x_4 = x_5 = 0.$$

Define a rational map $\varphi : P_1 \times P_2 \dashrightarrow X$ as follows: given $(p, q) \in P_1 \times P_2$ such that the line $L = \overline{pq}$ is not contained in $X$, Bezout's theorem ensures $L$ meets $X$ in a third point $\varphi(p, q)$. The rational map $\psi : X \to P_1 \times P_2$ sending $x \in X$ to

$(\pi_1(x), \pi_2(x))$ where $\pi_i : X \dashrightarrow P_i$ is the projection serves as an inverse to $\varphi$, so $X$ is birational to $\mathbb{P}^2 \times \mathbb{P}^2$ hence also to $\mathbb{P}^4$.

This example calls to mind the very similar proof that a smooth, complex cubic surface is rational, which uses the fact that such a surface always contains a pair of skew lines. In contrast, cubic fourfolds containing even one plane are relatively rare. More precisely, the moduli space $\mathcal{C}$ of cubic fourfolds is 20-dimensional, and the subspace of those containing at least one plane has codimension 1.

For a very general cubic fourfold $X$, the algebraic part of the middle cohomology of $X$, i.e.

$$H^{2,2}(X, \mathbb{Z}) = H^4(X, \mathbb{Z}) \cap H^2(X, \Omega_X^2)$$

is 1-dimensional, spanned by the square of the hyperplane class, $h^2$ [34]. In other words, any algebraic surface in a very general cubic fourfold is homologous to a complete intersection. On the other hand, if $X$ contains a plane $P$, then the classes $h^2, P \in H^{2,2}(X)$ are linearly independent, which one can see via the intersection pairing:

| | $h^2$ | $P$ |
|---|---|---|
| $h^2$ | 3 | 1 |
| $P$ | 1 | 3 |

This phenomenon motivates Hassett's definition of a *special* cubic fourfold.

**Definition 1.2.2.** A cubic fourfold $X$ is *special* if $\operatorname{rank} H^{2,2}(X, \mathbb{Z}) > 1$. A *labeling* of $X$ is a choice of rank-2 primitive sublattice $K \subset H^{2,2}(X, \mathbb{Z})$ containing $h^2$, and the *discriminant* of $X$ is the discriminant of the intersection form on $K$.

The moduli space $\mathcal{C}_d$ of cubic fourfolds of discriminant $d$ is an irreducible divisor in $\mathcal{C}$ when $d \geq 8$ and $d = 0, 2 \mod 6$. Otherwise, $\mathcal{C}_d$ is empty. The space $\mathcal{C}_8$ comprises cubic fourfolds containing at least one plane; the next simplest examples are cubics containing a cubic scroll, belonging to $\mathcal{C}_{12}$, cubics containing a quartic scroll, belonging to $\mathcal{C}_{14}$, and cubics containing a sextic del Pezzo surface, belonging to $\mathcal{C}_{18}$.

Given a labeling $K$ of a special cubic fourfold $X$, the sublattice $K^{\perp} \subset H^4(X, \mathbb{Z})$ has rank 21. In all known cases where $X$ is rational, there is a polarized K3 surface $(S, f)$ and a Hodge isometry

$$K^{\perp} \xrightarrow{\sim} f^{\perp} \subset H^2(S, \mathbb{Z})(-1).$$

When such an isometry exists, one says the pair $(S, f)$ is *associated* to $X$. It turns out that the cubic fourfolds with associated K3 surfaces are precisely those of discriminant $d$ where $d$ is *admissible*, meaning not divisible by 4, 9, or any odd prime congruent to 2 modulo 3.

The premier conjecture about cubic fourfolds, typically attributed to Hassett, is that a cubic fourfold is rational precisely when it has an associated K3 surface. If that is true, then the rational cubic fourfolds lie in the countably infinite collection of divisors $\mathcal{C}_d \subset \mathcal{C}$ for $d$ admissible, and a very general cubic fourfold is irrational.

A cubic fourfold containing a plane $P$ is of discriminant 8, and a general member $X \in \mathcal{C}_8$ does not have an associated K3 surface. Nevertheless, there is a K3 surface arising from a classical geometric construction used throughout what follows.

*Example* 1.2.3. Let $X$ be a smooth cubic fourfold containing a plane $P$. The projection from $P$ onto a complementary plane $P^{\perp}$ gives a quadric surface fibration

$q\colon \mathrm{Bl}_P X \to P^\perp$; the fibers are quadric surfaces residual to $P$ in the intersection of $X$ with each $\mathbb{P}^3 \supset P$. A generic cubic $X$ contains no other planes meeting $P$, so the fibers of $q$ are irreducible. Imposing coordinates on $\mathbb{P}^5$ so that $P$ is cut out by $x_0 = x_1 = x_2 = 0$, write the defining equation for $X$ as

$$L_1x_3^2 + L_2x_3x_4 + L_3x_3x_5 + L_4x_4^2 + L_5x_4x_5 + L_6x_5^2 + Q_1x_3 + Q_2x_4 + Q_5x_5 + C = 0$$

where $L_i, Q_i, C \in \mathbb{C}[x_0, x_1, x_2]$ are linear, quadratic, and cubic, respectively. Then the Gram matrix associated to $q^{-1}(x_3 : x_4 : x_5)$ is the following:

$$A = \frac{1}{2}\begin{pmatrix} 2L_1 & L_2 & L_3 & Q_1 \\ L_2 & 2L_4 & L_5 & Q_2 \\ L_3 & L_5 & 2L_6 & Q_3 \\ Q_1 & Q_2 & Q_3 & C \end{pmatrix}$$

The singular fibers of $q$ are parametrized by the sextic curve $\Delta = \det A$, which is smooth by [34], having assumed that $X$ contains no plane meeting $P$.

Let $F(X/P^\perp)$ be the relative variety of lines of $q$, a variety parametrizing the lines in fibers of $q$. There is an obvious morphism $\pi : F(X/P^\perp) \to P^\perp$ whose fiber over $p \in P^\perp$ is isomorphic to $\mathbb{P}^1 \sqcup \mathbb{P}^1$ if $p \notin \Delta$ and to $\mathbb{P}^1$ if $p \in \Delta$. The Stein factorization of $\pi$ consists of a $\mathbb{P}^1$-bundle $F(X/P^\perp) \to S$ and a double cover $f : S \to P^\perp$ branched over $\Delta$. Since $\Delta$ is a smooth sextic, $S$ is a K3 surface.

The sublattice $K = \langle h^2, P\rangle \subset H^{2,2}(X, \mathbb{Z})$ is a marking on $X$, and for a very general $X \in \mathcal{C}_8$, this is the only marking. In [33], van Geemen shows that there is an embedding

$$K^\perp \subset f^\perp \subset H^2(S, \mathbb{Z})$$

with index 2 but no isometry $K^\perp \to f^\perp$, so $(S, f)$ is not associated to $X$. However, there may still be another polarized K3 surface $(T, g)$ associated to $X$, as happens in various cases when the quadric surface fibration $q : Bl_P X \to P^{\perp}$ admits a section. In that case, $X$ is rational since the base and fibers of $q$ are rational.

### 1.3 The derived category of a cubic fourfold

The derived category of coherent sheaves on a cubic fourfold $X$ offers a second perspective on the connections between cubic fourfolds and K3 surfaces. Kuznetsov studied a semiorthogonal decomposition

$$D^b(X) = \langle \mathcal{A}_X, \mathcal{O}_X(-1), \mathcal{O}_X, \mathcal{O}_X(1) \rangle,$$

proving in [23] that $\mathcal{A}_X$ is a K3 category, i.e. an indecomposable category whose Serre functor is a shift by 2. When there is a K3 surface $S$ for which $\mathcal{A}_X \simeq D^b(S)$, one says that $\mathcal{A}_X$ is *geometric*; for cubics in $\mathcal{C}_{14}$ and cubics containing a plane $P$ for which the quadric surface fibration $q : Bl_P X \to \mathbb{P}^2$ admits a section, Kuznetsov showed that $\mathcal{A}_X$ is geometric [24]. In a similar vein, when $X$ is singular, Kuznetsov showed that there is a K3 surface $S$ and a crepant categorical resolution $D^b(S) \to \mathcal{A}_X$ [24, Thm. 5.2]. In all of these cases, $X$ is rational. Moreover, in these examples, the K3 surface realizing $\mathcal{A}_X$ as a derived category is precisely the K3 surface associated to $X$ via Hodge theory. This phenomenon led Kuznetsov to conjecture that $X$ is rational if and only if $\mathcal{A}_X$ is geometric, and by [3] and [5, Cor. 1.7], Kuznetsov's conjecture is equivalent to Hassett's.

*Example* 1.3.1. Recall from Example 1.2.3 that a general cubic fourfold $X$ containing a plane has no associated K3 surface, but there is nevertheless a K3 surface $S$ arising from the geometry of $X$ as well as a $\mathbb{P}^1$-bundle on $S$ giving a Brauer class $\alpha \in \mathrm{Br}(S)[2]$. Kuznetsov showed that $\mathcal{A}_X$ is equivalent to $D^b(S, \alpha)$, the derived category of $\alpha$-twisted sheaves on $S$. In Section 3.2, we carry out a simpler proof of this equivalence that highlights its highly geometric character, inspired by the techniques of Addington and Lehn in [2]. In the course of writing this thesis, it came to my attention that a similar proof will also appear in Huybrechts' forthcoming book on cubic hypersurfaces [21].

It is worth noting also that the derived category affords another perspective on the Fano variety $F$ of lines on a cubic fourfold $X$. Kuznetsov and Markushevich proved in [26] that $F$ is isomorphic to a moduli space of rank-3 reflexive sheaves on $X$: namely, to $L \subset X$ one associates the sheaf

$$\mathcal{F}_L = \ker(\mathcal{O}_X(-1)^4 \overset{\text{ev}}{\twoheadrightarrow} \mathcal{I}_{L/X}),$$

which is a left mutation of $\mathcal{I}_{L/X}$ through $\mathcal{O}_X(-1)$. Using the sequences

$$0 \to \mathcal{F}_L \to \mathcal{O}_X(-1)^4 \to \mathcal{I}_{L/X} \to 0$$

and

$$0 \to \mathcal{I}_{L/X} \to \mathcal{O}_X \to \mathcal{O}_L \to 0,$$

it is easy to verify that $\mathcal{F}_L \in \mathcal{A}_X$. When there is an equivalence $\Phi : \mathcal{A}_X \xrightarrow{\sim} D^b(S, \alpha)$ between the Kuznetsov component and a derived category of twisted sheaves on a K3 surface $S$, a point on $F$ specifies a twisted complex $\Phi(\mathcal{F}_L)$ on $S$. Huybrechts proved in [20] that in this case $F$ is birational to a moduli space of stable $\alpha$-twisted sheaves on $S$.

For a complex cubic fourfold $X \in \mathcal{C}_8$, Macrì and Stellari showed explicitly in [27] that the Fano variety $F$ is birational to a moduli space of twisted sheaves on the K3 surface from Example 1.3.1. Their approach illuminates which twisted sheaves on $S$ the Fano variety parametrizes: they are twisted sheaves supported on curves in the linear system $f^*\mathcal{O}_{\mathbb{P}^2}(1)$ where $f : S \to \mathbb{P}^2$ is the double cover. By describing a relatively simple equivalence $\mathcal{A}_X \to D^b(S, \alpha)$ in Section 3.2, we present a streamlined proof of this fact that works over any field of characteristic zero.

### 1.4 The Fano variety of a cubic fourfold as a hyperkähler variety

A smooth, compact, simply-connected, complex Kähler variety $M$ equipped with an everywhere nondegenerate holomorphic 2-form $\sigma$ spanning $H^0(M, \Omega_M^2)$ is called a *hyperkähler variety.* Note that the (complex) dimension of a hyperkähler variety is always even. Hyperkähler varieties of dimension 2 are also called K3 surfaces, and if $S$ is a K3 surface, then the Hilbert scheme of $n$ points $S^{[n]}$ on $S$ is a hyperkähler $2n$-fold; any hyperkähler variety deformation equivalent to $S^{[n]}$ is said to be of $\mathrm{K3}^{[n]}$ *type*, and these are some of the best-understood examples. The primary reference for facts mentioned in this section is [12, Ch. III].

Beauville and Donagi proved in [6] that if $X$ is a general cubic fourfold of discriminant 14 and $F$ is the Fano variety of lines on $X$, then $F \simeq S^{[2]}$ for a K3 surface $S$ associated to $X$. Using a deformation theoretic argument, they concluded that the Fano variety of lines on any smooth cubic fourfold is a hyperkähler fourfold of $\mathrm{K3}^{[2]}$ type. This section lays out some of the tools from hyperkähler geometry (chiefly the Beauville-Bogomolov form) that are useful for studying the Fano variety $F$ of lines on a smooth, complex cubic fourfold. The main question we address is when $F$ admits a fibration, or, more generally, when $F$ is birational to another hyperkähler fourfold admitting a fibration. The hyperkähler perspective motivates the content of Chapter III where we work over fields other than $\mathbb{C}$.

The (complex) dimension of a hyperkähler variety is always even. Moreover, if $M$ is a hyperkähler $2n$-fold, $B$ is a smooth variety, and $f : M \to B$ is a nontrivial surjective morphism with connected fibers which is not an isomorphism, then the following must be true:

(i) $B \simeq \mathbb{P}^n$, proved in [22], and

(ii) the restriction of $\sigma$ to any fiber of $f$ is zero, and the fibers of $f$ are abelian varieties, proved in [29].

Such morphisms are called *Lagrangian fibrations* and are central to hyperkähler geometry.

The available methods for testing whether a hyperkähler variety $M$ admits a Lagrangian fibration (i.e. the Beauville-Bogomolov form, described shortly) are birational invariants among hyperkähler varieties. For this reason, it is much more convenient to ask whether $M$ is birational to another hyperkähler variety $N$ admitting a Lagrangian fibration, and the composition $M \dashrightarrow N \to \mathbb{P}^n$ is called a *rational Lagrangian fibration.*

*Example* 1.4.1. Let $S$ be a K3 surface. Any two K3 surfaces that are birational are in fact isomorphic, so the question of whether $S$ admits a rational Lagrangian fibration is the same as whether $S$ admits a Lagrangian fibration, in which case one calls $S$ elliptic. A classical fact (see, for example, [19, Ch. 11]) is that $S$ is elliptic if and only if there is a class $E \in \mathrm{Pic}(S)$ with $E^2 = 0$ under the intersection form on $S$. However, a general K3 surface contains no such curve, so elliptic K3 surfaces are rather rare.

For a hyperkähler $2n$-fold $M$, there is a quadratic form called the *Beauville-Bogomolov form* $q_M$ on $H^2(M, \mathbb{Z})$ defined uniquely (up to sign) by the properties that for some $c \in \mathbb{R}$,

$$q_M(\alpha)^n = c \int_M \alpha^{2n}$$

for all $\alpha \in H^2(M)$, and $q_M$ restricts to a primitive integral quadratic form on $H^2(M, \mathbb{Z})$. Conjecturally, $q_M$ generalizes the intersection form on a K3 surface in the sense that $M$ admits a rational Lagrangian fibration if and only if $q_M$ vanishes on a nontrivial divisor class $D \in H^2(M, \mathbb{Z})$ in the birational Kähler cone of $M$. In that case, the rational Lagrangian fibration $M \dashrightarrow \mathbb{P}^n$ is induced by the complete linear system $|D|$. By results from [28, Sect. 6], the existence of any nontrivial isotropic integral divisor class implies the existence of one in the birational Kähler cone (for further discussion, see [31, Cor. 7.3]). In [30], Matsushita verified this conjecture for hyperkähler varieties of $\mathrm{K3}^{[n]}$-type, a class that includes Fano varieties of cubic fourfolds.

Let $X$ be a smooth, complex cubic fourfold and $F$ its Fano variety of lines. To test whether $q_F$ vanishes on a nontrivial class in $H^2(F, \mathbb{Z})$, it is useful to consider the Abel-Jacobi map

$$\alpha : H^4(X, \mathbb{Z}) \to H^2(F, \mathbb{Z}), \;\; \alpha(S) = \pi_{2*}\pi_1^*(S)$$

where $W \subset X \times F$ is the incidence correspondence and $\pi_1$ and $\pi_2$ are its projections. In [6], it is proved that $\alpha$ respects the Hodge filtrations, restricts to an isomorphism

$$H^4(X, \mathbb{Z})_{\mathrm{prim}} \xrightarrow{\sim} H^2(F, \mathbb{Z})_{\mathrm{prim}},$$

and is compatible with the quadratic forms on each group in the sense that

$$q_F(\alpha(S), \alpha(T)) = -S.T$$

for any $S, T \in H^4(X, \mathbb{Z})_{\mathrm{prim}}$. With these facts, it is possible to characterize exactly when $F$ admits a rational Lagrangian fibration.

**Proposition 1.4.2.** *$F$ admits a rational Lagrangian fibration if and only if $X \in \mathcal{C}_d$ for some $d$ which is twice a square.*

*Proof.* We have already seen that a necessary condition for $F$ to admit a rational Lagrangian fibration is that $X$ be special. Choose a primitive rank two sublattice $h^2 \in K \subset H^{2,2}(X, \mathbb{Z})$ and a class $S \in K \cap H^4(X, \mathbb{Z})_{\mathrm{prim}}$. Let $i$ be the index of the sublattice $K' = \mathbb{Z}h^2 + \mathbb{Z}S$ in $K$. The discriminant of the intersection form on

$K'$ is $3S^2$, so the discriminant of the intersection form on $K$ is $3S^2/i^2$, and $X \in \mathcal{C}_d$ for $d = 3S^2/i^2$. There is a class $D \in \alpha(K)$ isotropic with respect to $q_F$ if and only if there is such a class in $\alpha(K')$. We will show that such a class exists in $K'$ if and only if $d$ is twice a square.

Applying the compatibility of the Abel-Jacobi map with the various quadratic forms and using the fact from [14] that $q_F(\alpha(h^2)) = 6$,

$$q_F(\alpha(mh^2 + nS)) = 6m^2 - n^2S^2 = 6m^2 - i^2n^2d/3.$$

Thus there is a class $D \in \alpha(K')$ satisfying $q_F(D) = 0$ if and only if there are integers $m, n$ with $2(3m/(in))^2 = d$. □

Note that, as with K3 surfaces, Proposition 1.4.2 implies that the Fano variety of a cubic fourfold rarely admits a rational Lagrangian fibration. The "first" example is the Fano variety of lines on a cubic fourfold $X$ containing a plane, i.e. the case $d = 8$.

*Example* 1.4.3. Let $X \in \mathcal{C}_8$ be a cubic fourfold containing a plane $P$ and $F$ its Fano variety of lines, and let $g = \alpha(h^2)$ and $p = \alpha([P])$. For simplicity, assume $X$ is general, so $H^2(F, \mathbb{Z})$ is spanned by $g$ and $p$. Using the proof of Proposition 1.4.2, it is easy to see that the classes in $H^2(F, \mathbb{Z})$ isotropic with respect to the Beauville-Bogomolov form are the multiples of $g-p$ and $2g-3p$. A straightforward calculation shows that the nef cone of $F$ is the closed cone whose boundary contains the classes $g + p$ and $3g - p$, illustrated in Figure 1. There is an isomorphism $H^2(F, \mathbb{Z}) \simeq H^2(M, \mathbb{Z})$ where $M$ is the Mukai flop of $F$ along $P^*$; indeed, $F$ and $M$ become isomorphic after cutting out a codimension-two subvariety of each. The nef cone of $M$ is the closed cone whose boundary contains the classes $3g - p$ and $g - p$, also illustrated in Figure 1.

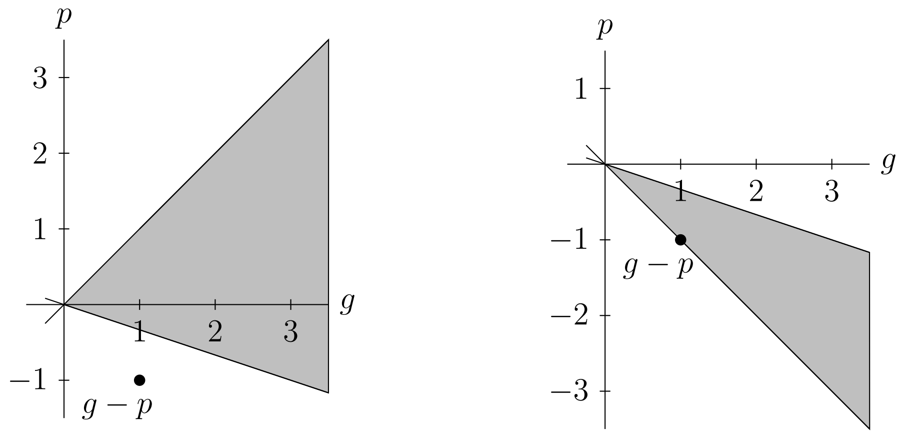


*Figure 1.* The shaded region the left illustrates the nef cone of $F$, the Fano variety of lines on a complex cubic fourfold containing a plane $P$. The shaded region on the right illustrates the nef cone of $M$, the Mukai flop of $F$ along the dual plane $P^*$.

Hence the complete linear system of $g - p$ induces a Lagrangian fibration $M \to \mathbb{P}^2$, but the complete linear system of $\pm(2g - 3p)$ does not induce a rational Lagrangian fibration on $F$.

Note that $[Q] = h^2 - [P]$ where $Q$ is a quadric surface residual to $P$, i.e. there is some $\mathbb{P}^3 \subset \mathbb{P}^5$ for which $\mathbb{P}^3 \cap X = Q \cup P$, so the complete linear system $|\alpha([Q])|$ induces the rational Lagrangian fibration $F \dashrightarrow \mathbb{P}^2$, which is the unique such rational Lagrangian fibration for a general member $X \in \mathcal{C}_8$.

However, the argument for the existence of the rational Lagrangian fibration in this case using Beauville-Bogomolov form is rather opaque and only works over $\mathbb{C}$, two shortcomings addressed via an explicit geometric construction given in Chapter III. To further motivate constructing the rational Lagrangian fibration in the discriminant 8 case, consider the following example where the linear system inducing a rational Lagrangian fibration is known but a geometric description of the Lagrangian fibration remains elusive.

*Example* 1.4.4. If $X \in \mathcal{C}_{18}$, then $X$ contains a sextic del Pezzo surface $S$, as explained in [1]. The intersection form on the sublattice spanned by $h^2$ and $S$ is given below:

| | $h^2$ | $S$ |
|---|---|---|
| $h^2$ | 3 | 6 |
| $S$ | 6 | 18 |

It is straightforward to check that the classes $\alpha(3h^2 - S)$ and $\alpha(S - h^2)$ are Kähler and isotropic with respect to the Beauville-Bogomolov form; however, even with these divisor classes in hand, it is unclear how to describe the associated rational Lagrangian fibrations from $F$ geometrically, essentially because it is more difficult to describe a divisor in either linear system.

### 1.5 Intermediate Jacobian torsor obstructions to rationality

To study the rational Lagrangian fibration from the Fano variety of lines of a cubic fourfold containing a plane, we will analyze the Fano variety $F(Y)$ of lines on a general cubic threefold $Y$ containing a plane $P$. Note that $Y$ is geometrically rational, which one can argue in two ways:

(i) $Y$ has four nodes along $P$, and projection from a node gives a birational equivalence onto $\mathbb{P}^3$;

(ii) projection from $P$ induces a quadric surface fibration $\mathrm{Bl}_P Y \to \mathbb{P}^1$ which geometrically has a section (for example, a line in $Y \setminus P$ gives a section).

Must $Y$ be $k$-rational, though? And is it possible for $Y$ to be $k$-rational even when $Y$ has no $k$-rational node and $\mathrm{Bl}_P T \to \mathbb{P}^1$ has no $k$-rational section? For a general cubic threefold over a field of characteristic zero, Chapter II Section 2.6 answers both questions in the negative by using intermediate Jacobian torsor obstructions, briefly described here.

Let $X$ be a smooth, projective threefold over a field $k$. How can one tell whether $X$ is rational over $k$? In the case $k = \mathbb{C}$, Clemens and Griffiths used the intermediate Jacobian $J^2(X)$, which for a rationally connected variety $X$ is

$$J^2(X) = H^3(X, \mathbb{C})/(H^3(X, \mathbb{Z}) \oplus H^{1,2}(X)),$$

to prove that smooth, complex cubic threefolds are irrational. When $k \subset \mathbb{C}$ is not algebraically closed, however, one can only use the intermediate Jacobian to determine whether $X$ is geometrically rational, not whether $X$ is rational over $k$. For other fields, it is not obvious what construction should replace $J^2(X)$.

Recent progress on a generalization of the intermediate Jacobian was made by Hassett and Tschinkel in [16] when $k \subset \mathbb{C}$ and by Benoist and Wittenberg in [7] for arbitrary fields. They construct a group scheme $\mathrm{CH}^2_{X/k}$ whose points agree with the codimension-2 Chow group, parametrizing rational equivalence classes of codimension-2 cycles. The identity component $(\mathrm{CH}^2_{X/k})^0$ is an abelian variety parametrizing classes that are algebraically trivial, and when $k = \mathbb{C}$, there is an isomorphism of abelian varieties $(\mathrm{CH}^2_{X/k})^0 \simeq J^2(X)$.

The quotient

$$\mathrm{NS}^2(X_{\bar{k}}) = \mathrm{CH}^2_{X/k} / (\mathrm{CH}^2_{X/k})^0$$

is a Galois module called the Néron-Severi group of algebraic equivalence classes of codimension-2 cycles. By [7, Thm 3.1(v)],

$$\mathrm{NS}^2(X_{\bar{k}})^{\mathrm{Aut}(\bar{k}/k)} = (\mathrm{CH}^2_{X/k} / (\mathrm{CH}^2_{X/k})^0)(k)$$

when $X$ is geometrically rational, so for each class $\gamma \in \mathrm{NS}^2(X_{\bar{k}})^{\mathrm{Aut}(\bar{k}/k)}$, there is a coset $(\mathrm{CH}^2_{X/k})^\gamma$ endowed with the structure of a $(\mathrm{CH}^2_{X/k})^0$-torsor. Moreover, Benoist and Wittenberg obtain the following when $X$ is a smooth, projective, $k$-rational threefold [7, Thm 3.11]:

(i) $(\mathrm{CH}^2_{X/k})^0 \simeq \mathrm{Pic}^0_C$ for some smooth, projective curve $C$, and

(ii) for all $\gamma \in \mathrm{NS}^2(X_{\bar{k}})^{\mathrm{Aut}(\bar{k}/k)}$, there is some $e_\gamma \in \mathbb{Z}$ and an equivalence of torsor classes $[(\mathrm{CH}^2_{X/k})^\gamma] = [\mathrm{Pic}^{e_\gamma}_C]$ in the Weil-Châtelet group $H^1(k, \mathrm{Pic}^0_C)$.

These conditions proved powerful in studying the rationality of a complete intersection of two quadrics in $\mathbb{P}^5$:

*Example* 1.5.1. Let $Y \subset \mathbb{P}^5$ be a complete intersection of two quadrics, and let $F$ be the Fano variety of lines in $Y$. By work of Wang in [35], there is a genus 2 curve $C$ for which $F$ is a $\mathrm{Pic}^0_C$-torsor, and

$$2[F] = [\mathrm{Pic}^1_C] \in H^1(k, \mathrm{Pic}^0_C).$$

When $F(k) \neq \emptyset$, the threefold $Y$ is $k$-rational by a classical construction. Conversely, if $Y$ is $k$-rational, then (i) implies $(\mathrm{CH}^2_{Y/k})^0 \simeq \mathrm{Pic}^0_C$ (see Lemma 2.6.2 for more details), and (ii) forces $[F] = 0$ since $2[\mathrm{Pic}^e_C] = 0$ for all $e \in \mathbb{Z}$. It follows that $F(k) \neq \emptyset$ if and only if $Y$ is $k$-rational. This argument allows Hassett and Tschinkel to characterize rationality for $Y$ when $k \subset \mathbb{C}$, and enables Benoist and Wittenberg to give an example over a imperfect field $k$ where $Y$ has $k$-points and is rational over a purely inseparable extension of $k$ but not over $k$ itself.

To study rationality of a cubic threefold $Y$ containing a plane $P$ using similar techniques, we need some understanding of the Chow variety of $\widetilde{Y} = \mathrm{Bl}_P Y$. We restrict to the case when the ground field has characteristic zero in order to take advantage of the fact that when $k = \mathbb{C}$, there is an isomorphism $J^2(\widetilde{Y}) \simeq (\mathrm{CH}^2_{\widetilde{Y}/k})^0$.

**Lemma 1.5.2.** *Over a field $k$ of characteristic zero, let $Y \subset \mathbb{P}^4$ be a cubic threefold containing a plane $P$ whose singular locus is a complete intersection of two conics in $P$. Let $f : \widetilde{Y} \to Y$ be the blowup along $P$. Then $(\mathrm{CH}^2_{\widetilde{Y}/k})^0$ is a surface.*

*Proof.* By the Lefschetz principle, we may assume $k \subset \mathbb{C}$, and moreover we may extend scalars to $\mathbb{C}$. Thus we are afforded an isomorphism $(\mathrm{CH}^2_{\widetilde{Y}_{\mathbb{C}}})^0 \simeq J^2(\widetilde{Y}_{\mathbb{C}})$, so we need only calculate the Hodge numbers of $\widetilde{Y}_{\mathbb{C}}$. We suppress the base change from the notation in what follows.

First, we claim $\widetilde{Y}$ embeds in $\mathbb{P}^4 \times \mathbb{P}^1$ as a complete intersection of divisors of type $(1,1)$ and $(2,1)$. Let $E$ be the exceptional divisor of the blowup $\widetilde{\mathbb{P}}^4 = Bl_P(\mathbb{P}^4)$. Using the embedding $\widetilde{\mathbb{P}}^4 \subset \mathbb{P}^4 \times \mathbb{P}^1$, consider the following diagram:

$$\begin{array}{ccccc} E & \hookrightarrow & \widetilde{\mathbb{P}}^4 & \xrightarrow{q} & \mathbb{P}^1 \\ \downarrow & & \downarrow{\scriptstyle p} & & \\ P & \hookrightarrow & \mathbb{P}^4 & & \end{array}$$

The classes $p^*[H]$ and $[E]$ generate $\mathrm{Pic}(\widetilde{\mathbb{P}}^4)$, where $[H] \in \mathrm{Pic}(\mathbb{P}^4)$ is the hyperplane class. Let $[h] \in \mathrm{Pic}(\mathbb{P}^1)$ be the other hyperplane class, so $q^*[h] = p^*[H] - [E]$. We have

$$\widetilde{Y} = p^*(3[H]) - [E] = 2p^*[H] + q^*[h],$$

which shows a $(2,1)$-divisor cuts $\widetilde{Y}$ from $\widetilde{\mathbb{P}}^4$. The Lefschetz hyperplane theorem gives

$$h^{p,q}(\widetilde{Y}) = h^{p+2,q+2}(\mathbb{P}^4 \times \mathbb{P}^1) = \begin{cases} 1 & p = q = 3 \\ 2 & p = q = 2 \\ 0 & \text{otherwise, for } p + q \geq 3, \end{cases}$$

where $h^{p,q} = \dim H^{p,q}(X, \mathbb{C})$. Let $f : \widetilde{Y} \to Y$ be the blowup. Since $Rf_* \mathcal{O}_{\widetilde{Y}} = \mathcal{O}_Y$, $h^{3,0}(\widetilde{Y}) = \dim H^3(Y, \mathcal{O}_Y) = 0$. Using the Hodge symmetry and Serre duality,

$$\chi(\widetilde{Y}) = 6 - 2h^{2,1}(\widetilde{Y})$$

On the other hand, if $Y'$ is a smooth cubic threefold, we have $\chi(Y') = -6$. Topologically, $\widetilde{Y}$ can be obtained from $Y'$ by cutting out small neighborhoods of four points and gluing in four copies of $\mathbb{P}^1$, so

$$\chi(\widetilde{Y}) = \chi(Y') - 4\chi(S^3) + 4\chi(\mathbb{P}^1) = 2.$$

It follows that $h^{2,1}(\widetilde{Y}) = 2$, giving the Hodge diamond below.

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & 2 & & 0 & \\ 0 & & 2 & & 2 & & 0 \end{array}$$

In particular, $\dim J^2(\widetilde{Y}) = 2$, as needed. $\square$

# CHAPTER II

# CUBIC THREEFOLDS CONTAINING A PLANE

This chapter describes the Fano scheme of lines on a cubic threefold containing a plane. The topic calls to mind Clemens and Griffiths's proof that a smooth, complex cubic threefold is irrational, which relies on the intermediate Jacobian, an abelian variety parametrizing lines contained in the threefold. Cubic threefolds containing a plane, however, are singular and therefore geometrically rational (for example, via projection from a singular point). The main result in this chapter is that one component of the Fano scheme of lines on a general cubic threefold containing a plane is birational to a torsor of an abelian surface. This torsor exhibits interesting behavior in the Weil-Châtelet group. Remarkably, in characteristic zero, whether the torsor is trivial over the base field $k$ controls whether the threefold is rational over $k$.

Moreover, the geometry outlined in this chapter foreshadows the next chapter, in which we study the rational Lagrangian fibration from the Fano variety of lines on a general cubic fourfold containing a plane. The fibers of the Lagrangian fibration described there are precisely the torsors examined in this chapter.

Section 2.1 outlines the basic geometry of a cubic threefold $Y$ containing a plane, Section 2.2 outlines the irreducible components of the Fano scheme $F(Y)$ of lines on $Y$, and Section 2.3 describes how these irreducible components intersect. One irreducible component $\overline{\mathcal{U}} \subset F(Y)$ is birational to a geometrically abelian surface $T$, and Section 2.4 introduces (rational) involutions on $\overline{\mathcal{U}}$ and $T$ used to equip $T$ with the structure of a torsor of an abelian surface. Building on this, Section 2.5 describes the arithmetic of $T$, and Section 2.6 describes an obstruction to the $k$-rationality of $Y$ coming from $T$.

## 2.1 Generalities

Let $Y \subset \mathbb{P}^4$ be a cubic threefold containing a plane $P$. Suppose also that $Y$ is *general*, by which we mean that $Y$ contains only one plane, and the subscheme $Z = \operatorname{Sing}(Y) \cap P$ is zero-dimensional. While we do not require that $\operatorname{Sing}(Y)$ be zero-dimensional, this will turn out to be the case. In Chapter III, we show that if $X$ is a smooth cubic fourfold containing exactly one plane $P$, then any section of $X$ by a hyperplane containing $P$ is general, motivating our definition here.

Fix coordinates on $\mathbb{P}^4$ so that $P = \{x_0 = x_1 = 0\}$, and write the defining equation for $Y$ as

$$\begin{aligned} f = {} & a_{22}x_2^2 + a_{23}x_2x_3 + a_{24}x_2x_4 + a_{33}x_3^2 + a_{34}x_3x_4 + a_{44}x_4^2 \\ & + b_2x_2 + b_3x_3 + b_4x_4 + c \end{aligned}$$

where $a_{ij}, b_i, c \in k[x_0, x_1]$ are linear, quadratic, and cubic forms, respectively. The projection $Y \dashrightarrow \mathbb{P}^1$ given by $(x_0 : \cdots : x_4) \mapsto (x_0 : x_1)$ induces a quadric surface fibration $q : \operatorname{Bl}_P Y \to \mathbb{P}^1$. Since $Y$ contains only one plane, the degenerate fibers of $q$ are no worse than cones. The matrix below is the Gram matrix for the fiber of $q$ over $(x_0 : y_0)$.

$$A = \begin{pmatrix} a_{22} & \frac{1}{2}a_{23} & \frac{1}{2}a_{24} & \frac{1}{2}b_2 \\ \frac{1}{2}a_{23} & a_{33} & \frac{1}{2}a_{34} & \frac{1}{2}b_3 \\ \frac{1}{2}a_{24} & \frac{1}{2}a_{34} & a_{44} & \frac{1}{2}b_4 \\ \frac{1}{2}b_2 & \frac{1}{2}b_3 & \frac{1}{2}b_4 & c \end{pmatrix}$$

The sextic equation $\det A = 0$ cuts out the discriminant locus of $q$, a degree six divisor on $\mathbb{P}^1$ parametrizing the singular fibers of $q$.

Geometrically, each fiber of $q$ has either one or two rulings depending on whether the fiber is smooth or singular. A double cover $C$ of $\mathbb{P}^1$ branched over the divisor $\det A = 0$ parametrizes rulings on fibers of $q$. Note that by the Riemann-

Hurwitz formula, the arithmetic genus of $C$ is 2. Over $k^s$, the curve $C$ is cut out of weighted projective space by the equation $y^2 = \det A$, but over a nonclosed field, a hyperelliptic curve is only defined up to quadratic twist by its branch locus. The following lemma clarifies the equation for $C$ over a nonclosed field, which will not be referenced aagain but is useful for working out explicit examples.

**Lemma 2.1.1.** *In* $\mathbb{P}(1,1,3)$*, the equation for* $C$ *can be written* $y^2 = \det A$*.*

*Proof.* A quadric surface $Q$ over $k$ has a $k$-rational ruling if and only if the determinant of the Gram matrix for $Q$ is a square in $k$. Since the matrix $A$ is the Gram matrix for the fiber of $q$ over $(x_0 : x_1)$, the solutions over $k$ to the equation $y^2 = \det A$ parametrize $k$-rational rulings on fibers of $q$. □

We will reference the curve $C$ frequently through the chapter. The following lemma clarifies the geometry of $Y$ and the quadric surface fibration $q : \mathrm{Bl}_P Y \to \mathbb{P}^1$.

**Lemma 2.1.2.** *Let* $Y$ *be a general cubic threefold containing a plane* $P$*, and let* $Z = \mathrm{Sing(Y)} \cap \mathrm{P}$*. Then*

*(i)* $Z$ *is a complete intersection of two conics in* $P$*,*

*(ii) there is a bijection between quadric surfaces in* $Y$ *and conics in* $P$ *containing* $Z$*, given by* $Q \mapsto Q \cap P$*,*

*(iii) each* $k$*-point of* $Z$ *gives a section of* $q$*,*

*(iv)* $Y$ *has isolated singularities, and*

*(v) if the discriminant of* $q$ *is reduced, then* $Y$ *is smooth away from* $P$*.*

*Proof.* Write the equation for $Y$ as as $x_0Q_0 + x_1Q_1 = 0$ where the $Q_i$ are quadratic in $x_0, \ldots, x_4$, and let $q_i = Q_i(0, 0, x_2, x_3, x_4)$. The Jacobian criterion shows that $Y$ is singular along the locus

$$
\begin{aligned}
Q_0 + x_0\frac{dQ_0}{dx_0} + x_1\frac{dQ_1}{dx_0} &= 0\\
Q_1 + x_0\frac{dQ_0}{dx_1} + x_1\frac{dQ_1}{dx_1} &= 0\\
x_0\frac{dQ_0}{dx_i} + x_1\frac{dQ_1}{dx_i} &= 0 \text{ for } i = 2,3,4.
\end{aligned}
$$

On $P$, the equations become $q_0 = q_1 = 0$, i.e. the intersection of two conics. Then (i) follows from the assumption that $Y$ is general, which entails $\dim Z = 0$.

For (ii), note that each quadric surface in $Y$ is cut out by the equations

$$sx_1 - tx_0 = sQ_0 + tQ_1 = 0$$

for $(s : t) \in \mathbb{P}^1$, and each point in $Z$ is belongs to the locus $Q_i = 0$ for $i = 1, 2$ so satisfies these equations. Hence each quadric surface in $Y$ contains $Z$, and the map $Q \mapsto Q \cap P$ identifies the pencil of quadrics in $X$ with the pencil of conics through $Z$. Moreover, each point in $Z$ gives a section of $q$ since it is contained in each quadric surface in $Y$, proving (iii).

Now we consider the singularities of $\mathrm{Bl}_P Y$. For $y \in \mathrm{Bl}_P Y$, we have

$$T_y(\mathrm{Bl}_P Y) = T_y Q + T_{q(y)}\mathbb{P}^1$$

where $Q \ni y$ is the quadric surface containing $y$. Hence if $\dim T_y(\mathrm{Bl}_P Y) > 3$, then $\dim T_y Q > 2$, which happens for at most six points, i.e. the cone points of each of the finitely many singular fibers of $q$. Since the blowup $\mathrm{Bl}_P Y \to Y$ is an isomorphism away from $P$, there are only finitely many singular points on $Y$, proving (iv).

A local computation shows that $\mathrm{Bl}_P Y$ is smooth if and only if the discriminant of $q$ is reduced. When $\mathrm{Bl}_P Y$ is smooth, $Y$ is smooth away from $P$, proving (v). □

Finally, notice that the lines contained in quadric surfaces in $Y$ form a conic bundle $p : \mathcal{F} \to C$, each line belonging to at most one quadric surface. Importantly, sections of $p$ are in bijection with sections of $q$: specifying a point in each smooth fiber of $q$ is the same as specifying a line in each ruling of that fiber, explained further in [15, Sect. 3]. By Lemma 2.1.2, each (geometric) point $z \in Z$ gives a section of $q$, and $\tau_z$ denotes the corresponding (geometric) section of $p$. In a slight abuse of notation, for $c \in C$ we variously regard $\tau_z(c)$ as a line in $Y$ and as the corresponding point in $\mathcal{F}$.

## 2.2 The Fano scheme

This section describes the irreducible components of the Fano scheme of lines on $Y$, denoted $F(Y)$. These lines come in three types: those contained in $P$, those meeting $P$ once, and those disjoint from $P$. A line that meets $P$ once belongs to $\mathcal{F}$, and a line contained in $P$ belongs to the dual plane $P^*$; let $\mathcal{U} \subset F(Y)$ be the open subsceheme composed of lines in $Y$ not meeting $P$.

**Theorem 2.2.1.** $F(Y)$ *contains the following components:*

1. *the plane* $P^*$ *dual to* $P$*,*

2. *a ruled surface* $\mathcal{F}$ *over a curve* $C$ *of arithmetic genus* 2*,*

3. *and a singular surface* $\overline{\mathcal{U}}$*, geometrically birational to* $\operatorname{Sym}^2 C$*.*

*Moreover, each of the components above is irreducible if* $C$ *is irreducible.*

*Proof.* All that requires proof is the birational geometry of $\overline{\mathcal{U}}$ over the separable closure $k^s$.

Choose a point $z \in Z$. For a line $L \subset Y \setminus P$, let $P'$ be the plane spanned by $L$ and $z$. Since $Y$ is singular at $z$, so too is the curve $P' \cap Y$. Thus $P' \cap Y$

consists of three lines: $L$ and two other lines $M$ and $N$ (which may coincide with one another), both containing $z$. Note $[M], [N] \in \mathcal{F}$ since $M$ and $N$ meet but are not contained in $P$. The equation $\varphi_z([L]) = \{p([M]), p([N])\}$ defines a morphism $\varphi_z : \mathcal{U} \to \operatorname{Sym}^2 C$.

To show $\varphi_z$ is a birational equivalence, we construct its rational inverse. Let $V \subset \operatorname{Sym}^2 C$ be the open set consisting of pairs $\{c, d\}$ of distinct points such that the lines $\tau_z(c)$ and $\tau_z(d)$ are not both contained in $P$. For such a pair, the plane $P'$ spanned by $\tau_z(c)$ and $\tau_z(d)$ meets $Y$ in a third line $L$, and we define a morphism $\psi_z : V \to F(Y)$ by $\psi_z(c, d) = [L]$. It is straightforward that $\psi_z \circ \varphi_z = \mathrm{id}_{\mathcal{U}}$, as needed. $\square$

*Remark* 2.2.2. Let $S_z \subset Y$ be the union over $c \in C$ of the lines $\tau_z(c)$, which is a cone over $C$. An unreduced length-2 subscheme of $C$ specifies a line $\tau_z(c) \subset S_z$ and a normal direction to that line in $S_z$, and together these span a plane $P'$ meeting $Y$ in a third line. In this way, one can instead define $\psi_z$ as a rational map $\operatorname{Hilb}^2 C \dashrightarrow \mathcal{U}$. The domain of $\psi_z$ under this new definition is the complement of at most three points, i.e. the pairs $\{c, d\}$ of distinct points with $\tau_z(c), \tau_z(d) \subset P$.

*Remark* 2.2.3. One can regard $\mathcal{U}$ as an open subscheme of the space of sections of the quadric surface fibration $q : \operatorname{Bl}_P Y \to \mathbb{P}^1$. In [15], Hassett and Tschinkel study spaces of sections of quadric surface bundles over curves, ordering these sections by a height which in this case is given by the formula

$$h(\sigma) := \deg N_{\sigma(\mathbb{P}^1)/\operatorname{Bl}_P Y}.$$

For each $h \in \mathbb{Z}$, let $\operatorname{Sect}(q, h)$ denote the space of sections of $q$ of height $h$, which can be regarded as a subscheme of the Hilbert scheme of $\operatorname{Bl}_P Y$. If $L \subset Y \setminus P$ is a line, then $q|_L$ is an isomorphism, so $q|_L^{-1}$ is a section, and

$$h(q|_L^{-1}) = \deg N_{L/Y} = 0.$$

Hence there is an open immersion $\mathcal{U} \to \mathrm{Sect}(q, 0)$. The authors prove that when the base field is algebraically closed or finite (so that $\mathrm{Br}(C) = 0$), for each $h \gg 0$, there exists an integer $d$ and a composition

$$\mathrm{Sect}(q, h) \to \mathcal{S} \to \mathrm{Pic}_C^d$$

where the first morphism is an open immersion and the second is a projective bundle [15, Prop. 2]. This provides a second perspective on Theorem 2.2.1, at least when $k$ is algebraically closed or finite, keeping in mind that $\mathrm{Sym}^2 C$ is a blowup of $\mathrm{Pic}_C^2$ when $C$ is smooth.

A line $[L] \in \mathcal{U}$ gives a section of $q$ and hence a section $\sigma_L$ of $p$. When $C$ is smooth, so too is $\mathcal{F}$, and the following lemma describes the numerics of the two types of sections of $p$ mentioned so far.

**Lemma 2.2.4.** *Suppose $C$ is smooth, and let $[L], [M] \in \mathcal{U}$ and $z, w \in Z$. Under the intersection pairing on $\mathcal{F}$,*

*(i)* $\sigma_L.\tau_z = 2$,

*(ii)* $\sigma_L.\sigma_M = 3$, *and*

*(iii)* $\tau_z.\tau_w = 1$.

*Proof.* We may work over $k^s$. The equation (i) follows from the proof of Theorem 2.2.1: $[M] \in \tau_z(C)$ if and only if $z \in M$, $[M] \in \sigma_L(C)$ if and only if $L \cap M = \emptyset$, and there are two lines meeting $z$ and $L$. For (ii), it is enough to calculate $\sigma_L.\sigma_M$ for a particular choice of $L$ and $M$: indeed, Theorem 2.2.1 implies $\mathcal{U}$ is irreducible when $C$ is smooth, and a curve in $\mathcal{U}$ connecting two points gives an algebraic (hence also numerical) equivalence between the corresponding sections of $\mathcal{F}$. A generic hyperplane section of $Y$ is a smooth cubic surface $S$ and contains skew lines not meeting $P$, which we may assume for the purpose of our calculation are $L$ and $M$.

Note that the span of $L$ and $M$ intersects $P$ in a line $\ell$, and the intersection number $\sigma_L.\sigma_M$ counts how many lines in $Y$ meet $L$, $M$, and $P$, which are exactly the lines in $S$ meeting $L$, $M$, and $\ell$. Given three disjoint lines in a cubic surface, there are three other lines meeting each of those.

Because $\mathrm{Num}(\mathcal{F})$ is generated by the class $f$ of a fiber and the class of any section (see, for example, [13, V.2]), there are integers $a, b \in \mathbb{Z}$ with $\tau_z \sim a\sigma_L + bf$. Pairing with $f$ yields $a = 1$, and pairing with $\sigma_L$ yields $b = -1$. Thus $\tau_z \sim \sigma_L - f$, and the same is true for $\tau_w$, so (iii) follows from (i) and (ii). □

## 2.3 Intersections of the components of the Fano variety

This section analyzes the pairwise intersections of the components of $F(Y)$, enabling a description of the boundary of $\overline{\mathcal{U}}$.

**Lemma 2.3.1.** *$\mathcal{F} \cap P^*$ is zero-dimensional.*

*Proof.* If $[L] \in \mathcal{F} \cap P^*$, then $L$ is contained in a quadric surface $Q$ meeting $P$ in a degenerate conic (one component of which is $L$). There are at most three degenerate conics containing $Z$, so there are finitely many quadrics $Q$ for which $Q \cap P$ is degenerate by by Lemma 2.1.2 (ii). Each of these quadrics contributes two points to $\mathcal{F} \cap P^*$. □

*Remark* 2.3.2. Note that since $\mathcal{F}$ and $P^*$ meet in codimension 2, Hartshorne's Connectedness Theorem [10, Thm. 18.12] implies that

$$\overline{F(Y) \setminus \overline{\mathcal{U}}} = \mathcal{F} \cup P^*$$

is not Cohen-Macaulay. By [10, Thm. 21.23], $\overline{\mathcal{U}}$ is not Cohen-Macaulay. In particular, $\overline{\mathcal{U}}$ is not smooth.

Recall from Proposition 1.1.3 that $F(Y)$ is smooth at $[L]$ if and only if $Y$ is smooth along $L$. The following two lemmas also use the basic fact that a scheme is singular along the intersection of any two irreducible components.

**Lemma 2.3.3.** *$\overline{\mathcal{U}} \cap P^*$ is the union of the pencils $z^* \subset P^*$ of lines in $P$ through each of the points $z \in Z$.*

*Proof.* A line in $P$ is a singular point of $F(Y)$ if and only if it intersects $Z$ nontrivially, so

$$P^* \cap \operatorname{Sing}(F(Y)) = \bigcup_{z \in Z} z^*.$$

Moreover, $P^*$ is smooth, so any point in $P^* \cap \operatorname{Sing}(F(Y))$ belongs to a second component of $F(Y)$. Then

$$\bigcup_{z \in Z} z^* = (P^* \cap \overline{\mathcal{U}}) \cup (P^* \cap \mathcal{F}).$$

The left-hand side is purely 1-dimensional, and $P^* \cap \mathcal{F}$ is 0-dimensional by Lemma 2.3.1, so

$$\bigcup_{z \in Z} z^* = \overline{\mathcal{U}} \cap P^*,$$

as needed. □

**Lemma 2.3.4.** *Suppose $z \in Z$ is a rational point, so the morphism $\psi_z : \operatorname{Hilb}^2 C \dashrightarrow \overline{\mathcal{U}}$ from Theorem 2.2.1 and Remark 2.2.2 is defined. Let $\Gamma$ be the graph of $\psi_z$. Then*

*(i) the projection $\pi_2 : \Gamma \to \overline{\mathcal{U}}$ is an isomorphism except over the zero-dimensional subscheme $\overline{\mathcal{U}} \cap \mathcal{F} \cap P^*$, over which the fibers contain two points,*

*(ii) $\Gamma$ is the blowup of $\overline{\mathcal{U}}$ along $\overline{\mathcal{U}} \cap P^*$,*

*(iii) the projection* $\pi_1 : \Gamma \to \operatorname{Hilb}^2 C$ *is the blowup of* $\operatorname{Hilb}^2 C$ *at the finitely many points along* $E$ *corresponding to the pairs* $\{c, d\}$ *for which* $\tau_z(c), \tau_z(d) \in P^*$*.*

*Proof.* We assume $Z$ is reduced, though it is straightforward to adapt the argument in each of the cases where $Z$ is nonreduced. For each $c \in C$, let $\bar{c}$ denote the image of $c$ under the hyperelliptic involution. Let $E = \{\{c, \bar{c}\} \mid c \in C\} \subset \operatorname{Hilb}^2 C$, let $s_1, s_2, s_3 \in \operatorname{Hilb}^2 C$ be the points corresponding to the pairs $\{c, d\}$ for which $\tau_z(c), \tau_z(d) \in P^*$, and let $E_i = \pi_1^{-1}(s_i)$. Note that $\psi_z$ is defined except at $s_1, s_2, s_3$.

First, we claim $\pi_2$ is an isomorphism away from $E_1 \sqcup E_2 \sqcup E_3$. Since $\psi_z \circ \varphi_z = \mathrm{id}_{\mathcal{U}}$, we know $\mathcal{U}$ it contained in the image of $\psi_z$ hence also of $\pi_2$, and so $\pi_2$ is surjective. Moreover, $\psi_z$ is injective on its domain, so $\pi_2$ is injective away from $E_1 \sqcup E_2 \sqcup E_3$, as needed.

It is straightforward to check that for each point $w \neq z \in Z$, the pencil $w^*$ is not contained in the image of $\psi_z$. Therefore, $\pi_2$ projects $E_1 \sqcup E_2 \sqcup E_3$ onto the pairwise-intersecting triple of lines

$$\bigcup_{w \neq z \in Z} w^*.$$

Also noticing that $\pi_2(\pi_1^{-1}(E)) = z^*$, we obtain (i).

Using the universal property of the blowup, $\pi_2$ factors through a morphism $\rho : \Gamma \to \mathrm{Bl}_{\overline{\mathcal{U}} \cap P^*} \overline{\mathcal{U}}$. The morphism $\mathrm{Bl}_{\overline{\mathcal{U}} \cap P^*} \overline{\mathcal{U}} \to \overline{\mathcal{U}}$ fails to be an isomorphism over the same six points as $\pi_2$, i.e. the six pairwise intersections of lines in $\overline{\mathcal{U}} \cap P^*$ which together compose $\overline{\mathcal{U}} \cap \mathcal{F} \cap P^*$. Since $\pi_2$ is two-to-one over these points, $\rho$ must be an isomorphism.

For (iii), notice that the preimage $E_i$ of each point $s_i$ under $\pi_1$ is isomorphic to $\mathbb{P}^1$. By the universal property of the blowup, $\pi_1$ factors through a morphism $\mathrm{Bl}_{s_1, s_2, s_3} \operatorname{Hilb}^2 C \to \operatorname{Hilb}^2 C$ which must be an isomorphism. $\square$

Figure 2 illustrates the configurations of lines on $\Gamma$, $\operatorname{Hilb}^2 C$, and $\overline{\mathcal{U}}$.

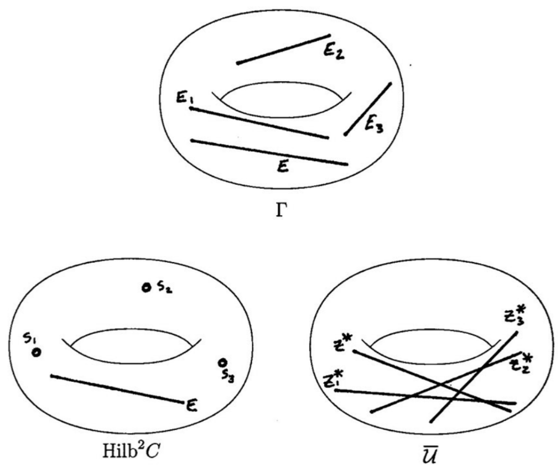


*Figure 2.* Configurations of lines on the schemes appearing in the proof of Lemma 2.3.4 in the case $C$ is smooth and $Z$ is reduced.

Note that the morphism $\Gamma \xrightarrow{\sim} \operatorname{Bl}_{\overline{\mathcal{U}}\cap P^*} \overline{\mathcal{U}}$ may not be defined over $k$ if none of the points in $Z$ are $k$-rational. In that case, $\operatorname{Bl}_{\overline{\mathcal{U}}\cap P^*} \overline{\mathcal{U}}$ still contains a quadruple of skew lines whose Galois orbits correspond to the Galois orbits of points in $Z$. When $C$ is smooth, $\operatorname{Hilb}^2 C \simeq \operatorname{Sym}^2 C \simeq \operatorname{Bl}_{\mathcal{O}_C} \operatorname{Pic}^0_C$, so the four lines on $\Gamma$ can be blown down to obtain the abelian surface $\operatorname{Pic}^0_C$. When $Z$ has no $k$-rational point, the quadruple of lines on $\operatorname{Bl}_{\overline{\mathcal{U}}\cap P^*} \overline{\mathcal{U}}$ can still be blown down to obtain a surface $T$ which is geometrically isomorphic to $\operatorname{Pic}^0_C$. This surface $T$ is the object of interest in the next section.

We summarize the discussion above by the following corollary.

**Corollary 2.3.5.** *The blowup* $\operatorname{Bl}_{\overline{\mathcal{U}}\cap P^*} \overline{\mathcal{U}} \to \overline{\mathcal{U}}$ *is an isomorphism away from the finitely many points* $\overline{\mathcal{U}} \cap \mathcal{F} \cap P^*$ *and two-to-one over* $\overline{\mathcal{U}} \cap \mathcal{F} \cap P^*$.

Finally, we work toward analyzing the intersection $\overline{\mathcal{U}} \cap \mathcal{F}$.

**Lemma 2.3.6.** *The intersection* $\overline{\mathcal{U}} \cap \mathrm{Sing}(\mathcal{F})$ *is zero-dimensional.*

*Proof.* We may work over $k^s$ and fix a point $z \in Z$. If $C$ is smooth, then so is $\mathcal{F}$, and there is nothing to prove. So, assume $C$ is singular. Then $\mathrm{Sing}(\mathcal{F})$ consists of a conic over each singular point of $C$. To show that such a conic does not belong to $\overline{\mathcal{U}}$, we first show that is contains a dense open set not belonging to the image of the rational map $\psi_z$.

Let $c$ be a singular point of $C$, let $Q = q^{-1}(c)$ be the quadric lying over $c$, and let $L \subset Q$ be a line not intersecting $Z$. Such a line $L$ is generic in $Q$. The cone point $y \in Q$ does not lie in $P$ or else $Y$ would have a fifth node along $P$.

The plane $P'$ spanned by $L$ and $z$ intersects $Y$ in two other lines: $\tau_z(c)$ and another line $L'$ containing $z$. Since $\tau_z(c)$ and $L$ intersect at $y \notin P$, the line $L'$ joins $z$ and $L \cap P$. Hence $L' \subset P$. As $L \cap P \notin Z$, and $\deg(L' \cap Q \cap P) = 2$, the line $L'$ does not contain a length-two subscheme of $Z$ so does not belong to the image of $\tau_z$. It follows that $L$ is not in the image of $\psi_z$.

Now, we apply Lemma 2.3.4, which shows that the only curves in $\overline{\mathcal{U}}$ without dense open sets lying in the image of $\psi_z$ are the pencils $w^*$ for $z \neq w \in Z$. Since $\mathrm{Sing}(\mathcal{F}) \not\subset P^*$ and $\mathrm{Sing}(\mathcal{F}) \cap \mathrm{im}(\psi_z)$ is finite, we arrive at the desired result. □

Let $C_z \subset \mathcal{F}$ denote the image of $\tau_z$.

**Lemma 2.3.7.** $\overline{\mathcal{U}} \cap \mathcal{F} = \cup_{z \in Z} C_z$.

*Proof.* First, we show $C_z \subset \overline{\mathcal{U}} \cap \mathcal{F}$ for each point $z \in Z$. Note $C_z \subset \mathrm{Sing}(F(Y))$ since each line in $C_z$ meets $Z$. Hence

$$C_z \subset \mathcal{F} \cap \mathrm{Sing}(F(Y)) = (\overline{\mathcal{U}} \cap \mathcal{F}) \cup (\mathcal{F} \cap P^*) \cup \mathrm{Sing}(\mathcal{F}).$$

But $\mathcal{F} \cap P^*$ is finite by Lemma 2.3.1, $\mathrm{Sing}(\mathcal{F})$ consists of finitely many conics, and the genus of $C_z$ is 2, so there is no embedding

$$C_z \to (\mathcal{F} \cap P^*) \cup \mathrm{Sing}(\mathcal{F}).$$

Therefore $C_z \subset \overline{\mathcal{U}} \cap \mathcal{F}$.

For the reverse inclusion, note that if $L \in \overline{\mathcal{U}} \cap \mathcal{F}$, then $L$ is a singular point of $F(Y)$ so passes through a singular point of $Y$. The lines in $\mathcal{F}$ passing through singular points of $Y$ are parametrized by the curves $C_z$ for $z \in Z$ and by $\mathrm{Sing}(\mathcal{F})$, so

$$\overline{\mathcal{U}} \cap \mathcal{F} \subset \bigcup_{z \in Z} C_z \cup \mathrm{Sing}(\mathcal{F}).$$

By Lemma 2.3.6, $\overline{\mathcal{U}} \cap \mathrm{Sing}(\mathcal{F})$ is zero-dimensional, so it remains to show that $\overline{\mathcal{U}} \cap \mathcal{F}$ is equidimensional. Indeed, $F(Y)$ is Gorenstein, and $P^*$ is Cohen-Macaulay, so

$$\overline{F(Y) \setminus P^*} = \mathcal{F} \cup \overline{\mathcal{U}}$$

is Cohen-Macaulay by [10, Thm. 21.23]. By [10, Cor. 18.11], $\mathcal{F} \cap \overline{\mathcal{U}}$ is purely 1-dimensional. □

**Proposition 2.3.8.** *We have the following description of the boundary of* $\overline{\mathcal{U}}$*:*

$$\overline{\mathcal{U}} = \mathcal{U} \cup \bigcup_{z \in Z} (z^* \cup C_z).$$

*Proof.* This follows directly from Lemmas 2.3.3 and 2.3.7. □

## 2.4 Rational Involutions on $\overline{\mathcal{U}}$ and $T$

Recall from Corollary 2.3.5 and the discussion preceding it that $\mathrm{Bl}_{\overline{\mathcal{U}} \cap P^*} \overline{\mathcal{U}}$ contains a set of skew lines in bijection with the subscheme $Z$, and that these lines can be contracted over $k$ to obtain a surface $T$. We can make the following identification:

$$T = \left( \overline{\mathcal{U}} \setminus \bigcup_{z \in Z} z^* \right) \cup Z.$$

When $C$ is smooth, $T$ is geometrically an abelian surface. In the next section, we equip $T$ with the structure of a $\mathrm{Pic}^0_C$-torsor, for which we will need a collection of involutions $j_c$ on $T$ to define an action by $\mathrm{Div}(C)$. First, we define a collection of rational involutions on $\overline{\mathcal{U}}$.

**Definition 2.4.1.** For $c \in C$, define a rational map $i_c : \overline{\mathcal{U}} \dashrightarrow F(Y)$ as follows: $c$ specifies a ruling in a quadric surface in $Y$, and a line $L \subset Y \setminus P$ meets a unique line $M$ in this ruling. The plane spanned by $L$ and $M$ meets $Y$ in a third line, $i_c(L)$.

We have defined $i_c$ on $\mathcal{U}$, though the domain turns out to be larger, as explained shortly.

**Lemma 2.4.2.** *For $c, d \in C$ and $z \in Z$,*

$$i_c(\tau_z(d)) = i_d(\tau_z(c))$$

*whenever both sides of the equation are defined.*

*Proof.* It suffices to check that the two rational maps $C \times C \to \overline{\mathcal{U}}$ defined by sending $(c, d)$ to $i_c(\tau_z(d))$ and to $i_d(\tau_z(c))$ agree on a dense open set. For the open set, take

$$V = \{(c, d) \mid \bar{c} \neq d \text{ and } \tau_z(c), \tau_z(d) \not\subset P\} \subset C \times C.$$

For $(c, d) \in V$, the lines $\tau_z(c)$ and $\tau_z(d)$ meet at $z$, and the third line in the plane spanned by $\tau_z(c)$ and $\tau_z(d))$ is, by definition, both $i_c(\tau_z(d))$ and $i_d(\tau_z(c))$. □

*Remark* 2.4.3. Lemma 2.4.2 affords another description of the birational maps $\psi_z \colon \mathrm{Hilb}^2 C \dashrightarrow \overline{\mathcal{U}}$ from the proof of Theorem 2.2.1:

$$\psi_z(\{c, d\}) = i_c(\tau_z(d)).$$

**Lemma 2.4.4.** *Each $i_c$ is a rational involution whose image is contained in $\overline{\mathcal{U}}$.*

*Proof.* It is easy to see that $i_c^2 = \mathrm{id}_{\overline{\mathcal{U}}}$. To show that the image of $i_c$ is contained in $\overline{\mathcal{U}}$, it is enough to show that the image contains $\mathcal{U}$.

Let $[L] \in \mathcal{U}$ and take $[M] \in \mathcal{U}$ be any line with $L \cap M \neq \emptyset$. The plane $P'$ spanned by $L$ and $M$ intersects $P$ at a single point $y$ and intersects $Y$ in the union of $L$, $M$, and a third line $N$. The point $y \in P$ must lie on $N$, so $[N] \in \mathcal{F}$. Taking $c \in C$ to be the ruling $N$ belongs to, $[L] = i_c([M])$. □

**Lemma 2.4.5.** *For all $c$, the domain of $i_c$ includes the following:*

*(i) the subscheme $\mathcal{U}$;*

*(ii) $\tau_z(d)$ for $z \in Z$ and $d \in C$ if $d \notin \{c, \bar{c}\}$ and $\tau_z(d) \not\subset P$;*

*(iii) $\tau_z(c)$.*

*If also $\tau_z(c) \not\subset P$ for all $z$, then the domain of $i_c$ includes*

*(iv) any line $\ell \in P^* \cap \overline{\mathcal{U}}$ not of the form $\tau_z(d)$;*

*(v) $\tau_z(\bar{c})$.*

*Proof.* Let $Q \subset Y$ be the quadric surface with ruling $c$. It is clear that $i_c$ is defined on $\mathcal{U}$ and on $\tau_z(d)$ meeting the conditions in (ii): each of these lines $L$ meets a unique line $M$ in the ruling $c$ of $Q$, and the plane $P'$ spanned by $L$ and $M$ is not $P$ so meets $Y$ in dimension one, giving a third line in $\overline{\mathcal{U}}$. For (iii), the same argument holds, but the plane $P'$ is spanned by $\tau_z(d)$ and a normal vector to that line in the surface

$$\bigcup_{d \in C} \tau_z(d) \subset Y.$$

as in Remark 2.2.2. A line $\ell$ of the form in (iv) also meets a unique line in the ruling $c$ of $Q$, namely $\tau_z(c)$. If $\tau_z(c) \not\subset P$, then the plane spanned by $\ell$ and $\tau_z(c)$ is not $P$ so meets $Y$ in dimension one, giving a third line in $\overline{\mathcal{U}}$. For (v), note that $\psi_z(\{c, \bar{c}\})$ is well-defined; it is the line $[T_z Q \cap P]$. Using $i_c(\tau_z(\bar{c})) = \psi_z(\{c, \bar{c}\})$, we see $\tau_z(\bar{c})$ is in the domain of $i_c$. □

**Definition 2.4.6.** For $c \in C$, let $j_c : T \dashrightarrow T$ be the rational map already defined on $\mathcal{U} \subset T$ by the rational involution $i_c$.

**Lemma 2.4.7.** *Each rational map $j_c$ extends to a morphism.*

When $C$ is smooth, Lemma 2.4.7 can been deduced from the fact that any rational map from a smooth variety to an abelian variety extends to a morphism. However, it is useful to have descriptions for how $j_c$ is defined at each point on $T$, motivating a more explicit proof.

*Proof of Lemma 2.4.7.* Let $Q$ be the quadric surface with a ruling specified by $c$. The following cases describe all types of points on $T$:

(i) $[L] \in \mathcal{U}$;

(ii) $\tau_z(d)$ for $z \in Z$ if $\bar{c} \neq d \in C$ and $\tau_z(d) \not\subset P$;

(iii) $\tau_z(\bar{c})$ for $z \in Z$ if $\tau_z(\bar{c}) \not\subset P$;

(iv) $z \in Z$.

On each of these types of points, $j_c$ is defined as follows.

(i) Note $i_c([L]) \notin P^*$, so $i_c([L])$ can be identified with a point of $T$, and $j_c([L]) = i_c([L])$.

(ii) The line in the $c$-ruling of $Q$ that $\tau_z(d)$ meets is $\tau_z(c)$. By Lemma 2.4.5, $i_c(\tau_z(d))$ is defined: it is the third line $\ell$ in the plane spanned by $\tau_z(c)$ and $\tau_z(d)$. We can identify $j_c(\tau_z(d)) = i_c(\tau_z(d))$ so long as the line $i_c(\tau_z(d))$ does not lie in $P$. To verify this, note that the rational map $\psi_z : \mathrm{Sym}^2 C \dashrightarrow \overline{\mathcal{U}}$ sending $\{a, b\}$ to $i_a(\tau_z(b))$ is injective, and it is straightforward to check that the pencil of lines in $P$ through $z$ is the image of the pencil of pairs $\{a, \bar{a}\}$. Having assumed $\bar{c} \neq d$, we obtain $i_c(\tau_z(d)) \not\subset P$.

(iii) Suppose $\tau_z(c) \not\subset P$. Then by Lemma 2.4.5, $i_c(\tau_z(\bar{c})) \in z^*$. Since $z^*$ is contracted in $T$ to a point identified with $z$, $j_c(\tau_z(\bar{c})) = z$.

If $\tau_z(c) \subset P$, then $i_c(\tau_z(\bar{c}))$ is not defined: $\tau_z(\bar{c})$ meets each line in the $c$-ruling of $Q$. Given a line $L \subset Q$ in the $c$-ruling, the plane $P'$ spanned by $L$ and $\tau_z(c)$ intersects $Y$ in a third line $M$. Noting that $P'$ contains $z$ and $L \cap P$, we see $[M] \in z^*$. Different choices of $L$ give different lines $[M] \in z^*$, but $z^*$ is contracted to $z$ in $T$, so we may define $j_c(\tau_z(\bar{c})) = z$.

(iv) Since $j_c$ is an involution, applying $j_c$ to both sides of (iii) gives $j_c(z) = \tau_z(\bar{c})$.

□

## 2.5 A torsor associated to the Fano variety

In this section, we restrict to the case that $C$ is smooth, so $T$ is geometrically isomorphic to the abelian surface $\mathrm{Pic}^0_C$. However, no isomorphism between $T$ and $\mathrm{Pic}^0_C$ need be available if none of the points in $Z$ are $k$-rational. This section describes how to endow $T$ with the structure of a $\mathrm{Pic}^0_C$ torsor, following the approach of Wang in [35]. The strategy is to put a group scheme structure on the disconnected group variety

$$\mathrm{Pic}^0_C \sqcup T \sqcup \mathrm{Pic}^1_C \sqcup T'$$

where $T \simeq T'$, and points in $T'$ are the "negatives" of points in $T$. Since the construction is rather technical, it is worth giving a simple description of the group law in a few cases.

- addition on $\operatorname{Pic}^0_C \sqcup \operatorname{Pic}^1_C$ is as in $\operatorname{Pic}_C / \omega_C$;
- for $[L] \in \mathcal{U} \subset T$ and $(c) \in \operatorname{Pic}^1_C$, the sum $[L] + (c)$ is the "negative" of the third line coplanar to $L$ and $\sigma_L(\bar{c})$;
- a pair of lines $[L], [M] \in \mathcal{U} \subset T$ that meet are coplanar to a third line which must meet $P$ so belongs to some ruling $c \in C$ of a quadric surface, and $[L] + [M] = (c) \in \operatorname{Pic}^1_C$;
- a general pair of skew lines $[L], [M] \in \overline{\mathcal{U}} \subset T$ span a $\mathbb{P}^3$ intersecting $Y$ in a smooth cubic surface, and in that surface there are three lines meeting all of $[L]$, $[M]$, and $P$. Those lines belong to $\mathcal{F}$ so specify three points on $C$, and the sum of these three points is $[L] + [M] \in \operatorname{Pic}^1_C$.

The proofs of Lemmas 2.5.2 and 2.5.4 below use information about the incidence relations between lines on cubic surfaces. We fix the following notation for the lines on a smooth cubic surface over an algebraically closed field, similar to the notational scheme used in [13, V.4].

*Notation* 2.5.1. A smooth cubic surface $S$ over an algebraically closed field is the blowup of $\mathbb{P}^2$ at six points $p_1, \ldots, p_6$ in general position and contains 27 lines:

(i) the six exceptional divisors $E_i$;

(ii) for each pair $\{i, j\}$, the proper transform $\ell_{ij}$ of the line joining $p_i$ to $p_j$;

(iii) and for each $i$, the proper transform $F_i$ of the conic passing through all the $p_j$ except $p_i$.

The incidence relations are as follows:

(i) $E_i$ meets $\ell_{ij}$ and $f_j$ for $i \neq j$;

(ii) $\ell_{ij}$ meets $E_i$, $E_j$, $F_i$, $F_j$, and $\ell_{hk}$ for $\{i, j\} \cap \{h, k\} = \emptyset$;

(iii) $F_i$ meets $E_j$ for $i \neq j$ and $\ell_{ij}$ for all $j$.

Moreover, given six pairwise skew lines $L_1, \ldots, L_6 \subset S$, the blowdown of the $L_i$ is isomorphic to $\mathbb{P}^2$, so one may assume $L_i = E_i$.

Let $T'$ be the variety isomorphic to $T$ whose points are written $-t$ for each $t \in T$.

**Lemma 2.5.2.** *The rules*

$$t + (c) = -j_{\bar{c}}(t)$$

*and*

$$-t + (c) = j_c(t)$$

*define an action of* $\mathrm{Div}(C)$ *on* $T \sqcup T'$.

*Proof.* For the action to be well-defined, $(c) + (d)$ must act the same as $(d) + (c)$, so we need to check $j_c \circ j_{\bar{d}} = j_d \circ j_{\bar{c}}$ for any $c, d \in C$.

Work over $k^s$. As $T$ is irreducible, it suffices to show that the two maps $C \times C \times T \to T$ sending $(c, d, t)$ to $j_d \circ j_{\bar{c}}(t)$ and $j_c \circ j_{\bar{d}}(t)$ agree on a nonempty open set. In particular, we can choose $t = [L] \in \mathcal{U}$ and $c \neq d \in C$ such that the $\mathbb{P}^3$ spanned by the lines $\sigma_L(\bar{c})$ and $\sigma_L(\bar{d})$ meets $Y$ in a smooth cubic surface S.

Label $\sigma_L(\bar{c}) = E_1$, $\sigma_L(\bar{d}) = E_2$, and $L = \ell_{12}$, using the notation for the lines on a smooth cubic surface. Notice that $S \cap P$ is a line meeting $\sigma_L(\bar{c})$ and $\sigma_L(\bar{d})$ but not $L$. Label this line $F_6$.

Now we calculate $j_{\bar{c}}([L]) = [F_2]$: indeed, only the line $F_2$ meets $E_1$ and $\ell_{12}$. The line $\ell_{26}$ meets $E_2$ and $F_6$ hence $\sigma_L(\bar{d})$ and $P$, so it is a line in the ruling of a quadric surface specified by $d$. As $j_{\bar{c}}([L]) = [F_2]$ meets $\ell_{26}$, $j_d \circ j_{\bar{c}}([L])$ is the line meeting $F_2$ and $\ell_{26}$, namely $E_6$. One calculates $j_c \circ j_{\bar{d}}([L]) = [E_6]$ similarly. □

**Proposition 2.5.3.** *The principal divisors act trivially on $T \sqcup T'$, so the action of $\mathrm{Div}(C)$ on $T \sqcup T'$ descends to an action by $\mathrm{Pic}_C$.*

*Proof.* First, notice that $T$ and $T'$ are in different orbits of the action by $\mathrm{Div}^0(C)$, and it suffices to show that principal divisors act trivially on $T$.

Every nontrivial divisor class in $\mathrm{Pic}^0_C$ can be represented as a difference of two effective divisors of degree 1 in exactly two ways: if $(c) - (d)$ represents $D$, then so does $(\bar{d}) - (\bar{c})$. By Lemma 2.5.2, $j_c \circ j_d = j_{\bar{d}} \circ j_{\bar{c}}$, so there is a well-defined morphism $\mathrm{Pic}^0_C \to \mathrm{Aut}(T)$ sending $(c) - (d)$ to $j_{\bar{d}} \circ j_{\bar{c}}$.

The map is automatically a homomorphism: its image is a commutative, projective subgroup scheme, and any unital morphism of abelian varieties is a homomorphism. Moreover, $j_{\bar{d}} \circ j_{\bar{c}}(t) = t + (c) - (d)$, so the homomorphism $\mathrm{Div}^0(C) \to \mathrm{Aut}(T)$ coming from the group action described earlier factors through the morphism $\mathrm{Pic}^0_C \to \mathrm{Aut}(T)$, i.e. principal divisors are in the kernel. □

Note that $\omega_C$ acts trivially on $T \sqcup T'$ since

$$t + (c) + (\bar{c}) = j_{\bar{c}}^2(t) = t,$$

so the action by $\mathrm{Pic}_C$ descends to an action by

$$\mathrm{Pic}_C / \omega_C \cong \mathrm{Pic}^0_C \sqcup \mathrm{Pic}^1_C.$$

Note that $\mathrm{Pic}^0_C$ acts on each of $T$ and $T'$, and an element of $\mathrm{Pic}^1_C$ exchanges the components $T$ and $T'$. The following lemma shows $T \sqcup T'$ is a torsor over $\mathrm{Pic}^0_C \sqcup \mathrm{Pic}^1_C$.

**Proposition 2.5.4.** *The action described above makes $T \sqcup T'$ a torsor over* $\mathrm{Pic}^0_C \sqcup \mathrm{Pic}^1_C$.

*Proof.* To show that the action of $\mathrm{Pic}^0_C \sqcup \mathrm{Pic}^1_C$ on $T \sqcup T'$ is transitive, we must check that for each $s, t \in T$,

(i) there is a divisor class $[D] \in \mathrm{Pic}^1_C$ so that $s + [D] = -t$;

(ii) there is a divisor class $[E] \in \mathrm{Pic}^0_C$ so that $s + [E] = t$.

Moreover, we verify

(iii) the action of $\mathrm{Pic}^0_C$ on $T$ is free;

(iv) the action of $\mathrm{Pic}^0_C \sqcup \mathrm{Pic}^1_C$ on $T \sqcup T'$ is free.

(i) First, we show that when $[L], [M] \in \mathcal{U}$ span a $\mathbb{P}^3$ meeting $Y$ in a smooth cubic surface, there exists $[D] \in \mathrm{Pic}^1_C$ so that $[L] + [D] = -[M]$.

Using Lemma 2.2.4, let $c, d, e \in C$ be the points lying below $\sigma_L \cap \sigma_M \subset \mathcal{F}$. No two of $\sigma_L(c)$, $\sigma_L(d)$, $\sigma_L(e)$ meet: if $\sigma_L(c)$ met $\sigma_L(d)$, then they would span a plane containing $L$ and $M$ which would not meet $X$ in degree three. Working over $k^s$ and using the standard notation for lines on the cubic surface $S$, assume $L = E_1$, $M = E_2$, $S \cap P = E_3$, $\sigma_L(c) = F_4$, $\sigma_L(d) = F_5$, and $\sigma_L(e) = F_6$. One calculates

$$j_{\bar{d}} \circ j_c([L]) = j_{\bar{d}}([\ell_{14}]) = [\ell_{26}] = j_e([M]),$$

so

$$(e) - [M] = [L] - (c) - (d),$$

and

$$[L] + (\bar{c}) + (\bar{d}) + (\bar{e}) = -[M].$$

Taking $[D] = (\bar{c}) + (\bar{d}) + (\bar{e}) - \omega_C$, we are done.

Now, for $[L] \in \mathcal{U}$, define a morphism $s_L : \mathrm{Pic}^1_C \to T'$ by $s_L([D]) = [L] + [D]$. By the above, the image of $s_L$ contains the dense open set

$$V = \{-[M] \mid [M] \in \mathcal{U} \text{ and } \mathrm{span}(L, M) \cap Y \text{ is a smooth cubic surface}\},$$

so $s_L$ is surjective.

Next, consider the summation map

$$\Sigma : T \times \mathrm{Pic}^1_C \to T', \quad (u, [D]) \mapsto u + [D].$$

Let $-t \in T'$. As $s_L$ is surjective for each $[L] \in \mathcal{U}$, we get

$$\mathcal{U} \subset \pi_1(\Sigma^{-1}(-t))$$

where $\pi_1$ is the projection $T \times \mathrm{Pic}^1_C \to T$. Since $\pi_1$ is closed,

$$\pi_1(\Sigma^{-1}(-t)) = T.$$

That is, for each $s \in T$ there exists $[D] \in \mathrm{Pic}^1_C$ so that $s + [D] = -t$.

(ii) Apply (i) to obtain $[D], [E] \in \mathrm{Pic}^1_C$ with $s + [D] = -t$ and $t + [E] = -t$. Then $s + [D - E] = t$.

(iii) Work over $k^s$. It is enough to check that if $(c) + (d)$ fixes a point $z \in Z \subset T$, then $(c) + (d) = \omega_C$. Indeed, if $z = z + (c) + (d)$, then

$$z + (\bar{d}) = z + (c)$$

$$j_d(z) = j_{\bar{c}}(z)$$

$$\tau_z(\bar{d}) = \tau_z(c),$$

so $\bar{d} = c$ since $\tau_z$ is an embedding.

(iv) The argument for (iii) also shows $\mathrm{Pic}^0_C$ acts freely on $T'$. Moreover, if $t + [D] = t + [E]$ for $t \in T \sqcup T'$ and $[D], [E] \in \mathrm{Pic}^1_C$, then $[D] - [E] \in \mathrm{Pic}^0_C$ fixes $t$, so $[D] - [E] = 0$ by (iii). ∎

We now prove our main structural results about $T$.

**Theorem 2.5.5.** *There is a commutative group law on*

$$G = \mathrm{Pic}^0_C \sqcup T \sqcup \mathrm{Pic}^1_C \sqcup T'$$

*extending the group law on* $\mathrm{Pic}^0_C \sqcup \mathrm{Pic}^1_C$ *such that for* $s, t \in T \sqcup T'$ *and* $c \in C$*,*

1. $s + (c) = -j_{\bar{c}}(s)$ *and* $-s + (c) = j_c(s)$*;*

2. $s + t$ *is the unique divisor class* $[D]$ *such that* $-s + [D] = t$ *under the action of* $\mathrm{Pic}^0_C \sqcup \mathrm{Pic}^1_C$ *on* $T \sqcup T'$ *defined prior.*

*Proof.* All that remains to check is that the group law is associative, i.e. the following all hold for for $s, t, u \in T \sqcup T'$ and $[D], [E], [F] \in \mathrm{Pic}^0_C \sqcup \mathrm{Pic}^1_C$.

(i) $([D] + [E]) + [F] = [D] + ([E] + [F])$

(ii) $(s + [D]) + [E] = s + ([D] + [E])$

(iii) $(s + t) + [D] = s + (t + [D])$

(iv) $(s + t) + u = s + (t + u)$

The first is inherited from the associativity of $\mathrm{Pic}^0_C \sqcup \mathrm{Pic}^1_C$, and the second follows from the fact that $\mathrm{Div}(C)$ acts on $T \sqcup T'$. For (iii), let $[E] = s + t$, meaning $[E]$ is the unique divisor class such that $-s + [E] = t$. Using (ii),

$$-s + ([E] + [D]) = t + [D],$$

i.e.

$$-s + ((s + t) + [D]) = t + [D].$$

Similarly, $s + (t + [D])$ is defined by the equation

$$-s + (s + (t + [D])) = t + [D],$$

so

$$-s + ((s+t) + [D]) = -s + (s + (t + [D])).$$

Since $\mathrm{Pic}^0_C \sqcup \mathrm{Pic}^1_C$ acts freely on $T \sqcup T'$, we deduce

$$(s+t) + [D] = s + (t + [D])$$

proving (iii). For (iv), let $K_1 = (s+t) + u$ and $K_2 = s + (t+u)$. Using (iii) and commutativity,

$$\begin{aligned} t + K_1 &= t + (u + (s+t)) \\ &= (t+u) + (s+t) \\ &= (t+s) + (t+u) \\ &= t + (s + (t+u)) \\ &= t + K_2, \end{aligned}$$

so again using (iii), the divisor class $t + K_1$ sends $-K_1$ and $-K_2$ both to $t$. As $\mathrm{Pic}^0_C \sqcup \mathrm{Pic}^1_C$ acts invertibly, $K_1 = K_2$. □

**Corollary 2.5.6.** *In* $H^1(k, \mathrm{Pic}^0_C)$, $4[T] = 0$ *and* $2[T] = [\mathrm{Pic}^1_C]$.

*Proof.* There is a short exact sequence of group schemes

$$0 \longrightarrow \mathrm{Pic}^0_C \longrightarrow G \xrightarrow{\pi} \mathbb{Z}/4\mathbb{Z} \longrightarrow 0$$

where $\pi(T) = 1$, $\pi(\mathrm{Pic}^1_C) = 2$, and $\pi(T') = 3$. This yields a short exact sequence

$$0 \longrightarrow \mathrm{Pic}^0_C(k^s) \longrightarrow G(k^s) \xrightarrow{\pi} \mathbb{Z}/4\mathbb{Z} \longrightarrow 0$$

of Galois modules. In the long exact sequence in Galois cohomology, the connecting homomorphism $H^0(k, \mathbb{Z}/4\mathbb{Z}) \to H^1(k, \mathrm{Pic}^0_C)$ sends $1 \mapsto [\pi^{-1}(1)] = [T]$ and $2 \mapsto [\mathrm{Pic}^1_C]$. □

This completes the proof of Theorem 1.0.2.

## 2.6 Connections to rationality

We retain earlier notation from the previous section for a general cubic threefold $Y$ containing a plane $P$. The threefold $\widetilde{Y} = \mathrm{Bl}_P Y$ is smooth and geometrically rational as the quadric surface fibration $q : \widetilde{Y} \to \mathbb{P}^1$ admits a section over $k^s$. If the torsor $T$ has a $k$-point, then there is a section of $q$ defined over $k$: either one of the points in $Z$ or a line in $Y \setminus P$. So, if $T$ is trivial, $\widetilde{Y}$ is rational over $k$. We will see, perhaps surprisingly, that when $\mathrm{char}(k) = 0$, the converse is also true, so the arithmetic of $T$ controls the rationality of $\widetilde{Y}$. To prove this, we use recent results about the Chow scheme of a smooth, $k$-rational threefold over an arbitrary field outlined in Section 1.5.

For the proof of our main theorem below, we argue as in [16, Thm 36] where Hassett and Tschinkel prove a similar result for an intersection of quadrics in $\mathbb{P}^5$.

**Theorem 2.6.1.** *Suppose $k$ is a field of characteristic zero. Then $\widetilde{Y}$ is $k$-rational if and only if $T(k) \neq \emptyset$.*

*Proof.* If $T(k) \neq \emptyset$, we have already seen $\widetilde{Y}$ is $k$-rational. So, suppose $\widetilde{Y}$ is $k$-rational, and let $\gamma \in \mathrm{NS}^2(\widetilde{Y}_{\bar{k}})^{\mathrm{Aut}(\bar{k}/k)}$ be the algebraic equivalence class of a line in $Y \setminus P$. By rigidity, the rational map $\mathcal{U} \to (\mathrm{CH}^2_{\widetilde{Y}/k})^{\gamma}$ sending a line in $Y \setminus P \subset \widetilde{Y}$ to its rational equivalence class extends to a morphism $T \to (\mathrm{CH}^2_{\widetilde{Y}/k})^{\gamma}$. Moreover, any two lines in $Y \setminus P$ are rationally inequivalent on $\widetilde{Y}$, for a rational equivalence between two lines determines a morphism $\mathbb{P}^1 \to T$, which must be constant. Hence $T$ embeds in $(\mathrm{CH}^2_{\widetilde{Y}/k})^{\gamma}$. By Lemma 1.5.2, $\dim((\mathrm{CH}^2_{\widetilde{Y}/k})^{\gamma}) = 2$, so this embedding is an isomorphism. Note that this is the only point at which we use the assumption on the characteristic of $k$.

Now, $T \simeq (\mathrm{CH}^2_{\widetilde{Y}/k})^{\gamma}$ is a torsor of $(\mathrm{CH}^2_{\widetilde{Y}/k})^0$ as well as a torsor of $\mathrm{Pic}^0_C$, so Lemma 2.6.2 below implies $(\mathrm{CH}^2_{\widetilde{Y}/k})^0 \simeq \mathrm{Pic}^0_C$. Under the assumption that $\widetilde{Y}$ is rational, [7, Thm. 3.11(iii)] affords some $e \in \mathbb{Z}$ for which

$$[T] = [(\mathrm{CH}^2_{\widetilde{Y}/k})^\gamma] = [\mathrm{Pic}^e_C] \in H^1(k, \mathrm{Pic}^0_C).$$

Since $C$ is hyperelliptic, $2[\mathrm{Pic}^e_C] = 0$ for all $e$, and applying Corollary 2.5.6,

$$[\mathrm{Pic}^1_C] = 2[T] = 2[\mathrm{Pic}^e_C] = 0 \in H^1(k, \mathrm{Pic}^0_C).$$

Hence $[\mathrm{Pic}^i_C] = 0$ for all $i$, and in particular $[T] = 0$, which means $T$ has a $k$-point. □

**Lemma 2.6.2.** *If $T$ is a torsor over two abelian varieties $A, B$, then $A \simeq B$.*

*Proof.* Over $k^s$, choose some point $t \in T$ and define an action of $A$ on $B$ by the following: $a \cdot b = b'$ where $b' \in B$ is the unique point with $b't = a(bt)$. The action is simply transitive and descends to $k$ since $(\sigma a) \cdot b = \sigma(a \cdot b)$; indeed,

$$\begin{aligned}
&(\sigma a) \cdot b = b' \\
&\iff b't = (\sigma a)(bt) \\
&\iff b't = \sigma(a(bt)) \\
&\iff \sigma^{-1}(b't) = a(bt) \\
&\iff (\sigma^{-1}b')t = a(bt) \\
&\iff a \cdot b = \sigma^{-1}b' \\
&\iff \sigma(a \cdot b) = b'.
\end{aligned}$$

Having equipped $B$ with the structure of an $A$-torsor, we conclude $A \simeq B$ since $B(k) \neq \emptyset$. □

Most likely, Theorem 2.6.1 holds for any field $k$, though the proof would require a finer understanding of the Chow scheme.

**Corollary 2.6.3.** *If $0 \neq [\mathrm{Pic}^1_C] \in H^1(k, \mathrm{Pic}^0_C)$, then $\widetilde{Y}$ is irrational over $k$.*

*Proof.* If $\mathrm{Pic}^1_C$ is nontrivial, then so is $T$, and we apply Theorem 2.6.1. □

**Corollary 2.6.4.** *If $Y$ contains a quadric surface $Q$ over $k \subset \mathbb{C}$ for which $Q(k) = \emptyset$, then $\widetilde{Y}$ is irrational over $k$.*

*Proof.* Since $Q(k) = \emptyset$, there can be no section of the quadric surface fibration $q$, so also $T(k) = \emptyset$, and we apply Theorem 2.6.1. □

Corollary 2.6.4 is useful for producing smooth, geometrically rational threefolds over $\mathbb{Q}$ that are $\mathbb{Q}$-irrational.

*Example* 2.6.5. Let $\widetilde{Y}$ be the blowup along $P = \{x_0 = x_1 = 0\}$ of the cubic threefold

$$Y = \{x_0 q_0 + x_1 q_1 = 0\}$$

where

$$q_0 = x_0^2 + x_1^2 + 2x_2^2 + 3x_3^2 + 5x_4^2$$

and

$$q_1 = x_0^2 + x_1^2 + x_2^2 + x_3^2 + x_4^2,$$

all of which is defined over $\mathbb{Q}$. Note $\widetilde{Y}$ is smooth and geometrically rational. However, $Y$ contains the quadric surface $Q = \{x_0 = q_1 = 0\}$, and $Q(\mathbb{R}) = \emptyset$, so Corollary 2.6.4 shows that $\widetilde{Y}$ is irrational over any subfield of $\mathbb{R}$.

It would be interesting to apply Theorem 2.6.1 to local-global questions about rationality. For example, consider the following question:

*Question* 2.6.6. Is there a general cubic threefold $Y$ containing a plane $P$ over a number field $K$ for which $\widetilde{Y}$ is $K_v$-rational for all places $v$ but irrational over $K$ itself?

In other words, can $\widetilde{Y}$ violate the Hasse principle with respect to rationality? There are as yet no examples of smooth threefolds violating the Hasse principle with respect to rationality. To answer Question 2.6.6 amounts to finding an example where $T$ is a nontrivial element of the Tate-Shafarevitch group (i.e. $T(K_v) \neq \emptyset$ for all $v$, but $T(K) = \emptyset$).

Toward producing an example, consider the following results.

**Proposition 2.6.7.** *If $C$ has good reduction at $p \neq 2$, then $\widetilde{Y}_{\mathbb{Q}_p}$ is rational.*

*Proof.* As $C$ has good reduction at $p$, so too do $\widetilde{Y}$ and $T$. By Lang's theorem, $T_p$ has an $\mathbb{F}_p$-point, and by Hensel's lemma, this lifts to a $\mathbb{Q}_p$ point of $T_{\mathbb{Q}_p}$. This point specifies a $\mathbb{Q}_p$-rational section of $q$, which makes $\widetilde{Y}_{\mathbb{Q}_p}$ rational. □

**Corollary 2.6.8.** *For all but finitely many places $v$ of $\mathbb{Q}$, $\widetilde{Y}_{\mathbb{Q}_v}$ is rational.*

*Proof.* Since $C$ has good reduction at all but finitely many primes $p$, this follows directly from Proposition 2.6.7. □

One obstacle in producing an example to answer Question 2.6.6 in the affirmative is the difficulty of proving that the torsor $T$ is nontrivial. One approach would be to show that $\mathrm{Pic}^1_C$ is nontrivial since the relation $2[T] = [\mathrm{Pic}^1_C]$ would then force $[T] \neq 0$. However, there are few tools to use to do this in general: we will sketch the premier approach for showing $[\mathrm{Pic}^1_C] \neq 0$, introduced by Poonen and Stoll:

Step 1. Count the number $N$ of "deficient places" of $C$, i.e. the number of places $v$ for which there is no odd $K_v$-rational divisor (not just divisor class!) on $C_v$.

Step 2. Under the Cassels-Tate pairing on the Tate-Shafarevitch group of $C$,

$$\langle \mathrm{Pic}^1_C, \mathrm{Pic}^1_C \rangle = N/2 \in \mathbb{Q}/\mathbb{Z}$$

by [32, Thm. 11]. If $N/2 \neq 0$, then conclude that $\mathrm{Pic}^1_C$ is nontrivial.

However, the following lemma and its corollary show that this strategy will not work in our examples.

**Lemma 2.6.9.** *Suppose $T(K) \neq \emptyset$ for some field extension $K/k$. Then $C$ has an odd $K$-rational divisor.*

*Proof.* The $K$-rational point on $T$ can be one of four types:

(i) a singular point $z \in P \subset Y$,

(ii) a line in a quadric surface $Q \subset Y$ passing through a singular point $z \in P \subset Y$, or

(iii) a line in $Y \setminus P$.

A line of type (ii) belongs to a ruling of a quadric surface in $Y$ so specifies a $K$-rational point in $C$. In case (i), suppose first that $Z = \{x, y, z, w\}$ is reduced, let $x$ be the $K$-rational point. Then the triple of lines $\overline{xy}, \overline{xz}, \overline{xw}$ is $K$-rational and belongs to the intersection $\mathcal{F} \cap P^*$ by Lemma 2.3.1. This triple specifies a $K$-rational divisor of odd degree on $C$.

For case (iii), we first prove that there is a hyperplane $H \supset L$ for which $Y \cap H$ is smooth. By Bertini's theorem, for a generic hyperplane containing $L$, the intersection $Y \cap H$ is smooth away from $L$. The hyperplanes for which $Y \cap H$ is singular along $L$ are $T_xY$ for $x \in L$, of which there is only a pencil, so a generic hyperplane containing $L$ is also smooth along $L$.

Now, $S = Y \cap H$ is a smooth cubic surface containing two $K$-rational lines: $L$ and $H \cap P$. There are five lines $M_1, \dots, M_5$ meeting these two, and each $M_i$ belongs to a ruling of a quadric surface so specifies a point $c_i \in C$. Over $K$, the divisor $c_1 + \cdots + c_5$ is defined. □

**Corollary 2.6.10.** *Suppose $T$ is locally solvable. Then $\langle \mathrm{Pic}^1_C, \mathrm{Pic}^1_C \rangle = 0$ under the Cassels-Tate pairing.*

*Proof.* By Lemma 2.6.9, $C_v$ is nowhere deficient, i.e. for each place $v$ of the ground field $K$, there is a $k$-rational odd divisor on $C$. Then apply [32, Thm. 11]. □

*Remark* 2.6.11. Lemma 2.6.9 echoes Bhargava, Gross, and Wang's result [8, Thm. 29], which makes the same claim when $T$ is the torsor studied by Wang in [35] whose points correspond to maximal linear spaces in a complete intersection of two quadrics. In fact, the result by Bhargava et al. is stronger, asserting that if a hyperelliptic curve $C$ has an odd $K$-rational divisor class, then there is a complete intersection of quadrics defined over $k$ whose Fano variety of maximal linear spaces is a torsor of $\mathrm{Pic}^0_C$.

# CHAPTER III

## CUBIC FOURFOLDS CONTAINING A PLANE

In this chapter, $X$ is a general cubic fourfold of discriminant 8 and $P \subset X$ is a plane; here, the term *general* refers to the open conditions that $X$ is smooth and contains no other plane meeting $P$. The ground field $k$ need not be closed but has characteristic not equal to 2. Our foundational geometric ingredient is the quadric surface fibration $q\colon \operatorname{Bl}_P X \to P^\perp$ obtained via projection to a complementary plane $P^\perp$, described in Example 1.2.3. A sextic curve $\Delta \subset P^\perp$ called the discriminant of $q$ parametrizes the singular fibers of $q$. Here, we use the assumption on the characteristic of $k$ as $\Delta$ is cubic when $\operatorname{char}(k) = 2$ by [17, Sect. 4.1]. The assumption that $X$ contains no plane meeting $P$ guarantees that the singular fibers of $q$ are no worse than cones and, by Lemma 2 in [34], that $\Delta$ is smooth.

A double cover $S$ of $P^\perp$ branched over $\Delta$ parametrizes rulings of the fibers of $q$, and since $\Delta$ is a smooth sextic, $S$ is a smooth K3 surface. The lines in the fibers of $q$ form a conic bundle over $S$ defining a Brauer class $\alpha \in \operatorname{Br}(S)[2]$. We have discussed the Hodge-theoretic relationship between $X$ and the twisted K3 surface $(S, \alpha)$ in Section 1.4.

Let $F$ be the Fano variety of lines on $X$, a smooth fourfold. Recall from Example 1.4.3 that in the case $k = \mathbb{C}$, $F$ is hyperkähler, and the Beauville-Bogomolov form detects that there is another hyperkähler fourfold $M$ birational to $F$ admitting a Lagrangian fibration $M \to \mathbb{P}^2$. This chapter describes how to construct the Lagrangian fibration explicitly in a way that works over an arbitrary field of characteristic different from 2. We then give another perspective on the Lagrangian fibration in characteristic zero by reproving a result from [27] that $F$ is birational to a moduli space of $\alpha$-twisted torsion sheaves on $S$ supported on curves in a 2-dimensional linear system, from this point of view, the Lagrangian fibration sends a twisted sheaf to its support.

Section 3.1 constructs the Lagrangian fibration explicitly using geometry from Chapter II. In Section 3.2, we reprove Kuznetsov's result that there is an equivalence $\mathcal{A}_X \simeq D^b(S, \alpha)$, at least in characteristic zero, giving a more explicit equivalence than the one in [24]. Using this equivalence, Section 3.3 outlines how to regard $F$ as birational to a moduli space of $\alpha$-twisted sheaves on $S$.

### 3.1 A rational Lagrangian fibration

For a line $L \subset X \setminus P$, the projection $q(L) \subset P^\perp$ is again a line, allowing us to define a rational map $\pi : F \dashrightarrow (P^\perp)^*$ by $\pi([L]) = [q(L)]$. In light of Example 1.4.3, the following lemma shows that when $k = \mathbb{C}$, $\pi$ is the rational Lagrangian fibration detected by the Beauville-Bogomolov form. This observation motivates the definition of $\pi$ but is not necessary for any results in what follows, so we are free to refer to results proved later in this section in order to verify some details in the proof.

**Lemma 3.1.1.** *The linear system $|\alpha(Q)|$ induces $\pi$, where $Q \subset X$ is a quadric surface and $\alpha : H^4(X, \mathbb{Z}) \to H^2(F, \mathbb{Z})$ is the Abel-Jacobi map.*

*Proof.* The class $\alpha([Q])$ is represented by a divisor $D \subset F$ parametrizing lines in $X$ meeting $Q$, and there is an open set $U \subset D$ parametrizing lines meeting $Q$ but not $P$. Let $p \in P^\perp$ be the image under $q$ of the proper transform of $Q$. For each $[L] \in U$, we have $p \in q(L)$, so

$$\pi([L]) \in p^* \subset (P^\perp)^*.$$

Hence $U \subset \pi^{-1}(p^*)$, so also $D \subset \pi^{-1}(p^*)$ where by $\pi^{-1}(p^*)$ we mean the closure of the set of lines in the domain of $\pi$ mapped to $p^*$.

We now measure the difference between $\pi^{-1}(p^*)$ and $D$. By Lemma 3.1.5 below, if $[L] \in \pi^{-1}(p^*)$ and $L \cap P = \{x\}$, then there is some $[\ell] \in p^*$ such that $X \cap H$ is singular at $x$ where $H$ is the hyperplane spanned by $P$ and $\ell$. By Lemma 3.1.3 below, $X \cap H$ has finitely many singular points along $P$, and through each point $x \in P$ there is a one-dimensional family of lines in $X$. Hence $\pi^{-1}(p^*)$ differs from $D$ by at most a 2-dimensional subscheme, so $[D] = [\pi^{-1}(p^*)]$ as divisor classes.

Now, $p^*$ is a line in $(P^\perp)^*$, so $\pi^{-1}(p^*)$ represents the divisor class $\pi^*\mathcal{O}_{(P^\perp)^*}(1)$, which proves that the complete linear system

$$|\alpha(Q)| = |[D]| = |[\pi^{-1}(p^*)]|$$

induces $\pi$. □

Before proving that $\pi$ is rational Lagrangian, it is worth explaining the relationship between (closures of) fibers of $\pi$ and the content of Chapter II. Note that if $L \subset P^\perp$, then $\pi^{-1}([L]) \subset F$ contains all the lines in $(X \setminus P) \cap H$ where $H$ is the hyperplane spanned by $P$ and $L$. In the language of Theorem 2.2.1, $\pi^{-1}([L])$ is the subscheme $\mathcal{U}$ of the Fano scheme of lines on $Y = X \cap H$.

We proceed as follows: first, we verify that $\pi$ extends to $F \setminus P^*$; second, we show that the blowup $\mathrm{Bl}_{P^*} F$ resolves the indeterminacy of $\pi$; finally, we show that $\pi$ factors through the contraction of the exceptional divisor

$$E = \{([L], x) \mid x \in L \subset P\} \subset \mathrm{Bl}_{P^*} F$$

via the second projection. From this, we obtain a Lagrangian fibration from the Mukai flop of $F$ along $P^*$ whose fibers exhibit the arithmetic of the torsors described in Section 2.5.

**Lemma 3.1.2.** *For each $x \in P$, there is a unique hyperplane $H \supset P$ for which $X \cap H$ is singular at $x$.*

*Proof.* Since $X$ is a hypersurface, $X \cap H$ is singular at $x$ if and only if $H = T_x H$. When $x \in P$, one also has $T_x P \supset P$. □

**Lemma 3.1.3.** *For each hyperplane $H \supset P$, the singular locus of $X \cap H$ along $P$ is zero-dimensional of length four.*

*Proof.* There is a morphism $g : P \to (P^{\perp})^*$ sending $x \mapsto ([T_x X \cap P^{\perp}])$, and the fiber of $g$ over $[L] \in (P^{\perp})^*$ is the singular locus of $X \cap H$ along $P$, where $H$ is the hyperplane spanned by $P$ and $L$. Since any cubic threefold containing a plane has singularities along the plane, $g$ is surjective. Hence $g$ is finite, as is any surjective endomorphism of $\mathbb{P}^2$. We end by applying Lemma 2.1.2, which says that a cubic threefold containing a plane $P$ with finitely many singularities along $P$ has four nodes on $P$, counting multiplicity. □

In other words, for each hyperplane $H \subset P$, the cubic threefold $Y = X \cap H$ containing $P$ is general in the sense of Chapter II. The analysis of the Fano variety of lines on each cubic threefold $Y$ obtained as such a hyperplane section helps to show to extend $\pi$ to its domain.

**Proposition 3.1.4.** *The domain of $\pi$ is $F \setminus P^*$.*

The claim follows immediately from the next two lemmas.

**Lemma 3.1.5.** *Let $M$ be a line in $X$ meeting $P$ in exactly one point $x$. Then $[M]$ belongs to the domain of $\pi$, and $\pi([M]) = [T_x X \cap P^{\perp}]$.*

*Proof.* Let

$$\mathcal{H} = \{(L, H) \mid L, P \subset H\} \subset F \times Gr(5, 6),$$

let $p_1$ be the first projection, and let $p_2 : \mathcal{H} \to (P^\perp)^*$ send $(L, H) \mapsto [H \cap P^\perp]$. If $L \subset X \setminus P$, then $p_1^{-1}([L]) = \{(L, H)\}$ where $H$ is the hyperplane spanned by $L$ and $P$, and $p_2(L, H) = \pi([L])$, which verifies that the diagram below commutes.

$$\begin{array}{ccc} & \mathcal{H} & \\ {}^{p_1}\swarrow & & \searrow^{p_2} \\ F & \overset{}{\underset{\pi}{\dashrightarrow}} & (P^\perp)^* \end{array}$$

Now, for each $\ell \subset P^\perp$, the restriction of $p_1$ to $p_2^{-1}([\ell])$ is an embedding, and moreover $p_2^{-1}([\ell])$ can be identified with the Fano scheme of lines on $Y = X \cap H$ where $H$ is the hyperplane spanned by $P$ and $\ell$. Let $Z = P \cap \operatorname{Sing}(Y)$, and let $P^*$, $\mathcal{F}$, and $\overline{\mathcal{U}}$ be the three components of this Fano scheme, as in Theorem 2.2.1. We have $\mathcal{U} = \pi^{-1}([\ell])$, so $[M] \in \overline{\pi^{-1}([\ell])}$ if and only if $[M] \in \overline{\mathcal{U}} \cap \mathcal{F}$, which by Lemma 2.3.7 holds if and only if $M \cap Z \neq \emptyset$. Since $M \cap P = \{x\}$, we have $M \cap Z \neq \emptyset$ if and only if $Y$ is singular at $x$, which occurs precisely when $H = T_x X$. Thus $[M]$ lies in the closure of the $\pi^{-1}([\ell])$ if and only if $\ell = T_x X \cap H$. Since $F$ is normal, $\pi$ extends to $[M]$. $\square$

**Lemma 3.1.6.** *Each point $[L] \in P^*$ lies in the closures of a pencil of fibers of $\pi$. Specifically, $[L] \in \overline{\pi^{-1}([M])}$ if and only if $M = T_x X \cap P^\perp$ for some $x \in L$.*

*Proof.* Suppose $M \subset P^\perp$, let $H$ be the hyperplane spanned by $P$ and $M$ let $Y = X \cap H$, and let $F(Y)$ be the Fano scheme of lines on $Y$. As usual, we refer to the components of $F(Y)$ as $\overline{\mathcal{U}}$, $\mathcal{F}$, and $P^*$. Since $L \subset P$, we have $[L] \in \overline{\mathcal{U}}$ if and only if $L$ passes through a node of $Y$ by Lemma 2.3.3, and this is the case if and only if $H = T_x X$ for some $x \in L$. This completes the argument since $\overline{\pi^{-1}([M])} = \overline{\mathcal{U}}$. $\square$

Lemma 3.1.6 helps to describe the resolution of indeterminacy of $\pi$:

**Lemma 3.1.7.** *The blowup of $F$ along $P^*$ resolves the indeterminacy of $\pi$.*

*Proof.* Let $\Gamma \subset F \times (P^{\perp})^*$ be the graph of $\pi$ along with its projections $p_1$ and $p_2$, and note that since $\pi$ does not extend to any point on $P^*$, the projection $p_1$ factors through $\mathrm{Bl}_{P^*} F$. By Lemma 3.1.6, the exceptional divisor $E'$ of $\Gamma \to F$ can be identified with

$$\{([L], [T_x X \cap P^{\perp}]) \mid x \in L\},$$

and the induced map

$$E' \to E = \{([L], x) \mid x \in L \subset P\} \subset \mathrm{Bl}_{P^*} F$$

is an isomorphism. Hence $\Gamma \simeq \mathrm{Bl}_{P^*} F$, as needed. □

As mentioned in Section 1.4, the exceptional divisor $E \subset \mathrm{Bl}_{P^*} F$ can be contracted via the projection $E \to P$ to give a blowdown $q : \mathrm{Bl}_{P^*} F \to M$ where $M$ is again a smooth variety, called the Mukai flop of $F$ along $P^*$. Over $\mathbb{C}$, $M$ is again hyperkähler.

**Proposition 3.1.8.** *The map* $p_2 : \mathrm{Bl}_{P^*} F \to (P^{\perp})^*$ *factors through* $q : \mathrm{Bl}_{P^*} F \to M$.

*Proof.* Let $[N] \in (P^{\perp})^*$, and let $H$ be the hyperplane spanned by $P$ and $N$. By Lemmas 3.1.6 and 3.1.7,

$$p_2^{-1}([N]) \cap E = \bigcup_{x \in P \cap \mathrm{Sing}(X \cap H)} \{([L], x) \mid x \in L \subset P\},$$

so $p_2^{-1}([N]) \cap E$ consists of a union of fibers of $q : \mathrm{Bl}_{P^*} F \to M$. □

We arrive at the following commutative square.

$$\begin{array}{ccc} Bl_{P^*}F & \xrightarrow{q} & M \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle \rho} \\ F & \overset{\pi}{\dashrightarrow} & \mathbb{P}^2 \end{array}$$

The smooth fibers of $\rho$ are geometrically abelian surfaces: in fact, they are the types of surfaces $T$ whose arithmetic we analyzed in Section 2.5:

**Proposition 3.1.9.** *Let $T = \rho^{-1}([L])$ be a smooth fiber of $\rho$. Then there is a double cover $C$ of $L$ branched over $L \cap \Delta$ for which $T$ is a $\mathrm{Pic}^0_C$-torsor. Moreover, $2[T] = [\mathrm{Pic}^1_C]$ in the Weil-Châtelet group.*

*Proof.* Let $H$ be the hyperplane spanned by $P$ and $L$. Then $Y = X \cap H$ is a general cubic threefold containing $P$ by Lemma 3.1.3. The quadric surface fibration $\mathrm{Bl}_P Y \to L$ has discriminant locus $L \cap \Delta$, and let $C$ be the double cover of $L$ parametrizing lines in the fibers of $q$. Let $F(Y)$ be the Fano variety of lines on $Y$, and let $\overline{\mathcal{U}}$ be the closure of the set of lines in $Y \setminus P$. Note that $\overline{\mathcal{U}} = \overline{\pi^{-1}([L])}$, and from the natural embedding $\mathrm{Bl}_{\overline{\mathcal{U}} \cap P^*} \overline{\mathcal{U}} \subset \mathrm{Bl}_{P^*} F$, we identify

$$(\pi \circ p)^{-1}([L]) = \mathrm{Bl}_{\overline{\mathcal{U}} \cap P^*} \overline{\mathcal{U}}.$$

By the proof of Proposition 3.1.8, the restriction of $q$ to $(\pi \circ p)^{-1}([L])$ contracts a line in $\overline{\mathcal{U}}$ for each point in $P \cap \mathrm{Sing}(X \cap H)$. The same lines are contracted by the defining morphism $\mathrm{Bl}_{\overline{\mathcal{U}} \cap P^*} \overline{\mathcal{U}} \to T$ from Lemma 2.3.4. Hence we can identify $\rho^{-1}([L])$ with the $\mathrm{Pic}^0_C$-torsor $T$ and apply Theorem 1.0.2. $\square$

This concludes the proof of Theorem 1.0.1. In particular, the arithmetic from Section 2.5 clarifies the geometry of the smooth fibers of $\rho$, which over $\mathbb{C}$ are already guaranteed by results about hyperkähler varieties to be abelian surfaces, as explained in Section 1.4.

### 3.2 The derived category

Recall that the Kuznetsov component $\mathcal{A}_X \subset D^b(X)$ is defined as the left orthogonal complement

$$\mathcal{A}_X = \langle \mathcal{O}_X(-1), \mathcal{O}_X, \mathcal{O}_X(1) \rangle^{\perp}.$$

As mentioned in Section 1.3, Kuznetsov proved in [24] that $\mathcal{A}_X$ is equivalent to the a the derived category of twisted coherent sheaves $D^b(S, \alpha)$, at least over the ground field $\mathbb{C}$.

Here, we reprove Kuznetsov's result for a general cubic fourfold $X$ of discriminant 8 over a field $k$ of characteristic zero, using an approach similar to Addington and Lehn's proof in [2] of Kuznetsov's equivalence for discriminant 14. The utility of our equivalence is that its geometric character allows us to recast the Fano variety $F$ of lines on $X$ as birational to a moduli space of rank zero sheaves on $S$ in the next section.

The first step is to construct a functor $\Phi : D^b(S, \alpha) \to D^b(X)$, which we do by specifying a Fourier-Mukai kernel $\mathcal{K} \in D^b(X \times S, 1 \boxtimes \alpha)$; then $\Phi$ is defined by

$$\Phi(-) = (\pi_1)_*(\pi_2^*(-) \otimes \mathcal{K})$$

where $\pi_i$ is the $i$th projection from $X \times S$. To choose an appropriate kernel $\mathcal{K}$, first note than a point $s \in S$ specifies a ruling of a quadric surface $i : Q \hookrightarrow X$ hence also a sheaf $i_*\mathcal{I}_{L/Q}$ which is independent of the choice $L$ of line in the ruling of $Q$. The sheaf $i_*\mathcal{I}_{L/Q}(1)$ is globally generated, so the canonical map

$$\mathcal{O}_X^2 \simeq H^0(X, i_*\mathcal{I}_{L/Q}(1)) \otimes \mathcal{O}_X \xrightarrow{\text{ev}} i_*\mathcal{I}_{L/Q}(1)$$

is surjective. Twisting and taking kernels, there are short exact sequences

$$0 \to K_{L/Q} \to \mathcal{O}_X(-1)^2 \xrightarrow{\text{ev}} i_*\mathcal{I}_{L/Q} \to 0$$

defining a collection of sheaves $K_{L/Q}$ on $X$. The Brauer class $\alpha$ obstructs the existence of a section $F_1(\mathrm{Bl}_P X/S)$ of the relative variety of lines of the quadric surface fibration, i.e. a choice of line in each ruling of each fiber of $q$. Similarly, the obstruction to a universal sheaf on $X \times S$ whose restriction to each slice $X \times \{s\}$ is the corresponding sheaf $\pi_1^* K_{L/Q}$ is the Brauer class $1 \boxtimes \alpha$, so such a universal sheaf $\mathcal{K}$ lies in $D^b(X \times S, 1 \boxtimes \alpha)$. Let $\Phi : D^b(S, \alpha) \to D^b(X)$ be the Fourier-Mukai transform with kernel $\mathcal{K}$.

**Theorem 3.2.1.** *The functor $\Phi$ takes values in $\mathcal{A}_X$ and is an equivalence onto its essential image.*

By the following lemma, we need to check that the image of $\Phi$ is $\mathcal{A}_X$ and that $\Phi$ is fully faithful.

**Lemma 3.2.2.** *Any fully faithful exact functor of triangulated categories $\Psi : \mathcal{C} \to \mathcal{A}_X$ admitting left and right adjoints is an equivalence.*

*Proof.* As $\Psi$ has left and right adjoints, its image is a component of a semi-orthogonal decomposition. The Serre functor on $\mathcal{A}_X$ is the shift $-[2]$, so the left and right orthogonal complements to the image of $\Psi$ coincide. As explained in Section 1.3, $\mathcal{A}_X$ is indecomposable, so by [18, Cor. 1.56], $\Psi$ must be essentially surjective. □

**Proposition 3.2.3.** *The image of $\Phi$ is contained in $\mathcal{A}_X$.*

The following lemmas leverage the proof of Proposition 3.2.3.

**Lemma 3.2.4.** *If $L$ is a line in a quadric surface $i : Q \hookrightarrow X$, then*

$$\dim \operatorname{Ext}^i_X(\mathcal{O}_X(j), i_*\mathcal{I}_{L/Q}) = \begin{cases} 0, & j = 0, 1 \\ 2, & j = -1, i = 0 \\ 0, & j = -1, i \neq 0. \end{cases}$$

*Proof.* Note $\operatorname{Ext}^i_X(\mathcal{O}_X(j), i_*\mathcal{I}_{L/Q}) \cong H^i(Q, \mathcal{I}_{L/Q}(-j))$. To calculate these cohomology groups, consider the exact sequence

$$0 \to \mathcal{I}_{L/Q}(-j) \to \mathcal{O}_Q(-j) \to \mathcal{O}_L(-j) \to 0$$

For $j = 0, 1$, the long exact sequence in cohomology forces the cohomology of $\mathcal{I}_{L/Q}(-j)$ to vanish. For $j = -1$, we have $h^0(Q, \mathcal{O}_Q(1)) = 4$ and $h^0(Q, \mathcal{O}_L(1)) = 2$, and the higher cohomology of these sheaves vanishes. Moreover, the map

$$H^0(Q, \mathcal{O}_Q(1)) \to H^0(L, \mathcal{O}_L(1))$$

is a surjection since any linear form on $L$ is the restriction of a linear form on $Q$ (in fact, the restriction of a linear form on $\mathbb{P}^3$). Hence

$$h^i(Q, \mathcal{I}_{L/Q}(1)) = \begin{cases} 2, & i = 0 \\ 0, & i \neq 0 \end{cases}$$

as needed. □

**Lemma 3.2.5.** *$K_{L/Q} \in \mathcal{A}_X$ for all $L \subset Q \subset X$.*

*Proof.* We must show $\mathrm{Ext}^i_X(\mathcal{O}_X(j), K_{L/Q}) = 0$ for $L \subset Q \subset X$, $j = -1, 0, 1$, and all $i$. From the short exact sequence

$$0 \to K_{L/Q} \to \mathcal{O}_X(-1)^2 \to i_*\mathcal{I}_{L/Q} \to 0,$$

we obtain a long exact sequence

$$\cdots \to \mathrm{Ext}^i_X(\mathcal{O}_X(j), \mathcal{O}_X(-1)^2) \to \mathrm{Ext}^i_X(\mathcal{O}_X(j), i_*\mathcal{I}_{L/Q}) \to \mathrm{Ext}^{i+1}(\mathcal{O}_X(j), K_{L/Q}) \\ \to \mathrm{Ext}^{i+1}_X(\mathcal{O}_X(j), \mathcal{O}_X(-1)^2) \to \dots$$

When $i > 0$, this yields isomorphisms

$$\mathrm{Ext}^{i+1}_X(\mathcal{O}_X(j), K_{L/Q}) \cong \mathrm{Ext}^i_X(\mathcal{O}_X(j), i_*\mathcal{I}_{L/Q}) = 0$$

for $j = -1, 0, 1$, using Lemma 3.2.4. Taking $i = 0$, we have

$$0 \to H^0(X, K_{L/Q}(-j)) \to H^0(X, \mathcal{O}_X(-j-1)^2) \to H^0(X, i_*\mathcal{I}_{L/Q}(-j)).$$

The middle term, hence also the first term, vanishes for $j = 0, 1$. For $j = -1$, we get

$$H^0(X, K_{L/Q}(1)) = \ker[H^0(X, \mathcal{O}_X^2) \xrightarrow{\text{ev}} H^0(X, i_*\mathcal{I}_{L/Q}(1))] = 0,$$

completing the calculations. □

*Proof of Proposition 3.2.3.* Let $\Phi^L$ be the left adjoint to $\Phi$, which exists by [18, Prop. 5.9]. By Lemma 3.2.5, $\Phi(\mathcal{O}_s) = K_{L/Q} \in \mathcal{A}_X$, so

$$\mathrm{Ext}^i(\Phi^L(\mathcal{O}_X(j)), \mathcal{O}_s) \cong \mathrm{Ext}^i(\mathcal{O}_X(j), K_{L/Q}) = 0$$

for $-1 \leq j \leq 1$ and all $i$. Hence $\Phi^L(\mathcal{O}_X(j)) = 0$ for $-1 \leq j \leq 1$. It follows that the image of $\Phi$ lies in $\mathcal{A}_X$. □

The remaining piece to the proof of Theorem 3.2.1 is the following proposition.

**Proposition 3.2.6.** *The Fourier-Mukai transform $\Phi : D^b(S, \alpha) \to \mathcal{A}_X$ is fully faithful.*

Bondal and Orlov provide the following criterion for $\Phi$ to be fully faithful, as outlined in [18, Ch. 7]:

$$\dim \mathrm{Ext}^i(\Phi(\mathcal{O}_s), \Phi(\mathcal{O}_t)) = \begin{cases} 1, & \text{if } s = t, i = 0 \\ 0, & \text{if } s \neq t \text{ or } i < 0 \text{ or } i > 2. \end{cases}$$

In other words, we must show

$$\dim \mathrm{Ext}^i(K_{L_1/Q_1}, K_{L_2/Q_2}) = \begin{cases} 1, & \text{if } \mathcal{I}_{L_1/Q_1} \simeq \mathcal{I}_{L_2/Q_2} \text{ and } i = 0 \\ 0, & \text{if } \mathcal{I}_{L_1/Q_1} \not\simeq \mathcal{I}_{L_2/Q_2} \text{ or } i < 0 \text{ or } i > 2. \end{cases}$$

We first prove a few lemmas.

**Lemma 3.2.7.** $\mathrm{Hom}_X(K_{L_1/Q_1}, K_{L_2/Q_2}) \cong \mathrm{Hom}_X(h_*\mathcal{I}_{L_1/Q_1}, i_*\mathcal{I}_{L_2/Q_2})$ *where $h$ and $i$ are the inclusions of $Q_1$ and $Q_2$, respectively.*

*Proof.* Applying $\mathrm{Hom}_X(-, K_{L_2/Q_2})$ to the sequence

$$0 \to K_{L_1/Q_1} \to \mathcal{O}_X(-1)^2 \to h_*\mathcal{I}_{L_1/Q_1} \to 0$$

yields

$$\mathrm{Hom}(K_{L_1/Q_1}, K_{L_2/Q_2}) \cong \mathrm{Ext}^1(h_*\mathcal{I}_{L_1/Q_1}, K_{L_2/Q_2}), \tag{3.1}$$

since $K_{L_2/Q_2} \in \mathcal{O}_X(-1)^{\perp}$. Applying $\mathrm{Hom}(h_*\mathcal{I}_{L_1/Q_1}, -)$ to the sequence

$$0 \to K_{L_2/Q_2} \to \mathcal{O}_X(-1)^2 \to j_*\mathcal{I}_{L_2/Q_2} \to 0$$

yields

$$\mathrm{Hom}(h_*\mathcal{I}_{L_1/Q_1}, j_*\mathcal{I}_{L_2/Q_2}) \cong \mathrm{Ext}^1(h_*\mathcal{I}_{L_1/Q_1}, K_{L_2/Q_2}) \tag{3.2}$$

since by Grothendieck-Verdier duality,

$$\mathrm{Ext}^i_X(h_*\mathcal{I}_{L_1/Q_1}, \mathcal{O}_X(-1)^2) \cong \mathrm{Ext}^{i-2}_{Q_1}(\mathcal{I}_{L_1/Q_1}, h^*\mathcal{O}_X(-1)^2 \otimes \omega_h) = 0$$

for $i = 0, 1$. Combining the equations (3.1) and (3.2) completes the proof. $\square$

**Lemma 3.2.8.** *Retaining notation from before,*

$$\dim \mathrm{Hom}_X(h_*\mathcal{I}_{L_1/Q_1}, i_*\mathcal{I}_{L_2/Q_2}) = \begin{cases} 1, & \text{if } \mathcal{I}_{L_1/Q_1} \simeq \mathcal{I}_{L_2/Q_2} \\ 0, & \text{otherwise.} \end{cases}$$

*Proof.* First, suppose $Q_1 \neq Q_2$. Then the image of a morphism $h_*\mathcal{I}_{L_1/Q_1} \to i_*\mathcal{I}_{L_2/Q_2}$ is supported on the 0-dimensional subscheme $Q_1 \cap Q_2$. As $\mathcal{I}_{L_2/Q_2}$ is torsion-free, the morphism is trivial.

Now suppose $Q_1 = Q_2$ which we now denote $i : Q \hookrightarrow X$. We have

$$\mathrm{Hom}_X(i_*\mathcal{I}_{L_1/Q}, i_*\mathcal{I}_{L_2/Q}) \cong \mathrm{Hom}_Q(\mathcal{I}_{L_1/Q}, \mathcal{I}_{L_2/Q}).$$

We claim that any nonzero morphism $\phi : \mathcal{I}_{L_1/Q} \to \mathcal{I}_{L_2/Q}$ is an isomorphism. Indeed, $\mathrm{im}\,\phi \subset \mathcal{I}_{L_2/Q_2}$ has rank 1 since $\mathcal{I}_{L_2/Q_2}$ is torsion-free. Hence $\mathrm{rank}(\ker\phi) = 0$, so $\ker\phi$ is torsion and therefore trivial. The additivity of the Hilbert polynomial applied to the sequence

$$0 \to \mathcal{I}_{L_1/Q_1} \xrightarrow{\phi} \mathcal{I}_{L_2/Q_2} \to \operatorname{coker}\phi \to 0$$

shows $\dim \operatorname{supp}(\operatorname{coker}\phi) = \deg(p_{\operatorname{coker}\phi}) = 0$, so $\phi$ is an isomorphism.

Hence $\operatorname{End}_Q(\mathcal{I}_{L_1/Q_1})$ is a division algebra, and the same is true after passing to the algebraic closure, so $\dim \operatorname{End}_Q(\mathcal{I}_{L_1/Q_1}) = 1$. □

**Lemma 3.2.9.** $\chi(K_{L_1/Q_1}, K_{L_2/Q_2}) = 0$.

*Proof.* As the Euler characteristic is deformation invariant, we may assume $Q_1 = Q_2$ and $L_1 = L_2$, and simply write $Q$ and $L$. We may also assume $Q$ is smooth. From the defining sequence for $K_{L/Q}$ and the fact that $K_{L/Q} \in \mathcal{A}_X$, we have

$$\chi_X(K_{L/Q}, K_{L/Q}) + \chi_X(i_*\mathcal{I}_{L/Q}, K_{L/Q}) = \chi_X(\mathcal{O}_X(-1)^2, K_{L/Q}) = 0$$

and

$$\chi_X(i_*\mathcal{I}_{L/Q}, K_{L/Q}) + \chi_X(i_*\mathcal{I}_{L/Q}, i_*\mathcal{I}_{L/Q}) = \chi_X(i_*\mathcal{I}_{L/Q}, \mathcal{O}_X(-1)^2),$$

so

$$\chi_X(K_{L/Q}, K_{L/Q}) = \chi_X(i_*\mathcal{I}_{L/Q}, i_*\mathcal{I}_{L/Q}) - 2\chi_X(i_*\mathcal{I}_{L/Q}, \mathcal{O}_X(-1)).$$

From Grothendieck-Verdier duality and the Künneth formula,

$$\chi_X(i_*\mathcal{I}_{L/Q}, \mathcal{O}_X(-1)) = \chi_Q(\mathcal{I}_{L/Q}, \mathcal{O}_Q) = \chi_Q(Q, \mathcal{O}_Q(1,0)) = 2.$$

It remains to show that $\chi_X(\mathcal{I}_{L/Q}, \mathcal{I}_{L/Q}) = 4$.

Since $i_*\mathcal{I}_{L/Q}$ is supported on $Q$, a codimension-2 subvariety, the classes $c_0(i_*\mathcal{I}_{L/Q})$ and $c_1(i_*\mathcal{I}_{L/Q})$ are both trivial. The Grothendieck-Riemann-Roch Theorem gives

$$\operatorname{ch}(i_*\mathcal{I}_{L/Q}) = i_* \operatorname{ch}\mathcal{I}_{L/Q} \cdot \operatorname{td}(N_{Q/X}),$$

which gives $c_2(i_*\mathcal{I}_{L/Q}) = i_*[1] = [Q]$. Similarly, $c_2(i_*\mathcal{I}^\vee_{L/Q}) = [Q]$. Using the Hirzebruch-Riemann-Roch formula,

$$\begin{aligned}\chi_X(i_*\mathcal{I}_{L/Q}, i_*\mathcal{I}_{L/Q}) &= \chi_X(X, i_*\mathcal{I}^\vee_{L/Q} \otimes i_*\mathcal{I}_{L/Q})\\ &= \int_X \operatorname{ch}(i_*\mathcal{I}^\vee_{L/Q}) \operatorname{ch}(i_*\mathcal{I}_{L/Q}) \operatorname{td}(X)\\ &= c_2(i_*\mathcal{I}^\vee_{L/Q}) \cdot c_2(i_*\mathcal{I}_{L/Q}) \cdot 1\\ &= [Q]^2 = 4,\end{aligned}$$

as needed. □

*Proof of Proposition 3.2.6.* As mentioned earlier, Bondal and Orlov's criterion reduces the argument to verifying

$$\dim \operatorname{Ext}^i_X(K_{L_1/Q_1}, K_{L_2/Q_2}) = \begin{cases} 1, & \text{if } h_*\mathcal{I}_{L_1/Q_1} \simeq i_*\mathcal{I}_{L_2/Q_2} \text{ and } i = 0 \\ 0, & \text{if } h_*\mathcal{I}_{L_1/Q_1} \not\simeq i_*\mathcal{I}_{L_2/Q_2} \text{ or } i < 0 \text{ or } i > 2. \end{cases}$$

Since the objects $K_{L_i/Q_i}$ are sheaves, this is true for $i < 0$, and Lemmas 3.2.7 and 3.2.8 verify the criterion for $i = 0$. Since the Serre functor on $\mathcal{A}_X$ is a shift by 2, the criterion is also satisfied for $i \geq 2$. The case $i = 1$ follows from Lemma 3.2.9. □

## 3.3 The Fano variety as a moduli space of sheaves

The equivalence $\Phi : D^b(S, \alpha)$ described above allows us to connect the explicit construction of a rational Lagrangian fibration $\pi : F \dashrightarrow \mathbb{P}^2$ from Section 3.1 to the literature. For example, at least over $\mathbb{C}$, Macrì and Stellari's proved in [27] that $F$ is birational to a moduli space of rank-zero $\alpha$-twisted sheaves on $S$ supported on curves in the linear system $|f^*\mathcal{O}_{\mathbb{P}^2}(1)|$ where $f : S \to \mathbb{P}^2$ is the double cover. Another description of the rational Lagrangian fibration, then, is to send $L$ to the support of the corresponding twisted sheaf on $S$. Here, we reprove

Macrì and Stellari's result over a field of characteristic zero, showing that the two descriptions of the rational Lagrangian fibration on $F$ agree.

In [26], Kuznetsov and Markushevich proved that the Fano variety $F$ of lines on $X$ is a moduli space of rank-three reflexive sheaves $\mathcal{F}_L$ on $X$, each of which is the left mutation of $\mathcal{I}_{L/X}$ through $\mathcal{O}_X(-1)$,

$$\mathcal{F}_L = \ker(\mathcal{O}_X(-1)^4 \stackrel{\text{ev}}{\twoheadrightarrow} \mathcal{I}_{L/X}).$$

Using the sequences

$$0 \to \mathcal{F}_L \to \mathcal{O}_X(-1)^4 \to \mathcal{I}_{L/X} \to 0$$

and

$$0 \to \mathcal{I}_{L/X} \to \mathcal{O}_X \to \mathcal{O}_L \to 0,$$

it is straightforward to verify that $\mathcal{F}_L \in \mathcal{A}_X$. Now, twisting the evaluation maps $\mathcal{O}_X^6 \to \mathcal{F}_L(2)$, taking cones, and shifting, we define a collection of complexes

$$\mathcal{G}_L = \text{cone}(\mathcal{O}_X(-1)^6 \stackrel{\text{ev}}{\to} \mathcal{F}_L(1))[-1]$$

which are (shifts of) the mutations of $\mathcal{F}_L(1)$ past $\mathcal{O}_X(-1)$. The functor $\mathcal{A}_X \to \mathcal{A}_X$ obtained by sending $C$ to the mutation of $C(1)$ past $\mathcal{O}_X(-1)$ is an autoequivalence (see [23, Sect. 4]), so we may also regard $F$ as a moduli space of complexes $\mathcal{G}_L \in \mathcal{A}_X$.

The functor $\Phi$ admits a left adjoint $\Phi^L \colon D^b(X) \to D^b(S, \alpha)$, as explained in [18, Prop. 5.9]. This functor restricts to an equivalence $\mathcal{A}_X \to D^b(S, \alpha)$, allowing us to regard $F$ as a moduli space of $\alpha$-twisted complexes $\Phi^L(\mathcal{G}_L)$ on $S$. We first calculate the support of $\Phi^L(\mathcal{G}_L)$.

**Lemma 3.3.1.** *If $i : L \hookrightarrow X$ is the inclusion of a line,* $\Phi^L(\mathcal{G}_L) \simeq \Phi^L(i_*\mathcal{O}_L(1))[-3]$.

*Proof.* In Proposition 3.2.3, we proved $\Phi^L(\mathcal{O}_X(j)) = 0$ for $j = -1, 0, 1$. The lemma therefore follows from applying $\Phi^L$ to the exact triangles

$$\mathcal{O}_X(-1)^6 \to \mathcal{F}_L(1) \to \mathcal{G}_L[1],$$

$$\mathcal{O}_X^4 \to \mathcal{I}_{L/X}(1) \to \mathcal{F}_L(1)[1],$$

and

$$\mathcal{O}_X(1) \to i_*\mathcal{O}_L(1) \to \mathcal{I}_{L/X}(1)[1].$$

□

**Lemma 3.3.2.** *Let $i\colon L \hookrightarrow X$ be the inclusion of a line, let $s \in S$, and let $M \subset Q \subset X$ be a line in the corresponding ruling of a quadric surface. Then $s \in \operatorname{supp}(\Phi^L(\mathcal{G}_L))$ if and only if $L \cap Q \neq \emptyset$.*

*Proof.* We will show that $\operatorname{Ext}^i_S(\Phi^L(\mathcal{G}_L), \mathcal{O}_s) = 0$ for all $i$ if and only if $L \cap Q = \emptyset$. By Lemma 3.3.1,

$$\begin{aligned} R\operatorname{Hom}_S(\Phi^L(\mathcal{G}_L), \mathcal{O}_s) &\cong R\operatorname{Hom}_S(\Phi^L(i_*\mathcal{O}_L(1)[-3], \mathcal{O}_s) \\ &\cong R\operatorname{Hom}_X(i_*\mathcal{O}_L(1), \Phi(\mathcal{O}_s)[3]), \end{aligned}$$

so

$$\operatorname{Ext}^i_S(\Phi^L(\mathcal{G}_L), \mathcal{O}_s) \cong \operatorname{Ext}^{i+3}_X(i_*\mathcal{O}_L, K_{M/Q}(-1)).$$

By Grothendieck-Verdier duality,

$$\operatorname{Ext}^i_X(i_*\mathcal{O}_L, \mathcal{O}_X(-2)) \cong \operatorname{Ext}^{i-3}_L(\mathcal{O}_L, \mathcal{O}_L(-1)) \cong H^{i-3}(L, \mathcal{O}_L(-1)) = 0$$

for all $i$. Hence the exact sequence

$$0 \to K_{M/Q}(-1) \to \mathcal{O}_X(-2)^2 \to \mathcal{I}_{M/Q}(-1) \to 0$$

begets isomorphisms

$$\operatorname{Ext}^i_S(\Phi^L(\mathcal{G}_L), \mathcal{O}_s) \cong \operatorname{Ext}^{i+2}_X(i_*\mathcal{O}_L, \mathcal{I}_{M/Q}(-1)).$$

The following table contains values for $\dim \mathrm{Ext}^i(i_*\mathcal{O}_L, \mathcal{I}_{M/Q}(-1))$ for various configurations of lines $L, M \subset X$.

| | $i=0$ | $i=1$ | $i=2$ | $i=3$ | $i=4$ |
|---|---|---|---|---|---|
| $L \cap Q = \emptyset$ | 0 | 0 | 0 | 0 | 0 |
| $L \cap Q = \{x\}$, and $x \notin \mathrm{Sing}(Q)$ | 0 | 0 | 1 | 1 | 0 |
| $L \subset Q$ | 0 | 0 | 1 | 1 | 0 |

The calculations in this table are carried out in the appendix to this document. It remains to show that if $L \cap Q = \{x\} = \mathrm{Sing}(Q)$, then $s \in \mathrm{supp}(\Phi^L(\mathcal{G}_L))$. Note that in this case, $L \subset X \setminus P$, and there is a genus 2 curve $C \subset S$ parametrizing quadric surfaces in $X$ that $L$ meets. We have shown that all but finitely many points of $C$ belong to $\mathrm{supp}(\Phi^L(\mathcal{G}_L))$, and the support of a complex is closed, so $s \in C \subset \mathrm{supp}(\Phi^L(\mathcal{G}_L))$, as needed. □

Let $f : S \to \mathbb{P}^2$ be the double cover. For a line $L \subset X \setminus P$, the genus-2 curve $f^{-1}(q(L))$ parametrizes rulings on quadrics that $L$ intersects. If instead $L \cap P = \{p\}$, then $L$ is contained in a quadric $Q \subset X$, and the other quadrics in $X$ meeting $L$ are those contained in $T_pX$; there is a pencil of such quadrics, and their rulings are parametrized by $f^{-1}(T_pX \cap P^{\perp})$. On the other hand, if $L \subset P$, then $L$ meets every quadric surface contained in $X$. We summarize this in the following corollary.

**Corollary 3.3.3.** *If $L \not\subset P$, then $\Phi^L(\mathcal{G}_L)$ is supported on a curve in the linear system $|f^*\mathcal{O}_{\mathbb{P}^2}(1)|$. Moreover, the rational map* $\mathrm{supp}\colon F \dashrightarrow |f^*\mathcal{O}_{\mathbb{P}^2}(1)|$ *sending $\Phi^L(\mathcal{G}_L)$ to its support is surjective.*

It is easy to see that the rational map supp agrees with the rational Lagrangian fibration $\pi$ studied in Section 3.1: indeed, if $L \subset X \setminus P$, then $\Phi^L(\mathcal{G}_L)$ is supported on $f^{-1}(q(L))$, and $\pi([L]) = [q(L)]$.

To compare our work with Macrì and Stallari's result from [27] that $F$ is birational to a moduli space of rank-zero $\alpha$-twisted sheaves on $S$, we prove that our complexes $\mathcal{G}_L$ are also rank-zero $\alpha$-twisted sheaves.

**Proposition 3.3.4.** *If $L \not\subset P$, and $C = \operatorname{supp} \Phi^L(\mathcal{G}_L)$ is an integral curve, then $\Phi^L(\mathcal{G}_L)$ is a twisted sheaf.*

*Proof.* Let $\mathcal{H}^i$ be the $i$th cohomology sheaf of $\Phi^L(\mathcal{G}_L)$, and let $m$ be the minimal integer for which $\mathcal{H}^m \neq 0$. Let $s \in \operatorname{supp} \mathcal{H}^m \subset C$. We will use the spectral sequence

$$E_2^{p,q} = \operatorname{Ext}^p(\mathcal{H}^{-q}, \mathcal{O}_s) \Rightarrow \operatorname{Ext}^{p+q}(\Phi^L(\mathcal{G}_L), \mathcal{O}_s).$$

Note that $E^{0,-m} \neq 0$, and $E^{p,q} = 0$ for $p < 0$ or $q < -m$, so

$$0 \neq E^{0,-m} \cong \operatorname{Ext}^{-m}(\Phi^L(\mathcal{G}_L), \mathcal{O}_s),$$

and $\operatorname{Ext}^i(\Phi^L(\mathcal{G}_L), \mathcal{O}_s) = 0$ for $i < -m$. From the proof of Lemma 3.3.2,

$$\dim \operatorname{Ext}^i(\Phi^L(\mathcal{G}_L), \mathcal{O}_s) = \begin{cases} 1 & \text{if } i = 0, 1 \\ 0 & \text{otherwise,} \end{cases}$$

so $m = 0$, and $\dim E^{0,0} = 1$. Since

$$\dim \operatorname{Hom}(\mathcal{H}^0, \mathcal{O}_s) = \dim E^{0,0} = 1,$$

for each $s \in C$, the sheaf $\mathcal{H}^0$ is a line bundle on $C$. Then for each $s \in C$, there is some neighborhood $U \ni s$ on which $\mathcal{H}^0|_U$ admits a resolution of the form

$$0 \to \mathcal{O}_{S \cap U}(-(C \cap U)) \to \mathcal{O}_{S \cap U} \to \mathcal{H}^0|_U \to 0,$$

from which we deduce

$$\dim E^{p,0} = \dim \operatorname{Ext}^p_S(\mathcal{H}^0, \mathcal{O}_s) = \dim \operatorname{Ext}^p_U(\mathcal{H}^0|_U, \mathcal{O}_s) = \begin{cases} 1 & p = 1 \\ 0 & p > 1. \end{cases}$$

Hence the differentials $E^{0,1} \to E^{2,0}$ and $E^{1,0} \to E^{3,-1}$ are both trivial, and

$$1 = \dim \operatorname{Ext}^1(\Phi^L(\mathcal{G}_L), \mathcal{O}_s) = \dim E^{0,1} + \dim E^{1,0} = \dim E^{0,1} + 1,$$

so $\operatorname{Hom}(\mathcal{H}^{-1}, \mathcal{O}_s) = E^{0,1} = 0$. In particular, $s \notin \operatorname{supp} \mathcal{H}^{-1}$, and as this is true for all $s \in C$, we conclude $\mathcal{H}^{-1} = 0$.

Finally, suppose there is some $i > 0$ for which $\mathcal{H}^{-i} \neq 0$, and let $i$ be minimal for this property. We have shown $i > 1$. Let $s \in \operatorname{supp} \mathcal{H}^{-i}$. The differential from $E^{0,i}$ is trivial, so $\operatorname{Ext}^i(\Phi^L(\mathcal{G}_L), \mathcal{O}_s) \neq 0$ which is impossible for $i > 1$. Hence the only nontrivial cohomology sheaf of $\Phi^L(\mathcal{G}_L)$ is $\mathcal{H}^0$, as needed. □

To finish, it is worth mentioning how the perspective on the rational Lagrangian fibration $F \dashrightarrow \mathbb{P}^2$ offered in this section could fuel future work. Recall from Proposition 1.4.2 that if $X$ is a cubic fourfold whose discriminant is twice a square, then its Fano variety of lines $F$ admits a rational Lagrangian fibration. Example 1.4.4 explored the case when $X$ is of discriminant 18: here, $F$ admits two rational Lagrangian fibrations, but it is not clear how to describe either one geometrically. There is a K3 surface $K$ with a Brauer class $\beta \in \operatorname{Br}(K)[3]$ arising naturally from the geometry of $X$, explained in [1], and it is reasonable to expect from [25] that there is an equivalence $\Psi : \mathcal{A}_X \to D^b(K, \beta)$. After constructing the functor $\Psi$ explicitly, one can regard $F$ as a moduli space of $\beta$-twisted complexes on $K$. As in discriminant 8, is it the case that $F$ is birational to a moduli space of rank-zero, $\beta$-twisted sheaves on $K$? If so, does describing the support of these sheaves make a rational Lagrangian fibration from $F$ more geometrically legible?

# APPENDIX

## SUPPORTING CALCULATIONS

Again, $X$ is a general cubic fourfold containing a plane $P$, and $S$ is the K3 surface parametrizing rulings in the fibers of $\mathrm{Bl}_P X \to \mathbb{P}^2$.

Let $j : Q \hookrightarrow X$ and $i : L \hookrightarrow X$ be the inclusions of a quadric surface and a line in $X$, respectively, and let $M \subset Q$ be a line. Let $s \in S$ be point corresponding to the ruling of $Q$ containing $M$. This appendix contains calculations of the groups

$$\mathrm{Ext}^i(i_*\mathcal{O}_L, j_*\mathcal{I}_{M/Q}(-1))$$

for various configurations of $L$ and $M$, used in Section 3.3.

**Lemma A.0.1.** *If $L \cap Q = \emptyset$, then $\mathrm{Ext}^i(i_*\mathcal{O}_L, j_*\mathcal{I}_{M/Q}(-1)) = 0$ for all $i$.*

*Proof.* As

$$\omega_i = i^*\omega_X^* \otimes \omega_L \simeq \omega_{\mathbb{P}^5}^*|_L \otimes N_{X/\mathbb{P}^5}^*|_L \otimes \mathcal{O}_L(-2) \cong \mathcal{O}_L(1),$$

Grothendieck-Verdier duality yields

$$\begin{aligned} R\mathcal{H}om_X(i_*\mathcal{O}_L, j_*\mathcal{I}_{M/Q}(-1)) &\simeq i_*R\mathcal{H}om_L(\mathcal{O}_L, i^*j_*\mathcal{I}_{M/Q}(-1) \otimes \omega_i[-3]) \\ &\simeq i_*(i^*j_*\mathcal{I}_{M/Q}(-1) \otimes \mathcal{O}_L(1))[-3] \end{aligned}$$

Since $L \cap Q = \emptyset$, the composition $i^*j_*$ is trivial. Applying $R\Gamma$, the result follows. □

**Lemma A.0.2.** *If $L \cap Q = \{p\}$, and $Q$ is smooth at $p$, then $\mathrm{Ext}^i(i_*\mathcal{O}_L, j_*\mathcal{I}_{M/Q}(-1))$ is 1-dimensional for $i = 2, 3$ and is trivial otherwise.*

*Proof.* As $i^*j_*\mathcal{F}$ is supported at $p$ for any sheaf $\mathcal{F}$ on $Q$, the complex

$$R\mathcal{H}om_X(i_*\mathcal{O}_L, j_*\mathcal{I}_{M/Q}(-1))$$

is supported at $p$. Hence we can calculate the cohomology sheaves of the above complex by first restricting to an open neighborhood $U \subset X$ of $p$ for which $Q \cap U$ is smooth, $\mathcal{I}_{M/Q}(-1)|_{Q\cap U} \simeq \mathcal{O}_{Q\cap U}$, and $\omega_i|_{L\cap U} \simeq \mathcal{O}_{L\cap U}$. Then

$$R\,\mathcal{H}om_X(i_*\mathcal{O}_L, j_*\mathcal{I}_{M/Q}(-1)) \simeq R\,\mathcal{H}om_{L\cap U}(\mathcal{O}_{L\cap U}, i^*j_*\mathcal{O}_{Q\cap U}[-3]),$$

and

$$\mathrm{Ext}^i_X(i_*\mathcal{O}_L, j_*\mathcal{I}_{M/Q}(-1)) \cong H^{i-3}(L\cap U, i^*j_*\mathcal{O}_{Q\cap U}).$$

Since $Q$ is locally a complete intersection in $X$, we can further restrict the open set $U$ to assume $Q$ is cut out of $X$ by two regular functions $f$ and $g$ of degrees $d$ and $e$, respectively. Moreover, we may assume $\mathcal{O}_U(d)$ and $\mathcal{O}_U(e)$ are trivial. Then there is a Koszul complex

$$0 \to \mathcal{O}_U \xrightarrow{(g,-f)} \mathcal{O}_U^2 \xrightarrow{[f,g]} \mathcal{O}_U \longrightarrow j_*\mathcal{O}_{Q\cap U} \to 0$$

which, after restriction to $L\cap U$, yields

$$0 \to \mathcal{O}_{L\cap U} \xrightarrow{(g,-f)} \mathcal{O}_{L\cap U}^2 \xrightarrow{[f,g]} \mathcal{O}_{L\cap U} \longrightarrow i^*j_*\mathcal{O}_{Q\cap U} \to 0.$$

Since $f$ and $g$ both vanish at $p$, this resolution of $i^*j_*\mathcal{O}_{Q\cap U}$ is quasi-isomorphic to the sequence

$$0 \longrightarrow \mathcal{O}_p \xrightarrow{0} \mathcal{O}_p \longrightarrow 0,$$

so

$$\mathcal{H}^i := \mathcal{H}^i(i^*j_*\mathcal{O}_{Q\cap U}) = \begin{cases} \mathcal{O}_p & i=-1,0 \\ 0 & \text{otherwise.} \end{cases}$$

From the spectral sequence

$$E_2^{p,q} = H^p(L, \mathcal{H}^q) \Rightarrow H^{p+q}(L, i^*j_*\mathcal{I}_{M/Q})$$

we obtain

$$h^i(L\cap U, i^*j_*\mathcal{O}_{Q\cap U}) = \begin{cases} 1 & i=-1,0 \\ 0 & \text{otherwise,} \end{cases}$$

completing the argument. □

In the the remaining cases, $L \subset Q$. We will first need the following lemma.

**Lemma A.0.3.** *If $L \subset Q$, but $L \not\subset P$, then $N_{Q/X}|_L \simeq \mathcal{O}_L \oplus \mathcal{O}_L(1)$.*

*Proof.* From the sequences

$$0 \to N_{L/X} \to N_{L/\mathbb{P}^5} \to N_{X/\mathbb{P}^5}|_L \to 0$$

and

$$0 \to N_{L/Q} \to N_{L/X} \to N_{Q/X}|_L \to 0,$$

we have $\deg(N_{Q/X}|_L) = 1$, so $N_{Q/X}|_L \simeq \mathcal{O}_L(a) \oplus \mathcal{O}_L(b)$ with $a + b = 1$. Applying $\mathcal{H}om_X(-, \mathcal{O}_L)$ to the sequence

$$0 \to \mathcal{I}_{Q \cup P/X} \to \mathcal{I}_{Q/X} \to \mathcal{I}_{Q/Q \cup P} \to 0$$

yields an injection

$$0 \to N_{Q/X}|_L \to \mathcal{O}_L(1)^2$$

since $\mathcal{I}_{Q/Q \cup P} \simeq \mathcal{I}_{Q \cap P/P} \simeq \mathcal{O}_P(-2)$, and $\mathcal{H}om_X(\mathcal{O}_P(-2), \mathcal{O}_L) = 0$. Thus $a, b \leq 1$, and the result follows. □

**Lemma A.0.4.** *If $L \subset Q$ and $L$ and $M$ lie in the same ruling of $Q$, then $\operatorname{Ext}^i(i_*\mathcal{O}_L, j_*\mathcal{I}_{M/Q}(-1))$ is 1-dimensional for $i = 2, 3$ and is trivial otherwise.*

*Proof.* Let $h : L \hookrightarrow Q$ be the inclusion. We proceed as before.

$$\begin{aligned} R\,\mathcal{H}om_X((jh)_*\mathcal{O}_L, j_*\mathcal{I}_{M/Q}(-1)) &\simeq (jh)_* R\,\mathcal{H}om_L(\mathcal{O}_L, h^*j^*j_*\mathcal{I}_{M/Q}(-1) \otimes \omega_i[-3]) \\ &\simeq (jh)_*(h^*j^*j_*\mathcal{I}_{M/Q})[-3] \end{aligned}$$

Taking global sections,

$$\operatorname{Ext}^i_X((jh)_*\mathcal{O}_L, j_*\mathcal{I}_{M/Q}(-1)) \simeq H^{i-3}(L, h^*j^*j_*\mathcal{I}_{M/Q}).$$

By [2, Proposition 11.8],

$$\mathcal{H}^i := \mathcal{H}^i(j^* j_* \mathcal{I}_{M/Q}) \simeq \mathcal{I}_{M/Q} \otimes \bigwedge^{-i} N^*_{Q/X}.$$

Since $L$ and $M$ lie in the same ruling of $Q$, $\mathcal{I}_{M/Q}|_L = \mathcal{O}_L$. By Lemma A.0.3 $N_{Q/X}|_L \simeq \mathcal{O}_L \oplus \mathcal{O}_L(1)$, so

$$\mathcal{H}^i \simeq \begin{cases} \mathcal{O}_L & i = 0 \\ \mathcal{O}_L \oplus \mathcal{O}_L(-1) & i = -1 \\ \mathcal{O}_L(-1) & i = -2 \\ 0 & \text{otherwise,} \end{cases}$$

Then

$$h^p(L, \mathcal{H}^q) = \begin{cases} 1 & p = 0, q \in \{-1, 0\} \\ 0 & \text{otherwise.} \end{cases}$$

and the spectral sequence

$$E_2^{p,q} = H^p(L, \mathcal{H}^q) \Rightarrow H^{p+q}(L, h^* j^* j_* \mathcal{I}_{M/Q})$$

gives

$$h^i(L, h^* j^* j_* \mathcal{I}_{M/Q}) = \begin{cases} 1 & i = -1, 0 \\ 0 & \text{otherwise,} \end{cases}$$

as needed. □

**Lemma A.0.5.** *If $L \subset Q$ and $L$ and $M$ lie in different rulings of $Q$, then $\operatorname{Ext}^i(i_* \mathcal{O}_L, j_* \mathcal{I}_{M/Q}(-1))$ is 1-dimensional for $i = 2, 3$ and is trivial otherwise.*

*Proof.* As before,

$$\operatorname{Ext}^i_X((jh)_* \mathcal{O}_L, j_* \mathcal{I}_{M/Q}(-1)) \cong H^{i-3}(L, h^* j^* j_* \mathcal{I}_{M/Q}),$$

and we let $\mathcal{H}^i = \mathcal{H}^i(h^* j^* j_* \mathcal{I}_{M/Q})$. Now, $\mathcal{I}_{M/Q}|_L \simeq \mathcal{O}_L(-1)$, so we instead have

$$\mathcal{H}^i \simeq \begin{cases} \mathcal{O}_L(-1) & i = 0 \\ \mathcal{O}_L(-1) \oplus \mathcal{O}_L(-2) & i = -1 \\ \mathcal{O}_L(-2) & i = -2 \\ 0 & \text{otherwise.} \end{cases}$$

Hence

$$h^p(L, \mathcal{H}^q) = \begin{cases} 1 & p = 1, q \in \{-2, -1\} \\ 0 & \text{otherwise.} \end{cases}$$

The spectral sequence from before yields the desired result. □

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