

CATEGORICAL INVARIANTS OF GRAPHS AND MATROIDS

by

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DISSERTATION ABSTRACT

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Title: Categorical Invariants of Graphs and Matroids

Graphs and matroids are two of the most important objects in combinatorics. We study invariants of graphs and matroids that behave well with respect to certain morphisms by realizing these invariants as functors from a category of graphs (resp. matroids).

For graphs, we study invariants that respect deletions and contractions of edges. For an integer $g > 0$, we define a category of $\mathcal{G}_{\leq g}^{\text{op}}$ of graphs of genus at most g where morphisms correspond to deletions and contractions. We prove that this category is locally Noetherian and show that many graph invariants form finitely generated modules over the category $\mathcal{G}_{\leq g}^{\text{op}}$. This fact allows us to exhibit many stabilization properties of these invariants. In particular we show that the torsion that can occur in the homologies of the unordered configuration space of n points in a graph and the matching complex of a graph are uniform over the entire family of graphs with genus g .

For matroids, we study valuative invariants of matroids. Given a matroid, one can define a corresponding polytope called the base polytope. Often, the base polytope of a matroid can be decomposed into a cell complex made up of base

polytopes of other matroids. A valutive invariant of matroids is an invariant that respects these polytope decompositions. We define a category $\mathcal{M}_{\text{id}}^{\wedge}$ of matroids whose morphisms correspond to containment of base polytopes. We then define the notion of a categorical matroid invariant which categorifies the notion of a valutive invariant. Finally, we prove that the functor sending a matroid to its Orlik-Solomon algebra is a categorical valutive invariant. This allows us to derive relations among the Orlik-Solomon algebras of a matroid and matroids that decompose its base polytope viewed as representations of any group Γ whose action is compatible with the polytope decomposition.

This dissertation includes previously unpublished co-authored material.

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CHAPTER I

INTRODUCTION

This dissertation includes previously unpublished co-authored material. Chapter II was written in collaboration with Eric Ramos. Chapter III was written in collaboration with Nicholas Proudfoot and Lorenzo Vecchi.

In this dissertation, we study invariants of graphs and matroids, two of the most important objects in combinatorics. We do this categorically by looking at graph or matroid invariants that are functors from a specific category of graphs or matroids. In both the graph and the matroid case, working categorically provides powerful machinery for studying these invariants.

In the graph case, our categorical setup will give us stabilization properties of various invariants that are uniform for all graphs of genus at most g . In the matroid case, our categorical setup will give us relations among group representations built out of matroids and their symmetries.

1.1. Graphs

Throughout the introduction, a **graph** will refer to an at most one-dimensional CW complex that is both connected and finite. We say that a graph G is a **minor** of a graph G' if G can be obtained from G' by a sequence of edge deletions and contractions. In Chapter II, we study a category $\mathcal{G}_{\leq g}$ where we have a morphism of graphs $G \rightarrow G'$ whenever G is isomorphic to a minor of G' . A precise definition of this category is given in 2.2.

Let \mathcal{C} be a category and k a Noetherian ring. A \mathcal{C} -module over k is a functor from \mathcal{C} to the category of k -modules. Let $\text{Rep}_k(\mathcal{C})$ be the category of \mathcal{C} -modules.

A \mathcal{C} -module \mathcal{M} is **finitely generated** if there is a finitely list of objects $\{X_i\}_i$ in \mathcal{C} , such that $\mathcal{M}(X_i)$ is a finitely generated k -module for each i and for any object X of \mathcal{C} , $\mathcal{M}(X)$ is generated as a k -module by images of the $\mathcal{M}(X_i)$ under maps coming from morphisms in \mathcal{C} . The category $\text{Rep}_k(\mathcal{C})$ is called **locally Noetherian** if every submodule of a finitely generated \mathcal{C} -module is itself finitely generated.

The theory of quasi-Gröbner categories in [SS17] provides a powerful tool for proving the category of modules over \mathcal{C} is locally Noetherian. In particular they define what it means for \mathcal{C} to be **quasi-Gröbner** and prove that the if \mathcal{C} is quasi-Gröbner, then $\text{Rep}_k(\mathcal{C})$ is locally Noetherian. This theory along with the results of [PR19b] allow us to show the following.

Theorem 1.1.1. *The category $\mathcal{G}_{\leq g}^{\text{op}}$ is quasi-Gröbner. Thus, the category of $\mathcal{G}_{\leq g}^{\text{op}}$ -modules is locally Noetherian which means every submodule of a finitely generated $\mathcal{G}_{\leq g}^{\text{op}}$ is itself finitely generated.*

Remark 1.1.2. In their seminal series of papers, Robertson and Seymour proved, among many other things, that the minor relation is actually a well-quasi-order [RS04, RS10]. That is to say, in *any* infinite collection of graphs, there must be a pair where one is a minor of the other. We will refer to this as the Graph Minor Theorem. Theorem 1.1.1 can be seen as a weakened categorical version of this theorem.

Modules over algebras over categories

In section 2.1 we generalize the Gröbner theory of categories in [SS17]. Let \mathcal{C} be a category and S a functor from \mathcal{C} to the category FI of finite sets with inclusions. There is a natural functor taking an object X in \mathcal{C} to the polynomial

ring

$$\mathcal{A}_S = k[t_e \mid e \in S(X)].$$

We define an \mathcal{A}_S -module to be a \mathcal{C} -module \mathcal{M} where for each object X of \mathcal{C} , $\mathcal{M}(X)$ is a module over the ring $\mathcal{A}_S(X)$ in a natural way. See 2.1 for a precise definition. Let $\text{Rep}_k(\mathcal{C}, S)$ be the category of \mathcal{A}_S -modules. We define what it means for a \mathcal{A}_S -module to be finitely generated and we say the category $\text{Rep}_k(\mathcal{C}, S)$ is **locally Noetherian** if every submodule of a finitely generated \mathcal{A}_S -module is itself finitely generated. We then define what it means for the (\mathcal{C}, S) to be **quasi-Gröbner** and prove that if the pair (\mathcal{C}, S) is quasi-Gröbner, then the category $\text{Rep}_k(\mathcal{C}, S)$ is locally Noetherian.

In the context of graphs, we have a functor $E : \mathcal{G}_{\leq g}^{\text{op}} \rightarrow \text{FI}$ taking a graph G to its set of edges. We prove the following.

Theorem 1.1.3. *The pair $(\mathcal{G}_{\leq g}^{\text{op}}, E)$ is quasi-Gröbner. Thus the category $\text{Rep}_k(\mathcal{C}, S)$ is locally Noetherian for any Noetherian ring k , which means every submodule of a finitely generated $\mathcal{G}_{\leq g}^{\text{op}}$ -module is itself finitely generated.*

One should think of Theorems 1.1.1 and 1.1.3 as a translation from Robertson and Seymour's Graph Minor Theorem in combinatorics, to a algebraic statement about the representations of the category $\mathcal{G}_{\leq g}^{\text{op}}$. We then leverage these theorems to prove a number of nontrivial stability results in three main applications, configuration spaces of graphs, simplicial complexes coming from graphs, and subspace arrangements coming from graphs.

Configuration spaces of graphs

For an integer $n \geq 0$ and a graph G viewed as a 1-dimensional CW-complex, the **unordered configuration space of n points** in G is the topological space

$$U_n(G) := \{(x_1, \dots, x_n) \in G^n \mid x_i \neq x_j\} / \mathfrak{S}_n.$$

The homology groups of these spaces have been extensively studied in [Abr00], [ADCK19], [KP12].

In [ADCK19], though they do not use this language, An, Drummond-Cole, and Knudsen prove that for any $i \geq 0$ the assignment of a graph G to the group

$$\mathcal{H}_i(G) := \bigoplus_{n \in \mathbb{N}} H_i(U_n(G); \mathbb{Z})$$

has the structure of a \mathcal{A}_E -module. We show the following.

Theorem 1.1.4. *The \mathcal{A}_E -module \mathcal{H}_i is finitely generated.*

This allows us to conclude, among other things, the following.

Corollary 1.1.5. *For any $i, g \in \mathbb{N}$ there exists an integer $\epsilon_{i,g}$ such that for any graph G of genus at most g , and any $n \in \mathbb{N}$, the torsion part of $H_i(U_n(G); \mathbb{Z})$ is annihilated by $\epsilon_{i,g}$.*

Previously, by the work of Ko and Park, it was known that the only torsion that can appear in $H_1(U_n(G); \mathbb{Z})$ is 2-torsion [KP12, Corollary 3.6]. Our result can be seen as a generalization of this result. However, our bound on the torsion does depend on the genus of the graph and the theory we have developed tells us nothing about what this bound is, only that the bound is uniform.

Homology of the matching complex

For a graph G , a **matching of G** is a collection of pairwise non-adjacent edges. The **matching complex $M(G)$** of G is the simplicial complex whose simplices are in bijection with matchings on G . Matching complexes of complete graphs and complete bipartite graphs have been studied extensively using a wide variety of techniques including discrete Morse theory. It is notable, however, that there is relatively little known about the matching complexes of general graphs. We aim to fill this gap by considering a new perspective on the problem: instead of focusing on the matching complex $M(G)$ for some particular graph G , we consider the matching complexes of all graphs of genus at most g at once.

The key observation that allows us to do this is that whenever G is a minor of G' there is a natural way to embed the edges of G into those of G' . Moreover, this embedding preserves the condition that the edges are disjoint, as any "undoing" of an edge deletion or contraction can only push things further apart. We therefore obtain simplicial complex maps $M(G) \rightarrow M(G')$ whenever G is a minor of G' , which induce maps on homology $H_i(M(G)) \rightarrow H_i(M(G'))$. In particular, for any fixed $i \geq 0$, the assignment

$$G \mapsto H_i(M(G))$$

is a well-defined $\mathcal{G}_{\leq g}^{\text{op}}$ -module over \mathbb{Z} . Using the categorical graph minor theorem, we will prove the following

Theorem 1.1.6. *The $\mathcal{G}_{\leq g}^{\text{op}}$ -module*

$$G \mapsto H_i(M(G); \mathbb{Z})$$

is finitely generated. This implies that there exists an integer $\epsilon_{i,g}$, depending only on i and g , such that the torsion part of $H_i(\mathcal{M}(G))$ is annihilated by $\epsilon_{i,g}$, for all graphs G .

The statement about torsion in the above theorem should be particularly interesting, as the torsion appearing in the homology of the matching complex has been a subject of considerable intrigue in recent years [SW07] [Jon10b] [Jon10a].

We also note that, by the very general nature of the weak categorical graph minor theorem, the conclusions of Theorem 1.1.6 will remain true when the matching complex is replaced by a large variety of graph-based complexes, such as those discussed in [Jon08]. See Section 2.4 for more on this.

Graphical linear subspace arrangements

To any graph G , one may define a linear subspace arrangement induced by the edge relation of its vertices. We define the complement of this arrangement in $(\mathbb{C}^d)^{V(G)}$ by $\text{Conf}(G, \mathbb{C}^d)$. More precisely, we have

$$\text{Conf}(G, \mathbb{C}^d) = \{(x_v)_{v \in V(G)} \in (\mathbb{C}^d)^{V(G)} \mid x_v \neq x_w \text{ if } \{v, w\} \text{ is an edge of } G\}.$$

For instance, if $G = K_n$ is the complete graph, $\text{Conf}(G, \mathbb{C}^d)$ is the classical configuration space of points in \mathbb{C}^d . If instead $G = K_{a,b}$ is the complete bipartite graph, then one recovers the colored configuration spaces $\tilde{\mathcal{Z}}_{a+b}^D$ as studied by Farb, Wolfson, and Wood [FWW19]. In this work we will specialize to the family of *line graph complements*.

Given a graph G , we write $\mathcal{L}^c(G)$ to denote the simple graph whose vertices are indexed by the edges of G , and whose edges indicate the corresponding edges in G do not share a vertex. These graphs have been called generalized Kneser

graphs by some authors [DMM20], as the usual Kneser graph $K(n, 2)$ is seen to be $\mathcal{L}^c(K_n)$.

The same observation made above tells us that whenever G is a minor of G' , one obtains a graph embedding

$$\mathcal{L}^c(G) \hookrightarrow \mathcal{L}^c(G').$$

This embedding induces a "forgetful" map

$$\text{Conf}(\mathcal{L}^c(G'), \mathbb{C}^d) \rightarrow \text{Conf}(\mathcal{L}^c(G), \mathbb{C}^d),$$

which when composed with cohomology yields

$$H^i(\text{Conf}(\mathcal{L}^c(G), \mathbb{C}^d)) \rightarrow H^i(\text{Conf}(\mathcal{L}^c(G'), \mathbb{C}^d)).$$

We then obtain the following.

Theorem 1.1.7. *For any fixed i, d, g , the assignment*

$$G \mapsto H^i(\text{Conf}(\mathcal{L}^c(G), \mathbb{C}^d))$$

defines a finitely generated $\mathcal{G}_{\leq g}^{\text{op}}$ -module over \mathbb{Z} .

One observes that the complete bipartite graph $K_{a,b}$ can (essentially) be realized as the line graph complement of some other graph. In particular, the finite generation result of Theorem 1.1.7 can be seen as a generalization of some results in [FWW19] (see Theorem 2.6.4).

1.2. Matroids

A **matroid** is an object that abstracts the notion of linear independence. Precisely, a matroid is a pair $M = (E, \mathcal{B})$ where \mathcal{B} is a collection of subsets of E satisfying the following

(B1) $\mathcal{B} \neq \emptyset$

(B2) If $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \setminus B_2$, then there exists $y \in B_2 \setminus B_1$ such that $(B_1 \setminus \{x\}) \cup \{y\} \in \mathcal{B}$.

The set E is called the **ground set** of M and elements of \mathcal{B} are called bases. Given a matroid $M = (E, \mathcal{B})$, one can define its **base polytope** as the convex hull of the indicator vectors of its bases in \mathbb{R}^E .

A **decomposition** of M is a decomposition of its base polytope P_M as a CW-complex where the closed cells of the complex are base polytopes of other matroids. See Example 3.2.3 and Figure 1. A **valuative invariant** of matroids, is an invariant taking values in an abelian group that is compatible with all such decompositions. See Section 3.2 for a precise definition.

Said another way, if E is a finite set and $\text{Mat}(E)$ is the free abelian group generated by matroids on E , then there is a homomorphism from $\text{Mat}(E)$ to \mathbb{Z} -valued functions on \mathbb{R}^E taking a matroid M to the indicator vector of its base polytope. The kernel $I(E)$ of this homomorphism is generated by elements corresponding to decompositions of matroids. Thus, a matroid invariant thought of as a homomorphism from $\text{Mat}(E)$ to an abelian group is **valuative** if it descends to the quotient $\text{Mat}(E)/I(E)$.

A number of interesting matroid invariants turn out to be valutive. For example, the Tutte polynomial and characteristic polynomial [Spe08, Lemma

6.4], the Kazhdan-Lusztig polynomial [AS23, Theorem 8.8], and Eur’s volume polynomial [Eur20, Proposition 4.6] just to name a few.

In Chapter III we define a category $\mathcal{M}_{\text{id}}^{\wedge}(E)$ whose objects are matroids on E where we have a morphism $M \rightarrow M'$ whenever the polytope P_M contains the polytope $P_{M'}$. See 3.3 for a precise definition. A **categorical matroid invariant**, is a functor from $\mathcal{M}_{\text{id}}^{\wedge}(E)$ to the category of finite dimensional (graded) vector spaces over \mathbb{Q} . This induces a functor from the homotopy category of bounded chain complexes $\mathcal{K}_{\text{id}}(E)$ of $\mathcal{M}_{\text{id}}^{\wedge}(E)$. The Grothendieck group of $\mathcal{K}_{\text{id}}(E)$ is the group $\text{Mat}(E)$ so in this way a categorical matroid invariant categorifies the induced matroid invariant on Grothendieck groups. We then define what it means for a categorical matroid invariant to be **valuative** and we see that the matroid invariant categorified by a valutive categorical invariant must itself be valutive in the non-categorical sense.

The benefit of working categorically, is that if Φ is a categorical valutive invariant that is equivariant with respect to permutations on E and we have a group Γ acting on the ground set E such that Γ preserves a decomposition of a matroid, then we will obtain relations among the $\Phi(M)$ for M in the decomposition viewed as Γ -representations.

The Orlik-Solomon algebra of a matroid

Our main application of the theory of valutive categorical invariants of matroids will be the Orlik-Solomon algebra. Let M be a matroid on the ground set E . The Orlik-Solomon algebra of M denoted $\text{OS}(M)$ is a quotient of the exterior algebra of \mathbb{Q}^E whose generators correspond to **circuits** of the matroid. Namely, the subsets of E that are not contained in a basis of M . The Orlik-Solomon algebra

of a matroid is a graded algebra and the dimensions of its graded pieces are equal (modulo signs) to the coefficients of the characteristic polynomial of the matroid. Furthermore, whenever P_M contains $P_{M'}$, we obtain a natural map of algebras $\text{OS}(M) \rightarrow \text{OS}(M')$. In this way OS can be viewed as a functor from the category $\mathcal{M}_{\text{id}}^{\wedge}(E)$ to finite dimensional graded vector spaces over \mathbb{Q} . Thus OS is a categorical invariant of matroids. In Section 3.4 we will prove the following.

Theorem 1.2.1. *OS is a categorical valuative invariant.*

Then in Section 3.5 we use the fact that OS is valuative to obtain relations among $\text{OS}(M)$ as Γ -representations for groups Γ acting on E . We look at specific a family of matroid decompositions in Subsection 3.2.2 coming from the operation of **relaxation of a stressed flat** introduced by Ferroni and Schröter in [FS]. Then, in 3.5 we obtain the following explicit relations among the \mathfrak{S}_n -representations $\text{OS}(M)$ for matroids M involved in the operation of relaxation of a stressed hyperplane.

Theorem 1.2.2. *Let M be a matroid equipped with an action of the group Γ . Let H be a stressed hyperplane of M , and let \tilde{M} be the matroid obtained from M by relaxing γH for all $\gamma \in \Gamma$. Then*

$$\text{OS}(\tilde{M}) = \text{OS}(M) + \left(\text{Ind}_{\Gamma_H}^{\Gamma} \text{Res}_{\Gamma_H}^{\mathfrak{S}_h} \wedge^{k-1} V_{[h-1,1]} \right) \otimes \left(\mathbb{Q}[1-k] \oplus \mathbb{Q}[-k] \right)$$

as virtual graded representations of Γ .

In this way, we see that the theory of valuative categorical matroid invariants provides a very powerful tool for studying matroid invariants equivariantly with respect to group actions that preserve certain matroid polytope decompositions.

CHAPTER II

THE CATEGORICAL GRAPH MINOR THEOREM

This chapter was written in collaboration with Eric Ramos.

2.1. Gröbner theory of categories

Let \mathcal{C} be an essentially small category and k a ring. A \mathcal{C} -**module** over k is a functor from \mathcal{C} to the category of k -modules. Let $\text{Rep}_k(\mathcal{C})$ be the category of \mathcal{C} -**modules** over k where a morphism of \mathcal{C} -modules is a natural transformation. The category $\text{Rep}_k(\mathcal{C})$ is abelian. Let \mathcal{M} be a \mathcal{C} -module. A **submodule** of \mathcal{M} is a \mathcal{C} -module \mathcal{M}' such that for each object X of \mathcal{C} , $\mathcal{M}'(X)$ is a k -submodule of $\mathcal{M}(X)$ and for any morphism $\varphi : X \rightarrow Y$ in \mathcal{C} , $\mathcal{M}'(\varphi)$ is the restriction of $\mathcal{M}(\varphi)$ to $\mathcal{M}'(X)$. Equivalently, \mathcal{M}' is a submodule of \mathcal{M} when there is a natural transformation \mathcal{M}' to \mathcal{M} whose components are inclusions of k -modules. If $x \in \mathcal{M}(X)$ for some object X of \mathcal{C} , we say x is an **element** of \mathcal{M} or that \mathcal{M} **contains** x . The \mathcal{C} -module \mathcal{M} is **finitely generated** if there exist finitely many elements x_1, \dots, x_r of \mathcal{M} such that the only submodule of \mathcal{M} containing x_1, \dots, x_r is \mathcal{M} itself. In this case, we say that x_1, \dots, x_r generate \mathcal{M} . If every submodule of \mathcal{M} is finitely generated, then \mathcal{M} is said to be **Noetherian**. If every finitely generated module is Noetherian, the category $\text{Rep}_k(\mathcal{C})$ is said to be **locally Noetherian**. Sam and Snowden have developed powerful machinery for proving that module categories are locally Noetherian which we summarize below.

Given an object X of \mathcal{C} , let \mathcal{C}_x be the set of equivalence classes of morphisms out of X where $\varphi \in \text{Hom}_{\mathcal{C}}(X, Y)$ is equivalent to $\psi \in \text{Hom}_{\mathcal{C}}(X, Z)$ if there exists an isomorphism $\rho \in \text{Hom}_{\mathcal{C}}(Y, Z)$ such that $\rho \circ \varphi = \psi$. There is a natural quasi-order

\leq on \mathcal{C}_X where $\varphi \leq \psi$ if and only if there exists a morphism $\rho \in \text{Hom}_{\mathcal{C}}(Y, Z)$ such that $\rho \circ \varphi = \psi$. Consider the following two conditions for the category \mathcal{C} .

- **Property (G1):** For every object X of \mathcal{C} there exists a linear order \prec on \mathcal{C}_X preserved under post composition. That is to say, given $\varphi_1, \varphi_2 \in \text{Hom}_{\mathcal{C}}(X, Y)$, if $\varphi_1 \prec \varphi_2$, then $\rho \circ \varphi_1 \prec \rho \circ \varphi_2$ for any $\rho \in \text{Hom}_{\mathcal{C}}(Y, Z)$.
- **Property (G2):** The quasi-order \leq is a **well-quasi-order** on \mathcal{C}_X . Namely, for any infinite sequence $\varphi_1, \varphi_2, \varphi_3, \dots$ of elements of \mathcal{C}_X , there exists a pair of indices $i < j$ such that $\varphi_i \leq \varphi_j$.

Note, the order \prec in property (G1) is independent from the order \leq . The category \mathcal{C} is **directed** if there are no nontrivial endomorphisms. The category \mathcal{C} is called **Gröbner** if \mathcal{C} is a directed category and satisfies properties (G1) and (G2).

Let \mathcal{C} and \mathcal{D} be categories and $\Phi : \mathcal{D} \rightarrow \mathcal{C}$ a functor. The functor Φ is said to satisfy **property (F)** if for any object X of \mathcal{C} there exist finitely many (not necessarily distinct) objects Y_1, \dots, Y_n of \mathcal{D} along with a morphism $\varphi_i \in \text{Hom}_{\mathcal{C}}(X, \Phi(Y_i))$ for each i , such that for any object Y of \mathcal{D} and any morphism $\varphi \in \text{Hom}_{\mathcal{C}}(X, \Phi(Y))$, there exists a morphism $\psi \in \text{Hom}_{\mathcal{D}}(Y_i, Y)$ such that $\varphi = \Phi(\psi) \circ \varphi_i$. The category \mathcal{C} is said to be **quasi-Gröbner** if there exists a Gröbner category \mathcal{D} and a functor $\Phi : \mathcal{D} \rightarrow \mathcal{C}$ satisfying property (F). Note that any Gröbner category is necessarily quasi-Gröbner since the identity functor satisfies property (F).

Theorem 2.1.1. *[SS17, Theorem 1.1.3] If \mathcal{C} is a quasi-Gröbner category and k is a Noetherian commutative ring, then $\text{Rep}_k(\mathcal{C})$ is locally Noetherian.*

2.1.1. Modules over algebras over categories

Let FI be the category of finite sets with injections. Let \mathcal{C} be a category equipped with a functor $S : \mathcal{C} \rightarrow \text{FI}$ and let k be a commutative ring. There is a natural functor from \mathcal{C} to k -algebras taking an object X to the polynomial ring

$$\mathcal{A}_S(X) := k[t_e \mid e \in S(X)].$$

Let us define the category $\text{Rep}_k(\mathcal{C}, S)$ to be the category whose objects are \mathcal{C} -modules \mathcal{M} such that $\mathcal{M}(X)$ is a module over the k -algebra $\mathcal{A}_S(X)$ for each X in a natural way. Concretely, for any morphisms $\tilde{\varphi} \in \text{Hom}_{\mathcal{C}}(\tilde{X}, X)$ and $\varphi \in \text{Hom}_{\mathcal{C}}(X, Y)$ and any $a \in \mathcal{A}_S(\tilde{X})$, the following diagram commutes

$$\begin{array}{ccc} \mathcal{M}(X) & \xrightarrow{\mathcal{M}(\varphi)} & \mathcal{M}(Y) \\ \mathcal{A}_S(\varphi)(a) \downarrow & & \downarrow \mathcal{A}_S(\varphi \circ \tilde{\varphi})(a) \\ \mathcal{M}(X) & \xrightarrow{\mathcal{M}(\varphi)} & \mathcal{M}(Y) \end{array}$$

where the vertical arrows are given by multiplication by the indicated element. We will refer to an object of $\text{Rep}_k(\mathcal{C}, S)$ as an \mathcal{A}_S -**module**. Note that the category $\text{Rep}_k(\mathcal{C}, S)$ is abelian.

For each graph G , $\mathcal{A}_E(G)$ is a graded k -algebra where t_e is in degree 1 for each edge $e \in E(G)$. A **graded** \mathcal{A}_E -module M is an \mathcal{A}_E -module such that $\mathcal{M}(G)$ is a graded $\mathcal{A}_E(G)$ -module and for any minor morphism $\varphi : G \rightarrow G'$, the image under $\mathcal{M}(\varphi)$ of the degree i part of $\mathcal{M}(G')$ is contained in the degree i part of $\mathcal{M}(G)$. For \mathcal{A}_S -modules \mathcal{M} and \mathcal{M}' we say that \mathcal{M}' is a **submodule** of \mathcal{M} if, when viewed as \mathcal{C} -modules, \mathcal{M}' is a submodule of \mathcal{M} . An \mathcal{A}_S -module \mathcal{M} is called **finitely generated** if there exist finitely many elements $x_1, \dots, x_r \in \mathcal{M}$ such that the only submodule of \mathcal{M} containing x_1, \dots, x_r is \mathcal{M} itself. In this case x_1, \dots, x_r

generate \mathcal{M} . If every submodule of \mathcal{M} is finitely generated, then \mathcal{M} is said to be **Noetherian**. If every finitely generated \mathcal{A}_S -module is Noetherian, the category $\text{Rep}_k(\mathcal{C}, S)$ is said to be **locally Noetherian**. Now, we outline some basic facts about finitely generated \mathcal{A}_S -modules and Noetherian \mathcal{A}_S -modules, the proofs of which are completely standard. Let \mathcal{C} be a category, $S : \mathcal{C} \rightarrow \text{FI}$ a functor, and k a commutative ring. For any object X of \mathcal{C} , define the **principal projective** $\mathcal{P}_X^S \in \text{Rep}_k(\mathcal{C}, S)$ to be the module that takes an object X' to the free $\mathcal{A}_S(X')$ -module spanned by the set $\text{Hom}_{\mathcal{C}}(X, X')$, with maps defined via composition.

Lemma 2.1.2. *A module $\mathcal{M} \in \text{Rep}_k(\mathcal{C}, S)$ is finitely generated if and only if there exists a surjection*

$$\bigoplus_{i=1}^r \mathcal{P}_{X_i}^S \twoheadrightarrow \mathcal{M}$$

for some list of (not necessarily distinct) objects X_1, \dots, X_r of \mathcal{C} .

Proof. Suppose that X_1, \dots, X_r are objects of \mathcal{C} and $x_i \in \mathcal{M}(X_i)$ for all i . These classes generate \mathcal{M} if and only if the map

$$\bigoplus_{i=1}^r \mathcal{P}_{X_i}^S \rightarrow \mathcal{M}$$

taking $\text{id}_{X_i} \in \mathcal{P}_{X_i}^S(X_i)$ to x_i is surjective. □

Lemma 2.1.3. *A module \mathcal{M} is Noetherian if and only if every ascending chain of submodules of \mathcal{M} eventually stabilizes.*

Proof. Suppose that \mathcal{M} is Noetherian and $(\mathcal{N}_i \mid i \in \mathbb{N})$ is an ascending chain of submodules of \mathcal{M} . Let $\mathcal{N} := \bigcup_{i \in \mathbb{N}} \mathcal{N}_i \subseteq \mathcal{M}$. Since \mathcal{M} is Noetherian, \mathcal{N} is finitely generated. If we choose i large enough so that \mathcal{N}_i contains all of the finitely many generators of \mathcal{N} , then we have $\mathcal{N}_j = \mathcal{N}$ for all $j \geq i$.

Conversely, suppose that \mathcal{M} has a submodule $\mathcal{N} \subseteq \mathcal{M}$ that is not finitely generated. We will define an ascending chain of finitely generated submodules $(\mathcal{N}_i \mid i \in \mathbb{N})$ as follows. Let $\mathcal{N}_0 = 0$. Once we have defined \mathcal{N}_i , the fact that \mathcal{N}_i is finitely generated means that $\mathcal{N}_i \subsetneq \mathcal{N}$, so we may choose an object X_i of \mathcal{C} such that we have $x_i \in \mathcal{N}(X_i) \setminus \mathcal{N}_i(X_i)$. Then, define \mathcal{N}_{i+1} to be the smallest submodule of \mathcal{N} containing both \mathcal{N}_i and x_i . This chain of submodules clearly does not stabilize. \square

Lemma 2.1.4. *Suppose that $0 \rightarrow \mathcal{M}' \rightarrow \mathcal{M} \rightarrow \mathcal{M}'' \rightarrow 0$ is short exact sequence in $\text{Rep}_k(\mathcal{C}, S)$. Then \mathcal{M} is Noetherian if and only if both \mathcal{M}' and \mathcal{M}'' are Noetherian.*

Proof. If \mathcal{M} is Noetherian, then \mathcal{M}' is Noetherian by definition. If $\mathcal{N}'' \subset \mathcal{M}''$ is a submodule, let $\mathcal{N} \subseteq \mathcal{M}$ be the preimage of \mathcal{N}'' in \mathcal{M} . Since \mathcal{M} is Noetherian, \mathcal{N} is finitely generated, thus so is \mathcal{N}'' by Lemma 2.1.2.

Conversely, suppose that both \mathcal{M}' and \mathcal{M}'' are Noetherian, and let $(\mathcal{N}_i \mid i \in \mathbb{N})$ be an ascending chain of submodules of \mathcal{M} . For each i , let $\mathcal{N}'_i := \mathcal{N}_i \cap \mathcal{M}'$ and let \mathcal{N}''_i be the image of \mathcal{N}_i in \mathcal{M}'' . Since \mathcal{M}' and \mathcal{M}'' are both Noetherian, Lemma 2.1.3 tells us that there is an index n such that, for all $i > n$, $\mathcal{N}'_i = \mathcal{N}'_{i+1}$ and $\mathcal{N}''_i = \mathcal{N}''_{i+1}$. We can then conclude that $\mathcal{N}_i = \mathcal{N}_{i+1}$ by applying the Five Lemma to the following diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{N}'_i & \longrightarrow & \mathcal{N}_i & \longrightarrow & \mathcal{N}''_i & \longrightarrow & 0 \\
& & \downarrow = & & \downarrow & & \downarrow = & & \\
0 & \longrightarrow & \mathcal{N}'_{i+1} & \longrightarrow & \mathcal{N}_{i+1} & \longrightarrow & \mathcal{N}''_{i+1} & \longrightarrow & 0
\end{array}$$

Thus \mathcal{M} satisfies the ascending chain condition, and is therefore Noetherian by Lemma 2.1.3. \square

Remark 2.1.5. The notion of a principal projective module of $\text{Rep}_k(\mathcal{C})$ was defined in [SS17]. Given an object X of \mathcal{C} , the **principal projective** module \mathcal{P}_X is the \mathcal{C} -module that sends an object X' to the free k -module generated by the set $\text{Hom}_{\mathcal{C}}(X, X')$. Note the difference in notion between the principal projective modules \mathcal{P}_X in $\text{Rep}_k(\mathcal{C})$ and \mathcal{P}_X^S in $\text{Rep}_k(\mathcal{C}, S)$. Furthermore, when working in $\text{Rep}_k(\mathcal{C})$, the analogues of Lemmas 2.1.2, 2.1.3, and 2.1.4 all hold in this case too.

In the following sections, we build on the work of [SS17] and define what it means for the pair (\mathcal{C}, S) to be **Gröbner** (resp. **quasi-Gröbner**) and prove the following generalization of Theorem 2.1.1.

Theorem 2.1.6. *Let \mathcal{C} be a category and $S : \mathcal{C} \rightarrow \text{FI}$ a functor. If the pair (\mathcal{C}, S) is quasi-Gröbner, then $\text{Rep}_k(\mathcal{C}, S)$ is locally Noetherian for any Noetherian commutative ring k .*

Remark 2.1.7. Theorem 2.1.6 is motivated by the work of Nagel and Römer [NR19]. Though they do not make these definitions in the same generality, they essentially prove that the pair (FI, id) is quasi-Gröbner, and they use this result to show that $\text{Rep}_k(\text{FI}, \text{id})$ is locally Noetherian for any Noetherian commutative ring k . Moreover, they show that if $S_d : \text{FI} \rightarrow \text{FI}$ is the functor taking a set T to the set of unordered d -tuples of distinct elements of T , then the pair (FI, S_d) is quasi-Gröbner and the category $\text{Rep}_k(\text{FI}, \text{id})$ is locally Noetherian if and only if $d \leq 1$ [NR19, Proposition 4.8].

Remark 2.1.8. Note that we have $\text{Rep}_k(\mathcal{C}) = \text{Rep}_k(\mathcal{C}, \emptyset)$, where $\emptyset : \mathcal{C} \rightarrow \text{FI}$ is the constant functor that takes every object of \mathcal{C} to the empty set. We will see that the category \mathcal{C} is Gröbner (resp. quasi-Gröbner) in the sense of [SS17] if and only if the pair (\mathcal{C}, \emptyset) is quasi-Gröbner.

2.1.2. Gröbner pairs

Let OI be the category whose objects are totally ordered finite sets and whose morphisms are ordered inclusions, and let $\Psi : \text{OI} \rightarrow \text{FI}$ be the functor that forgets the order on a finite set. Let \mathcal{D} be an essentially small category and $T : \mathcal{D} \rightarrow \text{OI}$ any functor. Recall the definition of $\mathcal{A}_{\Psi \circ T}$ as in the previous section. For convenience, we will write $\mathcal{A}_T := \mathcal{A}_{\Psi \circ T}$. The purpose of this section is to define what it means for the pair (\mathcal{D}, T) to be Gröbner.

A **quartet** for the pair (\mathcal{D}, T) is a quadruple $\mu = (D, D', \varphi, m)$, where D and D' are objects of \mathcal{D} , $\varphi \in \text{Hom}_{\mathcal{D}}(D, D')$, and $m : T(D') \rightarrow \mathbb{N}$ is a map of sets. For any morphism $\psi \in \text{Hom}_{\mathcal{D}}(D', D'')$, we will write

$$\psi(\mu) := (D, D'', \psi \circ \varphi, m_\psi),$$

where m_ψ is determined by the conditions that $m_\psi \circ T(\psi) = m$ and m_ψ is identically zero outside of the image of $T(\psi)$. For any map $n : T(D') \rightarrow \mathbb{N}$, we will write

$$\mu + n := (D, D', \varphi, m + n).$$

If $\mu_1 = (D, D'_1, \varphi_1, m_1)$ and $\mu_2 = (D, D'_2, \varphi_2, m_2)$, we say that $\mu_1 \leq \mu_2$ if there exists a morphism $\psi : D'_1 \rightarrow D'_2$ and a map $n : T(D'_2) \rightarrow \mathbb{N}$ such that $\mu_2 = \psi(\mu_1) + n$.

Remark 2.1.9. The motivation for these definitions is that, once we choose a commutative ring k , the quartet μ determines a monomial

$$t^\mu := \prod_{a \in T(D')} t_a^{m(a)} \in \mathcal{A}_T(D')$$

along with an element

$$b_\mu := t^m \cdot \varphi \in \mathcal{P}_D(D') \in \text{Rep}_k(\mathcal{D}, \Psi \circ T).$$

Then $\mu_1 \leq \mu_2$ if and only if φ_2 factors through φ_1 via a map ψ and we have

$$b_{\mu_2} = t^n \psi(b_{\mu_1})$$

for some monomial $t^n \in \mathcal{A}(D'_2)$.

We say that μ_1 and μ_2 are **equivalent** if $\mu_1 \leq \mu_2$ and $\mu_2 \leq \mu_1$. For each object D of \mathcal{D} , let $|\mathcal{D}_D^T|$ denote the poset of equivalence classes of quartets with first coordinate D . Given a quartet $\mu = (D, D', \varphi, m)$, we will write $[\mu]$ to denote its equivalence class in $|\mathcal{D}_D^T|$. A well-order \prec of $|\mathcal{D}_D^T|$ is called **admissible** if, given two quartets $\mu_1 = (D, D', \varphi_1, m_1)$ and $\mu_2 = (D, D', \varphi_2, m_2)$ with the same source and target along with a morphism $\psi : D' \rightarrow D''$ and a map $n : T(D'') \rightarrow \mathbb{N}$, we have that $[\mu_1] \prec [\mu_2]$ implies $[\psi(\mu_1) + n] \prec [\psi(\mu_2) + n]$.

We say that the pair (\mathcal{D}, T) satisfies **property (G1)** if, for every object D of \mathcal{D} , the poset $|\mathcal{D}_D^T|$ admits an admissible well-order. A poset P is said to be **Noetherian** if, for any sequence $(p_i \mid i \in \mathbb{N})$ in P , there exist natural numbers $i < j$ such that $p_i \leq p_j$. We say that the pair (\mathcal{D}, T) satisfies **property (G2)** if, for every object D of \mathcal{D} , the poset $|\mathcal{D}_D^T|$ is Noetherian. We call the pair (\mathcal{D}, T) **Gröbner** if \mathcal{D} is directed and (\mathcal{D}, T) satisfies properties (G1) and (G2).

Remark 2.1.10. Property (G1) for the pair (\mathcal{D}, \emptyset) is equivalent to property (G1) for \mathcal{D} as defined in [SS17, Section 1.1], and similarly property (G2) for the pair (\mathcal{D}, \emptyset) is equivalent to property (G2) for \mathcal{D} . Thus a directed category \mathcal{D} is Gröbner in the sense of [SS17] if and only if the pair (\mathcal{D}, \emptyset) is Gröbner.

The following Proposition says that the functor T does not add anything interesting to property (G1). In other words, the distinction between a Gröbner category and a Gröbner pair lies entirely in the property (G2).

Proposition 2.1.11. *The pair (\mathcal{D}, T) satisfies property (G1) if and only if the pair (\mathcal{D}, \emptyset) satisfies property (G1).*

Proof. If \prec is an admissible order of $|\mathcal{D}_D^T|$, then restriction to quartets with $m = 0$ gives an admissible order of $|\mathcal{D}_D^\emptyset|$. Conversely, if we have an admissible order of $|\mathcal{D}_D^\emptyset|$, we can compare the classes of two quartets $\mu_1 = (D, D'_1, \varphi_1, m_1)$ and $\mu_2 = (D, D'_2, \varphi_2, m_2)$ for (\mathcal{D}, T) by first comparing the classes of the quartets $(D, D'_1, \varphi_1, 0)$ and $(D, D'_2, \varphi_2, 0)$ for (\mathcal{D}, \emptyset) and then, if they are equal, breaking the tie by comparing m_1 and m_2 lexicographically. \square

2.1.3. Gröbner bases

Let \mathcal{D} be an essentially small category and $T : \mathcal{D} \rightarrow \text{OI}$ a functor such that the pair (\mathcal{D}, T) is Gröbner, and choose an admissible well-order \prec of $|\mathcal{D}_D^T|$ for each object D of \mathcal{D} as in the definition of property (G1). For any pair of objects D and D' in \mathcal{D} , let $Q_{D, D'}$ be the set of quartets of the form whose first two entries are D and D' respectively. The fact that \mathcal{D} is directed implies that the natural map from $Q_{D, D'}$ to $|\mathcal{D}_D^T|$ is injective, thus $Q_{D, D'}$ is well-ordered by \prec .

Fix a commutative ring k , so that we may define the representation category $\text{Rep}_k(\mathcal{D}, \Psi \circ T)$. For any nonzero element

$$p = \sum_{\mu \in Q_{D, D'}} \lambda_\mu b_\mu \in \mathcal{P}_D^T(D')$$

where b_μ is defined as in Remark 2.1.9, we define the **leading quartet** $\text{LQ}(p)$ to be the maximal μ with respect to the well-order \prec such that the coefficient $\lambda_\mu \in k$ is nonzero. If $\mu = \text{LQ}(p)$, we define the **leading term** $\text{LT}(p) := \lambda_\mu b_\mu \in \mathcal{P}_D^T(D')$ and the **leading coefficient** $\text{LC}(p) := \lambda_\mu \in k$.

Lemma 2.1.12. *Suppose we have a morphism $\psi : D' \rightarrow D''$, a map $n : T(D') \rightarrow \mathbb{N}$, and an element $0 \neq p \in \mathcal{P}_D^T(D')$. Then*

$$\text{LT}(x^n \psi(p)) = x^n \psi(\text{LT}(p)).$$

Proof. This is precisely the definition of admissibility of the well-order \prec . □

Given a submodule $\mathcal{N} \subset \mathcal{P}_D^T$, we define a subset

$$\text{LQ}(\mathcal{N}) := \{[\text{LQ}(p)] \mid 0 \neq p \in \mathcal{N}(d')\} \subset |\mathcal{D}_D^T|.$$

For each object D' of \mathcal{D} , we define

$$\text{LT}(\mathcal{N})(D') := \{0\} \cup \{\text{LT}(p) \mid 0 \neq p \in \mathcal{N}(D') \subset \mathcal{P}_D^T(D')\}.$$

For each quartet $\mu = (D, D', \varphi, m)$, we define the ideal

$$\mathcal{L}^c(\mathcal{N}, \mu) := \{0\} \cup \{\mathcal{L}^c(p) \mid 0 \neq p \in \mathcal{N}(D') \text{ and } \text{LQ}(p) = \mu\} \subset k.$$

Lemma 2.1.12 implies that $\text{LT}(\mathcal{N}) \subset \mathcal{P}_D^T$ is a submodule and that we have an inclusion of ideals $\mathcal{L}^c(\mathcal{N}, \mu_2) \subset \mathcal{L}^c(\mathcal{N}, \mu_1)$ whenever $[\mu_1] \leq [\mu_2]$.

Suppose we are given a finite set $B = \{(D'_1, p_1), \dots, (D'_r, p_r)\}$ of pairs with $0 \neq p_i \in \mathcal{N}(D'_i)$ for all i . We say that B is a **Gröbner basis** for \mathcal{N} if the module $\text{LT}(\mathcal{N})$ is generated by the classes $\text{LT}(p_i)$ for $1 \leq i \leq r$.

Lemma 2.1.13. *If B is a Gröbner basis for \mathcal{N} , then B generates \mathcal{N} .*

Proof. If not, choose an element $p \in \mathcal{N}(D')$ that is not in the submodule generated by B , and choose it in such a way that the leading quartet $\text{LQ}(p)$ is minimal with respect to the admissible well-order on $|\mathcal{D}_D^T|$. Since B is a Gröbner basis, we may choose an index i , a morphism $\psi : D'_i \rightarrow D'$, a function $n : T(D') \rightarrow \mathbb{N}$, and a scalar $\lambda \in k$ such that $\text{LT}(p) = \lambda x^n \psi(\text{LT}(p_i))$. By Lemma 2.1.12, this is equal to $\text{LT}(\lambda x^n \psi(p_i))$. But then $p - \lambda x^n \psi(p_i)$ is not in the submodule generated by B and has a leading quartet strictly smaller than $\text{LQ}(p)$, which gives a contradiction. \square

Proposition 2.1.14. *Suppose that the ring k is Noetherian. For every object D of \mathcal{D} , the principal projective $\mathcal{P}_D^T \in \text{Rep}_k(\mathcal{D}, \Psi \circ T)$ is Noetherian.*

Proof. By Lemma 2.1.13, it is sufficient to show that every submodule $\mathcal{N} \subset \mathcal{P}_D^T$ has a Gröbner basis. By property (G2), the set $\text{LQ}(\mathcal{N}) \subset |\mathcal{D}_D^T|$ has only finitely many minimal elements with respect to the partial order. Choose finitely many quartets μ_1, \dots, μ_r representing these minimal classes, and write $\mu_i = (D, D'_i, \varphi_i, m_i)$. For each i , the fact that k is Noetherian implies that the ideal $\mathcal{L}^c(\mathcal{N}, \mu_i)$ is generated by finitely many elements

$$\lambda_i^1, \dots, \lambda_i^{s_i} \in k.$$

For each $j \leq s_i$, choose an element $0 \neq p_i^j \in \mathcal{N}(D'_i)$ with $\text{LT}(p_i^j) = \lambda_i^j b_{\mu_i}$, and let

$$B := \{(D'_i, p_i^j) \mid 1 \leq i \leq r, 1 \leq j \leq s_i\}.$$

We claim that B is a Gröbner bases for \mathcal{N} .

Let $0 \neq p \in \mathcal{N}(D')$ be given; we will show that $\text{LT}(p)$ is in the submodule of \mathcal{P}_d^T generated by the classes $\text{LT}(p_i^j)$. Let $\nu := \text{LQ}(p)$. By definition of the quartets μ_1, \dots, μ_r , there exists an index i such that $[\mu_i] \leq [\nu]$. That means that we can choose a morphism $\psi : D'_i \rightarrow D'$ and a map $n : T(D') \rightarrow \mathbb{N}$ such that $\nu = \psi(\mu_i) + n$. Since $[\mu_i] \leq [\nu]$, we have $\mathcal{L}^c(\mathcal{N}, \nu) \subset \mathcal{L}^c(\mathcal{N}, \mu_i)$, and therefore there exist elements $\zeta_i^1, \dots, \zeta_i^{s_i} \in k$ such that

$$\mathcal{L}^c(p) = \zeta_i^1 \lambda_i^1 + \dots + \zeta_i^{s_i} \lambda_i^{s_i}.$$

Then

$$\begin{aligned} \text{LT}(p) &= \mathcal{L}^c(p) b_\nu \\ &= (\zeta_i^1 \lambda_i^1 + \dots + \zeta_i^{s_i} \lambda_i^{s_i}) b_{\psi(\mu_i) + n} \\ &= x^n \psi(\zeta_i^1 \lambda_i^j b_{\mu_i} + \dots + \zeta_i^{s_i} \lambda_i^j b_{\mu_i}) \\ &= x^n \psi(\zeta_i^1 \text{LT}(p_i^1) + \dots + \zeta_i^{s_i} \text{LT}(p_i^{s_i})) \end{aligned}$$

is in the submodule of \mathcal{P}_D^T generated by the classes $\text{LT}(p_i^j)$. □

Corollary 2.1.15. *Let \mathcal{D} be an essentially small category, $T : \mathcal{D} \rightarrow \text{OI}$ a functor, and k a Noetherian commutative ring. If the pair (\mathcal{D}, T) is Gröbner, then $\text{Rep}_k(\mathcal{D}, \Psi \circ T)$ is locally Noetherian.*

Proof. Suppose that $\mathcal{M} \in \text{Rep}_k(\mathcal{D}, \Psi \circ T)$ is finitely generated. By Lemma 2.1.2, \mathcal{M} is a quotient of a direct sum of principal projectives. Proposition 2.1.14 tells us that each of these principal projectives is Noetherian, and Lemma 2.1.4 then tells us that the same is true of \mathcal{M} . □

2.1.4. Quasi-Gröbner pairs

Let $\Phi : \mathcal{D} \rightarrow \mathcal{C}$ be a functor. Following Sam and Snowden [SS17, Definition 3.2.1], we say that Φ has **property (F)** if, for any object X of \mathcal{C} , there exist finitely many objects D_1, \dots, D_r of \mathcal{D} along with morphisms $\varphi_i : X \rightarrow \Phi(D_i)$ such that, for any object D of \mathcal{D} and any morphism $\psi : X \rightarrow \Phi(D)$, there exists an i and a morphism $\rho : D_i \rightarrow D$ such that $\psi = \Phi(\rho) \circ \varphi_i$. Given a functor $S : \mathcal{C} \rightarrow \text{FI}$, we say that the pair (\mathcal{C}, S) is **quasi-Gröbner** if there exists a Gröbner pair (\mathcal{D}, T) and an essentially surjective functor $\Phi : \mathcal{D} \rightarrow \mathcal{C}$ with property (F) such that $S \circ \Phi$ is naturally isomorphic to $\Psi \circ T$.

Remark 2.1.16. The pair (\mathcal{C}, \emptyset) is quasi-Gröbner if and only if the category \mathcal{C} is quasi-Gröbner in the sense of [SS17].

Let $\Phi : \mathcal{D} \rightarrow \mathcal{C}$ and $S : \mathcal{C} \rightarrow \text{FI}$ be any functors. For any commutative ring k , we have an exact functor $\Phi^* : \text{Rep}_k(\mathcal{C}, S) \rightarrow \text{Rep}_k(\mathcal{D}, S \circ \Phi)$ that takes a module $\mathcal{M} \in \text{Rep}_k(\mathcal{C}, S)$ to

$$\Phi^* \mathcal{M} := \mathcal{M} \circ \Phi \in \text{Rep}_k(\mathcal{D}, S \circ \Phi).$$

Proposition 2.1.17. *Let $\Phi : \mathcal{D} \rightarrow \mathcal{C}$ be a functor with property (F), let $S : \mathcal{C} \rightarrow \text{FI}$ be any functor, and let k be a commutative ring. If $\mathcal{M} \in \text{Rep}_k(\mathcal{C}, S)$ is finitely generated, then $\Phi^* \mathcal{M} \in \text{Rep}_k(\mathcal{D}, S \circ \Phi)$ is finitely generated.*

Proof. Since \mathcal{M} is finitely generated, Lemma 2.1.2 tells us that \mathcal{M} is a quotient of a direct sum of principal projectives. Since Φ^* is exact, $\Phi^* \mathcal{M}$ is a quotient of a direct sum of pullbacks of principal projectives. Thus, it is sufficient to show that, for any object X of \mathcal{C} , $\Phi^* \mathcal{P}_X^T$ is finitely generated. Choose finitely many objects D_1, \dots, D_r of \mathcal{D} along with morphisms $\varphi_i : X \rightarrow \Phi(D_i)$ as in the definition of

property (F). Consider the maps

$$\mathcal{P}_{D_i}^T \rightarrow \Phi^* \mathcal{P}_{\Phi(D_i)}^T \rightarrow \Phi^* \mathcal{P}_X^T,$$

where the first map is induced by Φ and the second is induced by φ_i . Property (F) says precisely that the direct sum map

$$\bigoplus_{i=1}^r \mathcal{P}_{D_i}^T \rightarrow \Phi^* \mathcal{P}_X^T$$

is surjective, which implies that $\Phi^* \mathcal{P}_X^T$ is finitely generated. \square

Proof of Theorem 2.1.6. Let (\mathcal{C}, S) be quasi-Gröbner pair. That means that there exists a Gröbner pair (\mathcal{D}, T) , an essentially surjective functor $\Phi : \mathcal{D} \rightarrow \mathcal{C}$ with property (F), and a natural isomorphism $\Psi \circ T \cong S \circ \Phi$. Fix a commutative ring k , a finitely generated module $\mathcal{M} \in \text{Rep}_k(\mathcal{C}, S)$, and a submodule $\mathcal{N} \subset \mathcal{M}$. We need to prove that \mathcal{N} is finitely generated, as well.

Proposition 2.1.17 tells us that $\Phi^* \mathcal{M} \in \text{Rep}_k(\mathcal{D}, S \circ \Phi) \simeq \text{Rep}_k(\mathcal{D}, \Psi \circ T)$ is finitely generated, and Corollary 2.1.15 then implies that $\Phi^* \mathcal{N} \subset \Phi^* \mathcal{M}$ is also finitely generated. Choose a finite generating set x_1, \dots, x_r of $\Phi^*(\mathcal{M})$ where $x_i \in \Phi^* \mathcal{N}(D_i)$. This means that, for any object D of \mathcal{D} , $\Phi^* \mathcal{N}(D)$ is spanned over $\mathcal{A}_{S \circ \Phi}(D)$ by the images of the elements x_i along the maps induced by all possible morphisms $\varphi_i : D \rightarrow D_i$. This is equivalent to saying $\mathcal{N}(\Phi(D))$ is spanned over $\mathcal{A}_S(\Phi(D))$ by the elements $\Phi(x_i) \in \mathcal{N}(\Phi(D_i))$ along the maps induced by all morphisms $\Phi(\varphi_i) : \Phi(D) \rightarrow \Phi(D_i)$. Since Φ is essentially surjective, this means that \mathcal{N} is finitely generated. \square

2.2. Graphs

We define the category $\mathcal{G}_{\leq g}$ of genus at most g along with the edge functor $E : \mathcal{G}_{\leq g} \rightarrow \text{FI}$ and use the tools in Section 2.1 to prove that $\mathcal{G}_{\leq g}$ is quasi-Gröbner and the pair $(\mathcal{G}_{\leq g}, E)$ is quasi-Gröbner.

2.2.1. Defining the graph categories

A **directed graph** is a quadruple (V, A, h, t) , where V and A are finite sets (vertices and arrows), and h and t are each maps from A to V (head and tail). A **graph** is a quintuple (V, A, h, t, σ) , where (V, A, h, t) is a directed graph and σ is a fixed-point-free involution of A with the property that $h = t \circ \sigma$. If (V, A, h, t, σ) is a graph, elements of the quotient A/σ are called **edges**. Given a directed graph $D = (V, A, h, t)$, we define the **underlying graph** $\bar{D} = (V, \bar{A}, h, t, \sigma)$, where $\bar{A} = A \times \{\pm 1\}$, $h(a, 1) = h(a) = t(a, -1)$, $t(a, 1) = t(a) = h(a, -1)$, and σ acts by toggling the second factor. We will usually suppress h and t from the notation and simply write (V, A, σ) for a graph.

Remark 2.2.1. This might seem to be an unnecessarily complicated definition of a graph. For example, one might try defining a graph to consist of a vertex set, and edge set, and a map from edges to unordered pairs of vertices. However, we want a graph with a loop to have a nontrivial automorphism that reverses the orientation of the loop. It is difficult to formalize this with the unordered pair definition.

If $G = (V, A, \sigma)$ is a graph and $v, v' \in V$, a **walk** in G from v to v' is a finite sequence (a_1, \dots, a_n) of arrows with $t(a_1) = v$, $h(a_n) = v'$, and $h(a_i) = t(a_{i+1})$ for all $1 \leq i < n$. A **path** in G from v to v' is a walk from v to v' such that removing any collection of a_i from the walk is no longer a walk. We say that G is **connected**

if there exists at least one path between any pair of vertices, and we say that G is a **forest** if there exists at most one path between any pair of vertices. A nonempty connected forest is called a **tree**.

Let $G = (V, A, \sigma)$ and $G' = (V', A', \sigma')$ be graphs. A **minor morphism** from G to G' is a map

$$\varphi : V \sqcup A \sqcup \{\star\} \rightarrow V' \sqcup A' \sqcup \{\star\}$$

satisfying the following properties:

- $\varphi(\star) = \star$.
- For every vertex $v \in V$, $\varphi(v) \in V'$.
- For every arrow $a \in A$, $\varphi \circ \sigma(a) = \sigma' \circ \varphi(a)$ where σ' acts trivially on $V' \sqcup \{\star\}$.
- For every arrow $a' \in A'$, there exists a unique arrow $a \in A$ with $\varphi(a) = a'$.
- If $\varphi(a) \in A$, then $\varphi \circ h(a) = h' \circ \varphi(a)$ and $\varphi \circ t(a) = t' \circ \varphi(a)$.
- If $\varphi(a) \in V$, then $\varphi \circ h(a) = \varphi(a) = \varphi \circ t(a)$.
- For every $v' \in V'$, $\varphi^{-1}(v')$ consists of the edges and vertices of a tree.

The edges that map to vertices are called **contracted edges** and the edges that map to \star are called **deleted edges**. Note that the second condition and third conditions imply that the edges of G that are neither deleted nor contracted map bijectively to the edges of G' . In particular, φ induces an injection $\varphi^* : A'/\sigma' \rightarrow A/\sigma$.

Remark 2.2.2. If G and G' are connected graphs, then a minor morphism $\varphi : G \rightarrow G'$ is determined by its restriction to the set of arrows of G . (Connectedness

is necessary because the graph with two vertices and no arrows has a nontrivial automorphism swapping the vertices.) However, it is convenient to define φ on the whole set $V \sqcup A \sqcup \{\star\}$ so that minor morphisms can be composed simply by composing functions. Note that φ is *not* determined by the map on edges φ^* . To see this, consider the example where G has two vertices and two parallel edges between them, and G' consists of a single vertex with no edges. There are two minor morphisms from G to G' , corresponding to the choice of which edge is deleted and which edge is contracted.

Let \mathcal{G} denote the category whose objects are nonempty connected graphs and whose morphisms are minor morphisms.

Conjecture 2.2.3. *The category \mathcal{G}^{op} is quasi-Gröbner.*

Remark 2.2.4. The Gröbner cover of \mathcal{G}^{op} should be the category \mathcal{OD}^{op} of directed graphs whose arrows are ordered with opposite minor morphisms that preserve the order of arrows. That is if $\varphi : (V, A, h, t) \rightarrow (V', A', h', t')$ is a minor morphism of directed graphs whose arrows are ordered, then we also require that whenever $a'_1 \leq a'_2$ in A' , then $\varphi^*(a'_1) \leq \varphi^*(a'_2)$ in A . However, to prove that \mathcal{OD}^{op} satisfies property (G2) one would need a stronger version of Robertson and Seymour's labeled graph minor theorem [RS10, 1.7] where the order of the labels on edges is preserved in the above sense. If Conjecture 2.2.3 is proven, then all of the results in later sections can be upgraded from statements about graphs with genus at most g to statements about all graphs with no restriction on genus.

In light of Remark 2.2.4, we restrict our attention to a family of full subcategories of \mathcal{G}^{op} . The **combinatorial genus** or simply **genus** of a graph $G = (V, A, \sigma)$ is the quantity $|A/\sigma| - |V| + 1$. Viewed as a 1-dimensional

CW-complex where the 1-cells correspond to edges and the 0-cells correspond to vertices, every graph is homotopy equivalent to a finite wedge product of g circles $\bigvee_{i=1}^g S^1$. For any $g \geq 0$, let $\mathcal{G}_{\leq g}$ be the full subcategory of \mathcal{G} whose objects are non-empty connected graphs of genus at most g .

Theorem 2.2.5. *For any positive integer g , $\mathcal{G}_{\leq g}^{\text{op}}$ is quasi-Gröbner.*

The proof of Theorem 2.2.5 follows mostly from the work in [PR19a]. We will need slight generalizations of the theory they present which we outline below.

A **rooted tree** is a pair (T, v) where T is a tree and v is a vertex of T called the **root**. There is a natural partial order on the vertex set of rooted tree where $w \leq u$ if and only if the unique path from w to the root passes through u . A **direct descendant** of a vertex w is a vertex covered by w with respect to this partial order. A **planar rooted tree** is a rooted tree equipt with a linear order on the set of direct descendants of each vertex. Note that this gives a depth first linear order on the set of vertices.

In a nonempty connected graph G , a **spanning tree** for G is a subgraph that is a tree and contains all vertices of G . Now, for each genus g , fix once and for all a graph R_g with one vertex and g loops. Define a **rigidified graph** of genus g to be a quadruple (G, T, v, τ) where G is a graph of genus g , (T, v) is a planar rooted spanning tree of G , and $\tau : G \rightarrow R_g$ is a minor morphism where the contracted edges are exactly the edges of T . Note that rigidified graphs come equipt with a linear order on the vertices coming from their planar rooted spanning trees. A **morphism of rigidified graphs** $(G, T, v, \tau) \rightarrow (G', T', v', \tau')$ between rigidified graphs of genus g is a minor morphism $\varphi : G \rightarrow G'$ that restricts to a minor morphism $T \rightarrow T'$ such that $\varphi(v) = v'$, and if $w' \leq u'$ under linear order on vertices of G' , then the smallest vertex in the preimage of w' comes before the

smallest vertex in the preimage on u' under the linear order on the vertices of G . Note that a minor morphism between graphs of the same genus necessarily has no deleted edges.

Let \mathcal{RG}_g be the category whose objects are rigidified graphs of genus g and whose morphisms are minor morphisms of rigidified graphs of genus g . Let $\mathcal{RG}_{\leq g}^{\text{op}}$ denote the category whose objects are rigidified graphs of genus at most g and whose morphisms are minor morphisms of rigidified graphs between graphs of the same genus. One can think of $\mathcal{RG}_{\leq g}^{\text{op}}$ as a kind of disjoint union of the finitely many categories $\mathcal{RG}_h^{\text{op}}$ for $h \leq g$.

Theorem 2.2.6 ([PR19a]). *For any $g \geq 0$, the category $\mathcal{RG}_g^{\text{op}}$ is Gröbner.*

Corollary 2.2.7. *For any $g \geq 0$, the category $\mathcal{RG}_{\leq g}^{\text{op}}$ is Gröbner.*

Proof. It is clear from the definitions that $\mathcal{RG}_{\leq g}^{\text{op}}$ satisfies properties (G1) and (G2) since $\mathcal{RG}_h^{\text{op}}$ satisfies both properties for each of the finitely many $h \leq g$ and there are no morphisms between rigidified graphs of different genus in $\mathcal{RG}_{\leq g}^{\text{op}}$. \square

There is a forgetful functor

$$\Phi_{\leq g} : \mathcal{RG}_{\leq g}^{\text{op}} \rightarrow \mathcal{G}_{\leq g}^{\text{op}}$$

which sends a rigidified graph to its underlying graph and a minor morphism of rigidified graphs to the underlying minor morphism of graphs.

Lemma 2.2.8. *The functor $\Phi_{\leq g}$ satisfies property (F).*

Proof. This proof is similar to the proof of [PR19a, Lemma 3.11]. The fact that $\Phi_{\leq g}$ is essentially surjective is clear as every graph can be given a rigidified graph structure.

Let G be a graph of genus $h \leq g$ with exactly n edges. To show $\Phi_{\leq g}$ has property (F), we need to produce a finite collection (G_i, T_i, v_i, τ_i) of Rigidified graphs of genus at most g along with minor morphisms $\varphi_i : G_i \rightarrow G$ such that, given any rigidified graph (G', T', v', τ') and minor morphism $\varphi : G' \rightarrow G$, there exists an index i and a morphism of rigidified graphs $\rho : (G', T', v', \tau') \rightarrow (G_i, T_i, v_i, \tau_i)$ with $\varphi = \varphi_i \circ \Phi_{\leq g}(\rho)$.

Consider all possible rigidified graphs with at most $n + 2g - h$ edges and genus at least h and at most g . For each isomorphism class of such rigidified graph, choose a representative (G'', T'', v'', τ'') and add it to our collection (G_i, T_i, v_i, τ_i) as many times as there are minor morphisms $G'' \rightarrow G$. Then, take our collection of minor morphisms φ_i so that for each i , every possible minor morphism $G_i \rightarrow G$ appears in our collection. Note, there are only finitely many isomorphism classes of rigidified graphs with at most $n + 2k - h$ in $\mathcal{RG}_h^{\text{op}}$ and for such a rigidified graph there are only finitely many minor morphisms from the underlying graph to G . Thus our collection of rigidified graphs and minor morphisms is indeed finite.

Now, fix a rigidified graph (G', T', v', τ') with genus k where $h \leq k \leq g$ and a minor morphism $\varphi : G' \rightarrow G$. Let E' be the set of edges that are contracted under φ . In a morphism of rigidified graphs, we are only allowed to contract edges in the designated spanning tree. Thus, let ρ be the morphism of rigidified graphs $\rho : (G', T', v', \tau') \rightarrow (G'/(E' \cap T'), T'/(E' \cap T'), v', \tau')$ corresponding to contracting the edges of $E' \cap T'$. Clearly φ factors through $\Phi_{\leq g}(\rho)$. All that we must verify is that $(G'/(E' \cap T'), T'/(E' \cap T'), v', \tau')$ is a member of our collection. In other words, we must show that $G'/(E' \cap T')$ has at most $n + 2g - h$ edges. Let n' be the number of edges of G' . Since deleting an edge lowers the genus of the graph by 1, we know that under φ , exactly $k - h$ edges are deleted. Thus, $n' = n + k - h + |E'|$ or

equivalently,

$$|E'| = n' - (n + k - h).$$

Furthermore, we know that the number of edges in T' is equal to $n' - k$. Thus,

$$|E' \cap T'| \geq n' - (n + k - h) - k = n' - n - 2k + h.$$

This implies that the number of edges in $G'/(E' \cap T')$ is at most

$$n' - (n' - n - 2k + h) = n + 2k + h$$

which is no more than $n + 2g + h$ since $k \leq g$. □

2.2.2. Edge functors

Recall that given a minor morphism of graphs $\varphi : (V, A, \sigma) \rightarrow (V', A', \sigma')$, we get an inclusion of edge sets $\varphi^* : A'/\sigma' \rightarrow A/\sigma$. Thus, we have an **edge functor**

$$E : \mathcal{G}_{\leq g}^{\text{op}} \rightarrow \text{FI}$$

taking a graph G to its set of edges. Similarly, the edges of a rigidified graph are totally ordered and given minor morphism of rigidified graphs, the pullback inclusion on edge sets preserves this order. This gives an ordered edge functor

$$\Omega : \mathcal{RG}_{\leq g}^{\text{op}} \rightarrow \text{OI}$$

taking a rigidified graph to its ordered set of edges. Recall from Section 2.1.2 that $\Psi : \text{OI} \rightarrow \text{FI}$ is the forgetful functor that sends an ordered set to its underlying (unordered) set.

Lemma 2.2.9. $\Psi \circ \Omega$ and $E \circ \Phi_{\leq g}$ are naturally isomorphic functors $\mathcal{RG}_{\leq g}^{\text{op}} \rightarrow \text{FI}$.

Proof. This is clear from the definitions of the functors. □

Theorem 2.2.10. *The pair $(\mathcal{RG}_{\leq g}^{\text{op}}, \Omega)$ is Gröbner.*

To prove Theorem 2.2.10 we will need a labeled version of a planar rooted tree. This is a slightly generalized version of the notion of an S -labeled planar rooted tree in [PR19a]. Let L be a set equipt with a well quasi order \leq . An L -labeled planar rooted tree is a triple (T, v, ℓ) where (T, v) is a planar rooted tree and ℓ is a function from the set of vertices of T to L . An **L -labeled minor morphism of planar rooted trees** or simply **L -labeled minor morphism** $(T, v, \ell) \rightarrow (T', v', \ell')$ is a minor morphism $\varphi : (T, v) \rightarrow (T', v')$ of rigidified graphs of genus 0 (planar rooted trees are exactly the rigidified graphs of genus 0) such that $\ell'(w') \leq \ell'(\max \varphi^{-1}(w'))$ where $\max \varphi^{-1}(w')$ is the first vertex in the preimage of w' under φ with respect to the natural depth first order on the vertices of T . Let \mathcal{PT}_L denote the category whose objects are L -labeled planar rooted trees and whose morphisms are L -labeled minor morphisms. For a fixed L -labeled planar rooted tree (T, v, ℓ) , we may give a quasi order \leq_L to the L -labeled minor morphisms into (T, v, ℓ) . Namely, if $\varphi' : (T', v', \ell') \rightarrow (T, v, \ell)$ and $\varphi'' : (T'', v'', \ell'') \rightarrow (T, v, \ell)$ are L -labeled minor morphisms, then $\varphi' \leq_L \varphi''$ if and only if there exists an L -labeled minor morphism $\psi : (T'', v'', \ell'') \rightarrow (T', v', \ell')$ such that $\varphi'' = \varphi' \circ \psi$. Let $|(\mathcal{PT}_L^{\text{op}})_{(T, v, \ell)}|$ denote the poset of equivalence classes of L -labeled minor morphisms into (T, v, ℓ) under this quasi order.

Lemma 2.2.11. *Let L be a well-quasi-ordered set. Partially order the isomorphism classes of L -labeled planar rooted trees where $[(T', v', \ell')] \leq [(T, v, \ell)]$ if and only if there is a minor morphism of L -labeled planar rooted trees $(T, v, \ell) \rightarrow (T', v', \ell')$. This is a well-quasi-order.*

Corollary 2.2.12. *Let L be a well-quasi-ordered set. Fix an L -labeled planar rooted tree (T, v, ℓ) . The poset $|(\mathcal{PT}_L^{\text{op}})_{(T, v, \ell)}|$ is a well-quasi-order.*

Remark 2.2.13. As stated before, the notions of an L -labeled planar rooted tree and L -labeled minor morphism are, respectively, slight generalizations of the notions of an S -labeled planar rooted tree and an S -labeled contraction defined in [PR19a] where the set of labels was an (unordered) finite set. Lemma 2.2.11 and Corollary 2.2.12 are the analogues of and have identical proofs to [PR19a, Theorem 3.6] and [PR19a, Corollary 3.7] respectively. Thus, we omit their proofs here.

Recall from Section 2.1.2 that a quartet $\mu = (R, R', \varphi, m)$ for the pair $(\bigsqcup_{h \leq g} \mathcal{RG}_h^{\text{op}}, OE)$ consists of two rigidified graphs of genus at most g , $R = (G, T, v, \tau)$, $R' = (G', T', v', \tau')$, a minor morphism of rigidified graphs $\varphi : R' \rightarrow R$, and a map of sets m from the edges of R' to \mathbb{N} . We should think of m as assigning to each edge of R' a natural number. For any fixed rigidified graph R , We also have a quasi order on quartets whose first coordinate is R where $(R, R', \varphi', m') \leq (R, R'', \varphi'', m'')$ if there exists a minor morphism of rigidified graphs $\psi : R'' \rightarrow R'$ such that $\varphi'' = \varphi' \circ \psi$ and if e' is an edge in R' , $m'(e') \leq m''(\varphi^*(e'))$ where φ^* is the natural inclusion of the edges of R' to the edges of R'' . Denote the corresponding poset of equivalence classes under this quasi order by $|(\mathcal{RG}_{\leq g}^{\text{op}})_R^{OE}|$.

Fix a rigidified graph $R = (G, T, v, \tau)$ with genus $h \leq g$. Now for a suitable choice of L , we wish to encode each quartet with first coordinate R as an L -labeled planar rooted tree so that the poset $|(\mathcal{RG}_{\leq g}^{\text{op}})_R^{OE}|$ is equivalent to the poset

$|(\mathcal{PT}_L^{\text{op}})_{(T,v,\ell)}|$ where (T, v) is the planar rooted tree of R and ℓ is some suitable label. To this end, let $L = (\mathbb{N} \cup \{\star\})^{2h} \times \mathbb{N}$. Order $(\mathbb{N} \cup \{\star\})$ with the usual order on \mathbb{N} along with setting \star to be incomparable to all elements of \mathbb{N} . Then, give L the usual order on the cartesian product of posets.

Given a quartet of the form (R, R', φ, m) , where $R' = (G', T', v', \tau')$ the corresponding L -labeled planar rooted tree will be of the form (T', v', ℓ') for some labeling ℓ' . Note that R' must also have genus h since we only have morphisms of rigidified graphs between rigidified graphs of the same genus. Thus, R' has h extra edges not in T' . The planar rooted structure of (T', v') gives an orientation and ordering to these h extra edges (we orient them from smaller to larger vertex and then order them by the order on their terminating vertex). Call these extra edges e'_1, \dots, e'_h . For each $1 \leq i \leq h$, let w'_{2i-1} be the vertex at which e'_i originates and let w'_{2i} be the vertex at which e'_i terminates. Then for each vertex w' of T' , and each $1 \leq j \leq 2h$, define the j th coordinate of $\ell'(w')$ to be $m(e'_j)$ if $w' \geq w_j$ and \star otherwise. Then for each edge e' in T' , if w' is the vertex of e' further from the root, set the last coordinate of w' to $m(e')$. Finally set the last coordinate of $\ell'(v')$ to 0. The intuition behind this labeling is that the first $2h$ coordinates encode the location and weights given by m for the h edges not in T' . The last coordinate encodes the weights given by m of the edges in T' . Finally, let ℓ be the fixed labeling of (T, v) corresponding to the quartet $(R, R, \text{Id}_R, 0)$.

Lemma 2.2.14. *Let (R, R', φ, m) be a quartet and (T', v', ℓ') be the corresponding L -labeled planar rooted tree as defined above. Then φ induces an L -labeled morphism $\varphi_L : (T', v', \ell') \rightarrow (T, v, \ell)$.*

Proof. This follows from the proof of Lemma [PR19a, Lemma 3.8]. □

Lemma 2.2.15. *Let $\mu' = (R, R', \varphi', m')$ and $\mu'' = (R, R'', \varphi'', m'')$ be quartets with corresponding L -labeled planar rooted trees (T', v', ℓ') and (T'', v'', ℓ'') and define φ'_L and φ''_L as in Lemma 2.2.14. Then $\mu' \leq \mu''$ if and only if $\varphi'_L \leq \varphi''_L$ in $|\mathcal{PT}_L^{\text{op}}(T, v, \ell)|$*

Proof. Note that $\mu' \leq \mu''$ means there exists a minor morphism of rigidified graphs $\psi : R'' \rightarrow R'$ such that $\varphi'' = \varphi' \circ \psi$ and for any edge e' of R' , $m'(e') \leq m''(\psi^*(e'))$. The proof of [PR19a, Lemma 3.8] shows that such a minor morphism ψ is equivalent to an L -labeled minor morphism of L -labeled planar rooted trees $\psi_L : (T'', v'', \ell'') \rightarrow (T', v', \ell')$ such that $\varphi''_L = \varphi'_L \circ \psi_L$. This is equivalent to saying $\varphi'_L \leq \varphi''_L$. □

proof of Theorem 2.2.10. That the pair $(\mathcal{RG}_{\leq g}^{\text{op}}, \Omega)$ satisfies property (G1) follows from Corollary 2.2.7 and Proposition 2.1.11. The fact that $(\mathcal{RG}_{\leq g}^{\text{op}}, OE)$ satisfies property (G2) follows from Lemma 2.2.15 and Corollary 2.2.12. □

Theorem 2.2.16. *The pair $(\mathcal{G}_{\leq g}^{\text{op}}, E)$ is quasi-Gröbner*

Proof. This follows from Lemmas 2.2.8 and 2.2.9. □

2.2.3. Finite generation

Theorems 2.2.5 and 2.2.16 tell us that the categories of representations $\text{Rep}_k(\mathcal{G}_{\leq g}^{\text{op}})$ and $\text{Rep}_k(\mathcal{G}_{\leq g}^{\text{op}}, E)$ are locally Noetherian. That is, every submodule of any finitely generated module of either of these categories, is itself finitely generated. If \mathcal{M} is a module in one of these categories, this Noetherianity property allows us to conclude global properties about \mathcal{M} that only depend on the module \mathcal{M} itself and g . The first such property gives a bound on how fast $\mathcal{G}_{\leq g}^{\text{op}}$ -modules can grow. For the following statement, we write $e(G)$ for the number of edges of G , and $v(G)$ for the number of vertices and $\text{gen}(G)$ for the genus of G .

Theorem 2.2.17. *Let \mathcal{M} be a finitely generated $\mathcal{G}_{\leq g}^{\text{op}}$ -module over a field k . Say \mathcal{M} is generated by the graphs G_1, \dots, G_r . Let $N = \max_i |e(G_i)|$. Then there exists a polynomial $P_g \in \mathbb{Q}[x, y]$ of degree at most N such that, for any graph G of genus at most g ,*

$$\dim_k \mathcal{M}(G) \leq \tau(G)P(e(G), v(G)),$$

Proof. By 2.1.2, it will suffice to prove the theorem in the case where \mathcal{M} is the principal projective $\mathcal{G}_{\leq g}^{\text{op}}$ -module $\mathcal{P}_{G'}$, for some fixed graph G' of genus $h' \leq g$. In this case, for any graph G of genus $h \leq g$, the dimension of $\mathcal{P}_{G'}(G)$ is precisely the number of minor morphisms from G to G' . Any such morphism, up to an automorphism of G' , can be determined in the following way: First one picks a spanning tree of G in which all of the contractions and none of the deletions will take place. There are exactly h edges not in this spanning tree and we need to select $h - h'$ of them to delete, or equivalently h' of them not to delete. There are $e(G) - h$ in the spanning tree and we need to contract all but $e(G') - h'$ of those edges. Thus, the number of minor morphisms from G to G' , or equivalently $\dim \mathcal{P}_{G'}(G)$, has the following bound.

$$\dim_k \mathcal{P}_{G'}(G) \leq \tau(G) \binom{h}{h'} \binom{e(G) - h}{e(G') - h'} |\text{Aut}(G')|.$$

This implies

$$\dim_k M(G) \leq \tau(G) h^{h'} (e(G) - h)^{e(G') - h'} |\text{Aut}(G')|.$$

We can rewrite this using the fact that $h = e(G) - v(G) + 1$ and $h' = e(G') - v(G') + 1$, which gives

$$\dim_k M(G) \leq \tau(G) (e(G) - v(G) + 1)^{e(G') - v(G') + 1} (v(G) - 1)^{e(G') - h'} |\text{Aut}(G')|.$$

Since G' is fixed, the product of terms after $\tau(G)$ is a polynomial in $e(G)$ and $v(G)$ of degree at most $e(G')$. Since $\mathcal{P}_{G'}$ is generated by the single element $\text{id}_{G'} \in \mathcal{P}_{G'}(G')$, the result holds. \square

Remark 2.2.18. The bound of Theorem 2.2.17 is an improvement of a bound found in [PR19a]., and can be seen to be sharp. Consider the principal projective module P_{G_0} , over the graph G_0 consisting of a single vertex with no edges. A minor morphism from a graph G to G_0 is precisely determined by a choice of spanning tree for G . In particular,

$$\dim_k P_{G_0}(G) = \tau(G)$$

Our second global property shows the kinds of torsion that can appear in $\mathcal{M}(G)$ for \mathcal{M} a finitely generated modules in $\text{Rep}_k(\mathcal{G}_{\leq g}^{\text{op}})$ or $\text{Rep}_k(\mathcal{G}_{\leq g}^{\text{op}}, E)$ is independent of the graph itself.

Theorem 2.2.19. *Let \mathcal{M} be a finitely generated $\mathcal{G}_{\leq g}^{\text{op}}$ -module (resp. \mathcal{A}_E -module) over \mathbb{Z} . Then, there exists a non-negative integer ϵ_g such that for all graphs G of genus at most g , the torsion part of $\mathcal{M}(G)$ is annihilated by ϵ_g .*

Proof. If $t \in \mathcal{M}(G)$ is a torsion element, then for any minor morphism $\varphi : G' \rightarrow G$, the induced map $\mathcal{M}(G) \rightarrow \mathcal{M}(G')$ must send t to a torsion element. Thus, we have a submodule \mathcal{T} of \mathcal{M} , where \mathcal{T} sends a graph G to the torsion part of $\mathcal{M}(G)$. By assumption $\mathcal{M}(G)$ is a finitely generated, so Theorem 2.2.5 (resp. Theorem 2.2.16) tells us that \mathcal{T} is also finitely generated, by say the elements x_1, \dots, x_n . We may take ϵ_g to be the least common multiple of the annihilators of x_1, \dots, x_n . \square

Remark 2.2.20. It is important to note that Theorems 2.2.17 and 2.2.19 are very much not constructive. Thus, they give us no control over the polynomial that

bounds dimension growth or the torsion that can appear. They simply tell us that these properties are uniform for all graphs of genus $\leq g$.

In the remaining sections of Chapter II we give examples of finitely generated $\mathcal{G}_{\leq g}^{\text{op}}$ -modules and \mathcal{A}_E -modules and leverage Theorems 2.2.17 and 2.2.19 to prove certain graph invariants are uniform over all graphs with genus at most g for any fixed g .

2.3. Homology of configuration spaces of graphs

Let G be a graph and n a natural number. We can think of G as a 1-dimensional CW-complex whose 0-cells correspond to the vertices of G and whose 1-cells correspond to the edges of G . The **unordered configuration space** of n points in G , is the topological space

$$U_n(G) := \{(x_1, \dots, x_n) \in G^n \mid x_i \neq x_j\} / \mathfrak{S}_n$$

where \mathfrak{S}_n acts by permuting the coordinates of $(x_1, \dots, x_n) \in G^n$.

Fix an integer $g \geq 0$. Let $E : \mathcal{G}_{\leq g}^{\text{op}} \rightarrow \text{FI}$ be the edge functor sending a graph G to its set of edges and let \mathcal{A}_E be the associated $\mathcal{G}_{\leq g}^{\text{op}}$ -algebra for the pair $(\mathcal{G}_{\leq g}^{\text{op}}, E)$ over the ring $k = \mathbb{Z}$. That is, given a graph G of genus $\leq g$,

$$\mathcal{A}_E(G) = \mathbb{Z}[t_e \mid e \in E(G)] = A/\sigma.$$

For any graph G , let

$$\mathcal{H}_i(G) := \bigoplus_{n \in \mathbb{N}} H_i(U_n(G); \mathbb{Z}).$$

Our goal is to show that \mathcal{H}_i can be given the structure of a finitely generated graded \mathcal{A}_E -module.

2.3.1. The reduced Świątkowski complex

Let $G = (V, A, \sigma)$ be a graph. For a vertex $v \in V$, let $\mathcal{M}_v(G)$ be the free $\mathcal{A}_E(G)$ -module generated by the formal symbol \emptyset along with the set

$$h_v := \{a \in A \mid h(a) = v\}.$$

Equip a bigrading on $\mathcal{M}_v(G)$ where $\deg t_e = (0, 1)$ for all $e \in E(G)$, $\deg \emptyset = (0, 0)$, and $\deg a = (1, 1)$ for any $a \in h_v$. Let $\widetilde{\mathcal{M}}(G) \subseteq \mathcal{M}(G)$ be the submodule generated by \emptyset and elements of the form $a - a'$ for $a, a' \in h_v = \subseteq A$, let $[a], [a'] \in E(G)$ be their respective images in $E(G)$ and give $\widetilde{\mathcal{M}}(G)$ the differential ∂_v of degree $(-1, 0)$ by setting

$$\partial(a - a') := (t_{[a]} - t_{[a']})\emptyset \text{ and } \partial\emptyset = 0$$

and extending $\mathcal{A}_E(G)$ -linearly to all of $\widetilde{\mathcal{M}}(G)$. The **reduced Świątkowski complex** of G is

$$\widetilde{S}(G) := \bigotimes_{v \in V} \widetilde{\mathcal{M}}_v(G)$$

where the tensor product is taken over $\mathcal{A}_E(G)$.

In [ADCK19, Lemma C.7], An, Drummond-Cole, and Knudson show that whenever $\varphi : G \rightarrow G'$ is a minor morphism, there is an induced map $\tilde{\varphi}^* : \widetilde{S}(G') \rightarrow \widetilde{S}(G)$ of $\mathcal{R}(G')$ -modules. which gives \widetilde{S} the structure of an \mathcal{A}_E -module. Moreover, the \mathcal{A}_E -module structure on \widetilde{S} respects the bigrading and differential. Thus, for

each $i \in N$, $\tilde{S}_{i,\bullet}$ also has the structure of an \mathcal{A}_E -module and the differential in \tilde{S} can be seen as a morphism $\partial_i : \tilde{S}_{i,\bullet} \rightarrow \tilde{S}_{i-1,\bullet}$ of \mathcal{A}_E -modules. Taking homology in degree i , we obtain another \mathcal{A}_E -module

$$H_i(\tilde{S}) := \ker(\partial_i) / \text{im}(\partial_{i+1}).$$

Note that $H_i(\tilde{S})$ is graded via the second component of the grading on \tilde{S} . The following theorem appears in [ADCK19, Theorem 4.5 and Proposition 4.9].

Theorem 2.3.1. *Let G be a graph. For all $i > 0$, there exists a canonical isomorphism graded \mathcal{A}_E -modules $H_i(\tilde{S}(G)) \cong \mathcal{H}_i(G)$.*

2.3.2. Finite generation

Identifying $H_i(\tilde{S}(G))$ with $\mathcal{H}_i(G)$ under the isomorphism given in Theorem 2.3.1 gives us an \mathcal{A}_E -module structure on \mathcal{H}_i . In [ADCK20, Theorem 2.10], it is shown that for a graph G , the action of $\mathcal{A}_E(G)$ on $\mathcal{H}_i(G)$ comes from a map of topological spaces. Specifically, for an edge $e \in E(G)$, there is a **stabilization map** of topological spaces $\Sigma_{n,e} : U_n(G) \rightarrow U_{n+1}(G)$ which adds a point to the edge e and the generator $t_e \in \mathcal{A}_E(G)$ acts on $\mathcal{H}_i(G)$ via the induced map $(\Sigma_{n,e})_*$ on homology. Thus, the \mathcal{A}_E -module structure on \mathcal{H}_i can be fully described without the use of \tilde{S} . However, to prove finite generation of \mathcal{H}_i , we will need to take advantage of the isomorphism with $H_i(\tilde{S})$.

Proposition 2.3.2. *The \mathcal{A}_E -module $\tilde{S}_{i,\bullet}$ is generated by $\tilde{S}_{i,i}(G)$ for all graphs G of genus $\leq g$ with at most $2i$ edges. More precisely, let G_1, \dots, G_r be representatives of each isomorphism class of graph with at most $2i$ edges. For each $1 \leq s \leq r$,*

choose a basis for the finite rank free abelian group $\tilde{\mathcal{S}}_{i,i}(G_s)$. Taken all together, these finitely many classes generate $\tilde{\mathcal{S}}_{i,\bullet}$.

Proof. For each graph $G = (V, A, \sigma)$, $\mathcal{A}(G)$ -module $\tilde{\mathcal{S}}_{i,\bullet}(G)$ is generated by classes of the form

$$\xi := \bigotimes_{j=1}^i (a_j - a'_j) \otimes \bigotimes_{v \notin \{v_1, \dots, v_i\}} \emptyset,$$

where v_1, \dots, v_i are distinct vertices and, for each j , a_j and a'_j are arrows with

$$h(a_j) = v_j = h(a'_j).$$

We will call $v_1, \dots, v_i \in V$ **distinguished vertices** and $[a_1], [a'_1], \dots, [a_i], [a'_i] \in E(G)$ **distinguished edges**. If G has an edge e such that at least one vertex incident to e is not distinguished and e is not a loop, then the class ξ may be pulled back from the graph obtained from G by contracting e [PR19a, Section 5.3]. Furthermore, if e is not distinguished and not a bridge, then it is clear that the class ξ may be pulled back from the graph obtained from G by deleting e . Note that contracting an edge preserves the genus of a graph and deleting an edge decreases the genus, thus the graphs obtained either by deleting or contracting an edge e are objects in $\mathcal{G}_{\leq g}^{\text{op}}$. Thus, if we are looking for generators for $\tilde{\mathcal{S}}_{i,\bullet}$, we may restrict our attention to the graphs G in $\mathcal{G}_{\leq g}^{\text{op}}$ and classes $\xi \in \tilde{\mathcal{S}}_{i,\bullet}(G)$ for which every edge of G is either a distinguished edge or a bridge between distinguished vertices. Finally, it is straightforward to check that ξ may be written as a linear combination of classes for which every bridge between distinguished vertices is itself a distinguished edge. Since there are at most $2i$ distinguished edges, it is sufficient to consider those graphs with at most $2i$ edges. \square

Theorem 2.3.3. *For any $i \in \mathbb{N}$, the \mathcal{A}_E -module \mathcal{H}_i is finitely generated.*

Proof. When $i = 0$, the theorem is trivial. If $i > 0$, we know that \mathcal{H}_i is isomorphic to a subquotient of $\tilde{\mathcal{S}}_{i,\bullet}$ by Theorem 2.3.1. Theorems 2.1.6 and 2.2.5 imply that subquotients of finitely generated \mathcal{A}_E -modules are themselves finitely generated. Proposition 2.3.2 shows that $\tilde{\mathcal{S}}_{i,\bullet}$ is finitely generated so the result follows. \square

By Theorem 2.2.19, we immediately have the following Corollary.

Corollary 2.3.4. *For any $i \in \mathbb{N}$ and $g > 0$, there is an integer $\epsilon_{i,g}$ depending only on i and g such that for any graph G of genus $\leq g$ and any $n \in \mathbb{N}$, the torsion part of $H_i(U_n(G); \mathbb{Z})$ is annihilated by $\epsilon_{i,g}$.*

Note that $\tilde{\mathcal{S}}_{i,\bullet}$ is finitely generated as a \mathcal{A}_E -module but not as a $\mathcal{G}_{\leq g}^{\text{op}}$ -submodule of \mathcal{H}_i . In particular, $\tilde{\mathcal{S}}_{i,\bullet}(G)$ is infinite dimensional over \mathbb{Z} for any graph G . However, let us now fix $n \in \mathbb{N}$ and consider the $\mathcal{G}_{\leq g}^{\text{op}}$ -submodule of $\tilde{\mathcal{S}}_{i,n}$ of $\tilde{\mathcal{S}}_{i,\bullet}$.

Proposition 2.3.5. *The $\mathcal{G}_{\leq g}^{\text{op}}$ -module $\tilde{\mathcal{S}}_{i,n}$ is finitely generated. Moreover, finite list of generators can be taken from $\tilde{\mathcal{S}}_{i,n}(G)$ for graphs G with at most $2i$ edges.*

Proof. By Proposition 2.3.2, $\tilde{\mathcal{S}}_{i,\bullet}$ is generated as an \mathcal{A}_E -module by the finitely many generators of $\tilde{\mathcal{S}}_{i,i}(G)$ for the finitely many graphs G with at most $2i$ edges. Let x_1, \dots, x_r be this finite list of generators where for each $1 \leq j \leq r$, $x_j \in \tilde{\mathcal{S}}_{i,i}(G_j)$ for the graph G_j with $2i$ edges. Thus, as a $\mathcal{G}_{\leq g}^{\text{op}}$ -module, $\tilde{\mathcal{S}}_{i,n}$ is generated by all elements of the form $ax_j \in \tilde{\mathcal{S}}_{i,n}(G_j)$ where $a \in \mathcal{A}_E(G_j) = \mathbb{Z}[t_e \mid E(G_j)]$ is a monic monomial of degree $n - i$ of which there are finitely many. \square

Now applying Theorem 2.2.17 we obtain the following.

Corollary 2.3.6. *For any natural number i , and any natural number n , there exists a polynomial $P(x, y)$ of degree at most $2i$ such that for any graph of genus at most*

G , if V is the set of vertices of G ,

$$\dim H_i(U_n(G); \mathbb{Z}) \leq \tau(G)P(|E(G)|, |V|).$$

Proof. By Theorem 2.3.1, $\dim H_i(U_n(G); \mathbb{Z}) \leq \dim \tilde{\mathcal{S}}_{i,n}(G)$. By Proposition 2.3.5, $\tilde{\mathcal{S}}_{i,n}$ is finitely generated by finitely many elements coming from $\tilde{\mathcal{S}}_{i,n}(G')$ for graphs G' with at most $2i$ edges. Thus 2.2.17 gives the desired result. \square

2.4. The matching complex and monotone graph properties

For a graph G , a **matching** on G is a subset $S \subseteq E(G)$ of non-loop edges such that no two edges in S share a vertex. Note that any subset of a matching is also a matching and the empty set, \emptyset , is a matching. Thus, the collection of matchings on G forms a simplicial complex $M(G)$ whose vertices are the edges of G , which we call the **matching complex on G** .

The topology of matching complexes on the complete graphs K_n and the complete bipartite graphs $K_{m,n}$ have been well studied. Much of what is known about the topology of these complexes is outlined in [Wac03] and [Jon08]. We outline a few notable results on these complexes below.

The rational homology of these complexes is known due to the work of Bouc [Bou92] and Friedman and Hanlon [FH98]. However, much is still unknown regarding the torsion in the integral homology of these complexes. Shareshian and Wachs show that the homology of $M(K_n)$ exhibits 3-torsion for sufficiently large n and the homology of $M(K_{m,n})$ also contains 3-torsion for certain (but infinitely many) values of m and n [SW07]. Later, Jonsson showed that 5-torsion is present in the homology of $M(K_n)$ for sufficiently large n and found that there are elements

of order 5, order 7, order 11, and order 13 appearing in the homology of $M(K_n)$ for varying values of n [Jon10a]. For graphs other than K_n and $K_{m,n}$, not much is known about the topology of their matching complex.

There is also a natural generalization of the matching complex. Note, that for a graph G , and any subset $F \subseteq E(G)$, we have the induced subgraph G_F of G . Namely, G_F is the graph with $V(G_F) = V(G)$ and $E(G_F) = F$. Thus, F is a matching on G if and only if each vertex of G_F has degree at most 1. More generally, for any integer $d \geq 1$ we can consider subsets $F \subseteq E(G)$ such that each vertex of G_F has degree at most d . Call such a subset a **d -matching of G** . Note that the collection of all d -matchings on G forms a simplicial complex which we will denote $M_d(G)$. In particular, $M_1(G) = M(G)$, the matching complex. Complexes of d -matchings of graphs have been studied in [Sin20] and [Jon13].

We will study the homology of these complexes $M_d(G)$ by showing that M_d has the structure of a finitely generated $\mathcal{G}_{\leq g}^{\text{op}}$ -module.

2.4.1. The edge module

Fix the base ring $k = \mathbb{Z}$. Let \mathcal{E} be the $\mathcal{G}_{\leq g}^{\text{op}}$ -module which assigns to each graph G the free \mathbb{Z} -module with basis indexed by the edges of G . Given a minor morphism $\varphi : G \rightarrow G'$, the natural inclusion $\varphi^* : E(G') \rightarrow E(G)$ gives us a natural inclusion $\mathcal{E}(\varphi) : \mathcal{E}(G') \rightarrow \mathcal{E}(G)$. Thus, \mathcal{E} is a $\mathcal{G}_{\leq g}^{\text{op}}$ -module which we will call the **edge module**. It satisfies a very important property.

Lemma 2.4.1. *For any i , the i th tensor power of the edge module, $\mathcal{E}^{\otimes i}$, is a finitely generated $\mathcal{G}_{\leq g}^{\text{op}}$ -module generated by graphs with at most i edges.*

Proof. For any graph G , $\mathcal{E}^{\otimes i}(G)$ has basis given by i -tuples of edges of G . Let G be a graph with strictly more than i edges. Then for any tuple (e_1, \dots, e_i) of edges

of G , one may find an edge e of G that is not among the e_j . By contracting e , or deleting it in the case where e is a loop, one obtains a minor morphism $\varphi : G \rightarrow G'$ for a new graph G' . It is clear that the tuple (e_1, \dots, e_i) will be in the image of the map $\mathcal{E}(G') \rightarrow \mathcal{E}(G)$ induced by φ . Thus, $\mathcal{E}^{\otimes i}$ is generated by graphs with at most i edges, of which there are only finitely many up to isomorphism in $\mathcal{G}_{\leq g}^{\text{op}}$. \square

2.4.2. Finite generation

Now we are ready to prove that $H_i(M_d(-); \mathbb{Z})$ is a finitely generated $\mathcal{G}_{\leq g}^{\text{op}}$ -module.

Theorem 2.4.2. *For any $i \geq 0$ and any $d \geq 1$, $H_i(M_d(-); \mathbb{Z})$ is a finitely generated $\mathcal{G}_{\leq g}^{\text{op}}$ -module.*

Proof. For this proof, we are working with \mathbb{Z} -coefficients, though we will suppress this from the notation. Also, for a graph G , let $V(G)$ denote its set of vertices.

First, we argue that $H_i(M_d(-))$ is in fact a $\mathcal{G}_{\leq g}^{\text{op}}$ -module. Given a minor morphism $\varphi : G \rightarrow G'$ of connected graphs, we have an induced inclusion on edge sets $\varphi^* : E(G') \rightarrow E(G)$. Now, let $F' \subset E(G')$. If some vertex $v \in V(G)$ were incident to more than d edges in $\varphi^*(F')$, then $\varphi(v) \in V(G')$ would be incident to more than d edges in F' . Thus, if F' is a d -matching on G' , then $\varphi^*(F')$ is a d -matching on G . We may therefore, consider φ^* as a function

$$\varphi^* : \{d\text{-matchings on } G'\} \rightarrow \{d\text{-matchings on } G\}.$$

Furthermore, this map is compatible with taking boundaries of simplices and so we get a map of chain complexes

$$\varphi_{\bullet} : C_{\bullet}(M_d(G')) \rightarrow C_{\bullet}(M_d(G))$$

where $C_{\bullet}(M_d(G))$ denotes the simplicial chain complex of $M_d(G)$. This induces a map $H_i(M_d(G')) \rightarrow H_i(M_d(G))$. The above construction is compatible compositions of minor morphisms and so indeed $H_i(\mathcal{M}_d(-))$ is a $\mathcal{G}_{\leq g}^{\text{op}}$ -module.

To see that it is finitely generated, note that following the above construction, for any i , $C_i(M_d(-))$ also forms a $\mathcal{G}_{\leq g}^{\text{op}}$ -module and $H_i(M_d(-))$ is a subquotient of $C_i(M_d(-))$. Notice that $C_i(M_d(-))$ is a submodule of $\wedge^i \mathcal{E}$, the i th wedge power of the edge module, which is a quotient of $\mathcal{E}^{\otimes i}$. Therefore $H_i(M_d(-))$ is a subquotient of $\mathcal{E}^{\otimes i}$. By Lemma 2.4.1, $\mathcal{E}^{\otimes i}$ is a finitely generated $\mathcal{G}_{\leq g}^{\text{op}}$ -module and so Theorem 2.2.5 tells us $H_i(M_d(-))$ is itself finitely generated. \square

Corollary 2.4.3. *For any $i \geq 0$ and $d \geq 1$, there exists a polynomial $P_{d,g}(x, y)$ of degree at most i , such that for any connected graph G with genus $\leq G$,*

$$\dim H_i(M_d(G), ; \mathbb{Z}) \leq \tau(G)P(|E(G)|, |V(G)|).$$

Proof. By the proof of Theorem 2.4.2, $\dim H_i(M_d(G); \mathbb{Z}) \leq \dim \mathcal{E}^{\otimes i}$ which is finitely generated by graphs with at most i edges. Thus applying Theorem 2.2.17 gives the desired result. \square

Applying Theorem 2.2.19 we also have the following.

Corollary 2.4.4. *For any $i \geq 0$ and any $d \geq 1$, there exists a positive integer $\epsilon_{i,d,g}$ such that, for any connected graph G with genus at most g , the torsion part of $H_i(M_d(-))$ is annihilated by $\epsilon_{i,d,g}$.*

2.4.3. Other graph complexes

The matching complex of a graph is just one example of a simplicial complex on the edges of a graph. More generally, one can consider what is called a **monotone graph property**. This is a collection \mathcal{P} of (not necessarily connected) graphs, closed under isomorphisms, such that if $G \in \mathcal{P}$ and G' is another graph with $V(G') = V(G)$ and $E(G') \subseteq E(G)$, then $G' \in \mathcal{P}$. In other words, \mathcal{P} is closed under edge deletions. Note that, for any graph G , we obtain a simplicial complex $\Delta_{\mathcal{P}}(G)$ on the edges of G where the n -simplices of $\Delta_{\mathcal{P}}(G)$ correspond to graphs $G' \in \mathcal{P}$ with $V(G') = V(G)$, $E(G') \subset E(G)$, and $|E(G')| = n + 1$. Intuitively if we identify a graph in \mathcal{P} with its set of edges, the n -simplices of $\Delta_{\mathcal{P}}(G)$ are just subsets of $E(G)$ of size $n + 1$ that are in \mathcal{P} . See [Jon08] for a comprehensive reference on graph complexes. In particular [Jon08, Table 7.1] gives a list of monotone graph properties and what is known about the homotopy type of their corresponding simplicial complexes.

A minor morphism $\varphi : G \rightarrow G'$ induces an inclusion $\varphi^* : E(G') \rightarrow E(G)$. Suppose that \mathcal{P} is a monotone graph property with the extra condition that, if $\varphi : G \rightarrow G'$ is a minor morphism and H' is a simplex in $\Delta_{\mathcal{P}}(G')$ (that is, $H' \in \mathcal{P}$, $V(H') = V(G')$, and $E(H') \subseteq E(G')$), then the subgraph of G induced by the image $\varphi^*(E(H'))$ is also in \mathcal{P} . Call such a \mathcal{P} a $\mathcal{G}_{\leq g}^{\text{op}}$ -**monotone graph property**. In rough terms, a $\mathcal{G}_{\leq g}^{\text{op}}$ -monotone graph property is a monotone graph property that is also preserved under “uncontracting” edges. Note that by construction, if \mathcal{P} is a

$\mathcal{G}_{\leq g}^{\text{op}}$ -monotone graph property, then for any minor morphism $\varphi : G \rightarrow G'$, the n -simplices of $\Delta_{\mathcal{P}}(G')$ naturally include in the n -simplices of $\Delta_{\mathcal{P}}(G)$. This observation and the argument used in the proof of Theorem 2.4.2 yields the following result.

Theorem 2.4.5. *Let \mathcal{P} be a $\mathcal{G}_{\leq g}^{\text{op}}$ -monotone graph property. For any i , the assignment of G to the i th simplicial homology group $H_i(\Delta_{\mathcal{P}}(G))$ forms a finitely generated $\mathcal{G}_{\leq g}^{\text{op}}$ -module.*

Thus, for any $\mathcal{G}_{\leq g}^{\text{op}}$ -monotone graph property, one would obtain an analogous result to Corollaries 2.4.3 and 2.4.4 about dimension of and the torsion that can appear in the i th homology group of $\Delta_{\mathcal{P}}(G)$ as G ranges over all graphs with genus at most g .

2.5. Commutative algebra of graph complexes

Famously, the study of simplicial complexes is, in a formal sense, dual to the commutative algebra of squarefree monomial ideals through the Stanley-Reisner correspondence. We consider this perspective in this section. Fix a field \mathbf{k} . Recall that given the edge functor $E : \mathcal{G}_{\leq g}^{\text{op}} \rightarrow \text{FI}$ gives rise to another functor \mathcal{A}_E from $\mathcal{G}_{\leq g}^{\text{op}}$ to \mathbf{k} -algebras where

$$\mathcal{A}_E(G) = \mathbf{k}[t_e \mid e \in E(G)].$$

In [NR19], Nagel and Römer show there are uniform bounds on the degrees for nonvanishing graded Betti numbers for certain families of ideals that form modules over a particular FI-algebra. Using similar techniques, we start by describing some of the homological algebra for \mathcal{A}_E -modules and prove analogous results about the graded Betti numbers for families of ideals forming modules over the edge algebra \mathcal{A}_E .

Lemma 2.5.1. *For any finitely generated \mathcal{A}_E -module \mathcal{M} there exists a projective resolution F_\bullet of \mathcal{M} by finitely generated \mathcal{A}_E -modules.*

Proof. By Lemma 2.1.2, since \mathcal{M} is finitely generated, there exists graphs G_1, \dots, G_n and a surjection

$$\bigoplus_{i=1}^n \mathcal{P}_{G_i}^E \rightarrow M$$

where $\mathcal{P}_{G_i}^E$ is the principal projective \mathcal{A}_E -module at G_i . Let $F_0 = \bigoplus_{i=1}^n \mathcal{P}_{G_i}^E$. The kernel K of this surjection must be finitely generated by Theorem 2.2.16 and so again, we may find finitely many graphs H_1, \dots, H_m such that there is a surjection

$$\bigoplus_{i=1}^m \mathcal{P}_{H_i}^E \rightarrow K.$$

Let $F_1 = \bigoplus_{i=1}^m \mathcal{P}_{H_i}^E$. Continuing in this fashion, we build the entire projective resolution F_\bullet of M . □

Given two finitely generated \mathcal{A}_E -modules, \mathcal{M} and \mathcal{N} , we may take their tensor product over \mathcal{A}_E by taking the pointwise tensor product. This gives an \mathcal{A}_E -module $\mathcal{M} \otimes_{\mathcal{A}_E} \mathcal{N}$. Explicitly, for any graph G ,

$$(\mathcal{M} \otimes_{\mathcal{A}_E} \mathcal{N})(G) = \mathcal{M}(G) \otimes_{\mathcal{A}_E(G)} \mathcal{N}(G),$$

where the morphisms are defined in the obvious way. Now, take a projective resolution F_\bullet of \mathcal{M} as in Lemma 2.5.1. Tensoring this resolution with \mathcal{N} gives a chain complex $F_\bullet \otimes_{\mathcal{A}_E} \mathcal{N}$ of \mathcal{A}_E -modules. We define $\text{Tor}_i(\mathcal{M}, \mathcal{N})$ to be the homology of this chain complex. Specifically,

$$\text{Tor}_i(\mathcal{M}, \mathcal{N}) = H_i(F_\bullet \otimes_{\mathcal{A}_E} \mathcal{N}).$$

Remark 2.5.2. Given a finitely generated graded \mathcal{A}_E -module \mathcal{M} , we may upgrade the surjection of Lemma 2.1.2 to a surjection of graded \mathcal{A}_E -modules in the following way. Suppose M is generated by the elements x_1, \dots, x_n where for each i , $x_i \in \mathcal{M}(G_i)$. We may assume that each x_i is a homogeneous element of $\mathcal{M}(G_i)$ of degree d_i . Then, for each i , we give a grading to the principal projective \mathcal{A}_E -module $\mathcal{P}_{G_i}^E$ the grading where $\varphi \in \mathcal{P}_{G_i}^E(G)$ is homogeneous of degree d_i for any graph G , and minor morphism $\varphi : G \rightarrow G_i$. Then, the morphism of \mathcal{A}_E -modules

$$\bigoplus_{i=1}^n \mathcal{P}_{G_i}^E \rightarrow M$$

that sends id_{G_i} to x_i is a surjection of graded \mathcal{A}_E -modules. With this fact, we see that the projective resolution of Lemma 2.5.1 may, in fact, be upgraded to a projective resolution of graded \mathcal{A}_E -modules.

Let \mathcal{I}_E be the \mathcal{A}_E -submodule of \mathcal{A}_E itself where $\mathcal{I}_E(G)$ is the ideal $\langle t_e \mid e \in E(G) \rangle \subset \mathcal{A}_E(G)$ for each graph G . Call \mathcal{I}_E the **edge ideal of \mathcal{A}_E** . Then, $\mathbf{k}_{\mathcal{A}_E} = \mathcal{A}_E/\mathcal{I}_E$ is the graded \mathcal{A}_E -module taking each graph to the base field \mathbf{k} in degree 0, where all morphisms get sent to the identity and for each graph G , monomials of $\mathcal{A}_E(G)$ act by zero, and elements of $\mathbf{K} \subseteq \mathcal{A}_E(G)$ act by multiplication.

Lemma 2.5.3. *If \mathcal{M} is a finitely generated graded \mathcal{A}_E -module, then $\text{Tor}_i(M, \mathbf{k}_{\mathcal{A}_E})$ is also a finitely generated graded \mathcal{A}_S -module for all i .*

Proof. Let F_\bullet be a projective resolution of \mathcal{M} where each term is finitely generated as in Lemma 2.5.1. By Remark 2.5.2 we may assume F_\bullet is a projective resolution by graded \mathcal{A}_E -modules. For each i , $F_i \otimes \mathbf{k}_{\mathcal{A}_E}$ is isomorphic as an \mathcal{A}_E -module to the module $F_i/\mathcal{I}_E F_i$ taking a graph G to the $\mathcal{A}_E(G)$ -module $F_i(G)/\mathcal{I}_E(G)F_i(G)$.

Thus, $\text{Tor}_i(\mathcal{M}, \mathbf{k}_{\mathcal{A}_E})$ is a subquotient of a finitely generated \mathcal{A}_E -module and is thus, finitely generated by Theorem 2.2.10. \square

2.5.1. Betti numbers

Given a finitely generated graded \mathcal{A}_E -module M , a graph G , and integers $i, a \geq 0$, the i th graded Betti number of M in degree a with respect to G is defined to be

$$\beta_{i,a}^G(\mathcal{M}) := \dim_{\mathbf{k}}(\text{Tor}_i(M, \mathbf{k}_{\mathcal{A}_E})(G))_a,$$

the dimension of the degree a part of $\text{Tor}_i(\mathcal{M}, \mathbf{k}_{\mathcal{A}_E})(G)$.

Theorem 2.5.4. *Let \mathcal{M} be a finitely generated graded \mathcal{A}_E -module and fix $i \geq 0$. Then, there exists an integer n_i such that for any graph G of genus at most g ,*

$$\beta_{i,a}^G(\mathcal{M}) = 0 \quad \text{for all } a > n_i.$$

Proof. By Lemma 2.5.3, we know that $T_i := \text{Tor}_i(\mathcal{M}, \mathbf{k}_{\mathcal{A}_E})$ is a finitely generated \mathcal{A}_E module. Therefore, we can find a finite list x_1, \dots, x_n of generators of T_i where $x_j \in M(G_j)$. Moreover, we may assume that each x_j is homogeneous of degree d_j . Then, define $n_i := \max\{d_j\}_{j=1}^n$. We know that $\beta_{i,a}^G(M) = \dim_{\mathbf{k}}(T_i(G))_a$ is the number of generators of degree a in a minimal homogeneous generating set of $T_i(G)$ (see for instance [MS05, Lemma 1.32]). Thus, since the images of the x_j under maps induced by minor morphisms $G_j \rightarrow G$ contain a minimal homogeneous generating set of $T_i(G)$, if $a > n_i$, we must have $\beta_{i,a}^G(M) = 0$. \square

For any fixed graph G , we can consider $\mathcal{A}_E(G) = K[t_e | e \in E(G)]$ as an $\mathbb{N}^{E(G)}$ -graded K -algebra where t_e is in degree $\mathbf{v}_e \in \mathbb{N}^{E(G)}$ which has a 1 in the e coordinate and 0's elsewhere. We call this grading the **edge-grading on $\mathcal{A}_E(G)$** . Note that

$\mathbf{k}_{\mathcal{A}_E}$ is an edge-graded \mathcal{A}_E -module where $\mathbf{k}_{\mathcal{A}_E}(G)$ is concentrated in degree 0 $\in \mathbb{N}^{E(G)}$. Thus, if \mathcal{M} is an edge-graded \mathcal{A}_E -module, this induces an edge-graded \mathcal{A}_E -module structure on $\text{Tor}_i(\mathcal{M}, \mathbf{k}_{\mathcal{A}_E})$ for any i .

Now, given an edge-graded \mathcal{A}_E -module \mathcal{M} , a graph G , an integer $i \geq 0$, and $\mathbf{a} \in \mathbb{N}^{E(G)}$, we can consider the i^{th} **edge-graded Betti number of $\mathcal{M}(G)$ in degree \mathbf{a}** denoted $\beta_{i,\mathbf{a}}(\mathcal{M}(G))$ where

$$\beta_{i,\mathbf{a}}(\mathcal{M}(G)) = \dim_{\mathbf{k}}(\text{Tor}_i(\mathcal{M}, \mathbf{k}_{\mathcal{A}_E})(G))_{\mathbf{a}}.$$

Let $\text{sum}(\mathbf{a})$ denote the sum of the entries of \mathbf{a} . We see that

$$\beta_{i,\mathbf{a}}^G(\mathcal{M}) = \sum_{\text{sum}(\mathbf{a})=a} \beta_{i,\mathbf{a}}(\mathcal{M}(G)).$$

By Theorem 2.5.4, we have the following immediate result.

Corollary 2.5.5. *Let M be a finitely generated edge-graded \mathcal{A}_E -module and fix $i \geq 0$. Then, there exists an integer n_i such that for any graph G with genus at most g ,*

$$\beta_{i,\mathbf{a}}(\mathcal{M}(G)) = 0 \quad \text{whenever} \quad \text{sum}(\mathbf{a}) > n_i.$$

2.5.2. Squarefree monomial ideals

Let us now consider the case where \mathcal{M} is a finitely generated edge-graded \mathcal{A}_E -module such that for each graph G , $\mathcal{M}(G)$ is a squarefree monomial ideal of $\mathcal{A}_E(G) = \mathbf{k}[t_e \mid e \in E(G)]$. In this case, using the Stanley-Reisner construction, one can associate to $\mathcal{M}(G)$ a simplicial complex $\Delta_{\mathcal{M}}(G)$ on the set $E(G)$. Namely, the simplices of $\Delta_{\mathcal{M}}(G)$ are given by squarefree monomials of $\mathcal{A}_E(G)$ not contained in $\mathcal{M}(G)$. Now, let us identify each subset $\sigma \subseteq E(G)$ with its indicator vector in $\mathbb{N}^{E(G)}$

so that each such subset of $E(G)$ corresponds to a squarefree degree in the edge-grading of $\mathcal{A}_E(G)$. The commutative algebra of $\mathcal{M}(G)$ is intimately linked with the topology and combinatorics of $\Delta_{\mathcal{M}}(G)$. See [Sta96] for an in-depth treatment of this relationship. In particular, there is a nice relationship between the edge-graded Betti numbers of $\mathcal{M}(G)$ and the simplicial complex $\Delta_{\mathcal{M}}(G)$ due to Hochster (see [MS05, Corollary 5.12]).

Theorem 2.5.6 (Hochster’s formula). *The nonzero Betti numbers of $\mathcal{M}(G)$ lie only in square free degrees, namely degrees corresponding to subsets $\sigma \subset E(G)$. Furthermore,*

$$\beta_{i,\sigma}(\mathcal{M}(G)) = \dim_K \tilde{H}^{|\sigma|-i-2}(\Delta_{\mathcal{M}}(G)|_{\sigma}; K)$$

where $\Delta_{\mathcal{M}}(G)|_{\sigma} = \{\tau \in \Delta_{\mathcal{M}}(G) \mid \tau \subseteq \sigma\}$.

Hochster’s formula together with Corollary 2.5.5 give the following.

Corollary 2.5.7. *Fix $i \geq 0$. There exists an integer n_i such that for any graph G with genus at most g , if $\sigma \subseteq E(G)$ with $|\sigma| > n_i$, then*

$$\dim_K \tilde{H}^{|\sigma|-i-2}(\Delta_{\mathcal{M}}(G)|_{\sigma}; K) = 0.$$

As an application of the above results, we first define a specific \mathcal{A}_E -module which will be denoted $I_{\mathcal{L}^c}$. For a graph G , the *edge ideal* of G is the ideal $I \subseteq \mathbf{k}[t_v \mid v \in V(G)]$ generated by monomials $t_v t_w$ where v and w are connected by an edge in $E(G)$. Given a graph G , define the **line graph** of G denoted $\mathcal{L}(G)$ to be the simple graph whose vertices are the edges of G and two vertices of $\mathcal{L}(G)$ are adjacent if and only if they share a vertex in G . Then, define the **line graph complement** of G , denoted $\mathcal{L}^c(G)$, to be the complement of $\mathcal{L}(G)$. Namely, the

vertices of $\mathcal{L}^c(G)$ are the edges of G and two vertices of $\mathcal{L}^c(G)$ are adjacent if and only if they do not share a vertex in G .

Example 2.5.8. If G is the star graph - i.e. the tree with one vertex of degree n and all other vertices of degree 1 - then $\mathcal{L}^c(G)$ is immediately seen to be a disjoint collection of points. On the other hand, if G is the complete graph K_n , then $\mathcal{L}^c(G)$ is the Kneser graph $K(n, 2)$. indeed, certain authors refer to line graph complements as being generalized Kneser graphs for this reason [DMM20].

Now, let $I_{\mathcal{L}^c}$ be the \mathcal{A}_E -module taking a graph G to the edge ideal of $\mathcal{L}^c(G)$. We see that

$$I_{\mathcal{L}^c}(G) = \langle t_e t_f \mid e, f \in E(G) \text{ don't share a vertex} \rangle \subseteq \mathcal{A}_E(G).$$

Indeed, $I_{\mathcal{L}^c}(G)$ is a squarefree monomial ideal of $\mathcal{A}_E(G)$. Furthermore, if $\varphi : G \rightarrow G'$ is a minor morphism, and $e, f \in E(G)$ don't share a vertex, then $\varphi^*(e), \varphi^*(f) \in E(G')$ don't share a vertex. Thus $I_{\mathcal{L}^c}$ does, in fact, give an \mathcal{A}_E -module of squarefree monomial ideals.

Proposition 2.5.9. *The \mathcal{A}_E -module $I_{\mathcal{L}^c}$ is finitely generated.*

Proof. Note that $I_{\mathcal{L}^c}$ is a submodule of the edge ideal \mathcal{I}_E . We see \mathcal{I}_E is finitely generated by the graphs G_1 and G_2 , where G_1 consists of two vertices and a single edge connecting them and the graph G_2 consists of a single vertex and a loop at that vertex. This is because for any graph G and any edge $e \in E(G)$, we can find a minor morphism $G \rightarrow G_i$ for some $i \in \{1, 2\}$ such that e gets sent to the single edge in G_i . Thus, Theorem 2.2.16 tells us that $I_{\mathcal{L}^c}$ is a finitely generated \mathcal{A}_E -module. \square

Applying Corollary 2.5.5 for any fixed i , we immediately get the existence of a bound on the degrees of the nonzero edge-graded Betti numbers $\beta_{i, \mathbf{a}}(I_{\mathcal{L}^c}(G))$

that is uniform as G ranges over all graphs. Moreover, Corollary 2.5.7 tells us about the cohomology of some subcomplexes of the simplicial complexes $\Delta_{\mathcal{L}^c}(G)$ as G ranges over all graphs. We note that $\Delta_{\mathcal{L}^c}(G)$ is precisely what is known as the flag complex - or clique complex - of the line graph $\mathcal{L}(G)$.

2.6. Linear subspace arrangements of line graph complements

In this section we study the cohomology of a certain family of hyperplane arrangement complements. In particular, we prove a finite generation result that recovers and expands upon similar results present in [FWW19].

For a graph G let $V(G)$ denote its set of vertices and as usual, let $E(G)$ denote its set of edges. Let d be a positive integer, and let \mathbf{k} be either \mathbb{C} or \mathbb{R} . For each $e \in E(G)$, if $v, w \in V(G)$ are the endpoints of e , let $W_e \subseteq (\mathbf{k}^d)^V$ be the subspace

$$W_e := \{x \in (\mathbf{k}^d)^V \mid x_v - x_w = 0\}.$$

In the definition of W_e , note that x_v and x_w are elements of \mathbf{k}^d . The collection of all W_e as e ranges over the edges of G is a subspace arrangement called the **graphical arrangement of G** denoted $\mathcal{A}(G)$. In the case where $d = 1$, each W_e is a hyperplane of \mathbf{k}^V . In general, W_e is a codimension d subspace of $(\mathbf{k}^d)^V$. Now, let $\text{Conf}(G, K^d) = (K^d)^V - \mathcal{A}(G)$ be the space obtained by removing the subspaces of $\mathcal{A}(G)$. In the case where G is the complete graph K_m , $\text{Conf}(G, K^d)$ is the usual ordered configuration space of m points in \mathbf{k}^d .

To each graph G , let us assign it to the space $\text{Conf}(\mathcal{L}^c(G), K^d)$ where $\mathcal{L}^c(G)$ is the line graph complement defined in the previous section. We see that if $\varphi : G \rightarrow G'$ is a minor morphism, then the inclusion $\varphi^* : E(G') \rightarrow E(G)$ is an inclusion

$V(\mathcal{L}^c(G')) \rightarrow V(\mathcal{L}^c(G))$. Thus, φ^* induces a natural projection

$$\bar{\varphi} : (\mathbf{k}^d)^{V(\mathcal{L}^c(G))} \rightarrow (\mathbf{k}^d)^{V(\mathcal{L}^c(G'))}$$

Furthermore, φ^* preserves pairs of edges that do not share a vertex. This yields a natural inclusion $E(\mathcal{L}^c(G')) \rightarrow E(\mathcal{L}^c(G))$. Thus, if $x \in \text{Conf}(\mathcal{L}^c(G), \mathbf{k}^d)$, then $\bar{\varphi}(x) \in \text{Conf}(\mathcal{L}^c(G'), \mathbf{k}^d)$ so $\bar{\varphi}$ restricts to a map

$$\text{Conf}(\mathcal{L}^c(G), \mathbf{k}^d) \rightarrow \text{Conf}(\mathcal{L}^c(G'), \mathbf{k}^d).$$

The construction of the above map is functorial with respect to compositions of minor morphisms. Thus, we have a functor from $\mathcal{G}_{\leq g}$ to the category of topological spaces sending a graph G to $\text{Conf}(\mathcal{L}^c(G), \mathbf{k}^d)$. Then, taking the cohomology with \mathbb{Z} coefficients, gives a functor $\mathcal{G}_{\leq g}^{\text{op}}$ to the category of abelian groups.

Theorem 2.6.1. *For any i , the $\mathcal{G}_{\leq g}^{\text{op}}$ -module $H^i(\text{Conf}(\mathcal{L}^c(-), \mathbf{k}^d); \mathbb{Z})$ is a finitely generated $\mathcal{G}_{\leq g}^{\text{op}}$ -module where \mathbf{k} is either \mathbb{C} or \mathbb{R} .*

Proof. For a graph G , we know that topologically $\text{Conf}(\mathcal{L}^c(G), \mathbf{k}^d)$ is the complement of a subspace arrangement over \mathbb{R} where each subspace has real codimension $r = d$ when $\mathbf{k} = \mathbb{R}$ and $r = 2d$ when $K = \mathbb{C}$. Thought of as a real vector space, let U denote the ambient vector space for $\text{Conf}(\mathcal{L}^c(G), \mathbf{k}^d)$. For each subspace $\tilde{U} \subseteq U$ that is an intersection of some of the subspaces of the form W_f for $f \in E(G)$, fix an orientation of the quotient space U/\tilde{U} . In [dLS01, Corollary 5.6], de Loungueville and Schultz give a presentation for the cohomology ring of the complement of a subspace arrangement over \mathbb{R} where each subspace has the same codimension. This result tells says that the cohomology ring $H^\bullet(\text{Conf}(\mathcal{L}^c(G), \mathbf{k}^d); \mathbb{Z})$ has the following presentation:

If r is even

$$H^\bullet(\text{Conf}(\mathcal{L}^c(G), \mathbf{k}^d); \mathbb{Z}) \cong \bigwedge^\bullet \mathbb{Z}^{E(\mathcal{L}^c(G))} / I$$

where the ideal I is generated by elements of the form

$$\sum_{i=0}^k (-1)^i \epsilon(e_1, \dots, \widehat{e}_i, \dots, e_k) e_1 \wedge \dots \wedge \widehat{e}_i \wedge \dots \wedge e_k$$

for all subsets $\{e_1, \dots, e_k\} \subseteq E(\mathcal{L}^c(G))$ that form a cycle in $\mathcal{L}^c(G)$. Here, $\epsilon(e_1, \dots, e_k)$ is the sign of the determinant, with respect to our choices of orientations, of the isomorphism

$$U / (W_{e_1} \cap \dots \cap W_{e_k}) \rightarrow U / W_{e_k} \oplus \dots \oplus U / W_{e_1}$$

given by the canonical projections onto each summand. If r is odd

$$H^\bullet(\text{Conf}(\mathcal{L}^c(G), \mathbf{k}^d); \mathbb{Z}) \cong \text{Sym}^\bullet \mathbb{Z}^{E(\mathcal{L}^c(G))} / I$$

where the ideal I is generated by elements of the form

$$e^2 \quad \text{and} \quad \sum_{i=0}^k (-1)^i \epsilon(e_1, \dots, \widehat{e}_i, \dots, e_k) e_1 \cdots \widehat{e}_i \cdots e_k$$

for any $e \in E$ and all subsets $\{e_1, \dots, e_k\} \subseteq E(\mathcal{L}^c(G))$ that form a cycle in $\mathcal{L}^c(G)$. Here, $\epsilon(e_1, \dots, e_k)$ is defined as in the even case. Whether r is even or odd, the generator e is in degree $r - 1$ for any edge $e \in E(G)$.

We know cycles of $\mathcal{L}^c(G)$ correspond to minimal sets $\{f_1, \dots, f_k\}$ of edges in $E(G)$ that satisfying the condition that f_i and f_{i+1} do not share a vertex and f_1 and f_k do not share a vertex. Given a minor morphism $\varphi : G \rightarrow G'$, the inclusion

$\varphi^* : E(G') \rightarrow E(G)$ preserves pairs of edges that don't share a vertex, and so the induced map $E(\mathcal{L}^c(G')) \rightarrow E(\mathcal{L}^c(G))$ sends cycles to cycles. Furthermore, the inclusion $\varphi^* : E(G') \rightarrow E(G)$ gives a natural inclusion of the ambient real vector space U' for $\text{Conf}(\mathcal{L}^c(G'), K^d)$ into the ambient real vector space U for $\text{Conf}(\mathcal{L}^c(G), \mathbf{k}^d)$. Thus, if $\varphi^*(f') = f$ for $f' \in E(G')$, then under this inclusion $W_{f'} \subseteq W_f$. Thus, given orientations on all the quotient spaces U/\tilde{U} where \tilde{U} is an intersection of some of the W_f for $f \in E(G)$, fixing an orientation of U/U' determines all of the orientations of U'/\tilde{U}' . Therefore, the presentation for $H^\bullet(\text{Conf}(\mathcal{L}^c(G)), \mathbb{C}^d; \mathbb{Z})$ given above, is compatible with minor morphisms.

From this presentation, we see that for each i , the $\mathcal{G}_{\leq g}^{\text{op}}$ module $H^{i(r-1)}(\text{Conf}(\mathcal{L}^c(-), \mathbb{C}^d); \mathbb{Z})$ is a quotient of the i th tensor power of $\mathbb{Z}^{E(\mathcal{L}^c(G))}$,

$$\bigotimes_{j=1}^i \mathbb{Z}^{E(\mathcal{L}^c(-))}.$$

Recall the edge module \mathcal{E} from 2.4. Notice that $\mathbb{Z}^{E(\mathcal{L}^c(-))}$ is the submodule of the second tensor power of the edge module $\mathcal{E}^{\otimes 2}$, where $\mathbb{Z}^{E(\mathcal{L}^c(G))}$ is generated by elements of the form $f_i \otimes f_j$ where $f_i, f_j \in E(G)$ do not share a vertex.

We know that $\mathcal{E}^{\otimes 2}$ is finitely generated by 2.4.1. We have shown that, for any i , $H^i(\text{Conf}(\mathcal{L}^c(-), \mathbf{k}^d); \mathbb{Z})$ is a subquotient of a finitely generated $\mathcal{G}_{\leq g}^{\text{op}}$ -module so the result follows from Theorem 1.1.1. □

Remark 2.6.2. If $d = 1$, $\text{Conf}(\mathcal{L}^c(G), \mathbf{k})$ is the complement of a hyperplane arrangement in \mathbf{k}^V . When $\mathbf{k} = \mathbb{C}$, $H^\bullet(\text{Conf}(\mathcal{L}^c(G), K))$ is just the Orlik-Solomon algebra of the complex hyperplane arrangement where the generators live in degree 1. If $d > 1$, the above presentation shows that $H^\bullet(\text{Conf}(\mathcal{L}^c(G), K^d))$ is still isomorphic to the aforementioned Orlik-Solomon algebra, only now the generators

live in degree $2d - 1$. When $\mathbf{k} = \mathbb{R}$, $H^\bullet(\text{Conf}(\mathcal{L}^c(G), K))$ is the Cordovil algebra of the real hyperplane arrangement. If d is odd, then $H^\bullet(\text{Conf}(\mathcal{L}^c(G), K^d))$ is isomorphic to $H^\bullet(\text{Conf}(\mathcal{L}^c(G), K))$, only now the generators live in degree $d - 1$.

Example 2.6.3. Let $a, b \geq 1$ and let G be the graph with two vertices of degrees $a + 1$ and $b + 1$, respectively, connected to one another by a single edge. Put another way, G is two copies of star graphs, of degrees a and b , whose central vertices have been glued together by a new edge. Then $\mathcal{L}^c(G)$ is easily seen to be a complete bipartite graph $K_{a,b}$, disjoint union a point. In this case, we have

$$\text{Conf}(\mathcal{L}^c(G), \mathbb{C}^d) = \tilde{\mathcal{Z}}_{a+b}^D(\mathbb{C}^d) \times \mathbb{C}^d$$

where $\tilde{\mathcal{Z}}_{a+b}^D(\mathbb{C}^d)$ are the colored configuration spaces considered by Farb Wolfson and Wood in [FWW19], with D being the vertices of the complete bipartite graph colored in the obvious way. Observe moreover that the graph G can be seen as an edge with a and b leaves sprouted on its two vertices, respectively. Therefore, Proposition 2.6.4 below implies that Theorem 2.6.1 can be seen as a generalization of the stabilization phenomena observed by [FWW19].

The following proposition follows directly from [PR19a, Corollaries 4.5 and 4.7, [PR19a]].

Proposition 2.6.4. *Let M be a finitely generated $\mathcal{G}_{\leq g}^{\text{op}}$ -module over a field k , and let G be a fixed graph, with a distinguished collection of vertices v_1, \dots, v_r , and edges e_1, \dots, e_s . We write $G^{(n_1, \dots, n_r)}$ for the graph obtained from G by attaching n_i leaves to the vertex v_i , and $G_{(m_1, \dots, m_s)}$ for the graph obtained from G by subdividing the edge e_j , m_j times. Then there exist polynomials $P_1(n_1, \dots, n_r)$ and*

$P_2(m_1, \dots, m_s)$ such that

$$\dim_k M(G^{(n_1, \dots, n_r)}) = P_1(n_1, \dots, n_r)$$

$$\dim_k M(G_{(m_1, \dots, m_s)}) = P_2(m_1, \dots, m_s),$$

for all vectors (n_1, \dots, n_r) and (m_1, \dots, m_s) whose each component is sufficiently large.

CHAPTER III

CATEGORICAL VALUATIVE INVARIANTS OF MATROIDS

This chapter was written in collaboration with Nicholas Proudfoot and Lorenzo Vecchi.

3.1. Background on matroids

The primary objects of study in this section are **matroids**. Let E be a finite set. Often we will consider $E = [n] := \{1, 2, \dots, n\}$ as we only care about E up to bijections of sets. A matroid M on E is a pair $M = (E, \mathcal{B})$ where \mathcal{B} is a collection of subsets of E satisfying the following properties:

(B1) $\mathcal{B} \neq \emptyset$

(B2) If $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \setminus B_2$, then there exists $y \in B_2 \setminus B_1$ such that $(B_1 \setminus \{x\}) \cup \{y\} \in \mathcal{B}$.

The set E is called the **ground set** of M and the elements of \mathcal{B} are the **bases** of M .

Example 3.1.1. Let V be a vector space and $E = \{v_1, \dots, v_n\}$ be any spanning set of vectors in V . Then, taking \mathcal{B} to be the subsets of E that are bases of the vector space V , we see that \mathcal{B} satisfies property (B1) since E is a spanning set and satisfies property (B2) by the Steinitz exchange lemma so $M = (E, \mathcal{B})$ indeed forms a matroid on E .

It is a well known fact that properties (B1) and (B2) imply that all elements of \mathcal{B} have the same cardinality. The **rank** of a matroid M is the cardinality of any basis of M .

Example 3.1.2. Let E be finite set and fix a nonnegative integer $r \leq |E|$. Let $\mathcal{B} = \binom{E}{r}$, that is, \mathcal{B} consists of all subsets of E with cardinality r . Clearly \mathcal{B} satisfies property (B1) since $r \leq |E|$. Furthermore for any $B_1, B_2 \in \mathcal{B}$, if $x \in B_1 \setminus B_2$, we can pick any $y \in B_2 \setminus B_1$. Then, $(B_1 \setminus \{x\}) \cup \{y\}$ will be a subset of E with size r and therefore in \mathcal{B} . Thus, \mathcal{B} satisfies property (B2) so (E, \mathcal{B}) is a matroid called the **uniform matroid of rank r on E** and it is denoted $U_{r,E}$. In the case where $E = [n]$ for some positive integer n , we will denote this matroid as $U_{r,n}$.

Given a matroid $M = (E, \mathcal{B})$, a subset $I \subseteq E$ is called **independent** if $I \subseteq B$ for some basis $B \in \mathcal{B}$. In the case where M is a matroid as in Example 3.1.1, I is independent if and only if it is a linearly independent set of vectors. A subset $D \subseteq E$ is called dependent if it is not independent. A **circuit** C of M is a minimal dependent subset of E . Given a subset $S \subseteq E$ the **rank of S in M** denoted $\text{rk}_M(S)$ or just $\text{rk } S$ when the matroid in question is clear, is the cardinality of the largest independent subset of S . Note that the rank of a matroid M is equivalent to the rank of the entire groundset E . A **flat** of M is a subset $F \subseteq E$ that is maximal for its rank. That is, F is a flat of M if for any $x \in E \setminus F$, $\text{rk}(F) < \text{rk}(F \cup \{x\})$. The collection of flats of a matroid is closed under intersections and the poset of flats of a matroid ordered by inclusion forms a lattice graded by rank.

We have chosen to defined a matroid on the ground set E in terms of its bases, but it turns out that one can define a matroid in terms of any one of its independent sets, its dependent sets, its circuits, its rank function, or its flats. For simplicity we will stick with our definition in terms of bases.

3.1.1. Matroid operations

We now recall a few well known ways of building new matroids from existing ones. Let $M = (E, \mathcal{B})$ be a matroid. For a subset $S \subseteq E$ the **restriction** of M to S , denoted M^S is the matroid whose ground set is S and whose set of bases is $\{B \cap S \mid B \in \mathcal{B}\}$. Fix a maximal independent subset I of S . The **contraction** of M by S , denoted M_S is the matroid whose ground set is $E \setminus S$ and whose set of bases is $\{B' \subseteq E \setminus S \mid B' \cup I \in \mathcal{B}\}$. It turns out that M_S does not depend on our choice of I .

Given matroids $M_1 = (E_1, \mathcal{B}_1)$ and $M_2 = (E_2, \mathcal{B}_2)$, the **direct sum** of M_1 and M_2 , denoted $M_1 \oplus M_2$ is a matroid whose ground set is $E_1 \sqcup E_2$ and whose bases are of the form $B_1 \sqcup B_2$ for some $B_1 \in \mathcal{B}_1$ and $B_2 \in \mathcal{B}_2$. A matroid M is **disconnected** if $M = M_1 \oplus M_2$ for matroids M_1 and M_2 with nonempty ground sets. Otherwise, M is **connected**. Every matroid M can be written uniquely as a finite direct sum of connected matroids M_1, \dots, M_n with nonempty ground sets. Here, M_1, \dots, M_n are called the **connected components** of M . Notice that if M is a matroid on E such that $M = M_1 \oplus M_2$, if $S \subseteq E$ is the ground set of M_1 , then $M^S = M_1$ and $M_S = M_2$.

3.2. Matroid polytopes and valiative invariants

Let E be a finite set and fix a real vector space \mathbb{R}^E with a distinguished basis $\{e_i\}_{i \in E}$. A **polytope** in \mathbb{R}^E is defined to be the convex hull of finitely many points in \mathbb{R}^E , that is, the unique minimal convex subset of \mathbb{R}^E containing each of those finitely many points. Equivalently, the convex hull of p_1, \dots, p_n , denoted $\text{Conv}(p_1, \dots, p_n)$ is given by the set

$$\text{Conv}(p_1, \dots, p_n) = \left\{ \sum_{i=1}^n \lambda_i p_i \mid \lambda_i \geq 0 \text{ for each } i \text{ and } \sum_{i=1}^n \lambda_i = 1 \right\}.$$

The dimension $\dim P$ of a polytope P is the dimension of the smallest Euclidean subspace of \mathbb{R}^E that contains it. A polytope P is homeomorphic to a closed ball of the same dimension. A face of a polytope P is any intersection of P with a halfspace of \mathbb{R}^E that contains no points in the interior of P . Note that each face of a polytope is itself a polytope and each point on the boundary of a polytope is contained in the interior of a unique face of the polytope. The zero dimensional faces of a polytope are called **vertices**, the 1-dimensional faces are called **edges**, and the faces of codimension 1 are called **facets**.

Given a subset $S \subseteq E$, let $e_S = \sum_{i \in S} e_i \in \mathbb{R}^E$ be the indicator vector of S . For a matroid $M = (E, \mathcal{B})$, its **base polytope** denoted P_M is the set

$$P_M = \text{Conv}(\{e_B \mid B \in \mathcal{B}\}).$$

The collection of points e_B for $B \in \mathcal{B}$ are exactly the vertices of P_M . Thus, a matroid is determined by its base polytope. We can see that P_M lies in the hyperplane of \mathbb{R}^E where the coordinates sum to the rank of M . It is well known that the dimension of P_M is equal to the cardinality of E minus the number of connected components of M . Furthermore, each nonempty face of P_M is a polytope of the form $P_{M'}$ for some matroid M' .

Example 3.2.1. Consider the matroid $U_{2,4}$. Its ground set is $E = \{1, 2, 3, 4\}$ and every subset of E of size 2 is a basis so $P_{U_{2,4}}$ is the convex hull of the points $(1, 1, 0, 0)$, $(1, 0, 1, 0)$, $(1, 0, 0, 1)$, $(0, 1, 1, 0)$, $(0, 1, 0, 1)$, and $(0, 0, 1, 1)$ in \mathbb{R}^4 so $P_{U_{2,4}}$.

See Figure 1. Note that $U_{2,4}$ is a connected matroid, so its basis polytope is 3-dimensional.

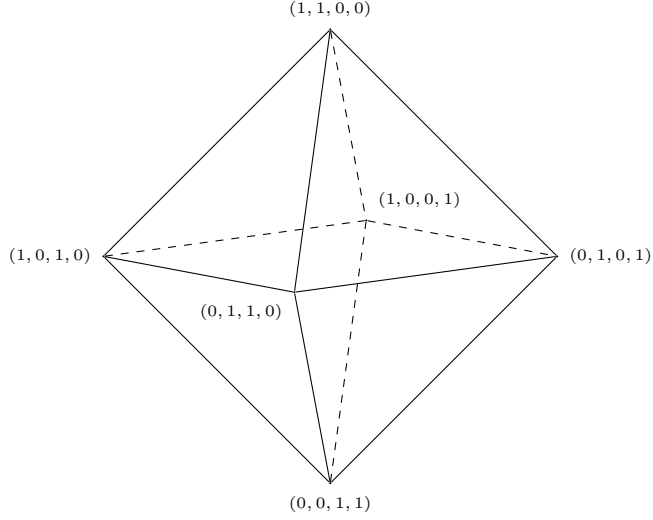


FIGURE 1. The base polytope for the matroid $U_{2,4}$ forms an octahedron.

Let $\text{Mat}(E)$ be the free abelian group generated by matroids on E . For each matroid M on E , let $1_M : \mathbb{R}^E \rightarrow \mathbb{Z}$ be the **indicator function** of P_M defined below:

$$1_M(x) = \begin{cases} 1 & \text{if } x \in P_M \\ 0 & \text{otherwise} \end{cases}.$$

We have a group homomorphism from $\text{Mat}(E)$ to the group of functions $\mathbb{R}^E \rightarrow \mathbb{Z}$ where M gets sent to 1_M . Let $I(E) < \text{Mat}(E)$ denote the kernel of this homomorphism and let $\text{Val}(E) = \text{Mat}(E)/I(E)$. We call $\text{Val}(E)$ the **valuative group** on E .

Let A be an abelian group. A homomorphism $\text{Mat}(E) \rightarrow A$ is called a **valuative matroid invariant** if f factors through $\text{Val}(E)$.

3.2.1. Matroid polytope decompositions

In order to determine whether a homomorphism $f : \text{Mat}(E) \rightarrow A$ is valiative, one needs to show that f sends $I(E)$ to 0. It turns out $I(E)$ is generated by decompositions of matroids which we now describe. Given a matroid M on the ground set E , a **decomposition** of M is a collection \mathcal{N} of matroids on E satisfying the following:

- If $N \in \mathcal{N}$, and N' is a matroid on E such that $P_{N'}$ is a face of P_N , then $N' \in \mathcal{N}$.
- If $N, N' \in \mathcal{N}$, then $P_N \cap P_{N'}$ is a face of both P_N and $P_{N'}$.
- $P_M = \bigcup_{N \in \mathcal{N}} P_N$.

Equivalently, a decomposition of M is a collection of matroids whose polytopes form the closed cells in a cell decomposition of M .

The matroids in \mathcal{N} are called **faces** of \mathcal{N} . A face $N \in \mathcal{N}$ is **internal** if P_N is not contained in the boundary of P_M . We write \mathcal{N}_{int} for the set of internal faces of \mathcal{N} . Note that \mathcal{N} can be recovered from \mathcal{N}_{int} by including all matroids which correspond to faces of P_N for each $N \in \mathcal{N}_{\text{int}}$.

Example 3.2.2. Given any matroid M , the collection \mathcal{N} consisting of M itself and all matroids that correspond to faces of P_M is a decomposition of M called the **trivial decomposition of M** . In this case $\mathcal{N}_{\text{int}} = \{M\}$.

Example 3.2.3. Here we describe the simplest nontrivial matroid decomposition.

Let $E = \{1, 2, 3, 4\}$, and consider $M = U_{2,4}$ the uniform matroid of rank 2 on E . Let N_1 be the matroid whose bases are all subsets of size 2 except for $\{3, 4\}$, let N_2 be the matroid whose bases are all subsets of size 2 except for $\{1, 2\}$ and

let $N_{1,2}$ be the matroid whose bases are all subsets of size 2 except for $\{1, 2\}$ and $\{3, 4\}$. Then, $\mathcal{N}_{\text{int}} = \{N_1, N_2, N_{1,2}\}$ form the internal faces for a decomposition \mathcal{N} of M . See Figure 2. In addition to the internal faces, \mathcal{N} also consists of the matroids corresponding to the eight triangular facets of P_M , the twelve edges of P_M , and the six vertices of P_M .

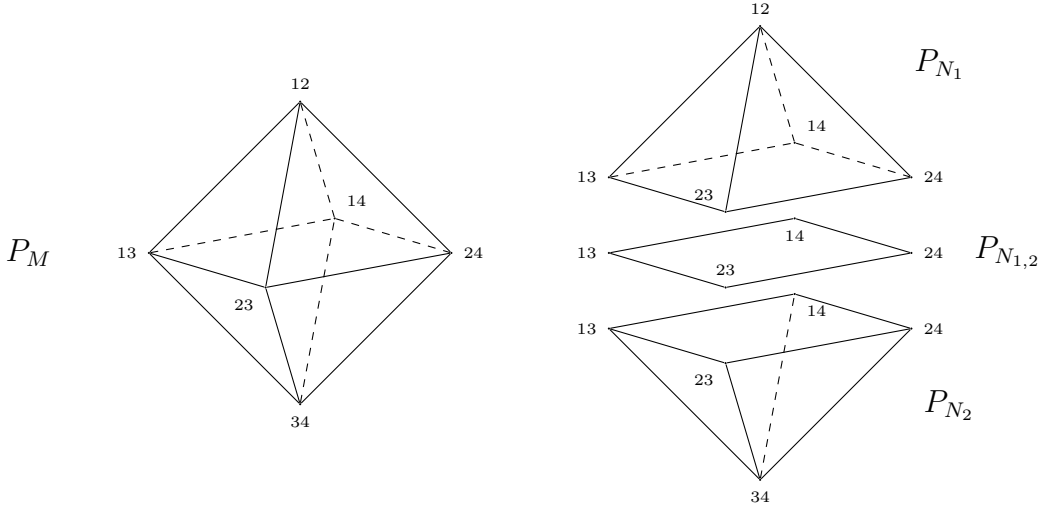


FIGURE 2. As in Example 3.2.3, the base polytope for the matroid M is given on the left. The base polytopes for the internal faces of a decomposition of M are given on the right. Here, the vertices are labeled with the basis that they correspond to.

The following theorem follows from [AFR10, Theorem 3.5] and [DF10, Corollary 3.9] and says that $I(E)$ is generated by decompositions of M .

Theorem 3.2.4. *The subgroup $I(E) < \text{Mat}(E)$ is the subgroup generated by elements of the form*

$$(-1)^{\dim P_M} M - \sum_{k=0}^{\dim P_M} (-1)^k \sum_{\substack{N \in \mathcal{N}_{\text{int}}, \\ \dim P_N = k}} N$$

as M ranges over all matroids on E and \mathcal{N} ranges over all decompositions of M .

Example 3.2.5. When \mathcal{N} is the trivial decomposition of a matroid M , the corresponding generator is $M - M = 0$ the identity in $\text{Mat}(E)$.

Example 3.2.6. Recall the decomposition \mathcal{N} described in Example 3.2.3. In this case, the corresponding generator of $I(E)$ is given by

$$-M + N_1 + N_2 - N_{1,2}.$$

3.2.2. Relaxation

We next review a large class of matroid decompositions that will be a source of examples in Section 3.5. Let M be a matroid on the ground set E . A flat $F \subseteq E$ is called **stressed** if the deletion M^F and the contraction M_F are both uniform matroids. Given a stressed flat F of rank r , Ferroni and Schröter define $\text{cusp}(F)$ to be the collection of k -subsets $S \subset E$ such that $|S \cap F| \geq r + 1$. If \mathcal{B} is the set of bases of M , they prove that $\mathcal{B} \cup \text{cusp}(F)$ is the set of bases for a new matroid $\text{Rel}(M, F)$, which they call the **relaxation** of M with respect to F [FS, Theorem 3.13]. If F is a circuit-hyperplane, then $\text{cusp}(F) = \{F\}$, and this coincides with the classical notion of relaxation in Matroid theory. See [Oxl11, Proposition 1.5.14]. If F is a hyperplane, this coincides with the notion of relaxation of a stressed hyperplane studied in [FNV22].

Let F be a rank r stressed of a matroid M on E . Let k be the rank of M . By definition of stressed, we necessarily have that $M_F = U_{k-r, E \setminus F}$ and $M^F = U_{r, F}$. Consider the matroid

$$\Pi_{r,k,E,F} := M_F \oplus M^F = U_{k-r, E \setminus F} \oplus U_{r, F}.$$

Notice that F is a stressed flat of F . Relaxing F in $\Pi_{r,k,E,F}$ gives the matroid

$$\Lambda_{r,k,E,F} := \text{Rel}(\Pi_{r,k,E,F}, F).$$

The following theorem is proven in [FS, Theorem 5.5].

Theorem 3.2.7. *Let M be a matroid of rank k on E and let F be a stressed flat of M of rank r . Then, the base polytope for $\Pi_{r,k,E,F}$ is the intersection of the base polytopes for $\Lambda_{r,k,E,F}$ and M and is a common face of both. Moreover, the base polytope for $\text{Rel}(M, F)$ is the union of the base polytopes for $\Lambda_{r,k,E,F}$ and M .*

In other words, Theorem 3.2.7 says that the matroids M and $\Lambda_{r,k,E,F}$ along with all matroids corresponding to faces of their base polytopes forms a decomposition of $\text{Rel}(M, F)$.

Example 3.2.8. Consider the matroid $N_{1,2}$ from Example 3.2.3. It has two stressed flats $F = 1, 2$ and $G = 3, 4$. In fact, $N_{1,2} = \Pi_{2,2,E,F} = \Pi_{2,2,E,G}$. The relaxation $\text{Rel}(N_{1,2}, F)$ is equal to the the matroid N_1 from Example 3.2.3 and the decomposition from Theorem 3.2.7 is trivial. The flat G remains a stressed flat in N_1 and the relaxation $\text{Rel}(N_1, G)$ gives the matroid $U_{2,4}$ and the decomposition coming from this relaxation is exactly the decomposition of $U_{2,4}$ from Example 3.2.3. Observe that starting with $N_{1,2}$ we could have relaxed G first and then relaxed F . Doing so would the same final decomposition of $U_{2,4}$ in the end.

Example 3.2.8 suggests a slight generalization of Theorem 3.2.7 in which we relax more than one stressed flat at once. Indeed, the following Proposition tells us we can do just that.

Proposition 3.2.9. *Let F_1 and F_2 be distinct rank r stressed flats of M . Then, the flat F_2 is a stressed flat of rank r in $\text{Rel}(M, F_1)$.*

Proof. Let \tilde{M} denote $\text{Rel}(M, F_1)$. Because $F_1 \neq F_2$, the intersection $F_1 \cap F_2$ must be a flat of M of rank $< r$. Since F_1 is stressed, every size r subset of F_1 is independent, so we conclude that $|F_1 \cap F_2| < r$. Thus, by [FS, Proposition 3.17], the rank of F_2 in \tilde{M} remains r .

Every basis of M is still a basis of \tilde{M} . Thus, every basis of M_{F_2} is still independent in \tilde{M}_{F_2} and every basis of M^{F_2} is independent in \tilde{M}^{F_2} . The fact that F_2 still has rank r in \tilde{M} implies that M_{F_2} and \tilde{M}_{F_2} have the same rank. Similarly M^{F_2} and \tilde{M}^{F_2} have the same rank. This means \tilde{M}_{F_2} and \tilde{M}^{F_2} must be uniform because M_{F_2} and M^{F_2} are uniform.

To see that F_2 is still a flat of M , observe that for any size r subset $S \subseteq F_2$ and any $i \in E \setminus F_2$, the set $S \cup i$ is independent in M and thus also in \tilde{M} . \square

Proposition 3.2.9 tells us that given a collection of stressed flats of M all with rank r , we may relax all of them one by one. Due to how relaxation works, the order in which we relax does not matter. This is because whenever F is a stressed flat of M , the elements of $\text{cusp}(F)$ are not bases in M . Thus, the sets of bases we add in our sequence of relaxations are all disjoint. Therefore, we may instead think of this process as relaxing all of these flats at once.

We will be interested in the following special case of this. Suppose that Γ is a finite group that acts on E by permutations, with the property that Γ fixes the matroid M . Let F be a stressed flat of M , and let $\mathcal{F} := \{\gamma F \mid \gamma \in \Gamma\}$ be the set of all stressed flats in the same orbit as F . Now define $\text{Rel}^\Gamma(M, F)$ to be the relaxation of M with respect to *all* of the elements of \mathcal{F} . More precisely, if \mathcal{B} is the collection of bases for M , then the collection of bases for $\text{Rel}^\Gamma(M, F)$ is

$$\mathcal{B} \cup \bigcup_{G \in \mathcal{F}} \text{cusp}(G).$$

Let \mathcal{N} be the collection of matroids consisting of M , $\Lambda_{r,k,E,G}$ for all $G \in \mathcal{F}$, and all matroids whose polytopes are faces of P_M or $P_{\Lambda_{r,k,E,G}}$. The following generalization of Theorem 3.2.7 follows from our discussion above along with repeated applications of Theorem 3.2.7, once for each $G \in \mathcal{F}$.

Theorem 3.2.10. *The collection \mathcal{N} is a decomposition of $\text{Rel}^\Gamma(M, F)$. If $M = \Pi_{r,k,E,F}$, then \mathcal{N} is the trivial decomposition of $\Lambda_{r,k,E,F}$. If not, then the only internal faces of \mathcal{N} are M , $\Lambda_{r,k,E,G}$ for all $G \in \mathcal{F}$, and $\Pi_{r,k,E,G}$ for all $G \in \mathcal{F}$.*

Example 3.2.11. Let $\Gamma = D_4$ act on the matroid $N_{1,2}$ from Example 3.2.3 by symmetries of the square $P_{N_{1,2}}$. For $F = \{1, 2\}$, $\mathcal{F} = \{F, G\}$ where $G = \{3, 4\}$. We could also achieve this by setting Γ equal to the subgroup $\mathfrak{S}_2 \subset D_4$, with the nontrivial element $\gamma \in \mathfrak{S}_2$ acting on E by the formula $\gamma(1) = 3$ and $\gamma(2) = 4$.

3.3. Matroid categories

Our goal in this section is to categorify the notion of a valiative matroid invariant.

Before we can define the categories we will work with, we need a notion of a map between matroids. Let E be a finite set. Given matroids M and M' on E , a **bijective weak map** from M to M' is a bijection $\sigma : E \rightarrow E$ such that whenever $S \subseteq E$ is an independent set of M' , we have that $\varphi^{-1}(S)$ is a independent set of M . In the case when M and M' have the same rank, this is equivalent to requiring that the preimage of a basis of M' is also a basis of M or said another way, that

$$P_{M'} \subseteq \varphi(P_M)$$

where $\varphi(P_M)$ is obtained from P_M by permuting the coordinates in \mathbb{R}^E via φ . We will only be concerned with bijective weak maps between matroids on the same rank, thus whenever we use the term *weak map*, we will assume that it is a weak map between two matroids of the same rank.

Let $\mathcal{M}(E)$ be the category whose objects are matroids on E and whose morphisms are bijective weak maps between matroids of the same rank. We will work with $\mathcal{M}(E)$ in Section 3.5. For now, let us consider the subcategory $\mathcal{M}_{\text{id}}(E)$ of $\mathcal{M}(E)$ whose objects are the same as those of $\mathcal{M}(E)$ but whose morphisms are only the weak maps between matroids of the same rank given by the identity map id_E on E . In other words, in $\mathcal{M}_{\text{id}}(E)$, given two matroids M and M' if $P_{M'} \subseteq P_M$ we have a unique morphism from M to M' . Let \mathcal{C} be the category of finite dimensional vector spaces over \mathbb{Q} or the category of finite dimensional graded vector spaces over \mathbb{Q} . A **categorical invariant** of matroids is any functor $\Phi : \mathcal{M}_{\text{id}}(E) \rightarrow \mathcal{C}$.

Example 3.3.1. Let \mathcal{C} be the category of finite dimensional vector spaces over \mathbb{Q} and let $\tau : \mathcal{M}_{\text{id}}(E) \rightarrow \mathcal{C}$ be the functor sending each matroid M to \mathbb{Q} and all morphisms to the identity map on \mathbb{Q} . We will call τ the **trivial categorical matroid invariant**.

Example 3.3.2. Let \mathcal{C} be the category of finite dimensional graded vector spaces over \mathbb{Q} . There is a functor $\text{OS} : \mathcal{M}_{\text{id}}(E) \rightarrow \mathcal{C}$ that takes a matroid M to its Orlik-Solomon algebra $\text{OS}(M)$. The generators of $\text{OS}(M)$ correspond to elements of the ground set E . id_E is a weak map of matroids $M \rightarrow M'$, we get a surjection of algebras (and thus a surjection of graded vector spaces) $\text{OS}(M) \rightarrow \text{OS}(M')$ sending the generator corresponding to an element $i \in E$ to the generator of $\text{OS}(M')$ also corresponding to i . See section 3.4 for a detailed treatment of this example.

3.3.1. The category $\mathcal{M}_{\text{id}}^{\wedge}(E)$

Let \mathcal{C} be the category of finite dimensional (graded) vector spaces over \mathbb{Q} and let $\Phi : \mathcal{M}_{\text{id}}(E) \rightarrow \mathcal{C}$ be a categorical matroid invariant. In order to define what it means for Φ to be valiative, we need a way to “upgrade” Φ to be a triangulated functor from a triangulated matroid category to $\mathcal{D}(\mathcal{C})$, the bounded derived category of \mathcal{C} .

If M is a matroid on E , consider the vector space over \mathbb{Q} given by

$$A_M = \text{Span}\{x - y \mid x, y \in P_M\} \cap \mathbb{Q}^E.$$

If id_E is a weak map $M \rightarrow M'$, then $P_{M'} \subseteq P_M$ and so $A_{M'}$ is a subspace of A_M .

Thus, we may define the 1-dimensional vector space

$$Q(M, M') = \bigwedge^{\dim P_M - \dim P_{M'}} (A_M / A_{M'}).$$

Notice that if M, M' , and M'' are matroids on E with $P_{M''} \subseteq P_{M'} \subseteq P_M$, then the wedge product gives a canonical isomorphism

$$\wedge : Q(M, M') \otimes Q(M', M'') \rightarrow Q(M, M''). \quad (3.3.1)$$

If id_E is not a weak map $M \rightarrow M'$, define $Q(M, M')$ to be the trivial vector space.

Now let us define a new category $\mathcal{M}_{\text{id}}^{\wedge}(E)$. The objects of $\mathcal{M}_{\text{id}}^{\wedge}$ will be formal direct sums of objects of $\mathcal{M}_{\text{id}}^{\wedge}(E)$. For matroids M, M' on E , take

$$\text{Hom}_{\mathcal{M}_{\text{id}}^{\wedge}(E)}(M, M') = Q(M, M').$$

A morphism between formal direct sums of matroids will be given by matrices whose entries consist of morphisms between the individual matroids and composition is given by matrix multiplication in conjunction with the wedge product in Equation 3.3.1. Notice that the category $\mathcal{M}_{\text{id}}^\wedge(E)$ is an additive category.

Remark 3.3.3. The reason we need the category $\mathcal{M}_{\text{id}}^\wedge(E)$, is because when we later define what it means for a categorical matroid invariant Φ to be valiative, we want our invariants to respect symmetries of M that preserve decompositions \mathcal{N} of M . That is, permutations of E that preserve the set of base polytopes of \mathcal{N} under the action on the ambient space \mathbb{R}^E . The issue that can arise is that such a symmetry may reverse the orientation on a base polytope and thus, if we are not careful with our definitions, we will be restricted to symmetries which also preserve orientations of the polytopes. Intuitively, the morphisms in $\mathcal{M}_{\text{id}}(E)$ keep track of the orientations of polytopes relative to each other. For instance, if $P_{N'}$ is a facet of P_N , then $Q(N, N')$ is naturally identified with $A_N/A_{N'}$ in which is naturally oriented by the class of the outward normal unit vector of $P_{N'}$ in P_N .

With Remark 3.3.3 in mind, in order to define what it means for Φ to be valiative, we want a canonical way to “upgrade” Φ to an additive functor from $\Phi^\wedge : \mathcal{M}_{\text{id}}^\wedge(E) \rightarrow \mathcal{C}$. At first, one may naively try to identify $\mathcal{M}_{\text{id}}(E)$ with a subcategory of $\mathcal{M}_{\text{id}}^\wedge(E)$ and extend Φ linearly to define Φ^\wedge . However, the 1-dimensional vector space $Q(M, M')$ only admits a canonical basis if $\dim M = \dim M'$ or if M' is a facet of M so most nontrivial weak maps in $\mathcal{M}_{\text{id}}(E)$ would need to be identified with trivial morphisms in $\mathcal{M}_{\text{id}}^\wedge(E)$. Thus, this strategy only works for if Φ is trivial on most morphisms. In the next section, we describe how define Φ^\wedge in general.

3.3.2. The determinant functor

Let \mathcal{C} be the category of finite dimensional (graded) vector spaces over \mathbb{Q} . In this section, we define an additive functor $\text{Det} : \mathcal{M}_{\text{id}}^{\wedge}(E) \rightarrow \mathbb{C}$. For any matroid M on E , let

$$\text{Det}(M) = \bigwedge^{\dim P_M} A_M^*.$$

For a nontrivial morphism σ in $\mathcal{M}_{\text{id}}^{\wedge}(E)$, i.e. a nontrivial element $\sigma \in Q(M, M')$, we have an isomorphism of vector spaces

$$(- \wedge \sigma) : \bigwedge^{\dim P_{M'}} A_{M'}^* \rightarrow \bigwedge^{\dim P_M} A_M^*$$

given by taking the wedge product with σ . Let $\text{Det}(\sigma) : \text{Det}(M) \rightarrow \text{Det}(M')$ be the dual of this isomorphism and extend linearly to define Det on all other morphisms of $\mathcal{M}_{\text{id}}^{\wedge}(E)$.

Let $\Phi : \mathcal{M}_{\text{id}}(E) \rightarrow \mathcal{C}$ be a categorical matroid invariant. Let $\Phi^{\wedge} : \mathcal{M}_{\text{id}}^{\wedge}(E) \rightarrow \mathcal{C}$ be the functor defined by $\Phi^{\wedge} = \Phi \otimes \text{Det}$. Concretely, for a matroid M on E , $\Phi^{\wedge}(M) = \Phi(M) \otimes \text{Det}(M)$ and given

$$\sigma \in \text{Hom}_{\mathcal{M}_{\text{id}}^{\wedge}(E)}(M, M'),$$

we set

$$\Phi^{\wedge}(\sigma) = \Phi(\text{id}_E) \otimes \text{Det}(\sigma) : \Phi^{\wedge}(M) \rightarrow \Phi^{\wedge}(M').$$

Given a categorical invariant $\Phi : \mathcal{M}_{\text{id}}(E) \rightarrow \mathcal{C}$, by tensoring with Det , we have built a functor $\Phi^{\wedge} : \mathcal{M}_{\text{id}}^{\wedge}(E) \rightarrow \mathcal{C}$. Let $\mathcal{K}_{\text{id}}(E)$ be the homotopy category of bounded chain complexes in $\mathcal{M}_{\text{id}}^{\wedge}(E)$. The category $\mathcal{K}_{\text{id}}(E)$ is triangulated and its Grothendieck group is naturally isomorphic to $\text{Mat}(E)$. We know that Φ^{\wedge}

naturally extends to a triangulated functor $\mathcal{K}_{\text{id}}(E) \rightarrow \mathcal{D}(\mathcal{C})$ which we will also call Φ^\wedge . Thus, passing to Grothendieck groups, this induces a group homomorphism $\Phi_* : \text{Mat}(E) \rightarrow \mathbb{Z}$. In this case, we will say that Φ **categorifies** the matroid invariant Φ_* .

Example 3.3.4. The trivial categorical invariant τ from example 3.3.1 categorifies the trivial invariant $\text{Mat}(E) \rightarrow \mathbb{Z}$ sending each matroid to $1 \in \mathbb{Z}$.

Example 3.3.5. The dimension of the i th graded piece of the Orlik-Solomon algebra $\text{OS}(M)$ is the coefficient of the degree i term in the Poincaré polynomial of a matroid. Thus, the categorical invariant OS from example 3.3.2 categorifies the invariant sending a matroid to its Poincaré polynomial in $\mathbb{Z}[t]$.

3.3.3. Valutive categorical matroid invariants

We are finally ready to define what it means for a categorical matroid invariant to be a valutive. Let \mathcal{N} be a decomposition of the matroid M on E and let $d = \dim P_M$. We define the chain complex $C_\bullet(\mathcal{N})$ in $\mathcal{M}^\wedge(E)$ as follows.

- Let $C_k(\mathcal{N}) = 0$ for $k < 0$ or $k > d + 1$.
- For $0 \leq k \leq d$, let $C_k(\mathcal{N}) = \bigoplus_{\substack{N \in \mathcal{N}_{\text{int}} \\ \dim P_N = k}} N$.
- Let $C_{d+1}(\mathcal{N}) = M$.

If $N \in \mathcal{N}_{\text{int}}$ with $\dim P_N = d$, then A_M/A_N is trivial so $Q(M, N)$ is canonically isomorphic to \mathbb{Q} . Thus, define the differential $C_{d+1}(\mathcal{N}) \rightarrow C_d(\mathcal{N})$ to be the diagonal map. For $N, N' \in \mathcal{N}_{\text{int}}$ with $\dim P_N = k$ and $\dim P_{N'} = k - 1$, if $P_{N'}$ is not a facet of P_N , then set the $N \rightarrow N'$ component of the differential $C_k(\mathcal{N}) \rightarrow C_{k-1}(\mathcal{N})$ to 0. If $P_{N'}$ is a facet of P_N , then $A_N/A_{N'}$ is 1-dimensional and thus equal to $Q(N, N')$.

In this case, set the $N \rightarrow N'$ component of the differential $C_k(\mathcal{N}) \rightarrow C_{k-1}(\mathcal{N})$ to be the class of the outward unit normal vector to $P_{N'}$ inside of P_N in the quotient space $A_N/A_{N'} = Q(N, N')$. It is straightforward to check that the differential squares to 0.

Example 3.3.6. Let \mathcal{N} be the trivial decomposition of M . The only element of \mathcal{N}_{int} is M itself. Thus, the chain complex $\mathbb{C}_\bullet(\mathcal{N})$ looks like

$$0 \rightarrow M \xrightarrow{1} M \rightarrow 0.$$

Recall that $Q(M, M)$ can be canonically identified with \mathbb{Q} .

Example 3.3.7. Consider the decomposition \mathcal{N} of $M = U_{2,4}$ given in Example 3.2.3. Then, the complex $C_\bullet(\mathcal{N})$ looks like

$$\begin{array}{ccccccc}
 & & & & N_1 & & \\
 & & & & \searrow & & \\
 0 & \longrightarrow & M & \xrightarrow{1} & \oplus & \longrightarrow & N_{1,2} \longrightarrow 0 \\
 & & & \searrow & \nearrow & & \\
 & & & & N_2 & &
 \end{array}$$

where the maps $N_1 \rightarrow N_{1,2}$ and $N_2 \rightarrow N_{1,2}$ correspond to the respective outward normal unit vector.

A categorical invariant $\Phi : \mathcal{M}_{\text{id}}(E) \rightarrow \mathcal{C}$ is **valuative** if for every decomposition \mathcal{N} of a matroid M on E , the chain complex $\Phi^\wedge(C_\bullet(\mathcal{N}))$ is an exact chain complex in \mathcal{C} . Equivalently, let $\mathcal{I}(E)$ be the full triangulated subcategory of $\mathcal{K}_{\text{id}}(E)$ spanned by the complexes $C_\bullet(\mathcal{N})$ for all decompositions \mathcal{N} of matroids on E . Let $\mathcal{V}(E)$ be the triangulated category quotient of $\mathcal{K}_{\text{id}}(E)$ by $\mathcal{I}(E)$. We see that a categorical invariant Φ is valuative if and only if $\Phi^\wedge : \mathcal{K}_{\text{id}}(E) \rightarrow \mathcal{D}(\mathcal{C})$ descends to a functor $\mathcal{V}(E) \rightarrow \mathcal{D}(\mathcal{C})$.

Recall that the Grothendieck group of $\mathcal{K}_{\text{id}}(E)$ can be identified with $\text{Mat}(E)$. Under this identification, the Euler characteristic of $C_{\bullet}(\mathcal{N})$ is exactly the generator of $I(E)$ corresponding to \mathcal{N} as in Theorem 3.2.4. Thus, the Grothendieck group of $\mathcal{I}(E)$ contains $I(E)$ which means that the Grothendieck group of $\mathcal{V}(E)$ is a quotient of $\text{Val}(E)$. If $\Phi : \mathcal{M}(E) \rightarrow \mathcal{C}$ is a valuative categorical invariant, then the induced matroid invariant $\Phi_* : \text{Mat}(E) \rightarrow \mathbb{Z}$ descends to $\text{Val}(E)$ and is therefore valuative.

Recall the trivial categorical invariant τ from Example 3.3.1.

Proposition 3.3.8. *The categorical invariant τ is valuative.*

Proof. Let $d = \dim P_M$ and let $C_{\leq d}(\mathcal{N})$ be the complex obtained from $C_{\bullet}(\mathcal{N})$ by setting the degree $d + 1$ term (i.e. the term corresponding to M itself) to 0. Recall that a decomposition of a matroid gives a cell decomposition of P_M and P_M is homeomorphic to a closed ball of dimension d . The terms in $C_{\leq d}(\mathcal{N})$ correspond to the internal faces of \mathcal{N} and we observe that $\tau^{\wedge}(C_{\leq d}(\mathcal{N}))$ is the cellular chain complex of a cell decomposition of a d -dimensional ball relative to its boundary. Thus, the homology of $\tau^{\wedge}(C_{\leq d}(\mathcal{N}))$ is one dimensional and concentrated in degree d .

We know that $\tau^{\wedge}(C_{\leq d}(\mathcal{N}))$ is a subcomplex of $\tau^{\wedge}(C_{\bullet}(\mathcal{N}))$ and the corresponding quotient complex is $\mathbb{Q}[-d - 1]$, the complex consisting of a single copy of \mathbb{Q} in degree $d + 1$. Thus, we have an exact sequence of chain complexes

$$0 \rightarrow \tau^{\wedge}(C_{\leq d}(\mathcal{N})) \rightarrow \tau^{\wedge}(C_{\bullet}(\mathcal{N})) \xrightarrow{q} \mathbb{Q}[-d - 1] \rightarrow 0$$

where the degree $d + 1$ part of q is an isomorphism $q_{d+1} : \tau^{\wedge}(M) \rightarrow \mathbb{Q}$. The map q_{d+1} determines an orientation Ω of P_M . We know that $\mathbb{Q}[-d - 1]$ is equal

to its homology. Meanwhile, we showed that the homology of $\tau^\wedge(C_{\leq d}(\mathcal{N}))$ can be identified with the relative homology of the pair $H_d(P_M, \partial P_M)$ which is 1-dimensional and concentrated in degree 1. In the long exact sequence on homology groups, the boundary map takes the generator of $\mathbb{Q}[-d-1]$ to the orientation class of $H_d(P_M, \partial P_M)$ determined by Ω . This map is an isomorphism which tells us that $\tau(C_\bullet^\Omega(\mathcal{N}))$ is trivial. \square

3.4. The Orlik–Solomon functor

One of the most important matroid invariants is the characteristic polynomial. The coefficients of the characteristic polynomial equal the dimensions of the graded pieces of Orlik-Solomon algebra. In this way, the Orlik-Solomon algebra categorifies the characteristic polynomial. In this section we recall the definition of the Orlik-Solomon algebra of a matroid and we prove that the categorical matroid invariant OS taking a matroid to its Orlik-Solomon algebra is valutive.

Let E be a finite and let M be a matroid on E . Fix a distinguished basis $\{x_i | i \in E\}$ of the vector space \mathbb{Q}^E and consider the exterior algebra $\Lambda(\mathbb{Q}^E)$. We will view $\Lambda(\mathbb{Q}^E)$ as a graded algebra over \mathbb{Q} where the generator x_e is in degree 1 for each $e \in E$. For a subset $S = \{s_1, \dots, s_k\}$ of E , define $w_S \in \Lambda(\mathbb{Q}^n)$ as

$$w_S = \sum_{i=1}^k (-1)^i x_{s_1} \cdots \widehat{x_{s_i}} \cdots x_{s_k}.$$

Let $J(M) \subseteq \Lambda(\mathbb{Q}^n)$ be the ideal generated by elements of the form w_S where S is a dependent set in M . Notice that the element w_S depends on the ordering of S only up to sign. Thus the ideal $J(M)$ is independent of the choices of orderings of the

dependent sets. The Orlik-Solomon algebra of M , denoted $\text{OS}(M)$, is the quotient $\Lambda(\mathbb{Q}^E)/J(S)$.

For matroids M and M' on E , notice that if id_E is a weak map from M to M' , then every independent subset of M' is also independent in M . Equivalently, every dependent set of M is also dependent in M' . This means $J(M)$ is contained in $J(M')$ and so we have a canonical surjection $\text{OS}(M) \rightarrow \text{OS}(M')$ sending the generator $x_e \in \text{OS}(M)$ to the generator $x_e \in \text{OS}(M')$ for each $e \in E$. Thus, we have a categorical invariant $\text{OS} : \mathcal{M}(E) \rightarrow \mathcal{C}$ where \mathcal{C} is the category of graded vector spaces over \mathbb{Q} which sends a matroid M to $\text{OS}(M)$ viewed as a graded vector space where the grading is inherited by the grading on $\Lambda(\mathbb{Q}^E)$.

By [OT92][Theorem 3.43], the Orlik-Solomon algebra of a matroid admits a concrete (non-canonical) basis depending on an ordering of the ground set which we now describe. Let us fix an identification of E with the set $[n] = \{1, \dots, n\}$ (i.e. fix a linear order on E). For any subset $S = \{i_1, \dots, i_k\} \subseteq E$ with $i_1 < \dots < i_k$, define the monomial $x_S \in \Lambda(\mathbb{Q}^E)$ as

$$x_S = x_{i_1} \cdots x_{i_k}.$$

Notice if S is a dependent set in M , then $x_S \in J(S)$ because

$$\begin{aligned} x_S &= x_S - x_{i_1} \sum_{j=2}^k (-1)^j (x_{i_1} \cdots \widehat{x_{i_j}} \cdots x_{i_k}) \\ &= -x_{i_1} \sum_{j=1}^k (-1)^j (x_{i_1} \cdots \widehat{x_{i_j}} \cdots x_{i_k}) \\ &= -x_{i_1} w_S. \end{aligned}$$

Thus, the image of x_S in $\text{OS}(M)$ is 0. A **broken circuit** of M is a subset $S \subseteq E = [n]$ that can be obtained by removing the minimal element from a circuit. A **non-**

broken circuit set (nbc-set for short) is a subset of E that does not contain a broken circuit. Let $\text{nbc}(M)$ denote the collection of nbc-sets of M . Note that both broken circuits and nbc-sets are necessarily independent sets as neither can contain a circuit of M . The images in $\text{OS}(M)$ of the set of monomials

$$\{x_S \in \Lambda(\mathbb{Q}^E) \mid S \in \text{nbc}(M)\}$$

forms a basis for $\text{OS}(M)$ as a vector space [OT92, Theorem 3.43]. We will refer to this basis as the **nbc-basis** of $\text{OS}(M)$.

Now, let us consider a new grading on $\Lambda(\mathbb{Q}^E)$ where the degree of x_i is i for each $i \in E$. This grading depends on the identification $E = [n]$. Also, by [OT92, Theorem 3.43], this grading induces a filtration on $\text{OS}(M)$ where, given $S \in \text{nbc}(M)$, the nbc-basis element x_S lies in the i th filtered piece if its degree in $\Lambda(\mathbb{Q}^E)$ is $\leq i$, and the associated graded ring $\text{gr OS}(M)$ under this filtration is isomorphic to $\Lambda(\mathbb{Q}^E)/J'(M)$ where $J'(M)$ is the ideal generated by $\{x_S \mid S \text{ is a circuit}\}$.

If id_E is a weak map from $M \rightarrow M'$, then any circuit of M is also dependent in M' . This means that every broken circuit in M contains a broken circuit of M' so if $S \in \text{nbc}(M')$, then $S \in \text{nbc}(M)$. Thus, the corresponding map $\text{OS}(M) \rightarrow \text{OS}(M')$ is filtered with respect to the filtration described above so it induces a map $\text{gr OS}(M) \rightarrow \text{gr OS}(M')$. For $S \in \text{nbc}(M)$, this map sends an nbc-basis element $x_S \in \text{gr OS}(M)$ to $x_S \in \text{gr OS}(M')$ if $S \in \text{nbc}(M')$ and to 0 otherwise.

Let \mathcal{N} be a decomposition of a matroid M on E , and let $d = \dim P_M$. For any $S \in \text{nbc}(M)$, let $\mathcal{N}_S \subseteq \mathcal{N}_{\text{int}}$ be the collection of internal faces $N \in \mathcal{N}_{\text{int}}$ such that $S \in \text{nbc}(N)$. The complex $\tau^\wedge(C_\bullet(\mathcal{N}))$ has a subcomplex consisting of the terms corresponding to matroids $N \in \mathcal{N}_{\text{int}} \setminus \mathcal{N}_S$ i.e. internal faces where

S is not an nbc-set. The fact that this is a subcomplex follows from the second sentence in the previous paragraph. Let $V_\bullet(\mathcal{N}, S)$ be the quotient of $\tau^\wedge(C_\bullet(\mathcal{N}))$ by this subcomplex. More concretely, $V_\bullet(\mathcal{N}, S)$ is defined by $V_{d+1}(\mathcal{N}, S) = \mathbb{Q}$ and

$$V_k(\mathcal{N}, S) := \bigoplus_{\substack{N \in \mathcal{N}_S \\ \dim P_N = k}} \mathbb{Q}$$

for all $0 \leq k \leq d$. The previous paragraph tells us we have the following isomorphism of complexes

$$\text{gr OS}^\wedge(C_\bullet(\mathcal{N})) \cong \bigoplus_{S \in \text{nbc}(M)} V_\bullet(\mathcal{N}, S) [-|S|]. \quad (3.4.2)$$

We will prove that $V_\bullet(\mathcal{N}, S)$ is exact, and use this to prove that OS is a valuative categorical invariant.

3.4.1. The nbc condition

Fix an identification $E = [n]$. For a matroid M on E , we will give equivalent notions of what it means for $S \in \text{nbc}(M)$. For each integer i , let $S_i = \{s \in S \mid s > i\}$, and let $H_{i,S}^+ \subseteq \mathbb{R}^E$ be the open half-space

$$H_{i,S}^+ = \left\{ x \in \mathbb{R}^E \mid \sum_{j \in S_i \cup \{i\}} x_j > |S_i| \right\}.$$

Lemma 3.4.1. *If M is a matroid on E , the following are equivalent:*

- (i). $S \in \text{nbc}(M)$.
- (ii). $S_i \cup \{i\}$ is independent for all $i \in E$.
- (iii). $P_M \cap H_{i,S}^+ \neq \emptyset$ for all $i \in E$.

(iv). $P_M \cap \bigcap_{i \in E}$

Proof. The fact that (i) and (ii) are equivalent follows immediately from the definition of a broken circuit. We know that $S_i \cup \{i\}$ is independent, if and only if it is contained in some basis B_i of M . This happens exactly when the indicator vector e_{B_i} is in the polynomial P_M . Since e_{B_i} lies in $H_{i,S}^+$, we have the equivalence of (ii) and (iii).

We have shown that (i), (ii), and (iii) are equivalent. It is obvious that (iv) implies (iii) and so to finish the proof, it suffices to show that (ii) implies (iv). Assume (ii) holds. For each $i \in E$ pick a basis B_i containing $S_i \cup \{i\}$. Then, choose real numbers $\epsilon_0 > \epsilon_1 > \dots > \epsilon_n$ such that $\epsilon_0 = 1$ and $\epsilon_n = 0$ and so that $\epsilon_i < \epsilon_{i-1}/(|S_i| + 1)$ for all $i \in E$. Define the point

$$x = \sum_{i \in E} (\epsilon_{i-1} - \epsilon_i) e_{B_i} \in \mathbb{R}^E.$$

Notice that x is a positive linear combination of the indicator vectors $\{e_{B_i} \mid i \in E\}$ whose coefficients sum to 1 so $x \in P_M$. We will show that $x \in H_{i,S}^+$ for all $i \in E$ which will complete the proof.

Let $i \in E$. We know that B_k contains S_i whenever $k \leq i$. Thus, for any $j \in S_i$, the j th coordinate of e_{B_k} is equal to 1. Therefore, the j th coordinate x_j of x is at least the sum of the coefficients of e_{B_k} for $k \leq i$. Concretely, for any $j \in S_i$,

$$x_j \geq \sum_{k \leq i} (\epsilon_{k-1} - \epsilon_k) = 1 - \epsilon_i.$$

Similarly, we know that B_i contains i and so $x_i \geq \epsilon_{i-1} - \epsilon_i$. Putting these together, we have

$$\begin{aligned}
\sum_{j \in S_i \cup i} x_j &= \sum_{j \in S_i} x_j + x_i \\
&\geq \sum_{j \in S_i} (1 - \epsilon_j) + (\epsilon_{i-1} - \epsilon_i) \\
&= |S_i|(1 - \epsilon_i) + (\epsilon_{i-1} - \epsilon_i) \\
&= |S_i| - (|S_i| + 1)\epsilon_i + \epsilon_{i-1} \\
&> |S_i|
\end{aligned}$$

where the last inequality follows from how we chose $\epsilon_0, \dots, \epsilon_n$. This shows that $x \in H_{i,S}^+$ as desired. \square

Proposition 3.4.2. *For any decomposition \mathcal{N} of M and any $S \in \text{nbc}(M)$, the complex $V_\bullet(\mathcal{N}, S)$ is exact.*

Proof. The strategy for this proof is similar to Proposition 3.3.8. Let $d = \dim P_M$ and let $V_{\leq d}(\mathcal{N}, S)$ be the complex obtained from $V_\bullet(\mathcal{N}, S)$ by setting the degree $d + 1$ to 0. Note that $V_{\leq d}(\mathcal{N}, S)$ is a subcomplex of $V_\bullet(\mathcal{N}, S)$ whose corresponding quotient complex is $\mathbb{Q}[-d - 1]$. Thus we have a short exact sequence of chain complexes

$$0 \rightarrow V_{\leq d}^\Omega(\mathcal{N}, S) \rightarrow V_\bullet^\Omega(\mathcal{N}, S) \rightarrow \mathbb{Q}[-d - 1] \rightarrow 0.$$

Just like in the proof of Proposition 3.3.8, we will show that $V_{\leq d}(\mathcal{N}, S)$ has 1-dimensional homology concentrated in degree d and that in the long exact sequence on homology, the generator of $\mathbb{Q}[-d - 1]$ gets sent to a generator of the homology of $V_{\leq d}(\mathcal{N}, S)$. This will imply that $V_\bullet(\mathcal{N}, S)$ is exact.

To do this, given a polytope $P \subseteq \mathbb{R}^E$, let \mathring{P} denote its relative interior. Let

$$U = \mathring{P}_M \cap \bigcap_{i \in E} H_{i,S}^+.$$

The equivalence of (i) and (iv) in Lemma 3.4.1 tells us that U is nonempty.

We see that U is an intersection of open convex subsets of a P_M so it is itself a nonempty open convex subset of P_M which implies U is homeomorphic to an open d -dimensional ball.

For each $N \in \mathcal{N}$ let $U_N = U \cap \mathring{P}_N$.

Notice that if N is not an interior face, then $U_N = \emptyset$ by definition of U . If N is an interior face, then $\mathring{P}_N \subseteq \mathring{P}_M$. In this case,

$$U_N = \mathring{P}_N \cap \bigcap_{i \in E} H_{i,S}^+$$

so Lemma 3.4.1 tells us that U_N is nonempty if and only if $S \in \text{NBC}(N)$. These U_N partition U so adding a single 0-cell, \star , and gluing ∂U to it, gives a cell decomposition of the quotient $\bar{U}/\partial U$. The (open) cells of this cell decomposition are given by the $\{U_N \mid N \in \mathcal{N}_{\text{int}} \text{ and } S \in \text{NBC}(N)\} \cup \{\star\}$. Therefore, the terms in $V_{\leq d}^{\Omega}(\mathcal{N})$ match the terms in the cellular chain complex used to compute the relative homology of the pair $(\bar{U}/\partial U, \star)$. To see that differential in these complexes match, it is enough to notice that if $P_{N'}$ is a face of P_N , then $U_{N'}$ is contained in the boundary of U_N and the outward normal direction of $U_{N'}$ relative to U_N is the same as that of $P_{N'}$ in P_N . Thus, $V_{\leq d}^{\Omega}(\mathcal{N}, S)$ can indeed be identified with the cellular chain complex for the pair $(\bar{U}/\partial U, \star)$ so its homology is identified with the reduced homology $\tilde{H}(\bar{U}/\partial U)$. Since $(\bar{U}/\partial U)$ is homeomorphic to the sphere S^d , the homology of this complex is one dimensional and concentrated in degree d .

To finish the proof, we simply observe that for the exact sequence

$$0 \rightarrow V_{\leq d}(\mathcal{N}) \rightarrow V_{\bullet}(\mathcal{N}) \xrightarrow{q} \mathbb{Q}[-d-1] \rightarrow 0,$$

the map q is an isomorphism in degree $d+1$. In other words we have an isomorphism $q_{d+1} : \tau^{\wedge}(M) \rightarrow \mathbb{Q}$. This isomorphism determines an orientation Ω of P_M . In the long exact sequence on homology, the boundary map sends the generator of $\mathbb{Q}[-d-1]$ to the orientation class of $H_d(\bar{U}/\partial U, \star)$ determined by Ω and so $V_{\bullet}^{\Omega}(\mathcal{N}, S)$ is exact. \square

Theorem 3.4.3. *The categorical invariant OS is valutive.*

Proof. We need to show that for any matroid M on E and any decomposition \mathcal{N} of M , the complex $\text{OS}^{\wedge}(C_{\bullet}^{\Omega}(\mathcal{N}))$ is exact. By Equation 3.4.2 and Proposition 3.4.2, $\text{OS}^{\wedge}(C_{\bullet}^{\Omega}(\mathcal{N}))$ admits a filtration whose associated graded is exact. The spectral sequence of the filtered complex has E_1 page equal to the homology of the associated graded and converges to the homology of the original complex. In this case, the E_1 page is zero, so the original complex must be exact as well. \square

3.5. Characters

Let \mathcal{C} be the category of finite dimensional vector spaces over \mathbb{Q} or the category of finite dimensional graded vector spaces over \mathbb{Q} . Consider the bounded derived category $\text{K}(\mathcal{C})$ of \mathcal{C} . If V_{\bullet} is an object of $\text{K}(\mathcal{C})$ and Γ is a finite group acting on V_{\bullet} via automorphisms in \mathcal{C} , then the **Euler characteristic** $\chi^{\Gamma}(V_{\bullet}) = \sum_k (-1)^k [V_k]$ is a graded virtual representation of Γ depending only on the Γ -equivariant isomorphism class of V .

Suppose $\Phi : \mathcal{M}(E) \rightarrow \mathcal{C}$ is valutive. Note, now Φ is a functor from $\mathcal{M}(E)$ not from $\mathcal{M}_{\text{id}}(E)$. Thus, we are now allowed weak maps that are non-identity permutations of the ground set E . When we say that Φ is valutive, we mean that Φ restricted to the subcategory $\mathcal{M}_{\text{id}}(E)$ is valutive. Then, for any decomposition \mathcal{N} of a matroid M on E , by definition of valutive, $\Phi^\wedge(C_\bullet(\mathcal{N}))$ is isomorphic to the zero object of $\text{K}(\mathcal{C})$. Thus, by our discussion in the previous paragraph, we immediately obtain the following result.

Proposition 3.5.1. *Let \mathcal{N} be a decomposition of a matroid M on the ground set E . Let Γ be a finite group acting on E which preserves \mathcal{N} . If $\Phi : \mathcal{M} \rightarrow \mathcal{C}$ is a valutive categorical invariant, then*

$$\chi^\Gamma(\Phi^\wedge(C_\bullet(\mathcal{N}))) = 0$$

as a virtual representation of Γ . More concretely, setting $d = \dim P_M$ and $\mathcal{N}_k = \{N \in \mathcal{N} \mid \dim P_N = k\}$, we have

$$\begin{aligned} (-1)^d \chi^\Gamma \left(\Phi(M) \otimes \text{Det}(M) \right) &= \sum_{k=0}^d (-1)^d \chi^\Gamma \left(\bigoplus_{N \in \mathcal{N}_k} \Phi(N) \otimes \text{Det}(N) \right) \\ &= \sum_{k=0}^d (-1)^d \chi^\Gamma \sum_{N \in \mathcal{N}_k/\Gamma} \text{Ind}_{\Gamma_N}^\Gamma \chi^{\Gamma_N} \left(\Phi(N) \otimes \text{Det}(N) \right) \end{aligned}$$

Example 3.5.2. Let \mathcal{N} be the decomposition of M in Example 3.2.3, and let $\Gamma = \mathfrak{S}_2$ act by swapping 1 with 3 and 2 with 4. Applying Proposition 3.5.1 to the trivial categorical invariant τ , we obtain the equation

$$-\text{Det}(M) = -\text{Det}(N_1) - \text{Det}(N_2) + \text{Det}(N_{12})$$

of virtual representations of \mathfrak{S}_2 . Since \mathfrak{S}_2 acts on P_M in an orientation preserving way, $\text{Det}(M)$ is the trivial representation. Since \mathfrak{S}_2 acts on $P_{N_{12}}$ in an orientation reversing way, $\text{Det}(N_{12})$ is the trivial representation. Finally, since \mathfrak{S}_2 swaps N_1 with N_2 , $\text{Det}(N_1) + \text{Det}(N_2)$ is the regular representation.

3.5.1. Equivariant relaxation

Let M be a matroid of rank k on the ground set E , equipped with an action of the group Γ . Let F be a stressed flat of rank r , and let $\mathcal{F} := \{\gamma F \mid \gamma \in \Gamma\}$. Let $\tilde{M} = \text{Rel}^\Gamma(M)$ be the matroid obtained by relaxing M with respect to every $G \in \mathcal{F}$, as described in Section 3.2.2. Let $\Gamma_F \subset \Gamma$ denote the stabilizer of F . Note that Γ_F acts on the sets F and $E \setminus F$, and therefore on the matroids $\Pi_{r,k,E,F}$ and $\Lambda_{r,k,E,F}$ from Section 3.2.2. Since r, k, E, F are fixed in this situation, let us simplify the notation and define

$$\Pi = \Pi_{r,k,E,F} \quad \text{and} \quad \Lambda = \Lambda_{r,k,E,F}.$$

Let \mathcal{C} be the category of finite dimensional graded vector spaces, and let $\Phi : \mathcal{M} \rightarrow \mathcal{C}$ be a valutive categorical invariant.

Proposition 3.5.3. *We have the following equality of virtual graded Γ -representations:*

$$\chi^\Gamma(\Phi^\wedge(\tilde{M})) = \chi^\Gamma(\Phi^\wedge(M)) + \text{Ind}_{\Gamma_F}^\Gamma \chi^{\Gamma_F}(\Phi^\wedge(\Lambda)) - \text{Ind}_{\Gamma_F}^\Gamma \chi^{\Gamma_F}(\Phi^\wedge(\Pi)).$$

Proof. If $M = \Pi$, then $\tilde{M} = \Lambda$ and the statement is trivial. Assume now that this is not the case. Let \mathcal{N} be the decomposition of \tilde{M} described in Theorem 3.2.10. For

any matroid N , let $d(N) := \dim P_N$. By Proposition 3.5.1, we have

$$\begin{aligned} (-1)^{d(\tilde{M})} \chi^\Gamma \left(\Phi^\wedge(\tilde{M}) \otimes \text{Det}(\tilde{M}) \right) &= (-1)^{d(M)} \chi^\Gamma \left(\Phi^\wedge(M) \otimes \text{Det}(M) \right) \\ &+ (-1)^{d(\Lambda)} \text{Ind}_{\Gamma_F}^\Gamma \chi^{\Gamma_F} \left(\Phi^\wedge(\Lambda) \otimes \text{Det}(\Lambda) \right) \\ &+ (-1)^{d(\Pi)} \text{Ind}_{\Gamma_F}^\Gamma \chi^{\Gamma_F} \left(\Phi^\wedge(\Pi) \otimes \text{Det}(\Pi) \right). \end{aligned}$$

To simplify this, we first note that

$$d(\tilde{M}) = d(M) = d(\Lambda) = d(\Pi) + 1.$$

Next, we observe that

$$A_{\tilde{M}} = A_M = A_\Lambda = A_\Pi \oplus \mathbb{Q} \cdot v_F$$

where v_F is given by the outward normal unit vector of P_Π in P_Λ . This implies that $\text{Det}(\tilde{M}) = \text{Det}(M)$ as representations of Γ . Since Γ_F fixes v_F , it also implies that $\text{Det}(\Lambda) = \text{Det}(\Pi) = \text{Res}_{\Gamma_F}^\Gamma \text{Det}(M)$ as representations of Γ_F . Thus

$$\begin{aligned} (-1)^{d(M)} \chi^\Gamma \left(\Phi^\wedge(\tilde{M}) \otimes \text{Det}(M) \right) &= (-1)^{d(M)} \chi^\Gamma \left(\Phi^\wedge(M) \otimes \text{Det}(M) \right) \\ &+ (-1)^{d(M)} \text{Ind}_{\Gamma_F}^\Gamma \chi^{\Gamma_F} \left(\Phi^\wedge(\Lambda) \otimes \text{Res}_{\Gamma_F}^\Gamma \text{Det}(M) \right) \\ &- (-1)^{d(M)} \text{Ind}_{\Gamma_F}^\Gamma \chi^{\Gamma_F} \left(\Phi^\wedge(\Pi) \otimes \text{Res}_{\Gamma_F}^\Gamma \text{Det}(M) \right) \\ &= (-1)^{d(M)} \chi^\Gamma \left(\Phi^\wedge(M) \otimes \text{Det}(M) \right) \\ &+ (-1)^{d(M)} \text{Ind}_{\Gamma_F}^\Gamma \chi^{\Gamma_F} \left(\Phi^\wedge(\Lambda) \right) \otimes \text{Det}(M) \\ &- (-1)^{d(M)} \text{Ind}_{\Gamma_F}^\Gamma \chi^{\Gamma_F} \left(\Phi^\wedge(\Pi) \right) \otimes \text{Det}(M). \end{aligned}$$

Dividing both sides by $(-1)^{d(M)} \text{Det}(M)$ yields the statement of the proposition. \square

3.5.2. Relaxing a stressed hyperplane

We conclude by applying Proposition 3.5.3 to the functor OS in the special case where the stressed flat $F = H$ is a hyperplane, meaning that $r = k - 1$. In this case, we have

$$\Pi_{k-1,k,E,H} = U_{1,E \setminus H} \oplus U_{k-1,H}.$$

The matroid $\Lambda_{k-1,k,E,H}$ obtained by relaxing $\Pi_{k-1,k,E,H}$ with respect to the stressed hyperplane H has as its bases those subsets $B \subset E$ of cardinality k with $|B \cap H| \geq k - 1$. It has no loops, and its simplification is the uniform matroid of rank k on the ground set $\bar{H} := H \sqcup \{*\}$ obtained from E by identifying all of the elements of $E \setminus H$.

For the rest of this section, we will write $h = |H|$ and identify H with $\{1, \dots, h\}$. If λ is a partition of h , we write V_λ to denote the corresponding Specht module for $\mathfrak{S}_h = \text{Aut}(H)$.

Lemma 3.5.4. *We have*

$$\text{OS}(\Lambda_{k-1,k,E,H}) - \text{OS}(\Pi_{k-1,k,E,H}) = \left(\bigwedge^{k-1} V_{[h-1,1]} \right) \otimes \left(\mathbb{Q}[1-k] \oplus \mathbb{Q}[-k] \right)$$

as graded virtual representations of \mathfrak{S}_h .

Proof. The matroids $\Lambda_{k-1,k,E,H}$ and $\Pi_{k-1,k,E,H}$ have the same independent sets of cardinality less than k , so their Orlik–Solomon algebras differ only in degrees $k - 1$ and k . Let us consider the degree k part first. We have

$$\text{OS}^k(\Pi_{k-1,k,E,H}) \cong \bigwedge^{k-2} V_{[h-1,1]}$$

and

$$\mathrm{OS}^k(\Lambda_{k-1,k,E,H}) \cong \bigwedge^{k-1} V_{[h-1,1]} \oplus \bigwedge^{k-2} V_{[h-1,1]},$$

which proves that the lemma is correct in degree k . The statement in degree $k - 1$ follows from the fact that, for any matroid M on a nonempty ground set, $\sum(-1)^i \mathrm{OS}^i(M) = 0$ as a virtual representation of the automorphism group of M . □

Proposition 3.5.3 and Lemma 3.5.4 have the following immediate corollary.

Corollary 3.5.5. *Let M be a matroid equipped with an action of the group Γ . Let H be a stressed hyperplane of M , and let \tilde{M} be the matroid obtained from M by relaxing γH for all $\gamma \in \Gamma$. Then*

$$\mathrm{OS}(\tilde{M}) = \mathrm{OS}(M) + \left(\mathrm{Ind}_{\Gamma_H}^{\Gamma} \mathrm{Res}_{\Gamma_H}^{\mathfrak{S}_h} \bigwedge^{k-1} V_{[h-1,1]} \right) \otimes \left(\mathbb{Q}[1-k] \oplus \mathbb{Q}[-k] \right)$$

as virtual graded representations of Γ .

Here, we use $\mathrm{Res}_{\Gamma_H}^{\mathfrak{S}_h}$ to denote the pullback of the natural homomorphism $\Gamma_H \rightarrow \mathfrak{S}_h$ given by the action of Γ_H on H .

3.5.3. The Vámos matroid

We conclude by showing a concrete application of Corollary 3.5.5. We will compute the Orlik-Solomon algebra of the **Vámos matroid** as a representation of its symmetry group. The Vámos matroid V_8 is the rank 4 matroid on the ground set $[8]$ whose set of bases are all cardinality 4 subsets of $[8]$ except for $H_1 = \{1, 2, 3, 4\}$, $H_2 = \{1, 4, 5, 6\}$, $H_3 = \{2, 3, 5, 6\}$, $H_4 = \{1, 4, 7, 8\}$, and

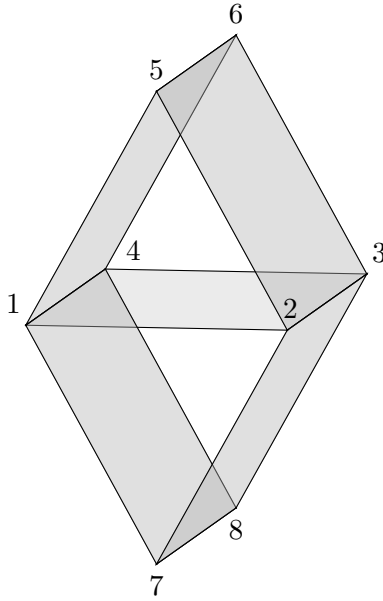


FIGURE 3. The Vámos matroid V_8 . The five shaded rectangles correspond to the five circuit hyperplanes of V_8 . Note, this does not depict the base polytope of V_8

$H_5 = \{2, 3, 7, 8\}$. The sets H_1, \dots, H_5 are all circuit-hyperplanes of V_8 and are therefore stressed hyperplanes. See Figure 3 for a visualization of V_8 .

The symmetry group Γ of V_8 is the subgroup of the symmetric group on $[8]$ generated by the elements (14) , $(12)(34)$, (56) , and $(57)(68)$. We see that $\Gamma \cong D_4 \times D_4$, where the first copy of D_4 is generated by the first two generators and the second copy of D_4 is generated by the second two generators. Under the action of Γ , the hyperplanes H_2, \dots, H_5 are all in the same orbit. Meanwhile, H_1 is fixed by Γ . Thus, to compute $\text{OS}(V_8)$ as a representation of Γ , we will apply Corollary 3.5.5 twice. First with $H = H_1$ and second with $H = H_2$. Doing so, we see that as a

virtual representation of Γ ,

$$\text{OS}^i(V_8) = \begin{cases} \text{OS}^i(U_{4,8}), & \text{for } i < 2 \\ \text{OS}^i(U_{4,8}) - \text{Ind}_{\Gamma_{H_1}}^{\Gamma} \text{Res}_{\Gamma_{H_1}}^{\mathfrak{S}_h} V_{[1,1,1,1]} \\ \quad - \text{Ind}_{\Gamma_{H_2}}^{\Gamma} \text{Res}_{\Gamma_{H_2}}^{\mathfrak{S}_h} V_{[1,1,1,1]}, & \text{for } i = 3, 4 \end{cases} \quad (3.5.3)$$

As a representation of \mathfrak{S}_8 , when $0 < i \leq 3$, we have

$$\begin{aligned} \text{OS}^i(U_{4,8}) &\cong \bigwedge^i \mathbb{Q}^8 \\ &\cong \bigwedge^i V_{[7,1]} \oplus \bigwedge^{i-1} V_{[7,1]} \\ &\cong V_{[8-i,1^i]} \oplus V_{[9-i,1^{i-1}]} \end{aligned}$$

and in degree 4, we have

$$\text{OS}^4(U_{4,8}) \cong \bigwedge^3 V_{[7,1]} \cong V_{[5,1,1,1]}.$$

Plugging this into Equation 3.5.3, as virtual representations of Γ , we have

$$\text{OS}^0(V_8) = \text{Res}_{\Gamma}^{\mathfrak{S}_8}(V_{[8]})$$

$$\text{OS}^1(V_8) = \text{Res}_{\Gamma}^{\mathfrak{S}_8}(V_{[7,1]} \oplus V_{[8]})$$

$$\text{OS}^2(V_8) = \text{Res}_{\Gamma}^{\mathfrak{S}_8}(V_{[6,1,1]} \oplus V_{[7,1]})$$

$$\begin{aligned} \text{OS}^3(V_8) &= \text{Res}_{\Gamma}^{\mathfrak{S}_8}(V_{[5,1,1,1]} \oplus V_{[6,1,1]}) - \text{Ind}_{\Gamma_{H_1}}^{\Gamma} \text{Res}_{\Gamma_{H_1}}^{\mathfrak{S}_h} V_{[1,1,1,1]} \\ &\quad - \text{Ind}_{\Gamma_{H_2}}^{\Gamma} \text{Res}_{\Gamma_{H_2}}^{\mathfrak{S}_h} V_{[1,1,1,1]}, \end{aligned}$$

$$\begin{aligned} \text{OS}^4(V_8) &= \text{Res}_\Gamma^{\text{Ss}}(V_{[5,1,1,1]}) - \text{Ind}_{\Gamma_{H_1}}^\Gamma \text{Res}_{\Gamma_{H_1}}^{\text{S}_h} V_{[1,1,1,1]} \\ &\quad - \text{Ind}_{\Gamma_{H_2}}^\Gamma \text{Res}_{\Gamma_{H_2}}^{\text{S}_h} V_{[1,1,1,1]}. \end{aligned}$$

Finally, we will compute these restrictions and inductions explicitly on characters. Note, $\Gamma_{H_1} = \Gamma$ as H_1 is fixed by the action of Γ . Meanwhile, Γ_{H_2} is the subgroup of Γ generated by (14), (56), (23) and (78). Since $\Gamma \cong D_4 \times D_4$, the irreducible characters of Γ are of the form $\chi_i \otimes \chi'_j$ where χ_i is an irreducible character of the copy of D_4 generated by (14) and (12)(34) and χ'_j is an irreducible character of the copy of D_4 generated by (56) and (57)(68). We use the following character tables for the two copies of D_4

| | e | (1, 4) | (14)(23) | (12)(34) | (1234) |
|----------|-----|--------|----------|----------|--------|
| χ_1 | 1 | 1 | 1 | 1 | 1 |
| χ_2 | 1 | -1 | 1 | -1 | 1 |
| χ_3 | 1 | -1 | 1 | 1 | -1 |
| χ_4 | 1 | 1 | 1 | -1 | -1 |
| χ_5 | 2 | 0 | -2 | 0 | 0 |

| | e | (5, 6) | (56)(78) | (57)(68) | (5768) |
|-----------|-----|--------|----------|----------|--------|
| χ'_1 | 1 | 1 | 1 | 1 | 1 |
| χ'_2 | 1 | -1 | 1 | -1 | 1 |
| χ'_3 | 1 | -1 | 1 | 1 | -1 |
| χ'_4 | 1 | 1 | 1 | -1 | -1 |
| χ'_5 | 2 | 0 | -2 | 0 | 0 |

Performing the restrictions and inductions, we have the following.

$$\text{char OS}^1(V_8) = \chi_1 \otimes \chi'_1 + \chi_1 \otimes \chi'_4 + \chi_1 \otimes \chi'_5 + \chi_3 \otimes \chi'_3 + \chi_4 \otimes \chi'_1 + \chi_5 \otimes \chi'_1.$$

$$\begin{aligned} \text{char OS}^2(V_8) &= \chi_1 \otimes \chi'_1 + \chi_1 \otimes \chi'_2 + 2\chi_1 \otimes \chi'_4 + 3\chi_1 \otimes \chi'_5 + \chi_2 \otimes \chi'_1 + 2\chi_4 \otimes \chi'_1 \\ &\quad + \chi_4 \otimes \chi'_4 + \chi_4 \otimes \chi'_5 + 3\chi_5 \otimes \chi'_1 + \chi_5 \otimes \chi'_4 + \chi_5 \otimes \chi'_5. \end{aligned}$$

$$\begin{aligned} \text{char OS}^3(V_8) &= 2\chi_1 \otimes \chi'_2 + \chi_1 \otimes \chi'_4 + 3\chi_1 \otimes \chi'_5 + 2\chi_2 \otimes \chi'_1 + \chi_2 \otimes \chi'_4 + \chi_2 \otimes \chi'_5 \\ &\quad + \chi_3 \otimes \chi'_1 + \chi_4 \otimes \chi'_1 + \chi_4 \otimes \chi'_2 + 2\chi_4 \otimes \chi'_4 + 3\chi_4 \otimes \chi'_5 \\ &\quad + \chi_5 \otimes \chi'_2 + 3\chi_5 \otimes \chi'_4 + 3\chi_5 \otimes \chi'_1 + 3\chi_5 \otimes \chi'_5. \end{aligned}$$

$$\begin{aligned}
\text{char OS}^4(V_8) &= \chi_1 \otimes \chi'_2 + \chi_1 \otimes \chi'_5 + \chi_2 \otimes \chi'_1 + \chi_2 \otimes \chi'_5 + \chi_2 \otimes \chi'_4 + \chi_3 \otimes \chi'_1 \\
&\quad + \chi_4 \otimes \chi'_2 + \chi_4 \otimes \chi'_4 + 2\chi_4 \otimes \chi'_5 + \chi_5 \otimes \chi'_1 \\
&\quad + \chi_5 \otimes \chi'_2 + 2\chi_5 \otimes \chi'_4 + 2\chi_5 \otimes \chi'_5.
\end{aligned}$$

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