A PARTIAL ORDER STRUCTURE ON THE SHELLINGS OF LEXICOGRAPHICALLY SHELLABLE POSETS.

by

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DISSERTATION ABSTRACT

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This dissertation has two main topics. The first is the introduction and in-depth study of a new poset theoretic structure designed to help us better understand the notion of lexicographic shellability of partially ordered sets (posets). Lexicographic shelling of posets was introduced by Björner via a type of poset labeling known as an EL-labeling and was generalized by Björner and Wachs to the notion of CL-labeling. We introduce and study a partial order structure on the maximal chains of any finite bounded poset P which has a CL-labeling λ. We call this partial order the maximal chain descent order induced by λ, denoted $P_\lambda(2)$. We show that this new partial order can be thought of as the structure of the set of shellings of P "derived from λ". A motivating example is the weak order of type A. Another especially interesting class of examples produces natural partial orders on standard Young tableaux. We prove several results about the cover relations of maximal chain descent orders in general. We characterize the EL-labelings whose maximal chain descent orders have the expected cover relations, and we prove that this is the case for many important families of EL-labelings.

The second main topic of this dissertation is that of determining the poset topology of two families of lattices known as s-weak order and the s-Tamari lattice.

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CHAPTER I

INTRODUCTION

Partially ordered sets (posets) arise naturally in many fields. For instance, face posets of simplicial complexes and cell complexes as well as intersection posets of hyperplane and subspace arrangements arise in topology and geometry (e.g. see Fomin and Shapiro (2000), Rietsch (2006), Rietsch and Williams (2008), Rietsch and Williams (2010), Hersh (2014), Orlik and Solomon (1980), Goresky and MacPherson (1988), and Knutson and Miller (2005)). Posets also arise in algebra where subgroup lattices of finite groups play an important role in characterizing which groups have properties such as solvability and supersolvability (e.g. see Stanley (1972), Quillen (1978), and Shareshian (2001)). The weak order and Bruhat order of any Coxeter group (e.g. see Dyer (1993)) and the Higher Bruhat orders in type A (e.g. see Elias (2016)) encode representation theoretic structure. The structure of these partial orders is often combinatorially interesting in its own right, and often provides useful information about the mathematical area from which the poset arose. For instance, the structure of the maximal chains of the weak order on any Coxeter group is closely related to the word problem for Coxeter groups (see Björner and Brenti (2010)) and encodes Matsumoto's theorem (see Matsumoto (1964) and Björner and Brenti (2010)).

Poset topology is a strong tool for studying the structure of partially ordered sets. It involves associating an abstract simplicial complex called the order complex to each poset. Then we can use the rich tools of topology to understand combinatorial structure of posets. For instance, poset topology is used in Björner, Lovász, and Yao (1992) to prove a complexity theory lower bound and is used in Shareshian (2001) to give a poset theoretic characterization of solvable finite groups.

This dissertation essentially focuses on two topics within poset topology. The first topic is a way to impose a useful new structure on the main types of lexicographic shellings of finite posets, thereby giving a clearer picture of the set of shellings of a poset. The second topic is a determination of the homotopy type of two families of partial orders known as s-weak order and the s-Tamari lattice lattice.

Recall that a shelling is a type of decomposition of the order complex of a poset, or of a simplicial complex more generally, specifically one which provides strong topological, combinatorial, and algebraic information about the complex. (See Chaper II background and precise definitions.) Not all simplicial complexes nor even all posets posses a shelling, but many natural and important posets arising from various areas of mathematics do. One powerful method of producing poset shellings is to assign labels to cover relations in the poset subject to certain conditions. While a class of edge labelings of posets known as R-labelings had previously been used in purely combinatorical ways to compute Möbius functions (see Stanley (1972) and Stanley (1974)), EL-labelings were introduced by Björner (1980) specifically to study poset topology by producing shellings of poset order complexes. This provided a topological way to compute the Möbius functions of many posets (via Hall's theorem) and to prove that the order complexes of many posets are Cohen-Macaulay. Björner and Wachs (1982) then generalized EL-labelings to CL-labelings which allow edge labels to depend on chains below the edge in the poset. They used this generalization to prove that every closed interval in the Bruhat order of any Coxeter group is shellable and further that every open interval in any Bruhat order is homeomorphic to a sphere. Björner and Wachs (1996) also extended these labeling techniques to non-graded posets. The shellings induced by a CL-labeling are called lexicographic shellings.

Lexicographic shellability of posets has proven itself to be one of the most effective tools for understanding the topological and combinatorial structure of many important families of posets . Such shellings may be used to compute the homotopy types and the h-vectors of poset order complexes and, in the graded case, to prove Cohen-Macaulayness of the associated Stanley-Reisner rings. For a thorough introduction to lexicographic shellability and many of its broad uses, see Lectures 3 and 4 of Wachs (2007).

The main theme of our work related to lexicographic shellability is to introduce a partial order $P_\lambda(2)$ on the maximal chains of any finite bounded poset P endowed with a CL-labeling λ. We prove that $P_\lambda(2)$ encodes the structure of the set of shellings induced by λ, including (but not limited to) the lexicographic shellings. We call the partial order $P_\lambda(2)$ the **maximal chain descent order** induced by λ.

The beginning of our work on the combinatorics of maximal chain descent orders is an in-depth analysis of their cover relations. This turns out to be more subtle than one might first expect. There are certain order relations which one might expect to be cover relations, but which are not. Nonetheless, we give two quite checkable sufficient conditions for an EL-labeling to induce a maximal chain descent order with the expected cover relations. We also prove that many well known families of EL-labelings satisfy these sufficient conditions, giving us a good handle on their maximal chain descent orders. Additionally, we give a more technical full characterization of the EL-labelings whose maximal chain descent orders have the expected cover relations. Our class of examples includes familiar EL-labelings of intervals in Young's lattice; in this case, the maximal chain descent orders turn out to be isomorphic to natural partial orders on standard Young tableaux or standard skew tableaux. We also prove that some well known posets may be realized as maximal chain descent orders of other posets. Specifically, we do this for type A and type B weak orders.

The second topic of this dissertation is to determine the homotopy type of the s-weak order and the s-Tamari lattice using another type of edge labeling known as an SB-labeling. Ceballos and Pons (2019) introduced two families of lattices on certain labeled trees called the s-weak order and the s-Tamari lattice as generalizations of the weak order on permutations and the classical Tamari lattice, respectively. These labeled trees are known as s-decreasing trees and were used in Ceballos and González D'León (2019) while studying Signature Catalan combinatorics. Ceballos and Pons also conjecture that the Hasse diagram of s-weak order is the 1-skeleton of a polytopal subdivision of polytope. In many cases, this subdivided polytope is conjectured to be the permutahedron. They also established that the Hasse diagram of the s-Tamari lattice is the 1-skeleton of a polytopal subdivision of polytope which, in many cases, is the associahedron. SB-labelings were introduced by Hersh and Mészáros (2017) to study the poset topology of not necessarily shellable finite lattices. Hersh and Mészáros proved that whenever a finite lattice possesses an SB-labeling, this implies that the order complex of every open interval is homotopy equivalent to a sphere or a ball of some dimension. We construct SB-labelings for s-weak order and the s-Tamari lattice. In this manner, we determine the poset topology of both families of lattices. This work appears in Lacina (2022).

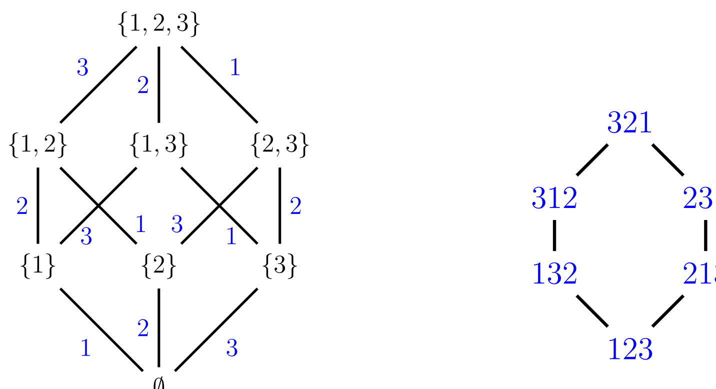
This dissertation is structured as follows: The remainder of Chapter I provides a more in-depth introduction to maximal chain descent orders and our results about them as well as a further introduction to our SB-labelings of s-weak order and the s-Tamari lattice. Chapter II provides the necessary background for the various topics in this thesis. Chapter III defines maximal chain descent orders and gives several fundamental properties of these partial orders. Chapter IV studies cover relations in maximal chain descent orders. In Chapter V, we give a broad range of examples of maximal chain descent orders constructed from well known families of posets and EL-labelings. Most of the examples in this chapter are fairly independent from the previous sections, so they can be read first if desired. In Chapter VI, we construct our SB-labeling of s-weak order and the s-Tamari lattice and use it to determine the homotopy type of each open interval in these lattices. Chapter VI is entirely independent of chapters III-V.

1.1 Maximal Chain Descent Orders

We begin by giving some sense of what a maximal chain descent order is through an example. To this end, we consider the Boolean lattice B_n with its standard EL-labeling. In this case, the maximal chain descent order is isomorphic to the weak order of type A (Theorem 3.2.1). Analyzing B_3 in detail below provides a running example with which to interpret our main results.

Consider the poset of all subsets of $[3] = \{1,2,3\}$ ordered by subset inclusion. This is the Boolean lattice B_3 which is shown in Fig. 1 (a). We label a cover relation $B \lessdot B \cup \{i\}$ in the Hasse diagram of B_3 by $\lambda(B, B \cup \{i\}) = i$. The labeling λ is a well known EL-labeling. We form a partial order on the maximal chains of B_3 by taking the transitive closure of moves of the form: $(\emptyset \lessdot \{1\} \lessdot \{1,2\} \lessdot \{1,2,3\}) \rightarrow (\emptyset \lessdot \{1\} \lessdot \{1,3\} \lessdot \{1,2,3\})$ where $(\emptyset \lessdot \{1\} \lessdot \{1,3\} \lessdot \{1,2,3\})$

contains exactly one element which is not in $(\emptyset \lessdot \{1\} \lessdot \{1,2\} \lessdot \{1,2,3\})$ and $(\emptyset \lessdot \{1\} \lessdot \{1,2\} \lessdot \{1,2,3\})$ is lexicographically first (thus, ascending) with respect to the labeling by λ in the interval $[\{1\}, \{1,2,3\}]$ on which the chains differ. We call such moves on maximal chains **polygon moves** (see Definition 3.1.2). The resulting maximal chain descent order $B_{3\lambda}(2)$ is shown in Fig. 1 (b) with the maximal chains of B_3 represented by their label sequences since the label sequences are all distinct. In this case, the resulting partial order is the weak order on the symmetric group S_3 as we expected.



(a) B_3 with EL-labeling λ. (b) Maximal chain descent order $B_{3\lambda}(2)$.

Figure 1. Boolean lattice with an EL-labeling and its maximal chain descent order.

Our first main theorem, Theorem 1.1.1, is that $P_\lambda(2)$ precisely encodes all of the shellings which can be "derived from λ."

Theorem 1.1.1. *Let P be a finite, bounded poset with a CL-labeling λ. For any total order $\Omega : m_1, m_2, \ldots, m_t$ on the maximal chains of P, the following are equivalent:*

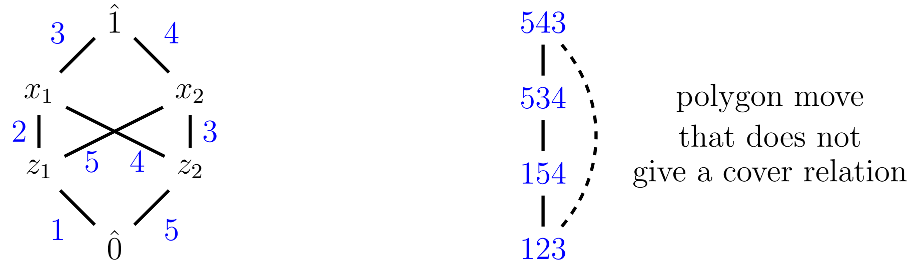
(1) Ω is a linear extension of the maximal chain descent order $P_\lambda(2)$.

(2) Ω induces a shelling order of the order complex $\Delta(P)$ with the property that for each $1 \le i \le t$ the restriction face $R(m_i)$ of m_i is precisely the face

$$R(m_i) = \Big\{ x \in m_i \;\Big|\; x \textit{ is a descent of } m_i \textit{ w.r.t. } \lambda \Big\}.$$

We prove this as Theorem 3.4.6. It is perhaps worth noting that the analogous statement for EL-labelings holds as well, by virtue of the fact that EL-labelings are instances of CL-labelings. The lexicographic shellings from a CL-labeling are given by ordering the maximal chains of P according to lexicographic order on their label sequences and breaking ties arbitrarily. We show that the linear extensions of $P_\lambda(2)$ give all of the lexicographic shellings as well as additional shellings. Theorem 1.1.1 allows us to recover the result (in type A) from Björner (1984) that any linear extension of the weak order on a Coxeter group gives rise to a shelling order of its Coxeter complex.

Maximal chain descent orders also possess seemingly interesting structure as posets in their own right. For instance, every cover relation in $P_\lambda(2)$ comes from a polygon move, a fact which might lead one to assume that all such polygon moves give cover relations. This is true for some examples such as the example above. However, we show that this is not always the case. For example, see the poset in Fig. 2 with an EL-labeling and its induced maximal chain descent order. Specifically, there is a polygon move between the maximal chain labeled 123 and the maximal chain labeled 543 despite the fact that 543 does not cover 123 in the maximal chain descent order.



(a) P with EL-labeling λ. (b) $P_\lambda(2)$

Figure 2. EL-labeling which is not polygon complete.

Informally, one might view this phenomenon as it sometimes being possible to "go around the back" to prevent a polygon move from giving a cover relation though it is still an order relation. We call a CL-labeling **polygon complete** if every polygon move gives a cover relation (see Definition 3.3.9). Our second main result is a characterization of the polygon complete EL-labelings in Theorem 4.1.19.

In addition, we give two quite checkable sufficient conditions for polygon completeness. These are much simpler than the full characterization of polygon completeness for EL-labelings. The first of these sufficient conditions is a property an EL-labeling may possess which we call being **polygon strong**. It is a relaxation of Björner's notion of strongly lexicographically shellable from Björner (1980). We prove that polygon strong implies polygon complete in Theorem 4.1.6. Several well known families of EL-labelings are shown to be polygon strong in Section 4.1.2. In order to describe the second sufficient condition, we introduce a notion of inversions for CL-labelings in Section 4.2. Then we formulate a condition on such inversions which we show to be sufficient for polygon completeness in Theorem 4.2.9. Theorem 4.2.9 also shows that this condition on inversions with respect to a CL-labeling λ implies that $P_\lambda(2)$ is ranked and that the homology facets of the induced shellings of the proper part of P are determined by rank in the maximal chain descent order. We also prove in Lemma 4.1.21 that a simple condition on labelings of certain induced subposets of P guarantees a CL-labeling is not polygon complete. We apply Theorem 4.1.19 in Section 4.1.4 to prove that a well known EL-labeling of the k-equal partition lattice is polygon complete despite being neither polygon strong nor satisfying the inversion condition.

We also develop the following structural properties of maximal chain descent orders. In Proposition 3.3.1, we prove that every maximal chain descent order has a unique minimal element given by the unique ascending maximal chain of P with respect to λ. We prove in Lemma 3.3.10 that the number of downward cover relations in $P_\lambda(2)$ from a maximal chain m of P is bounded above by the number of descents of m with respect to λ. In Corollary 3.4.4, we show that certain homology facets of the induced shellings of the proper part of P can be detected from the poset structure of $P_\lambda(2)$ and the lengths of maximal chains in P. We prove in Lemma 3.3.3 that maximal chain descent orders additionally satisfy a certain lifting property which reflects the recursive nature of CL-labelings. In Theorem 5.1.10, we show that lower intervals in $P_\lambda(2)$ which contain a maximal chain of length two have at most two coatoms. This implies such intervals have the homotopy type of either a ball or sphere.

Along with the weak order of type A, there are many more examples of maximal chain descent orders. We analyse the structure of a few such families in Section 5.2. We show that the type B weak order is a maximal chain descent order in Theorem 5.2.29. In Theorem 5.2.1, we prove that all intervals in any maximal chain descent order induced by Stanley's M-chain EL-labeling of any finite supersolvable lattice (see Stanley (1972)) are isomorphic to intervals in the weak order of type A, doing so via the map assigning to each maximal chain its label sequence. We use this to show that the linear extension EL-labelings of any finite distributive lattice produce maximal chain descent orders isomorphic to certain order ideals of the type A weak order. When the finite distributive lattice is any interval in Young's lattice, Theorem 5.2.7 shows that the resulting maximal chain descent orders are isomorphic to naturally defined partial orders on standard Young tableaux or standard skew tableaux. We also prove in Theorem 5.2.24 that the "min-max" EL-labeling of the partition lattice Π_n yields a maximal chain descent order isomorphic to a natural partial order on certain labeled trees.

1.2 An SB-labeling of s-Weak Order and the s-Tamari Lattice

Ceballos and Pons (2019) introduced a partial order called **s-weak order** on certain labeled trees known as **s-decreasing trees**. They observed that this partial order generalizes weak order on permutations. They proved s-weak order is a lattice. They also found a particular class of s-decreasing trees which play the role of 231-avoiding permutations. This led them to introduce a sublattice of s-weak order called the s-Tamari lattice, generalizing the Tamari lattice. The background on these lattices is in Section 2.2.4 and Section 2.2.5 while the necessary background on SB-labelings is in Section 2.1.3.

Our main result on this topic is the following theorem:

Theorem 1.2.1. *The lattices s-weak order and the s-Tamari lattice each admit an SB-labeling. Thus, the order complex of each open interval in s-weak order and the s-Tamari lattice is homotopy equivalent to a ball or sphere of some dimension.*

We prove this as Theorem 6.1.20 for s-weak order and Theorem 6.2.13 for the s-Tamari lattice. Our result generalizes another result of Hersh and Mészáros that weak order on permutations and the classical Tamari lattice admit SB-labelings, with our labelings specializing in those cases to SB-labelings distinct from theirs.

In s-weak order and the s-Tamari lattice, the spheres in Theorem 1.2.1 are not always top dimensional, demonstrating that these posets are not always shellable. We intrinsically characterize which intervals in s-weak order and the s-Tamari lattice are homotopy equivalent to spheres and which are homotopy equivalent to balls. We also determine the dimension of the spheres for the intervals yielding homotopy spheres. As a corollary, we deduce that the Möbius functions of s-weak order and the s-Tamari lattice only take values in $\{-1, 0, 1\}$. It is also known that the existence of an SB-labeling implies that distinct sets of atoms in an interval have distinct joins, giving another consequence of our results.

Part of Ceballos and Pons' interest in s-weak order came from geometry. They conjectured that the Hasse diagrams of s-weak order are the 1-skeleta of polytopal subdivisions of polytopes. They call these potential polytopal complexes **s-permutahedra**. They also conjecture that in particular cases the polytopes they are subdividing are classical permutahedra. Our result of an SB-labeling for s-weak order, though it considers these lattices from a topological perspective, seems to provide two pieces of evidence for Ceballos and Pons' conjecture. The first is that the Hasse diagrams of many lattices which admit SB-labelings can be realized as 1-skeleta of polytopes. The second comes from the fact that Ceballos and Pons' geometric perspective is somewhat similar in flavor to one point of view in Hersh (2018). Hersh studied posets which arise as the 1-skeleta of simple polytopes via directing edges by some cost vector. In particular, Theorem 4.9 in Hersh (2018) proves that all open intervals in lattices which are realizable as such 1-skeleta of simple polytopes are either homotopy balls or spheres.

Similarly, Ceballos and Pons' also took a geometric viewpoint on the s-Tamari lattice. They showed that the s-Tamari lattice is isomorphic to another generalization of the classical Tamari lattice, namely the ν-Tamari lattice introduced in Préville-Ratelle and Viennot (2017). The geometry of the ν-Tamari lattice was recently studied in Ceballos, Padrol, and Sarmiento (2019) where it was shown that the Hasse diagram of the ν-Tamari lattice is the 1-skeleta of a polytopal subdivision of a polytope.

CHAPTER II

BACKGROUND

2.1 Posets, Lexicographic Shellability, and SB-labelings

2.1.1 Posets and Simplicial Complexes.

A **partially ordered set** or **poset** is a pair $(P, \leq)$ of a set P and a binary relation $\leq$ on P which satisfies the following three conditions for all $x, y, z \in P$:

1. $x \leq x$ (reflexive)

2. $x \leq y$ and $y \leq z$ implies $x \leq z$ (transitive)

3. $x \leq y$ and $y \leq x$ implies $x = y$ (antisymmetric).

For $x, y \in P$ satisfying $x \leq y$, the **closed interval** from x to y is the set $[x, y] = \left\{z \in P \;\middle|\; x \leq z \leq y\right\}$. The **open interval** from x to y is defined analogously with strict inequalities and denoted (x, y). We say that y **covers** x, denoted $x \lessdot y$, if $x \leq z \leq y$ implies $z = x$ or $z = y$.

P is a **lattice** if each pair $x, y \in P$ has a unique least upper bound called the **join**, denoted $x \vee y$, and a unique greatest lower bound called the **meet**, denoted $x \wedge y$. We denote by $\hat{0}$ (respectively $\hat{1}$) the unique minimal (respectively unique maximal) element of P if such elements exist. If P has both a $\hat{0}$ and a $\hat{1}$, we say P is **bounded**. If P is bounded, we denote the proper part of P as $\overline{P} = P \setminus \left\{\hat{0}, \hat{1}\right\}$. We note that finite lattices are always bounded. The elements which cover $\hat{0}$ are called **atoms**, and the elements which are covered by $\hat{1}$ are called **coatoms**.

For $x, y \in P$ satisfying $x \leq y$, a **k-chain from x to y** in P is a subset $C = \{x_0, x_1, \ldots, x_k\} \subseteq P$ such that $x = x_0 < x_1 < \cdots < x_k = y$. A chain C is said to be **saturated** if $x_i \lessdot x_{i+1}$ for all i. A chain is said to be **maximal** if it is not properly contained in any other chain. We denote by $\boldsymbol{\mathcal{E}}(\mathbf{P})$ the set of edges in the Hasse diagram of P, that is, the set of cover relations of P. And we denote by $\boldsymbol{\mathcal{M}}(\mathbf{P})$ the set of maximal chains of P.

We will use the following four pieces of notation when working with chains. These notations will be ubiquitous in this work.

Definition 2.1.1. *If c_1 and c_2 are chains such that the largest element of c_1 is less than or equal to the smallest element of c_2 in P, we denote the concatenated chain $c_1 \cup c_2$ by $\mathbf{c_1} * \mathbf{c_2}$.*

For a chain c containing $x \in P$, we denote the subchain $c \cap [\hat{0}, x]$ by $\mathbf{c^x}$, i.e. everything not above x in c. We denote the subchain $c \cap [x, \hat{1}]$ by $\mathbf{c_x}$, i.e. everything not below x in c. Then for $y \in c$ satisfying $x \leq y$, the subchain $c \cap [x, y]$ is denoted $\mathbf{c_x^y}$.

If m and c are chains, then $\mathbf{m} \setminus \mathbf{c}$ denotes the subchain of m with all elements of c removed.

We say P is **ranked** if there is a function $rk : P \to \mathbb{N}$ such that $rk(x) = 0$ if x is minimal in P and $rk(y) = rk(x) + 1$ if $x \lessdot y$ in P. We say P is graded if all maximal chains of P have the same length. We note that for non-bounded posets, ranked and graded are distinct concepts. We recall this fact because the distinction appears in this work. A map $f : P \to Q$ between posets P and Q is called **order preserving** or a **poset map** if $x \leq y$ in P implies $f(x) \leq f(y)$ in Q. An order preserving bijection f is called a **poset** isomorphism if f^{-1} is also order preserving. An order preserving bijection $e : P \to [|P|]$ where $[|P|]$ has its usual total order is called a **linear extension** of P.

We will occasionally use the concept of subposet in this dissertation, so we point out a subtlety in the concept here. We say poset $(Q, \leq_Q)$ is a **subposet** of poset $(P, \leq_P)$ if $Q \subset P$ and for all $x, y \in Q$, $x \leq_Q y$ implies $x \leq_P y$. A stronger condition is the following: We say $(Q, \leq_Q)$ is an **induced subposet** of $(P, \leq_P)$ if $Q \subset P$ and for all $x, y \in Q$, $x \leq_Q y$ if and only if $x \leq_P y$.

An **abstract simplicial complex** is a collection Δ of subsets (the faces or simplices of Δ) of a finite set V such that $\{v\} \in \Delta$ for all $v \in V$ and if $F \subset G$ and $G \in \Delta$, then $F \in \Delta$. The dimension of a face F of Δ is $|F| - 1$. So, the empty set $\emptyset$, which is a face of every abstract simplicial complex has dimension -1. The dimension of Δ is the dimension of a maximal dimensional face of Δ. A face of Δ which is non-properly contained in any other face of Δ is called a **facet** of Δ. If all facets of Δ have the same dimension, the Δ is called pure. A d-dimensional **geometric simplex** in $\mathbb{R}^n$ is the convex hull of $d + 1$ affinely independent points in $\mathbb{R}^n$. A **geometric simplicial complex** in $\mathbb{R}^n$ is a collection D of geometric simplices in $\mathbb{R}^n$ such that every face of a simplex in D is also in D and the intersection of any two simplices is a face of each. An abstract simplicial complex Δ can be realized in many different ways as a geometric simplicial complex, but all such realizations are homeomorphic under the topology inherited from the usual topology on $\mathbb{R}^n$. Thus, we refer to this underlying topological space as the **geometric realization** of Δ, denoted $\|\Delta\|$. We are almost exclusively interested in abstract simplicial complexes in this dissertation.

The **order complex** of poset P, denoted $\Delta(P)$, is the abstract simplicial complex with vertices the elements of P and i-dimensional faces the i-chains of P. The maximal chains of P are precisely the facets of $\Delta(P)$. For $x, y \in P$, we denote by $\Delta(x, y)$ the order complex of the open interval (x, y) as an induced subposet of P. Thus, when we refer to topological properties of P, we mean the topological properties of any geometric realization of $\Delta(P)$. In particular, the homotopy type of P refers to the homotopy type of $\Delta(P)$. It is a well known theorem of P. Hall (see Rota (1964)) that μ_P, the Möbius function of P, satisfies $\mu_P(x, y) = \tilde{\chi}(\Delta(x, y))$. Here, $\tilde{\chi}$ is the reduced Euler characteristic. This provides one of the important connections between the combinatorial and enumerative structure of a poset and its topology.

A **shelling** of a simplicial complex Δ is a total order $F_1, F_2, \ldots, F_t$ of the facets of Δ such that $\overline{F_j} \cap (\cup_{i<j} \overline{F_i})$ is a pure codimension one subcomplex of $\overline{F_j}$ for each $1 < j \le t$. A simplicial complex which possesses a shelling is said to be **shellable**. A facet F_j such that $\overline{F_j} \cap (\cup_{i<j} \overline{F_i}) = \partial F_j$ is called a **homology facet**. Shellable simplicial complexes are homotopy equivalent to a (possibly empty) wedge of spheres with the spheres of each dimension indexed by the homology facets of that dimension. Let $\Delta_k = \bigcup_{1 \le i \le k} \overline{F_i}$ be the subcomplex of Δ formed by deleting the last $t - k$ facets in the shelling for each $1 \le k \le t$. A shelling induces a **restriction face** of each facet F_j which is the minimal face of F_j which is not contained in any facet prior to F_j in the shelling. Denoting the restriction face of F_j by $\mathbf{R(F_j)}$, we have $R(F_j) = \left\{ x \in F_j \;\middle|\; F_j \setminus \{x\} \in \Delta_{j-1} \right\}$. Facet F_j is a homology facet if and only if $R(F_j) = F_j$. Taking the restriction face of each facet defines the **restriction map** R of the shelling, a map from the facets to Δ. For a face f contained in facet F, we denote the set of faces of F which contain f by $[f, F]$. Then the following proposition gives a useful reformulation of a shelling.

Proposition 2.1.2 (Proposition 2.5 Björner and Wachs (1996))**.** *For a total ordering $F_1, F_2, \ldots, F_t$ of the facets of a simplicial complex Δ and a map $R : \{F_1, F_2, \ldots, F_t\} \to \Delta$, the following are equivalent:*

(1) $F_1, F_2, \ldots, F_t$ is a shelling and R is its restriction map.

(2) Δ is the disjoint union $\Delta = \bigsqcup_{1 \le i \le t} [R(F_i), F_i]$ and $R(F_i) \subseteq F_j$ implies $i \le j$ for all i, j.

Proposition 2.1.2 shows that shelling is a special case of another type of decomposition for certain simplicial complexes called partitioning. A simplicial complex Δ with facets $F_1, F_2, \ldots, F_t$ is called **partitionable** if it is the disjoint union $\Delta = \bigsqcup_{1 \le i \le t} [R_i, F_i]$ where $R_i \subseteq F_i$ is called the **restriction face** of facet F_i. This decomposition is called a **partitioning** of Δ. Proposition 2.1.2 obviously shows that a shelling of a Δ produces a partitioning of Δ where the restriction faces of the partitioning are simply the restriction faces of the shelling.

It is easy to construct partitionable simplicial complexes that are not connected, so partitionability is strictly more general than shellability because shellability implies connectedness.

2.1.2 Lexicographic Shellability. In this subsection, we recall the notion of lexicographic shellability, both EL-labelings and CL-labelings.

An **edge labeling** of poset P is a map $\lambda : \mathcal{E}(P) \to \Lambda$ for a poset Λ, that is, a label $\lambda(x, y)$ for each cover relation $x \lessdot y$ in P. An edge labeling λ induces a label sequence $\lambda(m)$ for each maximal chain $m \in \mathcal{M}(P)$ with $m : x_0 \lessdot x_1 \lessdot \cdots \lessdot x_{n-1} \lessdot x_n$ and $\lambda(m) : \lambda_1(m), \lambda_2(m) \ldots, \lambda_n(m)$ where $\lambda_i(m) = \lambda(x_{i-1}, x_i)$. We say $m \in \mathcal{M}(P)$ is an **ascending chain** with respect to λ if the label sequence $\lambda(m)$ is non-decreasing with respect to Λ. Further, we say that m has a **descent** at position i or we say that x_i is a **descent** of m if $\lambda_i(m) \not\le \lambda_{i+1}(m)$ in Λ. We say that m has an **ascent** at position i or that x_i is an **ascent** of m if $\lambda_i(m) \le \lambda_{i+1}(m)$ in Λ. Then Λ induces a **lexicographic order** on maximal chains with $m <_{lex} m'$ for $m, m' \in \mathcal{M}(P)$ if i is the first index of the label sequences of m and m' at which they disagree and $\lambda_i(m) < \lambda_i(m')$ in Λ. We break ties in the lexicographic order arbitrarily, that is, maximal chains with identical label sequences are thought of as incomparable in the lexicographic order until an arbitrary total order is assigned to the maximal chains which share the same label sequence. For our constructions, we use a well known kind of edge labeling called an **EL-labeling** and a widely used generalization called a **CL-labeling**. EL-labelings and the notion of lexicographic shellability were introduced in Björner (1980). CL-labelings were then introduced in Björner and Wachs (1982) and more deeply understood, particularly their recursive nature, by the same authors in Björner and Wachs (1983). Björner and Wachs' initial work on shelling was focused on graded posets (pure simplicial complexes), but they extended the ideas of shelling, including both EL-labelings and CL-labelings, to non-graded posets (non-pure simplicial complexes) in Björner and Wachs (1996) and Björner and Wachs (1997).

Definition 2.1.3 (Section 2 Björner (1980))**.** *An edge labeling λ of a finite, bounded poset P is an **EL-labeling** if for each pair $x, y \in P$ satisfying $x < y$, there is a unique ascending maximal chain with respect to λ in the closed interval $[x, y]$ and this ascending chain lexicographically precedes all other maximal chains in $[x, y]$.*

Theorem 2.1.4 (Proof of Theorem 2.3 Björner (1980) and Theorem 5.8 Björner and Wachs (1996))**.** *If finite, bounded poset P admits an EL-labeling λ, then any total order of the maximal chains of P which is compatible with the lexicographic order on maximal chains induced by λ is a shelling order of the order complex $\Delta(P)$. Moreover, the restriction map of any such shelling is given by $R(m) = \left\{x \in m \;\middle|\; x \text{ is a descent of } m \text{ w.r.t. } \lambda\right\}$ for any maximal chain $m \in \mathcal{M}(P)$, and the homology facets of the induced shelling of $\Delta(\overline{P})$ are given by the maximal chains $m \in \mathcal{M}(P)$ with descending label sequence with respect to λ.*

A generalization of an edge labeling is a **chain edge labeling**. Intuitively, a chain edge labeling is an edge labeling which depends on a choice of saturated chain from $\hat{0}$ to the bottom of the edge. Let $\mathcal{ME}(P)$ be the set of pairs $(m, x \lessdot y)$ for a maximal chain $m \in \mathcal{M}(P)$ and cover relation $x \lessdot y \in \mathcal{E}(P)$ such that $x \lessdot y$ is a cover relation in m. A **chain edge labeling** is a map $\lambda : \mathcal{ME}(P) \rightarrow \Lambda$ for a poset Λ such that if two maximal chains agree along their bottom d edges then their labels of those edges are the same. Just as with an edge labeling, a chain edge labeling induces a label sequence of each maximal chain. To make an analogy with EL-labeling, we must restrict the label sequences of maximal chains to closed intervals $[x, y]$, but their is no unique restriction to $[x, y]$. However, fixing a maximal chain r of $[\hat{0}, x]$ determines a unique restriction of λ to $\mathcal{ME}([x, y])$. We call r a **root** and the pair $[x, y]_r$ a **rooted interval**. Thus, we may refer to ascending and descending chains, the lexicographic order on maximal chains, and ascents and descents all with respect to λ and a root r.

Definition 2.1.5 (Definition 3.2 Björner and Wachs (1982))**.** *A chain edge labeling λ of a finite, bounded poset P is a **CL-labeling** if each rooted interval $[x, y]_r$ has a unique ascending maximal chain with respect to λ and this ascending chain lexcographically precedes all other maximal chains in $[x, y]_r$.*

EL-labelings are CL-labelings in which edge labels do not depend on roots. Just like EL-labelings, CL-labelings induce lexicographic shellings of order complexes.

Theorem 2.1.6 (Proof of Theorem 3.3 Björner and Wachs (1982) and Theorem 5.8 Björner and Wachs (1996))**.** *If finite, bounded poset P admits a CL-labeling λ, then any total order of the maximal chains of P which is compatible with the* *lexicographic order on maximal chains induced by λ is a shelling order of the order complex $\Delta(P)$. Moreover, the restriction map of any such shelling is given by $R(m) = \left\{x \in m \;\middle|\; x \text{ is a descent of } m \text{ w.r.t. } \lambda\right\}$ for any maximal chain $m \in \mathcal{M}(P)$, and the homology facets of the induced shelling of $\Delta(\overline{P})$ are given by the maximal chains $m \in \mathcal{M}(P)$ with descending label sequence with respect to λ.*

Another useful fact which we record here for later use is that EL-labelings and CL-labelings restrict to EL-labelings and CL-labelings of closed intervals and rooted intervals, respectively.

Proposition 2.1.7. *The restriction of an EL-labeling to any closed interval is an EL-labeling. The restriction of a CL-labeling to any closed rooted interval is a CL-labeling.*

Proof. These both follow from the recursive natures of Definition 2.1.3 and Definition 2.1.5. □

2.1.3 SB-labelings.

Hersh and Mészáros developed the notion of an SB-labeling in Hersh and Mészáros (2017) to show when certain lattices have open intervals which are homotopy balls or spheres.

Definition 2.1.8. *(Hersh & Mészáros, 2017, Definition 3.4) An* ***SB-labeling*** *is an edge labeling λ on a finite lattice L satisfying the following conditions for each $u, v, w \in L$ such that v and w are distinct elements which each cover u:*

(i) $\lambda(u, v) \neq \lambda(u, w)$

(ii) Each saturated chain from u to $v \vee w$ uses both of these labels $\lambda(u, v)$ and $\lambda(u, w)$ a positive number of times.

(iii) None of the saturated chains from u to $v \vee w$ use any other labels besides $\lambda(u, v)$ and $\lambda(u, w)$.

One of the main theorems in Hersh and Mészáros (2017) is the following characterization of the homotopy types of intervals in a lattice which admits an SB-labeling. We will use this theorem in Chapter III to draw our topological conclusions about s-weak order and the s-Tamari lattice.

Theorem 2.1.9. *(Hersh & Mészáros, 2017, Theorem 3.7) If L is a finite lattice which admits an SB-labeling, then each open interval (u, v) in L is homotopy equivalent to a ball or a sphere of some dimension. Moreover, $\Delta(u, v)$ is homotopy equivalent to a sphere if and only if v is a join of atoms of $[u, v]$, in which case it is homotopy equivalent to a sphere S^{d-2} where d is the number of atoms in $[u, v]$.*

2.2 Selected Families of Posets

2.2.1 Weak Order of Types A and B.

Here we recall the definition and basics of the weak order of type A, that is, the weak order on the symmetric group of permutations. We also recall a combinatorial representation for the type B Coxeter groups. In Section 5.2, we briefly mention weak order on a general Coxeter group. For general Coxeter groups and proofs of the facts presented here, see Björner and Brenti (2010) whose presentation we largely follow.

Let S_n be the symmetric group of permutations of $[n]$. Let S be the set of adjacent transpositions in S_n, that is, $s_i = (i, i+1)$ for $i \in [n-1]$. We note that (S_n, S) is isomorphic to the Coxeter system of type A_{n-1}. (W, S) be a Coxeter system. The **length** of a permutation $w \in S_n$, denoted $l(w)$, is the minimal k such that $w = s_{i_1} s_{i_2} \dots s_{i_k}$ for $s_{i_j} \in S$. We note that $l(w)$ is also the number of inversions of w, that is, the number of pairs of entries $i < j$ of W which appear in decreasing order in the one line notation of w. We denote the set of inversions of w by $\text{inv}\,(w)$, so $l(w) = |\text{inv}\,(w)|$. The **weak order** on S_n is the partial order $\leq_{wk}$ defined by $u \leq_{wk} w$ if and only if $w = us_{i_1}s_{i_2}\ldots s_{i_k}$ such that $s_{i_j} \in S$ and $l(us_{i_1}s_{i_2}\ldots s_{i_j}) = l(u) + j$ for each $1 \leq j \leq k$. We will use a subscript of "wk" to refer to objects in weak order on S_n. An important property of weak order is that cover relations have a very nice form. Namely, $u \lessdot_{wk} w$ if and only $l(u) = l(w) - 1$ and $u^{-1}w = s$ for some $s \in S$. Since we are dealing with right multiplication and right multiplication in the symmetric group corresponds to acting on the positions of permutations in one line notation, $u \lessdot_{wk} w$ for permutations $u, w \in S_n$ if and only if w is obtained by transposing an ascent of u in one line notation. The corresponding positions in w will thus have a descent. The weak order on the symmetric group is well known to be a lattice. There is also another useful description of weak order on S_n in terms of containment of inversion sets.

Proposition 2.2.1. *For $u, w \in S_n$, $u \leq_{wk} w$ if and only if $inv\,(u) \subseteq inv\,(w)$.*

We recall now some facts about the Type B Coxeter group, which we will denote S_n^B where n is a positive integer. For a more detailed account of the Type B Coxeter system, see Section 8.1 of Björner and Brenti (2010). S_n^B has a combinatorial representation as the signed permutations, that is, the bijections ω of the set $[-n, n] = \{-n, -n+1, \ldots, -1, 1, 2, \ldots, n-1, n\}$ with itself such that $\omega(-i) = -\omega(i)$ for all $i \in [n]$. The signed permutations $\omega \in S_n^B$ can be written in a modified one line notation. We will write $\omega = [\omega(1), \omega(2), \omega(3), \ldots, \omega(n)]$. For example, $\omega = [2, -1, 3]$ is the signed permutation with $\omega(1) = 2$, $\omega(2) = -1$, $\omega(3) = 3$, and $\omega(-i) = -\omega(i)$. In this combinatorial representation, S_n^B has a set of Coxeter generators given by $s_i = (i, i+1)$ for $1 \leq i \leq n-1$ and $s_0 = (-1, 1)$ where ωs_i is defined by swapping the ith and $i+1$th entries of ω, and consequently swapping the minus ith and minus $i+1$th entries, for $1 \leq i \leq n-1$ and ωs_0 is defined by changing the sign of the first, and consequently the minus first, entry of ω. We also recall that S_n^B is the symmetry group of the regular polytope the n-cube C_n and its dual the n-dimensional cross polytope.

2.2.2 Supersolvable Lattices.

To define supersolvable lattices, we first need to define distributive lattices. A lattice L is called **distributive** if it satisfies the following "distributivity" condition on meets and joins: $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$ for all $x, y, z \in L$. This condition is equivalent to the dual condition $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$ for all $x, y, z \in L$. For the purposes of this dissertation, the most important fact about distributive lattices is Birkhoff's well known Fundamental Theorem of Finite Distributive Lattices from Birkhoff (1937).

Theorem 2.2.2. *A poset L is a finite distributive lattice if and only if $L = J(P)$ for some finite poset P where $J(P)$ is the set of order ideals of P ordered by inclusion.*

Remark 2.2.3. Linear extensions of P give S_n EL-labelings of $J(P)$ where $n = |P|$. Fix e, a linear extension of P. For any cover relation $I \lessdot I'$ in $J(P)$ (I and I' are order ideals of P), $I' = I \cup \{x\}$ for some $x \in P$. We define an edge labeling λ_e of $J(P)$ by $\lambda_e(I \lessdot I') = e(x)$. This edge labeling gives an S_n EL-labeling since every element of P must be added exactly once at some point along each maximal chain in $J(P)$. Let $\boldsymbol{\mathcal{L}}(\mathbf{P}, \mathbf{e})$ be the set of permutations appearing as label sequences of maximal chains in $J(P)$ when labeled by λ_e.

In Björner and Wachs (1991), the sets of label sequences of distributive lattices defined by the EL-labelings of Remark 2.2.3 are studied. We use a special case of Proposition 4.1 from Björner and Wachs (1991). (Their proposition and the special case stated below can both be proven straightforwardly using Proposition 4.1.9 which occurs in Chapter 4.)

Proposition 2.2.4. *Let P be a finite poset with $|P| = n$. Let e be a linear extension of P. Then $\mathcal{L}(P, e)$, the set of permutations appearing as label sequences of maximal chains in $J(P)$ when labeled by λ_e, is an order ideal of the weak order on S_n.*

Supersolvable lattices were introduced in Stanley (1972) as generalizations of the subgroup lattices of supersolvable finite groups. A maximal chain $m : \hat{0} = x_1 \lessdot x_2 \lessdot x_3 \lessdot \cdots \lessdot x_n \lessdot x_{n+1} = \hat{0}$ in a finite lattice L is called a **modular chain** or **M-chain** if $x_i \wedge (y \vee z) = (x_i \wedge y) \vee (x_i \wedge z)$ for all $i \in [n-1]$ and for all $y, z \in L$, that is m is maximal chain of elements satisfying the distributive property. The M-chain condition is equivalent to the sublattice generated by m and any other chain of L being distributive. A finite lattice L is called a **supersolvable** lattice if it possesses an M-chain. In the case of the subgroup lattice of a supersolvable group, an M-chain is given by a maximal normal series with cyclic quotients.

Stanley (1972) also introduced natural edge labelings of supersolvable lattices to compute refinements of Möbius invariants. These labelings are based on the labelings of finite distributive lattices defined in Remark 2.2.3 and were called R-labelings by Stanley. These labelings were later shown to be EL-labelings in Björner (1980). Here we give a slightly different, but equivalent definition to that in Stanley (1972). Fix an M-chain $m : \hat{0} = x_1 \lessdot x_2 \lessdot x_3 \lessdot \cdots \lessdot x_n \lessdot x_{n+1} = \hat{0}$ of supersolvable lattice L. For any cover relation $y \lessdot z$ in L, define $\lambda(y, z) = i$ for the minimum $i \in [n]$ such that $x_i \wedge y = x_i \wedge z$ (see Section 3.14 of Stanley (2011)). Each M-chain thus gives a different edge labeling. However, in any such labeling, the label sequence of each maximal chain is a permutation of $[n]$. We will call these

M-chain EL-labelings. In the special case of distribitive lattices, M-chain EL-labelings are precisely the linear extension EL-labelings mentioned in Remark 2.2.3.

2.2.3 The Classical Tamari Lattice.

Here we recall some of the basics of the classical Tamari lattice. The **classical Tamari lattice** T_n was introduced in Tamari (1962) as a partial order on the proper parenthesizations of $n+1$ letters. One moves upward in the partial order by rightward applications of the associative law. Of course the number such parenthesizations is the nth Catalan number C_n and there are many bijections between parenthesizations and the other objects in the Catalan zoo. So, the Tamari lattice can be realized as a partial order on any of the Catalan objects and there are many interesting instances in which the partial order reflects particular structure in a given collection of Catalan objects.

For our purposes, mainly to see the s-Tamari lattice as a generalization of the classical Tamari lattice, we want to realize the Tamari lattice on the 231-avoiding permutations of $[n]$. A permutation is **231-avoiding** if its one-line notation does not contain any subsequence of the form yzx with $x < y < z$. There are C_n 231-avoiding permutations of $[n]$ and T_n is isomorphic to the weak order on permutations of $[n]$ restricted to the 231-avoiding permutations.

Fig. 3 shows the Tamari lattice T_3 on the 231-avoiding permutations of [3]. In this case, the only permutation of [3] which is not 231-avoiding is the permutation 231.

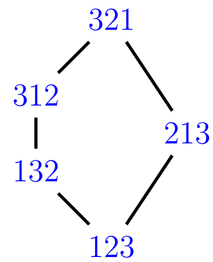
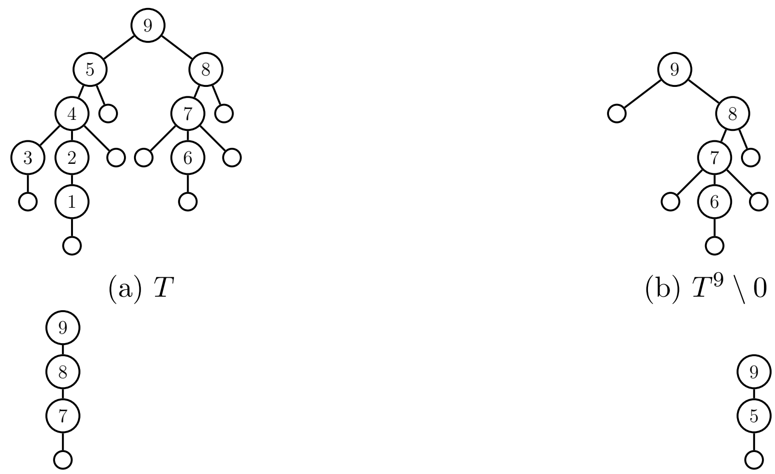


Figure 3. Classical Tamari lattice T_3

2.2.4 The s-weak order.

A **weak composition** is a sequence of non-negative integers $s = (s(1), \ldots, s(n))$ with $s(i) \in \mathbb{N}$ for all $i \in [n]$. We say the **length** of a weak composition s is $l(s) = n$. Let s be a weak composition. An **s-decreasing tree** is a planar rooted tree T with n internal vertices which are labeled 1 to n (leaves are not labeled and are the only unlabeled vertices) such that internal vertex i has $s(i) + 1$ children and all labeled descendants of i have labels less than i. The $s(i) + 1$ children of i are indexed by 0 to $s(i)$. We denote the full subtree of T rooted at i by T^i, and denote the full subtrees rooted at the $s(i) + 1$ children of i by $T^i_0, \ldots, T^i_{s(i)}$, respectively. For i and $0 \leq j \leq s(i)$, we denote by $T^i \setminus j$, the subtree of T obtained from T^i by replacing T^i_j with a leaf. Also, $T^i_{j_1,\ldots,j_k}$ will denote the forest of the full subtrees rooted at the $j_1, \ldots, j_k$ children of i. Let k be the jth child of i in T. We define the **jth left subtree of i in T**, denoted ${}_LT^i_j$, to be the subtree of T with root i obtained by walking from i to k and then down the left most subtree possible until reaching a leaf. Similarly, we define the **jth right most subtree of i in T**, denoted ${}_RT^i_j$, to be the subtree of T with root i obtained by walking from i to k and then down the right most subtree possible until reaching a leaf. We note that ${}_LT^i_j$ and ${}_RT^i_j$ are both always chains. Fig. 4 is an example of an s-decreasing tree with $s = (0, 0, 0, 2, 1, 3)$, along with some examples of the subtrees just defined.



(a) T (b) $T^9 \setminus 0$ (c) ${}_L T_1^9$ (d) ${}_R T_0^9$

Figure 4. An s-decreasing tree T with $s = (0, 0, 0, 2, 1, 0, 2, 1, 1)$ and examples of some defined subtrees.

Definition 2.2.5. *(Ceballos & Pons, 2019, Definition 2.1) Let T be an s-decreasing tree and $1 \leq x < y \leq n$. The **cardinality** of (y, x) in T, denoted $\#_{\boldsymbol{T}}\boldsymbol{(y, x)}$, is defined by the following rules:*

1. *$\#_T(y, x) = 0$ if x is left of y in T or $x \in T_0^y$;*

2. *$\#_T(y, x) = i$ if $x \in T_i^y$ with $0 < i < s(y)$; and*

3. *$\#_T(y, x) = s(y)$ if $x \in T_{s(y)}^y$ or x is right of y in T.*

*If $\#_T(y, x) > 0$, then (y, x) is said to be a **tree inversion** of T. We denote by **inv(T)** the multi-set of tree inversions of T counted with multiplicity their cardinality.*

Now we can also formally describe the jth left and right subtrees of i in T, examples of which are found in (c) and (d) of Fig. 4.

$$ {}_L T_j^i = \left\{ d \in T^i \;\middle|\; d = i, \text{ or } d \in T_j^i \text{ and } \#_T(e, d) = 0 \; \forall e \in T_j^i \text{ such that } d < e \right\}. $$

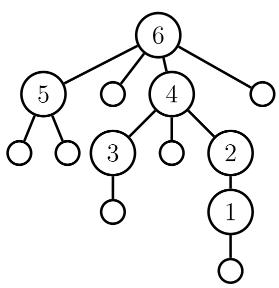
$$ {}_R T_j^i = \left\{ d \in T^i \;\middle|\; d = i, \text{ or } d \in T_j^i \text{ and } \#_T(e, d) = s(e) \; \forall e \in T_j^i \text{ such that } d < e \right\}. $$

Remark 2.2.6. For $s = (1, \ldots, 1)$, s-decreasing trees are in by bijection with permutations in $S_{l(s)}$ and tree inversions are precisely inversions of the corresponding permutation.

Remark 2.2.7. If T is an s-decreasing tree, $1 \leq a < b \leq n$, and $0 < \#_T(b,a) < s(b)$, then $a \in T^b_{\#_T(b,a)}$.

Remark 2.2.8. If $e \in T^a$ and $e \in T^b_i$ for some $a < b$, then $a \in T^b_i$. Further, if $e \in T^a$ and $a < b$, then $\#_T(b,e) = \#_T(b,a)$.

Fig. 5 is an s-decreasing tree with the cardinality of each pair of labeled vertices listed.



$\#_T(6,5) = 0$	$\#_T(6,4) = 2$	$\#_T(6,3) = 2$	$\#_T(6,2) = 2$	$\#_T(6,1) = 2$
	$\#_T(5,4) = 1$	$\#_T(5,3) = 1$	$\#_T(5,2) = 1$	$\#_T(5,1) = 1$
		$\#_T(4,3) = 0$	$\#_T(4,2) = 2$	$\#_T(4,1) = 2$
			$\#_T(3,2) = 0$	$\#_T(3,1) = 0$
				$\#_T(2,1) = 0$

Figure 5. An s-decreasing tree and its cardinalities for $s = (0,0,0,2,1,3)$.

Next we establish notation for sets of tree inversions examples of which follow Fig. 6 using s-decreasing trees from those examples of s-weak order.

Definition 2.2.9. *(Ceballos & Pons, 2019, Definition 2.2) A **multi-inversion set** on $[n]$ is a multi-set I of pairs (y,x) such that $1 \leq x < y \leq n$. We write $\boldsymbol{\#_I(y,x)}$ for the multiplicity of (y,x) in I so if (y,x) does not appear in I, $\#_I(y,x) = 0$.*

*Given multi-inversion sets I and J, we say I is **included** in J and write $I \subseteq J$ if $\#_I(y,x) \leq \#_J(y,x)$ for all $1 \leq x < y \leq n$. We also define the **multi-inversion set difference**, $\boldsymbol{J - I}$, to be the multi-inversion set with $\#_{J-I}(y,x) = \#_J(y,x) - \#_I(y,x)$ whenever this difference is non-negative and 0 otherwise.*

This leads to a characterization of those multi-inversion sets which are actually sets of tree inversions of s-decreasing trees. Further, it motivates the definition of s-weak order in analogy with the inversion set definition of weak order on permutations.

Proposition 2.2.10. *(Ceballos & Pons, 2019, Proposition 2.4) There is a bijection between s-decreasing trees and multi-inversion sets I satisfying $\#_I(y,x) \leq s(y)$ and the following two properties:*

- ***Transitivity***: *if $a < b < c$ and $\#_I(c,b) = i$, then $\#_I(b,a) = 0$ or $\#_I(c,a) \geq i$.*
- ***Planarity***: *if $a < b < c$ and $\#_I(c,a) = i$, then $\#_I(b,a) = s(b)$ or $\#_I(c,b) \geq i$.*

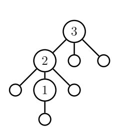
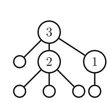
*Such multi-inversion sets are called **s-tree inversion sets**.*

Definition 2.2.11. *(Ceballos & Pons, 2019, Definition 2.5) Let s be a weak composition. The **s-weak order** is the partial order on s-decreasing trees given by $T \preceq Z$ if and only if $inv(Z) \subseteq inv(T)$ for s-decreasing trees T and Z using the inclusion of multi-inversion sets from Definition 2.2.9.*

Fig. 6 shows three examples of s-weak order. The labelings of the last two examples is our SB-labeling which is defined in Section 6.1.

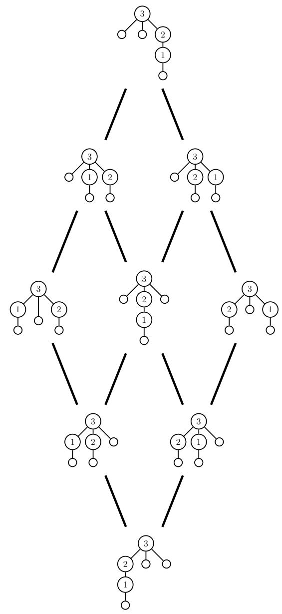
Below in Example 2.2.12, we illustrate Definition 2.2.9 and Proposition 2.2.10. We use subscripts on pairs (y,x) to indicate their multiplicity in a multi-inversion set.

Example 2.2.12. Illustrating Definition 2.2.9 and Proposition 2.2.10, we take

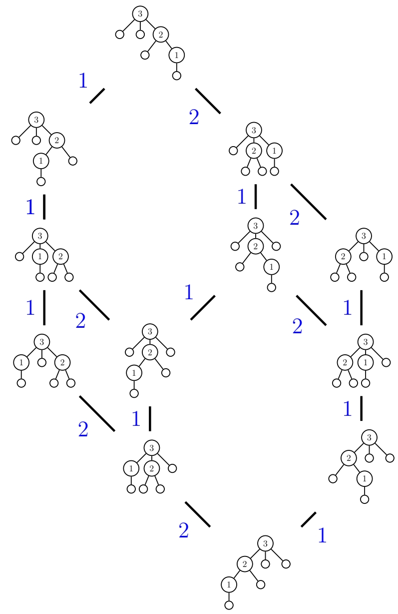
$T_1 =$  and $T_2 =$ 

and observe that $\text{inv}(T_1) = \{(2,1)_1\}$ and $\text{inv}(T_2) = \{(2,1)_2, (3,1)_2, (3,2)_1\}$. Thus, $\text{inv}(T_1) \subseteq \text{inv}(T_2)$ and $\text{inv}(T_2) - \text{inv}(T_1) = \{(2,1)_1, (3,1)_2, (3,2)_1\}$. Now we note that while $\text{inv}(T_1) = \{(2,1)_1\}$ is transitive, $I = \{(2,1)_1, (3,2)_1\}$ is not transitive because $\#_I(3,2) = 1$ while $\#_I(2,1) = 1 \neq 0$ and $\#_I(3,1) = 0 < \#_I(3,2)$. Similarly, $\text{inv}(T_2) = \{(2,1)_2, (3,1)_2, (3,2)_1\}$ is planar while $J = \{(2,1)_1, (3,1)_1\}$ is not planar because $\#_J(3,1) = 1$, but $\#_J(2,1) = 1 \neq 2 = s(2)$ and $\#_J(3,2) = 0 < \#_J(3,1)$.

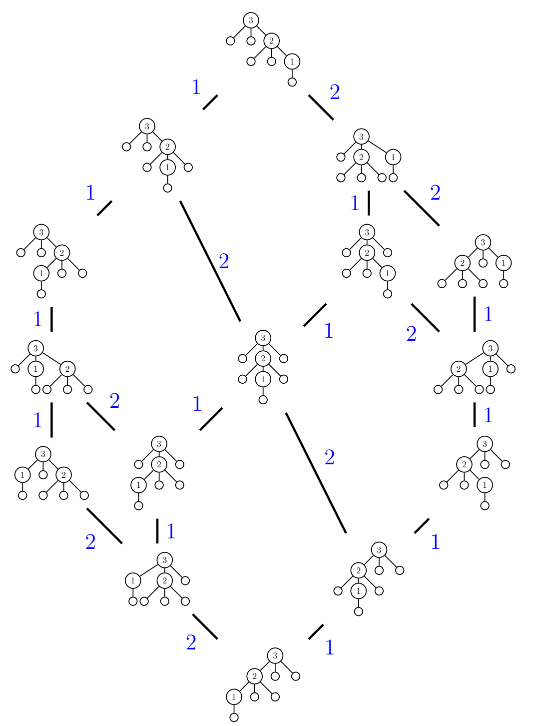
Remark 2.2.13. Taking $s = (1, \ldots, 1)$, s-weak order is isomorphic to weak order on the symmetric group $S_{l(s)}$.



(a) $s = (0, 0, 2)$



(b) $s = (0, 1, 2)$



(c) $s = (0, 2, 2)$

Figure 6. Examples of s-weak order. The labeling is our SB-labeling in Definition 6.1.1.

The following operations on multi-inversion sets are necessary to formulate the join in s-weak order which we will use in the course of our proofs. We give examples of these operations in Example 2.2.15 below.

- For weak composition s and multi-inversion sets I and J satisfying $\#_I(y,x), \#_J(y,x) \leq s(y)$ for all $1 \leq x < y \leq n$, the **union of I and J** is the smallest multi-inversion set by inclusion $\boldsymbol{I \cup J}$ such that $I, J \subseteq I \cup J$, that is $\#_{I\cup J}(y,x) = \max\{\#_I(y,x), \#_J(y,x)\}$ for all $1 \leq x < y \leq n$. Also, the **sum of I and J** is the multi-inversion set $\boldsymbol{I + J}$ with $\#_{I+J}(y,x) = \min\{\#_I(y,x) + \#_J(y,x), s(y)\}$ for all $1 \leq x < y \leq n$. If $J = \{(b,a)\}$, we write $\boldsymbol{I + (b,a)}$ for $I + J$.
- The **transitive closure**, denoted $\boldsymbol{I^{tc}}$, of a multi-inversion set I is the smallest transitive multi-inversion set, in terms of inclusion, containing I.

Theorem 2.2.14. *(Ceballos & Pons, 2019, Theorem 2.6) For any weak composition s, the s-weak order on s-decreasing trees is a lattice. The join of two s-decreasing trees T and Z is determined by*

$$inv\,(T \vee Z) = (inv\,(T) \cup inv\,(Z))^{tc}.$$

Example 2.2.15. This example illustrates the union and sum of multi-inversion sets as well as the transitive closure. Letting T_1 be the same s-decreasing tree as in Example 2.2.12, $\mathrm{inv}_=(T_1)\,\{(2,1)_1\}$. Now $\mathrm{inv}_\cup(T_1)\,\mathrm{inv}_=(T_1)\,\{(2,1)_1\}$ while $\mathrm{inv}_+(T_1)\,\mathrm{inv}_=(T_1)\,\{(2,1)_2\}$. In Example 2.2.12, we saw that the multi-inversion set $\{(2,1)_1, (3,2)_1\}$, which is also $\mathrm{inv}_+(T_1)\,(3,2)$, is not transitive. From our observations in Example 2.2.12, to satisfy the definition of transitivity in Proposition 2.2.10, $\{(2,1)_1, (3,2)_1\}^{tc}$ must contain $(3,1)$ with multiplicity at least 1. Thus, $\{(2,1)_1, (3,2)_1\}^{tc} = \{(2,1)_1, (3,1)_1,$

$(3,2)_1\}$. We can check that this is the multi-inversion set of one of the two s-decreasing trees covering T_1 in (c) of Fig. 6.

The cover relations in s-weak order are characterized as a certain type of operations known as tree rotations. We use this characterization heavily in our proofs. We first need a notion of an ascent in an s-decreasing tree. In the case $s = (1, \ldots, 1)$, this notion corresponds to the definition of ascents for permutations. Examples of tree ascents of the s-decreasing tree in Fig. 4 are given in Example 2.2.17.

Definition 2.2.16. *(Ceballos & Pons, 2019, Section 2.2) Let T be an s-decreasing tree and $1 \leq a < b \leq n$. The pair (a, b) is a* ***tree ascent*** *of T if the following hold:*

(i) $a \in T^b_i$ for some $0 \leq i < s(b)$,

(ii) if $a \in T^e_j$ for any $a < e < b$, then $j = s(e)$,

(iii) if $s(a) > 0$, then $T^a_{s(a)}$ is a leaf, that is, $T^a_{s(a)}$ contains no internal vertices.

Example 2.2.17. The tree ascents of the s-decreasing tree in (a) of Fig. 4 are as follows: $\{(1,4),(2,4),(3,4),(4,5),(5,9),(6,7),(7,8)\}$.

Remark 2.2.18. If $s(b) = 0$, then (a, b) with $a < b$ is not a tree ascent of any s-decreasing tree. This would contradict (i) of Definition 2.2.16.

Remark 2.2.19. An s-decreasing tree, T, cannot have tree ascents (a, b) and (a, c) with $b \neq c$. This would contradict condition (ii) of Definition 2.2.16 as either $a < b < c$ or $a < c < b$ while $a \notin T^b_{s(b)}, T^c_{s(c)}$ by condition (i) of Definition 2.2.16. We note that this implies that given an element $c \in [n]$ there is at most one $d \in [n]$ such that (c, d) is a tree ascent of T. Further, whenever (a, b) and (c, d) are distinct tree ascents of T, we may assume $a < c$. We make this assumption throughout our proofs.

Remark 2.2.20. We observe that by Remark 2.2.8, conditions (i) and (ii) of Definition 2.2.16 together are equivalent to $a \in {}_RT_i^b$ for some $0 \le i < s(b)$. The ith rightmost subtree of b in T, denoted ${}_RT_i^b$, is defined just after Definition 2.2.5.

Definition 2.2.21. *(Ceballos & Pons, 2019, Section 2.2) Let T be an s-decreasing tree with tree ascent (a, b). Then $(inv\,(T) + (b, a))^{tc}$ is an s-tree inversion set. We call the s-decreasing tree Z defined by $inv\,(Z) = (inv\,(T) + (b, a))^{tc}$ the **s-tree rotation of** T **along** (a, b). We denote this by $\mathbf{T} \xrightarrow{\mathbf{(a,b)}} \mathbf{Z}$.*

Ceballos and Pons characterized cover relations in s-weak order with the following theorem.

Theorem 2.2.22. *(Ceballos & Pons, 2019, Theorem 2.7) Let T and Z be s-decreasing trees. Then $T \prec Z$ if and only if there is a unique pair (a, b) which is a tree ascent of T such that $T \xrightarrow{(a,b)} Z$.*

Remark 2.2.23. $s(1)$ does not change the isomorphism type of s-weak order because no tree ascent of an s-decreasing tree may have larger element 1.

Remark 2.2.24. We describe an s-tree rotation in terms of an operation on the trees themselves. This is illustrated in Fig. 7. Suppose (a, b) is a tree ascent of T and $T \xrightarrow{(a,b)} Z$. Then $a \in {}_RT_j^b$ for some $j < s(b)$. Let g be the parent of a so $a \in T_{s(g)}^g$ and $g \in T_j^b$ or $g = b$ and a is the jth child of b. Let m be the smallest element of ${}_LT_{j+1}^b$ which is still larger than a. It is possible $m = b$. Then Z is the same as T except for the following changes: $Z_{s(g)}^g = T_0^a$ if $g \neq b$ and $Z_j^b = T_0^a$ if $g = b$ instead of T^a, $Z_i^a = T_i^a$ for $0 < i < s(a)$ if $s(a) > 0$, $Z_{s(a)}^a = T_0^m$ if $m \neq b$ and $Z_{s(a)}^a = T_{j+1}^b$ if $m = b$, Z_0^a is a leaf is a leaf if $s(a) > 0$, and $Z_0^m = Z^a$ if $m \neq b$ and $Z_{j+1}^b = Z^a$ if $m = b$.

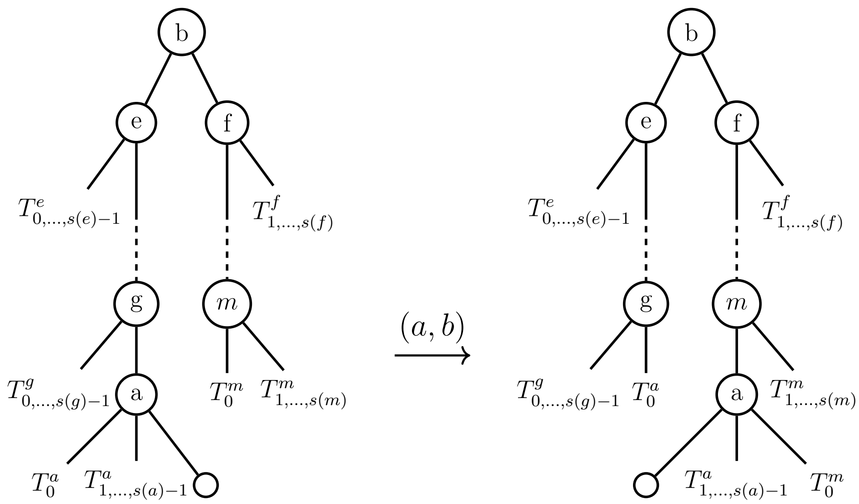


Figure 7. Illustration of the s-tree rotation along the tree ascent (a, b).

2.2.5 The s-Tamari lattice. The Tamari lattice is the sublattice of weak order on permutations generated by the 231-avoiding permutations. Similarly, the s-Tamari lattice is the sublattice of s-weak order generated by certain s-decreasing trees.

Definition 2.2.25. *(Ceballos & Pons, 2019, Definition 3.1) An s-decreasing tree T is called an* ***s-Tamari tree*** *if for any $a < b < c$, $\#_T(c, a) \leq \#_T(c, b)$ where $\#_T(c, a)$ is as defined in Definition 2.2.5. That is, all of the vertex labels in T^c_i are smaller than all of the vertex labels in T^c_j for $i < j$. The multi-inversion set of an s-Tamari tree is called an s-Tamari inversion set.*

We denote the partial order on s-Tamari trees induced by s-weak order by $\preceq_{Tam}$. Similarly, a subscript Tam will be used to denote objects in the s-Tamari lattice. For instance, $[T, Z]_{Tam}$ is the closed interval from T to Z in the s-Tamari lattice.

Theorem 2.2.26. *(Ceballos & Pons, 2019, Theorem 3.2) The collection of s-Tamari trees forms a sublattice of s-weak order, called the s-Tamari lattice.*

Remark 2.2.27. Taking $s = (1, \ldots, 1)$, the s-Tamari lattice is isomorphic to the classical Tamari lattice $T_{l(s)}$.

Similarly to s-weak order, there is a notion of ascent for s-Tamari trees and cover relations in the s-Tamari lattice are characterized as certain tree rotations along these ascents. For $a < b$, we say that (a, b) is a **Tamari tree ascent** of T if a is a non-right most child of b, that is, a is a direct descendant of b and $\#_T(b, a) < s(b)$. Note that in the s-Tamari lattice, $T^a_{s(a)}$ need not be a leaf for some (a, b) to be a Tamari tree ascent. We denote cover relations in the s-Tamari lattice by $\prec_{Tam}$.

Theorem 2.2.28. *(Ceballos & Pons, 2019, Section 3.1) Let T be an s-Tamari tree and let (a, b) be a Tamari tree ascent of T. Then $(inv\,(T) + (b, a))^{tc}$ is an s-Tamari inversion set. Let Z be the s-Tamari tree such that $inv\,(Z) = (inv\,(T) + (b, a))^{tc}$. We say Z is the s-**Tamari rotation of** T **along** (a, b) and write $\mathbf{T} \overset{\mathbf{Tam(a,b)}}{\longrightarrow} \mathbf{Z}$. Moreover, $T \prec_{Tam} Z$ if and only if there is a unique Tamari tree ascent (a, b) of T such that $T \overset{Tam(a,b)}{\longrightarrow} Z$.*

An s-Tamari rotation is essentially the same as an s-tree rotation except that the smaller element of the Tamari tree ascent may have right descendants and those right descendants are moved with along with a if $s(a) > 0$. An s-Tamari rotation is illustrated in Fig. 8.

Remark 2.2.29. Similarly to s-tree rotations, we describe s-Tamari rotations in terms of an operation on the trees themselves. Suppose that (a, b) is a Tamari tree ascent of T and $T \overset{Tam(a,b)}{\longrightarrow} Z$. Then $a \in T^b_j$ for some $j < s(b)$ and a is a child of b. Recall that every labeled vertex of T^b_{j+1} is greater than a since T is an s-Tamari tree. Let m be the smallest labeled vertex of ${}_LT^b_{j+1}$. Then Z is the same as T except for the following: $Z^b_j = T^a_0$ instead of T^a, $Z^a_i = T^a_i$ for $0 < i \leq s(a)$ if $s(a) > 0$, Z^a_0 is a leaf, $Z^m_0 = Z^a$.

Remark 2.2.30. An s-Tamari tree T cannot have Tamari tree ascents (a, b) and (a, c) with $b \neq c$. This follows from the fact that in a rooted tree, every non-root node has exactly one parent. Thus, whenever (a, b) and (c, d) are distinct Tamari tree ascents of T, we may assume $a < c$. We make this assumption throughout our proofs.

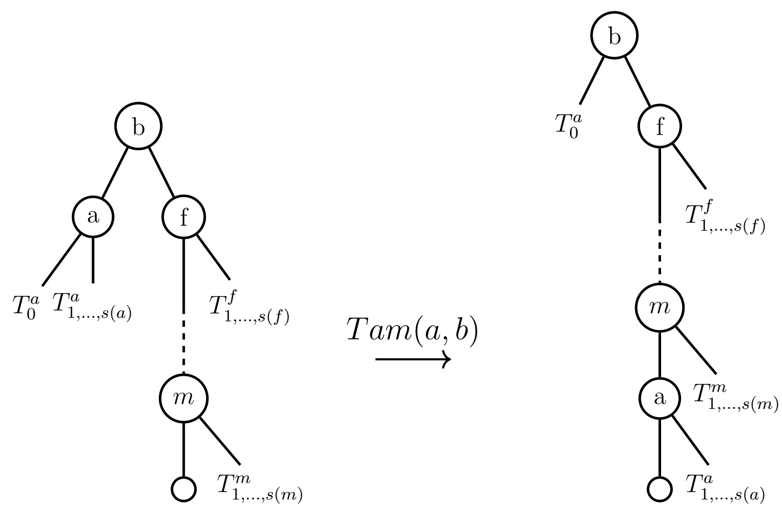


Figure 8. s-Tamari rotation along the Tamari tree ascent (a, b).

2.3 Young's Lattice and Standard Young Tableaux

We briefly recall the definitions of Young diagrams, Young tableau, and Young's lattice. For a thorough introduction to this topic see Fulton (1997) or Stanley (1999) Chapter 7, for instance. We use this background in Section 5.2.2 to analyze certain maximal chain descent orders of intervals in Young's Lattice.

A **Young diagram** α is a collection of rows of left justified boxes in which the ith row from the top has at most as many boxes as the $(i-1)$th row. Whenever convenient, we consider a Young diagram to include an arbitrary number of extra rows with zero boxes. We may refer to a Young diagram as the non-increasing tuple of the lengths of its rows, i.e. an integer partion. Fig. 9 shows the Young diagram

$\alpha = (3, 2, 1)$. We index the boxes in a Young diagram by matrix coordinates, so the coordinates (i, j) refer to the box in row i and column j.

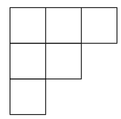


Figure 9. The Young diagram for $\alpha = (3, 2, 1)$.

We say that Young diagram α contains Young diagram μ if the Young diagram of α contains the Young diagram of μ. This is the same as each row of α being at least as large as the corresponding row of μ where we add as many 0s as necessary so that the diagrams have the same number of rows. If α contains μ, then we define a **skew diagram** α/μ as the diagram of the boxes contained in α but not in μ.

Young's Lattice, denoted $\mathcal{Y}$, is the partial order on all Young diagrams by diagram containment. We note that $\mathcal{Y}$ is infinite, has a unique minimal element given by the empty partition $\emptyset$, and $\mathcal{Y}$ is graded by the number of boxes in the diagram. Further, $\mathcal{Y}$ is a distributive lattice. For a fixed Young diagram α, we denote the principal order ideal of $\mathcal{Y}$ generated by α as $\mathcal{Y}(\alpha)$. For μ contained in α, we denote the closed interval $[\mu, \alpha]$ by $\mathcal{Y}(\mu, \alpha)$. Since $\mathcal{Y}$ is a distributive lattice, $\mathcal{Y}(\alpha)$ and $\mathcal{Y}(\mu, \alpha)$ are finite distributive lattices. Thus, by Theorem 2.2.2, $\mathcal{Y}(\alpha)$ and $\mathcal{Y}(\mu, \alpha)$ are the posets of order ideals of some finite posets. We let P_α be the partial order on the boxes in the Young diagram of α defined by the product order on the coordinates of the boxes, that is, box (i, j) is less than or equal to box (i', j') if and only if $i \leq i'$ and $j \leq j'$. Similarly, we let $P_{\alpha/\mu}$ be the partial order on the boxes of the skew diagram α/μ defined by the product order on the coordinates of the boxes. A box in P_α or $P_{\alpha/\mu}$ is exactly covered by the adjacent box to the east and the adjacent box to the south, if such boxes exist in the relevant diagram. Then $\mathcal{Y}(\alpha) \cong J(P_\alpha)$ and $\mathcal{Y}(\mu, \alpha) \cong J(P_{\alpha/\mu})$.

A **standard Young tableau** is an assignment of a positive integer from $[n]$ to each box of a Young diagram α with n boxes such that:

(1) The box fillings strictly increase across each row from left to right.

(2) The box fillings strictly increase down each column from top to bottom.

The Young diagram α is referred to as the **shape** of the Young tableau. Standard skew tableau are defined analogously as integer fillings of skew diagrams with the same row and column requirements. Fig. 10 shows an example of a standard Young tableau of shape $(3, 2, 1)$. In what follows, the arguments for Young diagrams and skew diagrams are the same, so we let α denote a Young diagram or a skew diagram. Thus, $\mathcal{Y}(\alpha)$ can denote any closed interval in $\mathcal{Y}$.

We will denote the collection of standard tableau of shape α by $\mathbf{ST}_\alpha$. For T, a standard tableau with n boxes, and a box b in T (possibly given by its coordinates or some other description), we denote the filling of box b in T by $T(b)$. For $i \in [n]$, denote the box of T whose filling is i by T^i.

1	2	3
4	5	
6		

Figure 10. A standard Young tableau of shape $\alpha = (3, 2, 1)$.

The **row word** of a tableau T is the word whose letters are the entries of T obtained by reading the rows of T from left to right where we read the rows from top to bottom. The row word of T is denoted $\mathbf{w(T)}$. For instance, the row word of the tableau in Fig. 10 is 123456. We may choose other reading orders of the boxes of a tableau to obtain other words. Reading the columns from top to bottom and the columns from left to right gives the **column word**.

Since $\mathcal{Y}(\alpha) \cong J(P_\alpha)$, the maximal chains of $\mathcal{Y}(\alpha)$ are in bijection with the linear extensions of P_α by taking the order in which the boxes are added in the maximal chain to form the diagram for α. By construction, the linear extensions of P_α precisely give the standard tableau of shape α by filling each box with its value under the linear extension. Thus, $\mathcal{M}(\mathcal{Y}(\alpha))$ is in bijection with ST_α. We denote the standard tableau corresponding to maximal chain $m \in \mathcal{M}(\mathcal{Y}(\alpha))$ by T_m. We denote the maximal chain in $\mathcal{M}(\mathcal{Y}(\alpha))$ corresponding to standard tableau T by m_T. Thus, $m_{T_m} = m$ and $T_{m_T} = T$.

Each linear extension of P_α gives an EL-labeling of $\mathcal{Y}(\alpha) \cong J(P_\alpha)$ as described in Remark 2.2.3. Fixing a linear extension of P_α is the same as fixing a standard tableau $T \in ST_\alpha$. We denote the EL-labeling of $\mathcal{Y}(\alpha)$ induced by T as λ_T.

CHAPTER III

MAXIMAL CHAIN DESCENT ORDERS

The structure of this chapter is as follows: In Section 3.1, we present the definition of a maximal chain descent order along with the natural example of the Boolean lattice and the weak order of type A. We also show several structural properties of these orders including proving Theorem 1.1.1.

3.1 Defining Maximal Chain Descent Orders

We will not assume our posets are graded. Since EL-labelings are instances of CL-labelings, the following constructions and properties work for EL-labelings as well. We begin with two definitions which together describe the polygon moves giving rise to maximal chain descent orders.

Definition 3.1.1. *Let P be a finite, bounded poset. Let $m, m' \in \mathcal{M}(P)$ be maximal chains of P with $m : x_0 = \hat{0} \lessdot x_1 \lessdot \cdots \lessdot x_{r-1} \lessdot x_r = \hat{1}$ and $m' : x'_0 = \hat{0} \lessdot x'_1 \lessdot \cdots \lessdot x'_{r'-1} \lessdot x'_{r'} = \hat{1}$. Suppose $r' \leq r$ since P is not necessarily graded. We say that m and m'* ***differ by a polygon*** *if there is some $1 \leq i \leq r-1$ such that $x'_j = x_j$ for all $j < i$, $x'_{i+1} = x_{i+l}$ for some $1 \leq l$, $x'_i \neq x_{i+k}$ for all $0 \leq k \leq l$, and $x'_{i+1+m} = x_{i+l+m}$ for all $0 \leq m \leq r - i - l = r' - i - 1$.*

Maximal chains differing by a polygon are illustrated in Fig. 11 below. Intuitively, m and m' differ by a polygon if they agree everywhere except on an interval where m' has length two.

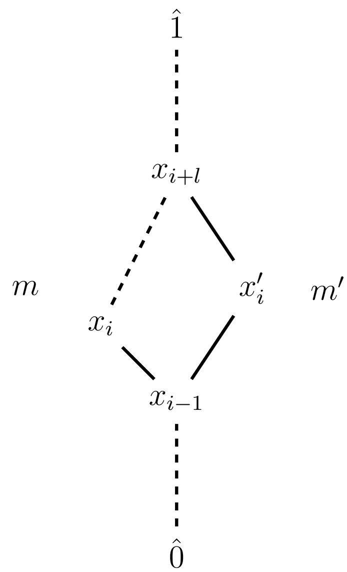


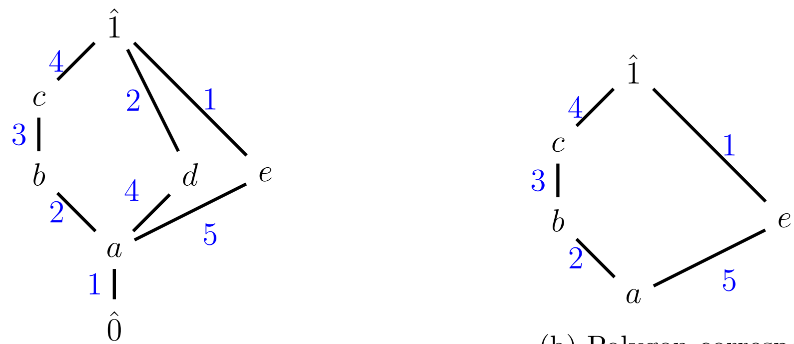
Figure 11. Maximal chains which differ by a polygon.

Definition 3.1.2. *Let P be a finite, bounded poset with a CL-labeling λ. We say that* $\mathbf{m}$ ***increases by a polygon move to*** $\mathbf{m}'$ *and write* $\mathbf{m} \to \mathbf{m}'$ *if m and m' differ by a polygon as in Definition 3.1.1 and m restricted to the rooted interval $[x_{i-1}, x'_{i+1}]_{m^{x_{i-1}}}$ is the unique ascending maximal chain with respect to λ while m' restricted to the rooted interval $[x_{i-1}, x'_{i+1}]_{m^{x_{i-1}}}$ is a descent with respect to λ. We refer to the pair of saturated chains $m_{x_{i-1}}^{x'_{i+1}}$ and $m'^{x'_{i+1}}_{x_{i-1}}$ as the* ***polygon corresponding to*** $\mathbf{m} \to \mathbf{m}'$*. Thus,* ***the bottom element of the polygon corresponding to*** $\mathbf{m} \to \mathbf{m}'$ *refers to x_{i-1} and* ***the top element of the polygon corresponding to*** $\mathbf{m} \to \mathbf{m}'$ *refers to x'_{i+1}.*

Intuitively, $m \to m'$ if m and m' differ by a polygon and m if $m \cap m'$ is codimension 1 in m' and m is the lexicographically least (hence, ascending) chain when restricted to the rooted interval where m and m' differ.

Example 3.1.3. Fig. 12 subfigure (a) shows a poset with an EL-labeling which has two increases by polygon moves: $(\hat{0} \lessdot a \lessdot b \lessdot c \hat{1}) \to (\hat{0} \lessdot a \lessdot d \lessdot \hat{1})$ and $(\hat{0} \lessdot a \lessdot b \lessdot c \lessdot \hat{1}) \to$

$(\hat{0} \lessdot a \lessdot e \lessdot \hat{1})$. The polygon corresponding to $(\hat{0} \lessdot a \lessdot b \lessdot c \lessdot \hat{1}) \to (\hat{0} \lessdot a \lessdot e \lessdot \hat{1})$ is depicted in subfigure (b). We emphasize that $(\hat{0} \lessdot a \lessdot d \lessdot \hat{1}) \not\to (\hat{0} \lessdot a \lessdot e \lessdot \hat{1})$ despite being lexicographically smaller because $(\hat{0} \lessdot a \lessdot d \lessdot \hat{1})$ is not the ascending (and thus, not the lexicographically least) saturated chain of $[a, \hat{1}]$.



(a) Poset with an EL-labeling.

(b) Polygon corresponding to $(\hat{0} \lessdot a \lessdot b \lessdot c \lessdot \hat{1}) \to (\hat{0} \lessdot a \lessdot e \lessdot \hat{1})$.

Figure 12. An EL-labeling with two increases by polygon moves.

Our polygon moves on maximal chains are somewhat reminiscent of the "polygon flips" between monotone paths in polytopes (oriented by a linear functional) which were employed in Athanasiadis, Edelman, and Reiner (2000). Similar moves on the maximal chains of a finite poset with an S_n EL-labeling were considered in McNamara (2003) to define a 0-Hecke algebra action on the maximal chains of such posets. Related "diamond moves" on thin posets were considered in Chandler (2019), but without the partial order structure on maximal chains from a CL-labeling which we introduce here.

We give two simple and useful propositions about increases by polygon moves before proceeding to the definition of a maximal chain descent order.

Proposition 3.1.4. *Let P be a finite, bounded poset with a CL-labeling λ. If $m \to m'$, then $\lambda(m) <_{lex} \lambda(m')$.*

Proof. This follows from the fact that the ascending chain in any rooted interval lexicographically precedes all other maximal chains in that rooted interval. $\square$

Proposition 3.1.5. *Let P be a finite, bounded poset with a CL-labeling λ. If $m' \in \mathcal{M}(P)$ has a descent at $x \in m'$ with respect to λ, then there is an unique $m \in \mathcal{M}(P)$ such that $m \to m'$ and $m' \setminus m = \{x\}$.*

Proof. Let w and z be the elements of m' satisfying $w \lessdot x \lessdot z$. Let c be the unique ascending maximal chain of the rooted interval $[w, z]_{m'^w}$ with respect to λ guarateed by the definition of a CL-labeling. Set $m = m'^w * c * m'_z$. Then by Definition 3.1.2, we have $m \to m'$ with $m' \setminus m = \{x\}$ and m is unique. $\square$

Next we introduce the notion of a maximal chain descent order.

Definition 3.1.6 (Hersh and L.)**.** *Let P be a finite, bounded poset with a CL-labeling λ. The **maximal chain descent order induced by $\boldsymbol{\lambda}$** is the partial order $\preceq_{\boldsymbol{\lambda}}$ on the maximal chains $\mathcal{M}(P)$ defined as the reflexive and transitive closure of the relation $m \to m'$, i.e. if m increases by a polygon move to m' with respect to λ. Denote the poset $(\mathcal{M}(P), \preceq_\lambda)$ by $\mathbf{P}_{\boldsymbol{\lambda}}(\mathbf{2})$.*

Remark 3.1.7. We note as above that Definition 3.1.6 applies to EL-labelings since they are instances of CL-labelings.

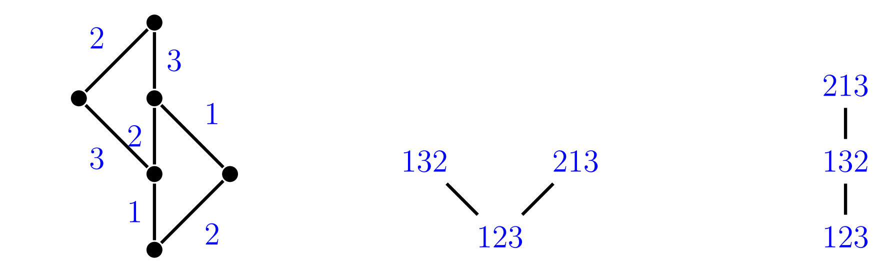
Remark 3.1.8. Let P be a finite, bounded poset with a CL-labeling λ. Since the proper part of P often has more interesting topology than P itself (a bounded poset is contractible), we often consider the lexicographic shelling induced on $\Delta(\overline{P})$ by simply deleting the cone points $\hat{0}$ and $\hat{1}$ from $\Delta(P)$. One might wonder if this lexicographic shelling of $\overline{P}$ gives a different maximal chain descent order than $P_\lambda(2)$. It does not. The facets of $\Delta(\overline{P})$ are simply the facets of $\Delta(P)$ with $\hat{0}$ and

$\hat{1}$ deleted while the shelling order, all codimension one intersections, and all face containments are preserved by deleting the cone points. Thus, the maximal chain descent orders are isomorphic via the map which deletes $\hat{0}$ and $\hat{1}$ from maximal chains of P.

A maximal chain descent order induced by an EL-labeling is shown in Example 3.1.10. Using Proposition 3.1.4, we can easily show that the relation of Definition 3.1.6 is antisymmetric, and so truly a partial order. We also easily have the corollary that lexicographic order is at least as fine as the corresponding maximal chain descent order.

Corollary 3.1.9. *Let P be a finite, bounded poset with a CL-labeling λ. If $m \prec_\lambda m'$, then $\lambda(m) <_{lex} \lambda(m')$.*

Example 3.1.10. Fig. 13 shows a poset P with an EL-labeling λ along with its induced maximal chain descent order $P_\lambda(2)$ and the induced lexicographic order on maximal chains. This example shows that $P_\lambda(2)$ may be strictly coarser than the induced lexicographic order.



(a) P with EL-labeling λ. (b) Consequent $P_\lambda(2)$. (c) Lex order from λ.

Figure 13. An EL-labeling with distinct lexicographic order and maximal chain descent order.

Now we turn to a motivating example of a maximal chain descent order before proving several fundamental properties.

3.2 A Motivating Example: Maximal Chain Descent Order for the Boolean Lattice is Weak Order on the Symmetric Group

Perhaps the most natural example of an EL-labeling is the labeling λ of the Boolean lattice B_n, the lattice of subsets of $[n]$ ordered by containment, in which $B \lessdot B'$ precisely when $B' = B \cup \{i\}$ for some $i \in [n] \setminus B$ and $\lambda(B, B') = i$. We will prove that $B_{n\lambda}(2)$ is isomorphic to the weak order on S_n via the map assigning each maximal chain to its label sequence with respect to λ.

Theorem 3.2.1. *Let B_n be the Boolean lattice of subsets of $[n]$ with its standard EL-labeling λ. Then the map $m \mapsto \lambda(m)$ is an isomorphism from the maximal chain descent order $B_{n\lambda}(2)$ to the weak order on S_n (the type A weak order).*

Proof. The label sequences of the maximal chains $\mathcal{M}(B_n)$ are precisely the permutations of $[n]$. Moreover, each permutation π occurs as the label sequence of exactly one maximal chain of B_n, namely, the maximal chain m_π whose rank i element is the union of the first i entries of π in one-line notation. For instance, $m_{3241} : \emptyset \lessdot \{3\} \lessdot \{2,3\} \lessdot \{2,3,4\} \lessdot \{1,2,3,4\}$ and $\lambda(m_{3241}) = 3241$.

Every rank two interval in B_n has the form shown in Fig. 14 where $A \subset [n]$ has $|A| \leq n-2$ and $i, j \in [n] \setminus A$ with $i < j$.

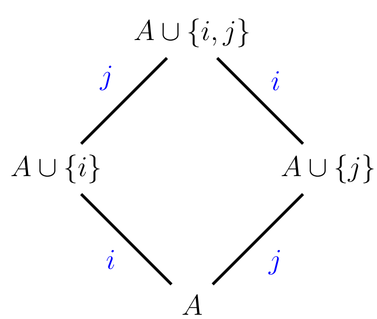


Figure 14. Typical rank two interval in the Boolean lattice.

Thus, if $m \to m'$ for maximal chains $m, m' \in \mathcal{M}(B_n)$, then $\lambda(m')$ is obtained from $\lambda(m)$ by transposing a unique pair of adjacent entries of $\lambda(m)$ which are in ascending order in $\lambda(m)$ and descending order in $\lambda(m')$. For example, $m_{3214} : (\emptyset \lessdot \{3\} \lessdot \{2,3\} \lessdot \{1,2,3\} \lessdot \{1,2,3,4\}) \to m_{3241} : (\emptyset \lessdot \{3\} \lessdot \{2,3\} \lessdot \{2,3,4\} \lessdot \{1,2,3,4\})$ and $\lambda(m_{3214}) = 3214$ while $\lambda(m_{3241}) = 3241$. Therefore, if $m \to m'$, then $\lambda(m)$ is covered by $\lambda(m')$ in weak order on the symmetric group S_n. Moreover, this implies that if $m \prec_\lambda m'$, then $\lambda(m) <_{wk} \lambda(m')$ in weak order on S_n. Hence, if $m \to m'$, then m is covered by m' in $P_\lambda(2)$; this follows by contradiction because supposing that $m \to m'$ and $m \to m'' \prec_\lambda m'$ for some other maximal chain m'' implies that $\lambda(m) <_{wk} \lambda(m'') <_{wk} \lambda(m')$ which contradicts the fact that $\lambda(m) \lessdot_{wk} \lambda(m')$. Lastly, it is clear by construction that if $\pi \lessdot_{wk} \sigma$ for permutations $\pi, \sigma \in S_n$, then $m_\pi \to m_\sigma$. Thus, $m \prec\!\!\cdot_\lambda m'$ if and only if $\lambda(m) \lessdot_{wk} \lambda(m')$. Therefore, $m \mapsto \lambda(m)$ gives a poset isomorphism from $B_{n\lambda}(2)$ to weak order on S_n. $\square$

In Section 5.2, we present more examples in depth, and for the most part, they do not require reading the remainder of this section. In particular, one example is a construction of the type B weak order as a maximal chain descent order.

3.3 Fundamental Properties of Maximal Chain Descent Orders

Here we prove several fundamental structural properties of maximal chain descent orders. We note as above that while these statements are written for CL-labelings, they also apply to EL-labelings since EL-labelings are instances of CL-labelings. The notations of Definition 2.1.1 are used ubiquitously in this section.

Proposition 3.3.1. *Let P be a finite, bounded poset with a CL-labeling λ. Then the maximal chain descent order $P_\lambda(2)$ has a unique minimal element, denoted by $\hat{0}$, given by the unique ascending maximal chain of P with respect to λ.*

Proof. This proof uses the same central idea as the proof that a lexicographic order from a CL-labeling induces a shelling order of $\Delta(P)$. Let $m_1, \ldots, m_t$ be a total order of the maximal chains $\mathcal{M}(P)$ which is compatible with the lexicographic order induced by λ. By definition of a CL-labeling, m_1 is the unique ascending maximal chain with respect to λ. We will proceed by induction on the index in this total order to show that $m_1 \preceq_\lambda m_i$ for all $1 \leq i \leq t$. This is trivial when $i = 1$. Assume it holds for all $1 \leq k \leq i$ for some $i \geq 1$. Since $i + 1 > 1$, m_{i+1} is not the unique ascending maximal chain of P, so m_{i+1} has a descent at some position j. Thus, $m_{i+1} : x_0 \lessdot x_1 \lessdot \cdots \lessdot x_{j-1} \lessdot x_j \lessdot x_{j+1} \lessdot \cdots \lessdot x_{n-1} \lessdot x_n$ with $\lambda(x_{j-1}, x_j) \not\leq \lambda(x_j, x_{j+1})$. Hence, m_{i+1} restricted to the rooted interval $[x_{j-1}, x_{j+1}]_{m_{i+1}^{x_{j-1}}}$ is not the unique ascending maximal chain of the rooted interval $[x_{j-1}, x_{j+1}]_{m_{i+1}^{x_{j-1}}}$. Then since λ is a CL-labeling, there is a unique ascending saturated chain c in the rooted interval $[x_{j-1}, x_{j+1}]_{m_{i+1}^{x_{j-1}}}$ with respect to λ which lexicographically precedes m_{i+1} restricted to the rooted interval $[x_{j-1}, x_{j+1}]_{m_{i+1}^{x_{j-1}}}$. Then $m_{i+1}^{x_{j-1}} * c * {m_{i+1}}_{x_{j+1}} \to m_{i+1}$. Thus, $m_{i+1}^{x_{j-1}} * c * {m_{i+1}}_{x_{j+1}} \prec_\lambda m_{i+1}$. And $m_{i+1}^{x_{j-1}} * c * {m_{i+1}}_{x_{j+1}}$ lexicographically precedes m_{i+1}, so $m_1 \preceq_\lambda m_{i+1}^{x_{j-1}} * c * {m_{i+1}}_{x_{j+1}}$ by the inductive hypothesis. Hence, $m_1 \prec_\lambda m_{i+1}$, so m_1 is the $\hat{0}$ of $P_\lambda(2)$ by induction. □

On the other hand, descending label sequences give maximal elements of the maximal chain descent order. They do not necessarily give all maximal elements though as witnessed by Example 3.1.10.

Proposition 3.3.2. *Let P be a finite, bounded poset which admits a CL-labeling λ. If maximal chain $m \in \mathcal{M}(P)$ has a descending label sequence with respect to λ, then m is a maximal element of $P_\lambda(2)$.*

Proof. Since m is descending with respect to λ, m has no ascents. Thus, there are no chains $m' \in \mathcal{M}(P)$ such that $m \to m'$ by Definition 3.1.2. So, m is a maximal element of $P_\lambda(2)$. $\square$

Since a CL-labeling restricted to a rooted closed interval is a CL-labeling of that interval (see Proposition 2.1.7), we can consider the maximal chain descent order induced by λ on the maximal chains of the interval. The next lemma shows that order relations in the maximal chain descent order of a rooted closed interval in a sense lift to order relations in the maximal chain descent order on the entire poset. This is an expression of the recursive nature of CL-labelings. We use this lemma ubiquitously.

Lemma 3.3.3. *Let P be a finite, bounded poset which admits a CL-labeling λ. Let $m = x_0 \lessdot x_1 \lessdot \cdots \lessdot x \lessdot \cdots \lessdot y \lessdot \cdots \lessdot x_{n-1} \lessdot x_n$ be a maximal chain of P and let c and c' be maximal chains of the rooted interval $[x, y]_{m^x}$. If $c \preceq_\lambda c'$ in the maximal chain descent order of the rooted interval $[x, y]_{m^x}$ induced by the restriction of λ, then $m^x * c * m_y \preceq_\lambda m^x * c' * m_y$ in $P_\lambda(2)$.*

Proof. First, we show that if $c \to c'$, then $m^x * c * m_y \to m^x * c' * m_y$. Then the result follows since $c \to c'$ are precisely the relations whose reflexive and transitive closure give $[x, y]_\lambda(2)$ and $m^x * c * m_y \to m^x * c' * m_y$ are among the relations whose reflexive and transitive closure give $P_\lambda(2)$. If c and c' differ by a polygon in $[x, y]$, then $m^x * c * m_y$ and $m^x * c' * m_y$ also differ by a polygon in P. Also, the restriction of λ to the rooted interval $[x, y]_{m^x}$ is compatible with λ on P. Thus, an ascent or descent in c or c' gives a an ascent or descent in $m^x * c * m_y$ or $m^x * c' * m_y$, respectively. Thus, $c \to c'$ implies $m^x * c * m_y \to m^x * c' * m_y$. $\square$

Remark 3.3.4. Lemma 3.3.3 implies that an isomorphic copy of $[x,y]_{m^x}{}_\lambda(2)$ appears as a subposet of $P_\lambda(2)$ for each extension of m^y to a maximal chain of P. However, Fig. 2 shows that cover relations in $[x,y]_{m^x}{}_\lambda(2)$ need not lift to cover relations in $P_\lambda(2)$. Further, Fig. 15 exhibits an EL-labeling in which $[x,y]_\lambda(2)$ lifts to a non-induced subposet of $P_\lambda(2)$. In particular, the chains of $[x,y]$ labeled 32 and 54 are not comparable in $[x,y]_\lambda(2)$, while the extended chains of P labeled 323 and 543 are comparable in $P_\lambda(2)$.

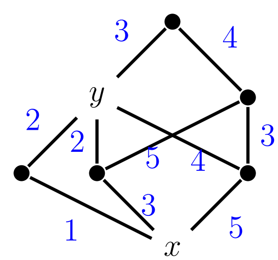


Figure 15. EL-labeling λ of P with $[x,y]_\lambda(2)$ lifting to a non-induced subposet of $P_\lambda(2)$.

As a corollary of Lemma 3.3.3 and Proposition 3.3.1 we describe certain general order relations in a maximal chain descent order. In particular, if maximal chain m is ascending on each interval where it differs from maximal chain m', then $m \prec_\lambda m'$. This corollary also provides the key fact used in our later proofs about shelling orders.

Corollary 3.3.5. *Let P be a finite, bounded poset with a CL-labeling λ. Let $m, m' \in M(P)$ be distinct maximal chains of P and let $y_0 < y_1 < y_2 < \cdots < y_k$ be all of the elements of m satisfying $y_l \in m \cap m'$ for each $0 \leq l \leq k$ while $m' \cap (y_l, y_{l+1})$ and $m \cap (y_l, y_{l+1})$ are non-empty and disjoint for each $0 \leq l \leq k-1$. Suppose that for each $0 \leq l \leq k-1$ m is ascending with respect to λ when restricted to the rooted interval $[y_l, y_{l+1}]_{m^{y_l}}$. Then $m \prec_\lambda m'$ in $P_\lambda(2)$.*

Proof. We have $k \geq 1$ since $m \neq m'$. We induct on k. When $k = 1$, $m \prec_\lambda m'$ by Proposition 3.3.1 and Lemma 3.3.3 directly.

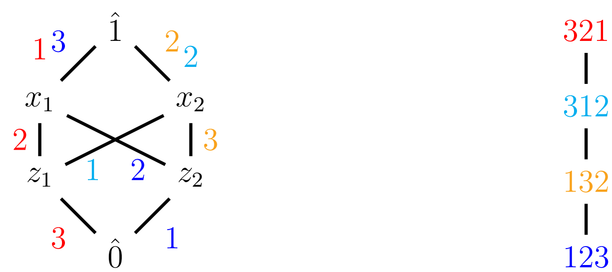
Assume that $m \prec_\lambda m'$ when $k = n$ for some $n \geq 1$. Suppose $k = n + 1$. By Proposition 3.3.1 and Lemma 3.3.3 $m \prec_\lambda m^{y_n} * m'^{y_{n+1}}_{y_n} * m_{y_{n+1}}$. Now by construction $m^{y_k} * m'^{y_{n+1}}_{y_n} * m_{y_{n+1}}$ differs from m' in n intervals. By definition of a CL-labeling the label sequences of m and $m^{y_n} * m'^{y_{n+1}}_{y_n} * m_{y_{n+1}}$ agree up to y_n. Thus, $m^{y_n} * m'^{y_{n+1}}_{y_n} * m_{y_{n+1}}$ is ascending with respect to λ when restricted to the n rooted intervals on which it differs from m'. Then the inductive hypothesis implies $m^{y_n} * m'^{y_{n+1}}_{y_n} * m_{y_{n+1}} \prec_\lambda m'$. Therefore, $m \prec_\lambda m'$, and the result holds by induction. □

Remark 3.3.6. Not every order relation in a maximal chain descent order is of the form in Corollary 3.3.5. We observe this in the example in Fig. 28. The maximal chains labeled 1265 and 3214 are comparable, but neither is the ascending maximal chain of the entire poset with respect to the labeling. This can also be observed in the minimal labeling of the partition lattice in Fig. 19 and Fig. 20.

We now observe several examples which begin to reveal the subtlety of cover relations and rank in maximal chain descent orders.

Remark 3.3.7. While every cover relation in a maximal chain descent order corresponds to an increase by a polygon move, it is perhaps surprising that not all increases by polygon moves result in cover relations. The example in Fig. 2 and Example 3.3.8 both exhibit this.

Example 3.3.8. Fig. 16 shows that an increase by a polygon move with respect to a CL-labeling which is not an EL-labeling need not give a cover relation in the induced maximal chain descent order. In this example, the only label which depends on the root is the label of $x_1 \lessdot \hat{1}$ which is the red 1 if the maximal chain labeled in red is used and the blue 3 otherwise. We observe that this is the dual of Björner and Wachs' CL-labeling of the Bruhat order of S_3 from Björner and Wachs (1982). We have that $(\hat{0} \lessdot z_2 \lessdot x_1 \lessdot \hat{1}) \to (\hat{0} \lessdot z_1 \lessdot x_1 \lessdot \hat{1})$ while $(\hat{0} \lessdot z_2 \lessdot x_1 \lessdot \hat{1})$ is not covered by $(\hat{0} \lessdot z_1 \lessdot x_1 \lessdot \hat{1})$ in $P_\lambda(2)$.



(a) P and CL-labeling λ. (b) $P_\lambda(2)$

Figure 16. A CL-labeling which is not polygon complete.

Fig. 2 and Example 3.3.8 lead us to introduce the following definition.

Definition 3.3.9. *Let λ be a CL-labeling of a finite, bounded poset P. We say λ is **polygon complete** if $m \to m'$ implies $m \prec_\lambda m'$ in $P_\lambda(2)$.*

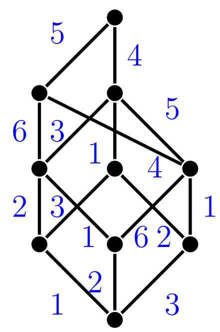
In Section 4.1, we give a rather technical characterization of polygon complete EL-labelings. In that section and in Section 4.2, we also give simpler conditions which are sufficient for polygon completeness or imply an EL-labeling is not polygon complete. But there is still more we can say about cover relations in general, namely the number of elements which a maximal chain covers in a maximal chain descent order is at most the number of descents of the maximal chain with respect to the labeling.

Lemma 3.3.10. *Let P be a finite, bounded poset with a CL-labeling λ. For a maximal chain $m \in \mathcal{M}(P)$, the number of maximal chains $m' \in \mathcal{M}(P)$ such that*

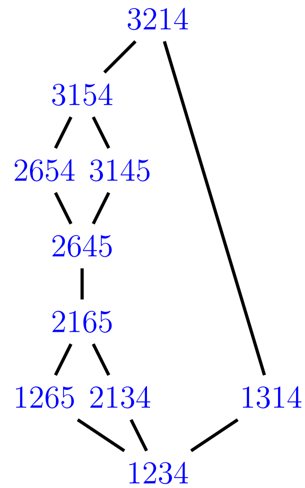
$m' \prec_\lambda m$ is at most the number of descents of m with respect to λ. Moreover, if λ is polygon complete in the sense of Definition 3.3.9, then the number of maximal chains $m' \in \mathcal{M}(P)$ such that $m' \prec_\lambda m$ is the number of descents of m with respect to λ.

Proof. By Definition 3.1.2 and Proposition 3.1.5 the number of maximal chains $m' \in \mathcal{M}(P)$ such that $m' \to m$ is exactly the number of descents of m with respect to λ. Then since $m' \to m$ does not necessarily give a cover relation, the number of downward cover relations from m is at most the number of descents of m with respect to λ. Further, if λ is polygon complete, then $m' \to m$ does give a cover relation. Thus, the number of downward cover relations from m is the number of descents of m with respect to λ. $\square$

Example 3.3.11. Fig. 28 shows a poset P with an EL-labeling λ and the resulting maximal chain descent order. The maximal chain descent order in this case is not ranked. Notice also that the number of downward cover relations in $P_\lambda(2)$ from each maximal chain of P is at most the number of descents in its label sequence. In particular, the upper bound is reached by all maximal chains of P except for the maximal chain labeled 2654. The label seqence 2654 has two descents, but $2134 \to 2654$ while $2134 \not\prec_\lambda 2654$ in $P_\lambda(2)$.



(a) P with EL-labeling λ.



(b) Consequent $P_\lambda(2)$.

Figure 17. A poset P with EL-labeling λ and its maximal chain descent order.

3.4 Equivalence of Linear Extensions of $\mathbf{P_\lambda(2)}$ and Shellings Induced by $\boldsymbol{\lambda}$

Now we show how maximal chain descent orders encode the shellings induced by λ by proving Theorem 1.1.1. Björner and Wachs' original proofs that EL-labelings and CL-labelings induce shellings essentially go through, but require a modification using properties of maximal chain descent orders which we showed previously. We emphasize the modification in the proof below. See their proof in Björner and Wachs (1996).

Lemma 3.4.1. *Let P be a finite, bounded poset which admits a CL-labeling λ. Then any linear extension of $P_\lambda(2)$ gives a shelling order of the order complex $\Delta(P)$ and the restriction map of any such shelling is given by $R(m) = \left\{x \in m \;\middle|\; x \text{ is a descent of } m \text{ w.r.t. } \lambda\right\}$ for any maximal chain $m \in \mathcal{M}(P)$. The homology facets of the shelling of $\Delta(\overline{P})$ induced by any linear extension of $P_\lambda(2)$ are given by the maximal chains $m \in \mathcal{M}(P)$ with descending label sequence with respect to λ.*

Proof. Let $m_1, \ldots, m_t$ be a linear extension of $P_\lambda(2)$. Consider $m_i \cap m_j$ for $i < j$. We will show that m_j has a descent at some $x \in m_j$ such that $x \notin m_i$. Then the fact that this is a shelling order follows simply from the proof of Theorem 5.8 in Björner and Wachs (1996), for instance, and the definition of $P_\lambda(2)$.

We consider the maximal intervals on which m_i and m_j differ. That is, let $y_0 < y_1 < y_2 < \cdots < y_k$ be all the elements of m_j such that $y_l \in m_i \cap m_j$ for each $0 \leq l \leq k$ while $m_j \cap (y_l, y_{l+1})$ and $m_i \cap (y_l, y_{l+1})$ are non-empty and disjoint for each $0 \leq l \leq k-1$. If m_j is ascending with respect to λ on each rooted interval $[y_l, y_{l+1}]_{m_j^{y_l}}$, then $m_j \prec_\lambda m_i$ by Corollary 3.3.5. However, this contradicts the fact that $m_1, \ldots, m_t$ is a linear extension of $P_\lambda(2)$ and $i < j$. Thus, m_j has a descent at some $x \in m_j$ such that $x \notin m_i$.

As for the restriction map, R as defined above gives the necessary partition of $\Delta(P)$ by Theorem 2.1.6 and Proposition 2.1.2. Now we verify the restriction set containment condition of Proposition 2.1.2. Assume $R(m_i) \subseteq m_j$ for some $i \neq j$. Assume that the maximal intervals on which m_i and m_j differ are the same as above. Then since $R(m_i)$ is exactly the set of elements of m_i at which m_i has descents and $R(m_i) \subseteq m_j$, $m_i \cap [y_l, y_{l+1}]$ is ascending with respect to λ and the root $m_i^{y_l}$ for each $1 \leq l \leq k-1$. Thus, $m_i \prec_\lambda m_j$ by Corollary 3.3.5. This implies $i < j$ because $m_1, m_2, \ldots, m_t$ is a linear extension of $P_\lambda(2)$. Therefore, R is the restriction map of this shelling.

Lastly, a maximal chain $m \in \mathcal{M}(P)$ has $m \setminus \{\hat{0}, \hat{1}\}$ a homology facet of the shelling of $\Delta(\overline{P})$ induced by $m_1, m_2, \ldots, m_t$ if and only if $R(m) = m \setminus \{\hat{0}, \hat{1}\}$. Thus, $m \setminus \{\hat{0}, \hat{1}\}$ is a homology facet if and only if m has a descent with respect to λ at each element of $m \setminus \{\hat{0}, \hat{1}\}$. $\square$

Remark 3.4.2. It is clear from Corollary 3.1.9 that the lexicographic shellings in the sense Björner and Wachs (see Section 2.1.2) are among the linear extension shellings of Lemma 3.4.1. On the other hand, one linear extension of the maximal chain descent order in Example 3.1.10 is $123, 213, 132$. This gives a shelling order of $\Delta(P)$ by Lemma 3.4.1. However, this total order is not compatible with the lexicographic order induced by the labeling. This shows that these linear extension shellings can give strictly more shelling orders of $\Delta(P)$ than the lexicographic ones. This can also be observed by applying Lemma 3.4.1 to the Boolean lattice with its standard EL-labeling as in Section 3.2 and any linear extension of the weak order on the symmetric group except for the lexicographic order on permutations.

Remark 3.4.3. We observe that Lemma 3.4.1 applied to the Boolean lattice and its EL-labeling from Section 3.2 recovers a special case of a result in Björner (1984). Namely, any linear extension of weak order on a Coxeter group induces a shelling of the corresponding Coxeter complex. In the case of type A, the Coxeter complex is the order complex of the proper part of the Boolean lattice, so Theorem 3.2.1 and Lemma 3.4.1 give the type A case of Björner's result.

Another consequence of Lemma 3.4.1 is that we can detect some homology facets (all the homology facets if the labeling is polygon complete) from the number of downward cover relations from a given element in a maximal chain descent order.

Corollary 3.4.4. *Let P be a finite, bounded poset which admits a CL-labeling λ. Let Ω be any linear extension of $P_\lambda(2)$. Let $m \in \mathcal{M}(P)$ be a maximal chain of length n. If the number of maximal chains $m' \in \mathcal{M}(P)$ such that $m' \prec_\lambda m$ is $n-1$, then $m \setminus \left\{\hat{0}, \hat{1}\right\}$ is a homology facet of the shelling of $\Delta(\overline{P})$ induced by Ω. Moreover, if λ is polygon complete, then $m \setminus \left\{\hat{0}, \hat{1}\right\}$ is a homology facet of the shelling of*

$\Delta(\overline{P})$ *induced by* Ω *if and only if the number of downward cover relations from* m *in* $P_\lambda(2)$ *is* $n-1$.

Proof. This follows from Lemma 3.3.10 and the homology facet statement of Lemma 3.4.1. $\square$

Conversely, the following lemma implies that, in a precise sense, maximal chain descent orders encode all shellings "derived from" a CL-labeling. This lemma also reinforces the name "maximal chain descent order."

Lemma 3.4.5. *Let* P *be a finite, bounded poset with a CL-labeling* λ. *Suppose a total order* $m_1, m_2, \ldots, m_t$ *on the maximal chains of* P *induces a shelling order of the order complex* $\Delta(P)$ *with the property that for each* $1 \le i \le t$ *the restriction face* $R(m_i)$ *of* m_i *is precisely the face* $R(m_i) = \left\{ x \in m_i \;\middle|\; x \text{ is a descent of } m_i \text{ w.r.t. } \lambda \right\}$. *Then* $m_1, m_2, \ldots, m_t$ *is a linear extension of* $P_\lambda(2)$.

Proof. It suffices to show that if $m_i \to m_j$ with respect to λ, then $i < j$. Let y be the unique element of m_j such that $m_j \setminus m_i = \{y\}$. Let x and z be the elements of m_j satisfying $x \lessdot y \lessdot z$. By definition of an increase by a polygon move (Definition 3.1.2), y is a descent of m_j and $m_i \cap [x,z]$ is ascending with respect to λ and the root m_i^x. Thus, $R(m_i) \subseteq m_j$. This implies $i < j$ by Proposition 2.1.2 since R is the restriction map of the shelling $m_1, m_2, \ldots, m_t$. $\square$

Combined, the previous two lemmas prove Theorem 1.1.1.

Theorem 3.4.6. *Let* P *be a finite, bounded poset with a CL-labeling* λ. *For any total order* $\Omega : m_1, m_2, \ldots, m_t$ *on the maximal chains of* P, *the following are equivalent:*

(1) Ω *is a linear extension of* $P_\lambda(2)$.

(2) Ω *induces a shelling order of the order complex* $\Delta(P)$ *with the property that for each* $1 \leq i \leq t$ *the restriction face* $R(m_i)$ *of* m_i *is precisely the face* $R(m_i) = \Big\{ x \in m_i \;\Big|\; x \text{ is a descent of } m_i \text{ w.r.t. } \lambda \Big\}$.

Proof. Statement (1) implies statement (2) by Lemma 3.4.1 and statement (2) implies statement (1) by Lemma 3.4.5. □

Since EL-labelings are instances of CL-labelings, the previous theorem holds for EL-labelings as well. We state this point as a corollary for emphasis since EL-labelings are possibly more widely known than CL-labelings.

Corollary 3.4.7. *Let* P *be a finite, bounded poset with an EL-labeling* λ. *For any total order* $\Omega : m_1, m_2, \ldots, m_t$ *on the maximal chains of* P, *the following are equivalent:*

(1) Ω *is a linear extension of* $P_\lambda(2)$.

(2) Ω *induces a shelling order of the order complex* $\Delta(P)$ *with the property that for each* $1 \leq i \leq t$ *the restriction face* $R(m_i)$ *of* m_i *is precisely the face* $R(m_i) = \Big\{ x \in m_i \;\Big|\; x \text{ is a descent of } m_i \text{ w.r.t } \lambda \Big\}$.

We also immediately have that if two different CL-labelings of the same poset give the same descent set for each maximal chain, then they induce the same maximal chain descent order.

Corollary 3.4.8. *Let* P *be a finite, bounded poset with two possibly different CL-labelings* λ *and* λ'. *Suppose that for each* $m \in \mathcal{M}(P)$ *and each* $x \in m$, m *has descent at* x *with respect to* λ *if and only if* m *has descent at* x *with respect to* λ'. *Then* $P_\lambda(2) = P_{\lambda'}(2)$.

Proof. Both $P_\lambda(2)$ and $P_{\lambda'}(2)$ are posets on $\mathcal{M}(P)$ and are finite. By Theorem 3.4.6 $P_\lambda(2)$ and $P_{\lambda'}(2)$ both have exactly the same set of linear extensions. A finite poset can be constructed simply from its set of linear extensions, so $P_\lambda(2) = P_{\lambda'}(2)$. $\square$

CHAPTER IV

COVER RELATIONS IN MAXIMAL CHAIN DESCENT ORDERS

This chapter proceeds as follows: Section 4.1 addresses cover relations of maximal chain descent orders and polygon completeness of EL-labelings. We prove the characterization of polygon complete EL-labelings in Theorem 4.1.19. We also give the sufficient condition for polygon complete EL-labelings called polygon strong. We then prove many well known EL-labelings are polygon strong, and so polygon complete. Section 4.2 presents a sufficient condition for polygon complete CL-labelings by introducing a notion of inversions for CL-labelings.

4.1 Understanding Cover Relations via "Polygon Completeness"

We characterize polygon completeness for EL-labelings (see Definition 3.3.9) with two technical conditions below in Theorem 4.1.19. However, we begin with a simpler concrete sufficient condition for polygon completeness of EL-labelings in Definition 4.1.5. The proof of this sufficient condition provides us the opportunity to get a taste for some of the proof techniques we will use for Theorem 4.1.19, but in a more constrained context where we may be less delicate.

4.1.1 Polygon Strong Implies Polygon Complete. We begin with several necessary lemmas some of which are possibly interesting in their own right. Then we present the definition of a polygon strong EL-labeling before we show that it is sufficient for polygon completeness. This first lemma is quite straightforward and provides the base cases for later induction arguments.

Lemma 4.1.1. *Let P be a finite, bounded poset in which the length of the longest maximal chain is either one or two. Let λ be an CL-labeling of P. Then λ is polygon complete.*

Proof. If the length of the longest maximal chain of P is one, then P has exactly one maximal chain and λ is vacuously polygon complete because there are no polygons to check.

Suppose the longest maximal chain of P has length two. Then every maximal chain of P has length two. Thus, every maximal chain of P, except for the unique ascending chain, is a descent. Thus, the increases by polygon moves of P with respect to λ are of the form $m_0 \to m$ where m_0 is the unique ascending chain of P and $m \neq m_0$ is any other maximal chain of P. Hence, every maximal chain of P, except for m_0, is a maximal element of $P_\lambda(2)$. Therefore, $P_\lambda(2)$ is ranked with rank one and λ is polygon complete. $\square$

The following lemma is quite useful for working with maximal chain descent orders, particularly for proofs by induction on chain length. Intuitively, the lemma says that if two maximal chains m and m' agree along an initial segment and are comparable in a maximal chain descent order, then each maximal chain between them in the maximal chain descent order agrees with m and m' on that same initial segment. The example in Fig. 2 shows, among other things, that the mirrored statement for final segments is not true.

Lemma 4.1.2. *Let P be a finite, bounded poset with a CL-labeling λ. Let m and m' be maximal chains $m : y_0 = \hat{0} \lessdot y_1 \lessdot \cdots \lessdot y_{i-1} \lessdot y_i \lessdot y_{i+1} \lessdot \cdots \lessdot y_{t-1} \lessdot y_t = \hat{1}$ and $m' : y_0 = \hat{0} \lessdot y_1 \lessdot \cdots \lessdot y_{i-1} \lessdot y'_i \lessdot y'_{i+1} \lessdot \cdots \lessdot y'_{t'-1} \lessdot y'_t = \hat{1}$. Suppose $m = m_0 \to m_1 \to m_2 \to \cdots \to m_k \to m_{k+1} = m'$. Then $m_j^{y_{i-1}} = m^{y_{i-1}} = m'^{y_{i-1}}$ for each $1 \leq j \leq k$.*

Proof. Suppose seeking a contradiction that there is some m_j such that $m_j^{y_{i-1}} \neq m^{y_{i-1}}$. We observe that for any of the maximal chains in the sequence to disagree with m at or below y_{i-1}, there must be some m_l such that the polygon corresponding to $m_l \rightarrow m_{l+1}$ has bottom element strictly less than y_{i-1}. Let $m_{j'} \rightarrow m_{j'+1}$ be the first increase by a polygon move in the sequence $m = m_0 \rightarrow m_1 \rightarrow m_2 \rightarrow \cdots \rightarrow m_k \rightarrow m_{k+1} = m'$ with corresponding polygon whose bottom element is strictly less than y_{i-1}. We must have $0 \leq j' \leq k$. Then $\lambda(m_{j'}) <_{lex} \lambda(m_{j'+1})$ by Proposition 3.1.4. The first entry at which the label sequences of $m_{j'}$ and $m_{j'+1}$ differ comes before the $(i-1)$th position. However, $m^{y_{i-1}} = m'^{y_{i-1}} = m_{j'}^{y_{i-1}}$ since $m_{j'} \rightarrow m_{j'+1}$ is the first increase by a polygon move with corresponding polygon whose bottom element is below y_{i-1}. So, the label sequences of $m_{j'}$ and m' agree in their first $i-1$ entries. Thus, $\lambda(m') <_{lex} \lambda(m_{j'+1})$ which contradicts Corollary 3.1.9 since $m_{j'+1} \preceq_\lambda m'$. Hence, $m_j^{y_{i-1}} = m^{y_{i-1}} = m'^{y_{i-1}}$ for each $1 \leq j \leq k$. $\square$

A consequence of Lemma 3.3.3 and Lemma 4.1.2 is that in a maximal chain descent order induced by an EL-labeling, whether or not an increase by a polygon move gives a cover relation only depends on what happens in the poset above the bottom of the corresponding polygon. In other words, whether or not an increase by a polygon move with respect to an EL-labeling gives a cover relation does not depend on the root used to get to the bottom of the polygon. Besides being useful for later arguments, it seems this fact may be of interest in its own right.

Corollary 4.1.3. *Let P be a finite, bounded poset which admits an EL-labeling λ. Let $m, m' \in \mathcal{M}(P)$ be maximal chains of P with*

$$m : x_0 = \hat{0} \lessdot x_1 \lessdot \cdots \lessdot x_{i-1} \lessdot \cdots \lessdot x_{i+l} \lessdot x_{r-1} \lessdot x_r = \hat{1}$$

and

$$m' : x_0 = \hat{0} \lessdot x_1 \lessdot \cdots \lessdot x_{i-1} \lessdot x_i' \lessdot x_{i+l} \lessdot \cdots \lessdot x_{r-1} \lessdot x_r = \hat{1}$$

such that $m \to m'$. Suppose $m \prec_\lambda m'$ in $P_\lambda(2)$. Then for any maximal chain c of $[\hat{0}, x_{i-1}]$, we have $c * m_{x_{i-1}} \prec_\lambda c * m'_{x_{i-1}}$ in $P_\lambda(2)$.

Proof. Since λ is an EL-labeling, the labels of $c * m_{x_{i-1}}$ and $c * m'_{x_{i-1}}$ agree with the labels of m and m', respectively, above x_{i-1}. Thus, $c * m_{x_{i-1}} \to c * m'_{x_{i-1}}$. Now suppose seeking a contradiction that $c * m_{x_{i-1}}$ is not covered by $c * m'_{x_{i-1}}$ in $P_\lambda(2)$. Then there are maximal chains $m_1, \ldots, m_k \in \mathcal{M}(P)$ with $k \geq 1$ such that $c * m_{x_{i-1}} \to m_1 \to \cdots \to m_k \to c * m'_{x_{i-1}}$. Then by Lemma 4.1.2 $m_j^{x_{i-1}} = c$ for each $1 \leq j \leq k$. Thus, $m_{x_{i-1}} \to m_{1x_{i-1}} \to \cdots \to m_{kx_{i-1}} \to m'_{x_{i-1}}$ in $[x_{i-1}, \hat{1}]$ with respect to λ. Then by Lemma 3.3.3 $m^{x_{i-1}} * m_{x_{i-1}} \prec_\lambda m^{x_{i-1}} * m_{1x_{i-1}} \prec_\lambda m^{x_{i-1}} * m'_{x_{i-1}}$ in $P_\lambda(2)$. However, since $m^{x_{i-1}} = m'^{x_{i-1}}$, we then have $m \prec_\lambda m^{x_{i-1}} * m_{1x_{i-1}} \prec_\lambda m'$ which contradicts that $m \prec_\lambda m'$ in $P_\lambda(2)$. Therefore, $c * m_{x_{i-1}} \prec_\lambda c * m'_{x_{i-1}}$ in $P_\lambda(2)$. □

Another consequence of Lemma 4.1.2 is the following lemma.

Corollary 4.1.4. *Let P be a finite, bounded poset with a CL-labeling λ. Suppose $m \to m'$ such that $m' \setminus m = \{x\}$ with $x \lessdot \hat{1}$. Then $m \prec_\lambda m'$ in $P_\lambda(2)$.*

Proof. This follows immediately from Proposition 3.1.5 and Lemma 4.1.2. □

Now we turn to a concrete sufficient condition for polygon completeness of EL-labelings which we call polygon strong. Polygon strong is simpler and easier to verify than the conditions in Theorem 4.1.19. Polygon strong is a weakening of Björner's notion of strongly lexicographically shellable from Björner (1980) which arose from studying admissible lattices.

Definition 4.1.5. *Let P be a finite, bounded poset with an EL-labeling λ. We say λ is a **polygon strong EL-labeling** if for each descent $x \lessdot y \lessdot z$, $\lambda(y \lessdot z) <$*

$\lambda(y' \lessdot z)$ *where* y' *is the coatom of* $[x,z]$ *contained in the unique ascending maximal chain of* $[x,z]$ *with respect to* λ.

The EL-labeling pictured in Fig. 18 is polygon strong.

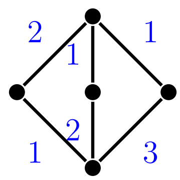


Figure 18. A polygon strong EL-labeling.

Theorem 4.1.6. *Let* P *be a finite, bounded poset with a polygon strong EL-labeling* λ. *Then* λ *is polygon complete.*

Proof. We first observe that the restriction of a polygon strong EL-labeling to any closed interval $[x,y]$ of P is also a polygon strong EL-labeling. This is straightforward because any interval of $[x,y]$ is also an interval of P. Next we observe that if $m \rightarrow m'$ for maximal chains m and m' with $x \lessdot \hat{1}$ contained in m and $x' \lessdot \hat{1}$ contained in m', then $\lambda(x,\hat{1}) \geq \lambda(x',\hat{1})$ since λ is polygon strong.

We proceed by induction on the length of the longest maximal chain of P. We have that λ is polygon complete if the length of the longest maximal chain of P is one or two by Lemma 4.1.1. Assume λ is polygon complete whenever the length of the longest maximal chain of P is any k with $1 \leq k \leq r$ for some $r \geq 2$. Let P have longest maximal chain of length $r+1$. Assume $m \rightarrow m'$. Since m and m' differ by a polygon, $m : \hat{0} = x_0 \lessdot \cdots \lessdot x_{i-1} \lessdot x_i \lessdot \cdots \lessdot x_{i+l} \lessdot \cdots \lessdot x_{r+1} = \hat{1}$ and $m' : \hat{0} = x_0 \lessdot \cdots \lessdot x_{i-1} \lessdot x_i' \lessdot x_{i+l} \lessdot \cdots \lessdot x_{r+1} = \hat{1}$ for some $l \geq 1$. So, $x_{i-1} \lessdot x_i \lessdot \cdots \lessdot x_{i+l}$ is the unique ascending maximal chain of $[x_{i-1}, x_{i+l}]$ with respect to λ while $x_{i-1} \lessdot x_i' \lessdot x_{i+l}$ is a descent with respect to λ.

Now, seeking a contradiction, suppose that $m \not\lessdot_\lambda m'$. By Corollary 4.1.4 $i + l < r + 1$, so $m^{x_r} \to m'^{x_r}$. Since $m \not\lessdot_\lambda m'$, there are maximal chains $m_1, m_2, \ldots, m_s \in \mathcal{M}(P)$ with $s \geq 1$ such that $m \to m_1 \to m_2 \to \cdots \to m_s \to m'$. Then by Lemma 4.1.2, $m_j^{x_{i-1}} = m^{x_{i-1}} = m^{x_{i-1}}$ for all $1 \leq j \leq s$. There are two cases we must consider. Either $x_r \in m_j$ for all $1 \leq j \leq s$ or there is some $1 \leq j \leq s$ such that $x_r \notin m_j$.

Suppose $x_r \in m_j$ for all $1 \leq j \leq s$. Then $m^{x_r} \to m_1^{x_r} \to m_2^{x_r} \to \cdots \to m_s^{x_r} \to m'^{x_r}$. Then since $s \geq 1$, $m^{x_r} \not\lessdot_\lambda m'^{x_r}$ in $[\hat{0}, x_r]_\lambda(2)$ despite the fact that $m^{x_r} \to m'^{x_r}$. However, this contradicts the fact that λ restricted to $[\hat{0}, x_r]$ is polygon complete by the inductive hypothesis because the length of the longest maximal chain in $[\hat{0}, x_r]$ is at most r and λ restricted to $[\hat{0}, x_r]$ is polygon strong.

Thus, there is a first m_t for $0 \leq t \leq s$ (say $m_0 = m$) such that $x_r \in m_t$, but $x_r \notin m_{t+1}$. Let $z_{t+1} \lessdot \hat{1}$ be contained in m_{t+1}. Then $\lambda(z_{t+1}, \hat{1}) < \lambda(x_r, \hat{1})$ since λ is polygon strong and $x_r \lessdot \hat{1}$ is contained in m_t while $x_r \notin m_{t+1}$. Now let $z_j \lessdot \hat{1}$ be contained in m_j. Since λ is polygon strong, we have

$$\lambda(x_r, \hat{1}) = \lambda(z_1, \hat{1}) = \lambda(z_2, \hat{1}) = \cdots = \lambda(z_t, \hat{1}) > \lambda(z_{t+1}, \hat{1}) \geq \cdots \geq \lambda(z_k, \hat{1}) \geq \lambda(x_r, \hat{1})$$

by our second observation in the first paragraph. But this implies the contradiction that $\lambda(x_r, \hat{1}) > \lambda(x_r, \hat{1})$. Therefore, λ is polygon complete, so the theorem holds by induction. $\square$

4.1.2 Examples of Polygon Strong EL-labelings.

Here we prove that many well known families of EL-labelings are polygon strong, and so polygon complete. This section is not entirely self contained, but we provide references and, when feasible, brief explanations. We expect that many more EL-labelings which "arise in nature" are polygon strong.

We begin with the examples where Theorem 4.1.6 applies most naturally which are the M-chain EL-labelings of finite supersolvable lattices due to Stanley (originally known as R-labelings by Stanley). See Section 2.2.2 for background on supersolvable lattices and M-chain EL-labelings.

Theorem 4.1.7. *Stanley's M-chain EL-labelings of any finite supersolvable lattice from Stanley (1972) are polygon strong. Thus, these EL-labelings are polygon complete.*

Proof. First, we observe that the label sequences of an M-chain EL-labeling λ are all permutations of $[n]$ where n is the rank of the supersolvable lattice. Thus, in any rank two interval of the lattice, the label sequence of each maximal chain is either a, b with $a < b$ if the chain is the unique ascending chain with respect to λ or b, a otherwise. Thus, λ is polygon strong, and so polygon complete by Theorem 4.1.6. □

Remark 4.1.8. In fact, our proof also works in the more general context of so called S_n EL-labelings of finite posets (not supersolvable lattices) in which the label sequence of every maximal chain is a permutation of $[n]$ (see McNamara (2003) for the terminology S_n EL-labeling). This is because our proof only relied on all label sequences being permutations of n. Thus, any S_n EL-labeling of a finite poset is polygon strong, and so polygon complete. As a note, McNamara (2003) showed that a finite lattice admitting an S_n EL-labeling is equivalent to that lattice being supersolvable.

We record this useful fact about the label sequences of rank two intervals from EL-labelings in which the label sequences of maximal chains are permutations because we will use it in other places.

Proposition 4.1.9. *Let P be a finite, bounded poset with an S_n EL-labeling λ. Suppose $m' \in \mathcal{M}(P)$ has a descent at $x \in m'$ with $rk(x) = i$. Then the unique $m \in \mathcal{M}(P)$ such that $m \to m'$ and $m' \setminus m = \{x\}$ guaranteed by Proposition 3.1.5 has $\lambda(m') = \lambda(m)(i, i+1)$.*

Proof. Let $\lambda(m) = \lambda_1\lambda_2 \dots \lambda_i\lambda_{i+1} \dots \lambda_n$. Then since $m' \setminus m = \{x\}$, $\lambda(m') = \lambda_1\lambda_2 \dots \lambda'_i\lambda'_{i+1} \dots \lambda_n$. Then since λ is an S_n EL-labeling, $\{\lambda_i, \lambda_{i+1}\} = \{\lambda'_i, \lambda'_{i+1}\}$. Lastly, since m has a descent at x and $m' \to m$, $\lambda_i > \lambda_{i+1}$ and $\lambda'_i < \lambda'_{i+1}$. Hence, $\lambda(m') = \lambda(m)(i, i+1)$. □

Another generalization of Theorem 4.1.7 is the following result for similar EL-labelings of upper-semimodular and lower-semimodular lattices.

Theorem 4.1.10. *Let L be a finite upper-semimodular or lower-semimodular lattice. Let λ be an EL-labeling of L induced by an admissible map on L as in Björner (1980). Then λ is polygon strong. Thus, λ is polygon complete.*

Proof. In Proposition 3.6 Björner (1980), it is shown that the labeling λ due to Stanley (1974) is an EL-labeling. In Theorem 3.7 Björner (1980), λ is shown to be an SL-labeling (strongly lexicographic) in the sense of Definition 3.4 Björner (1980). Such lattices are also graded. When restricted to intervals of length two in L, the defining condition of an SL-labeling is precisely the defining condition of a polygon strong EL-labeling. Then applying Theorem 4.1.6 completes the proof. □

Next we turn to the case of finite geometric lattices. A finite lattice L is called a **geometric lattice** if every element is a join of atoms of L (i.e. L is atomic) and if $x \wedge y \lessdot x, y$ implies $x, y \lessdot x \vee y$ for all $x, y \in L$ (i.e. L is upper semimodular). A finite lattice L is geometric if and only if it is the lattice of flats of a finite simply matroid.

Now we briefly recall the definition of a minimal labeling of a finite geometric lattice. For an element x in a geometric lattice L, we denote the set of atoms of L which are below x by $A(x)$. Let Ω be any total ordering of the atoms of L. Then the **minimal labeling induced by $\boldsymbol{\Omega}$** is the edge labeling λ_Ω of L given as follows: if $x \lessdot y$, then $\lambda_\Omega(x,y) = \min_\Omega(A(y) \setminus A(x))$. Minimal labelings were shown to be EL-labelings in Björner (1980) and were shown to characterize finite geometric lattices in Davidson and Hersh (2014). (We refer to those citations for more in-depth analysis of geometric lattices and minimal labelings.)

Theorem 4.1.11. *Every minimal labeling λ of a finite geometric lattice is polygon strong. Thus, λ is polygon complete.*

Proof. Let $x \lessdot y \lessdot z$ be an ascending saturated chain in a geometric lattice L with respect to a minimal labeling λ_Ω induced by a total atom order Ω. Then $\lambda_\Omega(x,y)$ is the minimal atom with respect to Ω which is below z, but not below x. This implies that for any $y' \neq y$ satisfying $x \lessdot y' \lessdot z$, $\lambda_\Omega(y',z) = \lambda_\Omega(x,y) < \lambda_\Omega(y,z)$. Thus, λ_Ω is polygon strong, and so polygon complete by Theorem 4.1.6. $\square$

Example 4.1.12. Fig. 19 shows the partition lattice Π_4, which is a geometric lattice, with a minimal labeling λ_Ω. The total atom order Ω is the one induced by the labels of the covers below the atoms. We label covers by the index of the corresponding atom with respect to Ω. Fig. 20 exhibits the induced maximal chain descent order $\Pi_{4_{\lambda_\Omega}}(2)$. The maximal chains of Π_4 are denoted as follows: m_{ijkl} denotes the chain $1|2|3|4 \lessdot ij|k|l \lessdot ijk|l \lessdot ijkl$ and m_{ij}^{kl} denotes the chain $1|2|3|4 \lessdot ij|k|l \lessdot ij|kl \lessdot ijkl$. This example illustrates that $P_\lambda(2)$ may have multiple maximal elements and that $P_\lambda(2)$ need not be ranked despite λ being polygon complete.

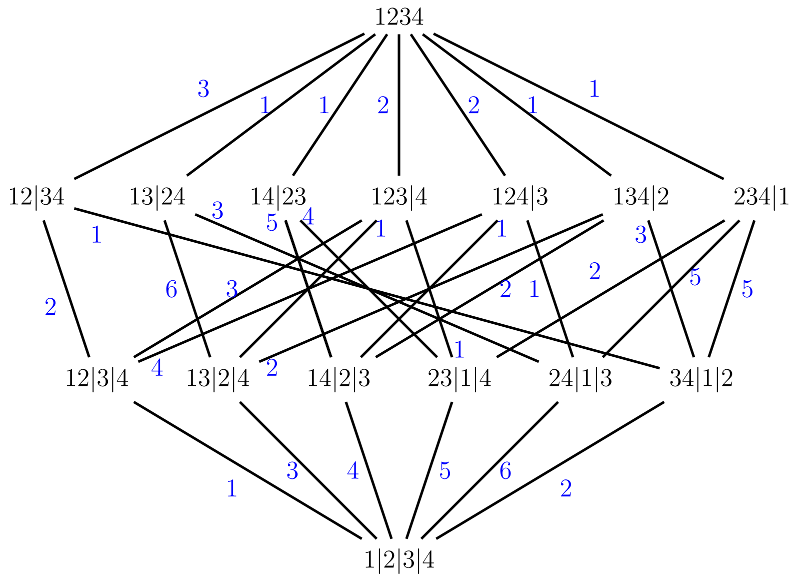


Figure 19. The partition lattice Π_4 with minimal labeling λ_Ω.

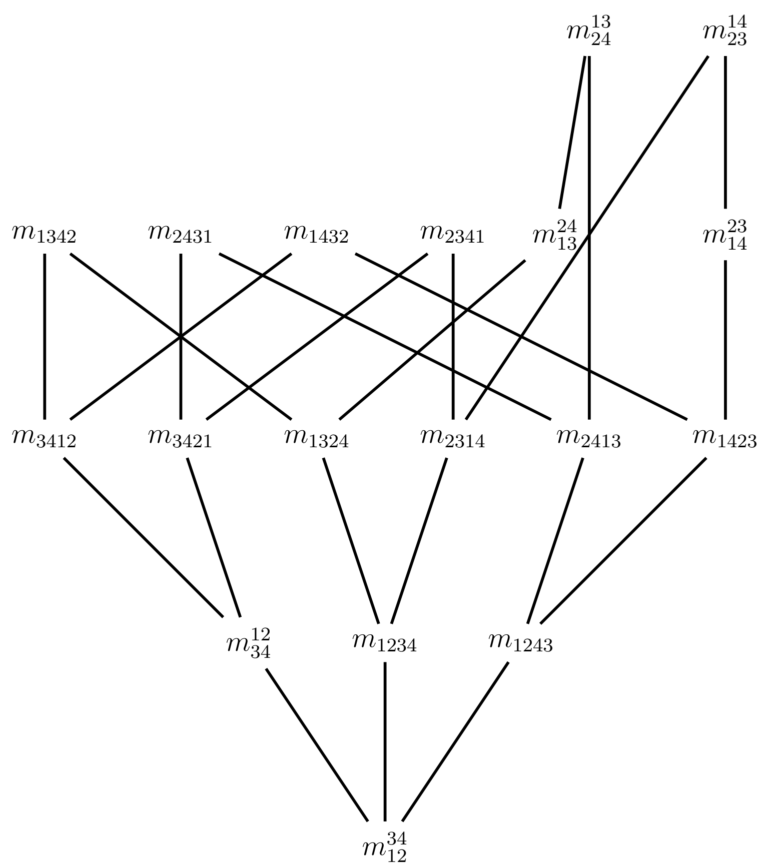


Figure 20. $\Pi_{4_{\lambda_\Omega}}(2)$ induced by the minimal labeling λ_Ω from Fig. 19.

We now consider certain EL-labelings from Dyer (1993) of closed intervals in the Bruhat order of any Coxeter group. (See Björner and Brenti (2010) for background on general Coxeter groups.) Let u and w be group elements of a Coxeter system (W, S). If $u \lessdot w$ in the Bruhat order on W, then $u^{-1}w = t$ for some reflection $t \in T$ where T is the set of reflections of (W, S). The cover relation $u \lessdot w$ is labeled by $\lambda(u, w) = t$ and the reflections T are totally ordered by any of the so called reflection orders introduced in Definition 2.1 Dyer (1993). The λ was shown to be an EL-labeling of any closed interval in the Bruhat order on W in Section 4 of Dyer (1993). We refer to these labelings as reflection order EL-labelings.

Theorem 4.1.13. *Every reflection order EL-labeling λ of a closed interval in the Bruhat order of any Coxeter group is polygon strong. Thus, λ is polygon complete.*

Proof. The fact that λ is polygon strong follows directly from the characterization of label sequences of rank two intervals in Lemma 4.1 (i) of Dyer (1993). Then we apply Theorem 4.1.6. □

Next we turn to a generalization of S_n EL-labelings. In McNamara and Thomas (2006), the authors generalized the notion of an S_n EL-labeling of a finite poset to the non-graded case with the notion of an interpolating EL-labeling. Interpolating EL-labelings were used to study modularity in non-graded lattices. These interpolating EL-labelings turn out to be polygon strong as well.

Theorem 4.1.14. *Any interpolating EL-labeling λ of a finite, bounded poset in the sense of McNamara and Thomas (2006) is polygon strong. Thus, λ is polygon complete.*

Proof. Condition (ii) of the definition of an interpolating EL-labeling (Definition 1.2 in McNamara and Thomas (2006)) allows us to use essentially the same reasoning as we used for the case of S_n EL-labelings. Thus, λ is polygon strong, and so polygon complete by Theorem 4.1.6. $\square$

Lastly, we show that an EL-labeling of the Tamari lattice and an EL-labeling of intervals in general Cambrian semilattices are polygon complete. Cambrian lattices were defined by Reading in Reading (2006) as certain lattice quotients of weak order with the Tamari lattice as a motivating special case.

Theorem 4.1.15. *Björner and Wachs' EL-labeling λ of the Tamari lattice defined in Section 9 of Björner and Wachs (1997) is polygon strong. Hence, λ is polygon complete.*

Proof. The fact that λ is polygon strong follows from the proof of Theorem 9.2 in Björner and Wachs (1997). Then we apply Theorem 4.1.6. $\square$

Theorem 4.1.16. *The EL-labeling λ of a closed interval in any Cambrian semilattice given in Section 3.1 of Kallipoliti and Mühle (2013) is polygon strong. Hence, λ is polygon complete.*

Proof. The fact that λ is polygon complete follows directly from Lemma 3.4 in Kallipoliti and Mühle (2013). Then we again apply Theorem 4.1.6. $\square$

4.1.3 Characterization of Polygon Complete EL-labelings. Here we characterize polygon complete EL-labelings in Theorem 4.1.19. Because the statements are slightly simpler, we actually characterize the EL-labelings which are not polygon complete which gives a characterization of polygon completeness for EL-labelings by negating the conditions in Theorem 4.1.19. The two technical conditions appearing in the following two lemmas provide the characterization. Fig. 21 is a schematic illustrating these conditions. It is clarifying to check these conditions in the example from Fig. 2 where the elements are labeled to match the statements of Lemma 4.1.17 and Lemma 4.1.18.

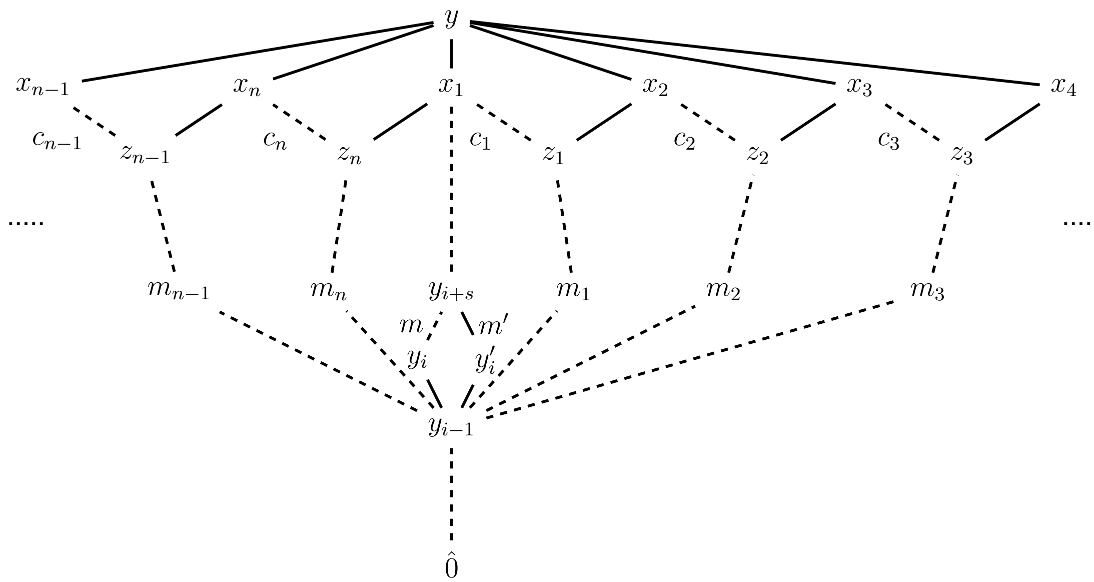


Figure 21. Illustration of conditions (i) and (ii) in Lemma 4.1.17 and Lemma 4.1.18.

Lemma 4.1.17. *Let P be a finite, bounded poset in which the length of some maximal chain is at least three. Let λ be an EL-labeling of P. Suppose that for some $n \geq 2$ there are elements $y, x_1, x_2, \ldots, x_n, z_1, z_2, \ldots, z_n \in P$ which, under the convention $x_{n+1} = x_1$, satisfy:*

(i) For $1 \leq i \leq n$, $z_i \lessdot x_{i+1} \lessdot y$ is a descent in $[z_i, y]$ and $x_i \lessdot y$ is contained in the unique ascending saturated chain c_i of $[z_i, y]$ with respect to λ.

(ii) There are the following saturated chains of length at least one: m, m' from $\hat{0}$ to x_1 such that $m \to m'$ and m_i from $\hat{0}$ to z_i for $1 \leq i \leq n$ which satisfy the relations $m \preceq_\lambda m_1 * c_1^{x_1}$ in $[\hat{0}, x_1]_\lambda(2)$, $m_i * z_i * x_{i+1} \prec_\lambda m_{i+1} * c_{i+1}^{x_{i+1}}$ in $[\hat{0}, x_{i+1}]_\lambda(2)$ for each $1 \leq i \leq n$, and $m_n * z_n * x_1 \preceq_\lambda m'$ in $[\hat{0}, x_1]_\lambda(2)$. (It is possible that m contains m_1 and $c_1^{x_1}$.)

Then λ is not polygon complete.

Proof. It is helpful to refer to Fig. 21 to visually follow the proof. We assume there exist $y, x_1, x_2, \ldots, x_n, z_1, z_2, \ldots, z_n \in P$ such that conditions (i) and (ii) of Lemma 4.1.17 are met. We know $z_1 \neq z_n$ since the chain c_1 is ascending and the chain $z_n \lessdot x_1 \lessdot y$ is a descent. By assumption in condition (ii), $m \to m'$. Thus, $m * y \to m' * y$. We will show that $m * y \not\lessdot_\lambda m' * y$ in $[\hat{0}, y]_\lambda(2)$. By the assumptions of condition (i), $c_i \to z_i \lessdot x_{i+1} \lessdot y$ for each $1 \leq i \leq n$. Thus,

$$\begin{aligned} m * y &\prec_\lambda m_1 * c_1 \to m_1 * z_1 * x_2 * y \prec_\lambda m_2 * c_2 \to m_2 * z_2 * x_3 * y \\ &\prec_\lambda m_3 * c_3 \to m_3 * z_3 * x_4 * y \prec_\lambda \ldots \\ &\prec_\lambda m_{n-1} * c_{n-1} \to m_{n-1} * z_{n-1} * x_n * y \prec_\lambda m_n * c_n \to m_n * z_n * x_1 * y \\ &\prec_\lambda m' * y. \end{aligned}$$

Hence, $m * y \not\lessdot_\lambda m' * y$ in $[\hat{0}, y]_\lambda(2)$ since $n \geq 2$. Moreover, letting c be any saturated chain from y to $\hat{1}$ we have that $m * y * c \to m' * y * c$, but $m * y * c \not\lessdot_\lambda m' * y * c$ in $P_\lambda(2)$ by Lemma 3.3.3. Therefore, λ is not polygon complete. □

In the next lemma, we show that conditions (i) and (ii) are also necessary for an EL-labeling to be not polygon complete. Despite the appearance that the conditions of Lemma 4.1.17 are very technical and that they might be saying no more than that there exists an increase by a polygon move which does not give a cover relation, they are actually useful for verifying some EL-labelings are polygon complete. Condition (i) is particularly useful if we have control over the top labels of saturated chains in the polygons corresponding to increases by polygon moves as we do with polygon strong EL-labelings.

Lemma 4.1.18. *Let P be a finite, bounded poset with an EL-labeling λ. Suppose that for maximal chains $m, m' \in \mathcal{M}(P)$, $m \to m'$ while $m \not\lessdot_\lambda m'$ in $P_\lambda(2)$. Then* *there are elements $y, x_1, x_2, \ldots, x_n, z_1, z_2, \ldots, z_n \in P$ for $n \geq 2$ which, under the convention $x_{n+1} = x_1$, satisfy:*

(i) For $1 \leq i \leq n$, $z_i \lessdot x_{i+1} \lessdot y$ is a descent in $[z_i, y]$ and $x_i \lessdot y$ is contained in the unique ascending saturated chain c_i of $[z_i, y]$ with respect to λ.

(ii) There are the following saturated chains of length at least one: m, m' from $\hat{0}$ to x_1 such that $m \to m'$ and m_i from $\hat{0}$ to z_i for $1 \leq i \leq n$ which satisfy the relations $m \preceq_\lambda m_1 * c_1^{x_1}$ in $[\hat{0}, x_1]_\lambda(2)$, $m_i * z_i * x_{i+1} \prec_\lambda m_{i+1} * c_{i+1}^{x_{i+1}}$ in $[\hat{0}, x_{i+1}]_\lambda(2)$ for each $1 \leq i \leq n$, and $m_n * x_1 \preceq_\lambda m'$ in $[\hat{0}, x_1]_\lambda(2)$. (It is possible that m contains m_1 and $c_1^{x_1}$.)

Proof. Again Fig. 21 guides and illuminates this proof. Let $m : y_0 = \hat{0} \lessdot y_1 \lessdot \ldots y_{i-1} \lessdot y_i \lessdot y_{i+1} \lessdot \cdots \lessdot y_{i+s} \lessdot \cdots \lessdot y_{t-1} \lessdot y_t = \hat{1}$ for some $s \geq 1$ with $y_i \lessdot y_{i+1} \lessdot \cdots \lessdot y_{i+s}$ an ascending saturated chain with respect to λ. Let $m' : y_0 = \hat{0} \lessdot y_1 \lessdot \ldots y_{i-1} \lessdot y_i' \lessdot y_{i+s} \lessdot \cdots \lessdot y_{t-1} \lessdot y_t = \hat{1}$ with $y_{i-1} \lessdot y_i' \lessdot y_{i+s}$ a descent with respect to λ. Assume $m \not\prec_\lambda m'$.

We will proceed by induction on the length of the longest maximal chain of P to show the elements and chains of conditions (i) and (ii) exist. If the longest maximal chain is length one or two, then the statement is vacuously true since λ is polygon complete by Lemma 4.1.1. We assume the statement holds for posets with longest maximal chain of any length at most $l-1$ for some $l \geq 3$.

Assume P has longest maximal chain of length $l \geq 3$. We first observe that if $y_{i+s} = \hat{1}$, then $m \prec_\lambda m'$ by Lemma 4.1.2. Thus, we may assume $y_{i+s} < \hat{1}$, so $i+s \leq t-1$. This implies $m^{y_{t-1}} \to m'^{y_{t-1}}$ which we will take advantage of repeatedly. Now since $m \not\prec_\lambda m'$, there must be maximal chains $d_0, d_1, d_2, \ldots, d_k, d_{k+1} \in \mathcal{M}(P)$ such that $m = d_0 \to d_1 \to d_2 \to \cdots \to d_k \to d_{k+1} = m'$ with $k \geq 1$. There are two cases we must consider. Either the top elements of the polygons corresponding to each of the increases by a polygon move $d_j \to d_{j+1}$ for $0 \leq j \leq k$ are strictly less than $\hat{1}$ in P or the top element of the polygon corresponding to some increase by a polygon move $d_j \to d_{j+1}$ is $\hat{1}$.

Suppose the top element of each polygon corresponding to the increases by a polygon move $d_j \to d_{j+1}$ for $0 \leq j \leq k$ is strictly less than $\hat{1}$ in P. Then $m^{y_{t-1}} \to d_1^{y_{t-1}} \to d_2^{y_{t-1}} \to \cdots \to d_k^{y_{t-1}} \to m'^{y_{t-1}}$. Thus, $m^{y_{t-1}} \not\lessdot_\lambda m'^{y_{t-1}}$ in $[\hat{0}, y_{t-1}]_\lambda(2)$ since $k \geq 1$. We previously observed that $m^{y_{t-1}} \to m'^{y_{t-1}}$. Now the length of the longest maximal chain in $[\hat{0}, y_{t-1}]$ is some $l' \leq l-1$ since the longest maximal chain in P has length l. If $l' \leq 2$, then $m^{y_{t-1}} \to m'^{y_{t-1}}$ with $m^{y_{t-1}} \not\lessdot_\lambda m'^{y_{t-1}}$ in $[\hat{0}, y_{t-1}]_\lambda(2)$ contradicts Lemma 4.1.1. Thus, we may assume $l' \geq 3$. Then by the inductive hypothesis there exist elements and chains of $[\hat{0}, y_{t-1}]$ satisfying conditions (i) and (ii). The same elements and chains satisfy conditions (i) and (ii) in P.

Next we consider the case that the top element of the polygon corresponding to some increase by a polygon move $d_j \to d_{j+1}$ is $\hat{1}$. We will construct the elements satisfying condition (i) and the chains of condition (ii) by considering the elements and chains involved each time the final edge of a chain changes in the sequence $m = d_0 \to d_1 \to d_2 \to \cdots \to d_k \to d_{k+1} = m'$. Let $y = \hat{1}$ and $x_1 = y_{t-1}$. Let d_{r_1} be the first maximal chain in the sequence $m = d_0 \to d_1 \to d_2 \to \cdots \to d_k \to d_{k+1} = m'$ such that $y_{t-1} \in d_{r_1}$, but $y_{t-1} \notin d_{r_1+1}$. Let $z_1 \lessdot x_2 \lessdot \hat{1}$ be the final three elements of d_{r_1+1} which are uniquely determined by d_{r_1+1} and must exist since $y_{t-1} \notin d_{r_1+1}$. Then d_{r_1} contains the ascending saturated chain c_1 from z_1 to $\hat{1}$ and $z_1 \lessdot x_2 \lessdot \hat{1}$ is a descent since $d_{r_1} \to d_{r_1+1}$ and $y_{t-1} \notin d_{r_1+1}$. The saturated chains c_1 and $z_1 \lessdot x_2 \lessdot \hat{1}$ form the polygon corresponding to the polygon move $d_{r_1} \to d_{r_1+1}$.

Now since $y_{t-1} \in d_{j'}$ for each $1 \leq j' \leq d_{r_1}$, we have $m^{y_{t-1}} \to d_1^{y_{t-1}} \to \cdots \to d_{r_1}^{y_{t-1}}$. Thus, $m^{y_{t-1}} \preceq_\lambda d_{r_1}^{z_1} * c_1^{y_{t-1}}$ in $[\hat{0}, y_{t-1}]_\lambda(2)$. Let $m_1 = d_{r_1}^{z_1}$.

Now since $d_{r_1+1} \to d_{r_1+2} \to \cdots \to d_k \to m'$ and $y_{t-1} \lessdot \hat{1}$ is contained in m', there is some d_{r_n} which is the last maximal chain in the sequence $m = d_0 \to d_1 \to d_2 \to \cdots \to d_k \to d_{k+1} = m'$ such that $y_{t-1} \notin d_{r_n}$, but $y_{t-1} \in d_{j'}$ for all $j' > r_n$. We do not yet know the value of n, but it is cleaner to introduce d_{r_n} at this point. We will see that the following process results in the value of n without depending on n. We note that $r_n < k+1$ since $y_{t-1} \in m'$. Let $x_n \in d_{r_n}$ be the unique element of the chain such that $x_n \lessdot \hat{1}$. Let z_n be the unique element of d_{r_n+1} such that $z_n \lessdot y_{t-1}$. Then to accomplish the increase by a polygon move $d_{r_n} \to d_{r_n+1}$ which removes x_n and replaces it with y_{t-1}, we must have that $z_n < x_n$, d_{r_n} contains the unique ascending chain c_n from z_n to $\hat{1}$ with $x_n \in c_n$, and $z_n \lessdot y_{t-1} \lessdot \hat{1}$ is a descent. Thus, c_n and $z_n \lessdot y_{t-1} \lessdot \hat{1}$ form the polygon corresponding to $d_{r_n} \to d_{r_n+1}$. Then since $y_{t-1} \in d_{j'}$ for all $j' > r_n$, we have $d_{r_n+1}^{y_{t-1}} \to d_{r_n+2}^{y_{t-1}} \to \cdots \to d_k^{y_{t-1}} \to d_{k+1}^{y_{t-1}} = m'^{y_{t-1}}$. Hence, we have $d_{r_n+1}^{z_n} * z_n * y_{t-1} \preceq_\lambda m'^{y_{t-1}}$ in $[\hat{0}, y_{t-1}]_\lambda(2)$. Lastly, $d_{r_n+1}^{z_n} = d_{r_n}^{z_n}$, so $d_{r_n}^{z_n} * z_n * y_{t-1} \preceq_\lambda m'^{y_{t-1}}$ in $[\hat{0}, y_{t-1}]_\lambda(2)$. Let $m_n = d_{r_n}^{z_n}$, and recall $x_{n+1} = x_1$ by convention, so $x_{n+1} = y_{t-1}$.

Now we move from d_{r_1} to d_{r_n} by the same process to produce the remaining elements of condition (i) and chains of condition (ii).

Let d_{r_2} be the first maximal chain after d_{r_1+1} in the sequence $m = d_0 \to d_1 \to d_2 \to \cdots \to d_k \to d_{k+1} = m'$ such that $x_2 \in d_{r_2}$, but $x_2 \notin d_{r_2+1}$. We note that $r_2 > r_1 + 1$ since $z_1 \lessdot x_2 \lessdot \hat{1}$ is contained in d_{r_1+1} and is a descent. Let x_3 be the unique element of d_{r_2+1} such that $x_3 \lessdot \hat{1}$ and let $z_2 \in d_{r_2+1}$ be the unique element such that $z_2 \lessdot x_3$. Thus, $z_2 \lessdot x_3 \lessdot \hat{1}$ is a descent and d_{r_2} contains the unique ascending chain c_2 from z_2 to $\hat{1}$ with $x_2 \in c_2$. The chains c_2 and $z_2 \lessdot x_3 \lessdot \hat{1}$ form the polygon corresponding to the polygon move $d_{r_2} \to d_{r_2+1}$. Since $x_2 \in d_{j'}$ for all $r_1 + 1 \le j' \le r_2$, we have $d_{r_1+1}^{x_2} \to \cdots \to d_{r_2}^{x_2}$. Thus, $d_{r_1+1}^{z_1} * z_1 * x_2 \preceq_\lambda d_{r_2}^{z_2} * c_2^{x_2}$ in $[\hat{0}, x_2]_\lambda(2)$. Lastly, $d_{r_1+1}^{z_1} = d_{r_1}^{z_1}$, so $d_{r_1}^{z_1} * z_1 * x_2 \preceq_\lambda d_{r_2}^{z_2} * c_2^{x_2}$ in $[\hat{0}, x_2]_\lambda(2)$. Let $m_2 = d_{r_2}^{z_2}$.

We continue this process until we reach d_{r_n}. Suppose we have constructed d_{r_i}, x_{i+1}, z_i, and c_i by the above process and set $m_i = d_{r_i}^{z_i}$. Then $d_{r_{i+1}}$ is the first maximal chain after d_{r_i+1} in the sequence $m = d_0 \to d_1 \to d_2 \to \cdots \to d_k \to d_{k+1} = m'$ such that $x_{i+1} \in d_{r_{i+1}}$, but $x_{i+1} \notin d_{r_{i+1}+1}$. We note that $r_{i+1} > r_i + 1$ since $z_i \lessdot x_{i+1} \lessdot \hat{1}$ is contained in d_{r_i+1} and is a descent by construction. Let x_{i+2} be the unique element of $d_{r_{i+1}+1}$ covered by $\hat{1}$ and let $z_{i+1} \in d_{r_{i+1}+1}$ be the unique element covered by x_{i+2}. Thus, $z_{i+1} \lessdot x_{i+2} \lessdot \hat{1}$ is a descent and $d_{r_{i+1}}$ contains the unique ascending saturated chain c_{i+1} from z_{i+1} to $\hat{1}$ with $x_{i+1} \in c_{i+1}$. The chains c_{i+1} and $z_{i+1} \lessdot x_{i+2} \lessdot \hat{1}$ form the polygon corresponding to the polygon move $d_{r_{i+1}} \to d_{r_{i+1}+1}$. Then since $x_{i+1} \in d_{j'}$ for all $r_i + 1 \le j' \le r_{i+1}$, we have $d_{r_i+1}^{x_{i+1}} \to \cdots \to d_{r_{i+1}}^{x_{i+1}}$. Thus, $d_{r_i+1}^{z_i} * z_i * x_{i+1} \preceq_\lambda d_{r_{i+1}}^{z_{i+1}} * c_{i+1}^{x_{i+1}}$ in $[\hat{0}, x_{i+1}]_\lambda(2)$. Lastly, $d_{r_i+1}^{z_i} = d_{r_i}^{z_i}$, so $d_{r_i}^{z_i} * z_i * x_{i+1} \preceq_\lambda d_{r_{i+1}}^{z_{i+1}} * c_{i+1}$ in $[\hat{0}, x_{i+1}]_\lambda(2)$. Let $m_{i+1} = d_{r_{i+1}}^{z_{i+1}}$.

We are guaranteed to reach d_{r_n} because the above process accounts for each change of the top edge in a chain in the sequence $m = d_0 \to d_1 \to d_2 \to \cdots \to d_k \to d_{k+1} = m'$. We know $n \ge 2$ because we are in the case where the top edge in some chain in the sequence $m = d_0 \to d_1 \to d_2 \to \cdots \to d_k \to d_{k+1} = m'$ changes from $y_{t-1} \lessdot \hat{1}$, and then must return to $y_{t-1} \lessdot \hat{1}$ because $y_{t-1} \lessdot \hat{1}$ is the top edge in both m and m'.

We have thus produced y, $z_1, \ldots, z_n$, and $x_1, \ldots, x_n$ satisfying condition (i) of the lemma. Letting $m_i = d_{r_i}^{z_i}$ for $1 \le i \le n$ and letting c_i be as constructed above for $1 \le i \le n$, as well as, $m' = m'^{x_1}$ and $m = m^{x_1}$ (slightly abusing notation) we have produced the saturated chains in condition (ii) of the lemma. The chains

$m = m^{x_1}$, $m' = m'^{x_1}$, and m_i all have length at least one since m and m' have length at least three because $y_{i+s} < \hat{1}$. This concludes the proof. $\square$

Theorem 4.1.19. *Let P be a finite, bounded poset with an EL-labeling λ. Then λ fails to be polygon complete if and only if P has some maximal chain of at least length three and there are elements $y, x_1, x_2, \ldots, x_n, z_1, z_2, \ldots, z_n \in P$ for $n \geq 2$ which, under the convention $x_{n+1} = x_1$, satisfy:*

(i) For $1 \leq i \leq n$, $z_i \lessdot x_{i+1} \lessdot y$ is a descent in $[z_i, y]$ and $x_i \lessdot y$ is contained in the unique ascending saturated chain c_i of $[z_i, y]$ with respect to λ.

(ii) There are the following saturated chains of length at least one: m, m' from $\hat{0}$ to x_1 such that $m \to m'$ and m_i from $\hat{0}$ to z_i for $1 \leq i \leq n$ which satisfy the relations $m \preceq_\lambda m_1 * c_1^{x_1}$ in $[\hat{0}, x_1]_\lambda(2)$, $m_i * z_i * x_{i+1} \prec_\lambda m_{i+1} * c_{i+1}^{x_{i+1}}$ in $[\hat{0}, x_{i+1}]_\lambda(2)$ for each $1 \leq i \leq n$, and $m_n * x_1 \preceq_\lambda m'$ in $[\hat{0}, x_1]_\lambda(2)$. (It is possible that m contains m_1 and $c_1^{x_1}$.)

Proof. The forward direction is Lemma 4.1.18 and the backward direction is Lemma 4.1.17. $\square$

This gives a slightly easier to check condition which ensures polygon completeness though it is still more difficult than polygon strong.

Lemma 4.1.20. *Let P be a finite bounded poset (possibly non-graded) with EL-labeling λ. Suppose that for each $y \in P$ and any elements $z_1, \ldots, z_n \in P$ and $x_1, \ldots, x_n \in P$ satisfying condition (i) of Lemma 4.1.17 and Lemma 4.1.18 we have $\lambda(x_i \lessdot y) \geq \lambda(x_{i+1} \lessdot y)$ for $1 \leq i \leq n$ and $\lambda(x_i \lessdot y) > \lambda(x_{i+1} \lessdot y)$ for at least one i. Then no elements exist in P satisfying the conditions Lemma 4.1.17, so λ is polygon complete.*

Proof. If we have such $y \in P$ and $z_1, \ldots, z_n \in P$ and $x_1, \ldots, x_n \in P$ with $\lambda(x_i \lessdot y) \geq \lambda(x_{i+1} \lessdot y)$ for $1 \leq i \leq n$ and $\lambda(x_j \lessdot y) > \lambda(x_{j+1} \lessdot y)$ for some j, then

$$\lambda(x_1 \lessdot y) \geq \lambda(x_2 \lessdot y) \geq \lambda(x_2 \lessdot y) \geq \cdots \geq \lambda(x_j \lessdot y) > \lambda(x_{j+1} \lessdot y) \geq \cdots \geq \lambda(x_n \lessdot y) \geq \lambda(x_1 \lessdot y).$$

Thus, $\lambda(x_1 \lessdot y) > \lambda(x_1 \lessdot y)$ which is a contradiction. Hence, no elements satisfying condition (i) of Lemma 4.1.17 and Lemma 4.1.18 exist, so λ is polygon complete.

□

This easily renders an alternate proof of Theorem 4.1.6. (Although, the below proof essentially has in the background the proof given immediately after Theorem 4.1.6.)

Proof. (of Theorem 4.1.6)

Assume seeking contradiction that some maximal chain increase does not give a cover relation in the maximal chain descent order. Then by Lemma 4.1.18 there are elements $y \in P$ and $z_1, \ldots, z_n \in P$ and $x_1, \ldots, x_n \in P$ satisfying condition (i) of Lemma 4.1.17 and Lemma 4.1.18. However, since λ is polygon strong we have $\lambda(x_i \lessdot y) > \lambda(x_{i+1} \lessdot y)$ for each $1 \leq i \leq n$. This contradicts Lemma 4.1.20. Therefore, P contains no such elements, and so has every maximal chain increase producing a cover relation in $P_\lambda(2)$. □

We now present a simple concrete condition on labelings of certain induced subposets of P which guarantees a CL-labeling is not polygon complete. Fig. 22 is a schematic which illustrates this condition. We may also observe this condition in the examples from Fig. 2 and Fig. 16.

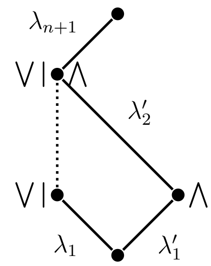


Figure 22. Illustration of the induced subposet condition in Lemma 4.1.21.

Lemma 4.1.21. *Let P be a finite, bounded poset with a CL-labeling λ. Suppose there are saturated chains in P of the form $c : x_1 \lessdot x_2 \lessdot \cdots \lessdot x_k \lessdot x_{k+1}$ with $k \geq 3$ and $c' : x_1 \lessdot x'_2 \lessdot x_k$ such that c is ascending with respect to λ, c' is a descent with respect to λ, and $\lambda(x_k, x_{k+1}) < \lambda(x'_2, x_k)$ all with respect to a root r from $\hat{0}$ to x_1. Then any maximal chain m containing c and r increases by a polygon move to the maximal chain m' obtained by replacing c in m with c', but $m \not\lessdot_\lambda m'$.*

Proof. First, for m and m' as defined above, $m \rightarrow m'$. Second, the chain c is the unique ascending maximal chain of the rooted interval $[x_1, x_{k+1}]_r$ with respect to λ. Also, $x'_2 \lessdot x_k \lessdot x_{k+1}$ is not the unique ascending maximal chain of the rooted interval $[x'_2, x_{k+1}]_{r*x'_2}$ since $\lambda(x_k \lessdot x_{k+1}) < \lambda(x'_2 \lessdot x_k)$. Thus, there is some saturated chain c_0 which is the unique ascending maximal chain of $[x'_2, x_{k+1}]_{r*x_2}$ with respect to λ. Now in the maximal chain descent order on $[x_1, x_{k+1}]_r$ induced by λ, we have that c is strictly less than $x_1 * c_0$ which is strictly less than c'. This follows from Proposition 3.3.1 and Lemma 3.3.3. Then Lemma 3.3.3 extends this to the entire poset P by producing a maximal chain which lies strictly between m and m' in the maximal chain descent order $P_\lambda(2)$. Thus, $m \not\lessdot_\lambda m'$. □

4.1.4 Application to the k-equal Partition Lattice.

We next show how to apply our characterization of polygon completeness to an important family of non-graded EL-shellable posets, namely the k-equal partition lattice. Here we really seem to need the full characterization itself, in that this EL-labeling is not polygon strong nor does it satisfy the condition on inversions introduced later in Section 4.2.

For positive integers $2 \leq k \leq n$, the k-equal partition lattice denoted $\Pi_{n,k}$ is the induced subposet of the set partition lattice Π_n on those set partitions of $[n]$ with no blocks of size $\{2, 3, \ldots, k-1\}$. An interval in $\Pi_{6,3}$ is shown below in Fig. 23. In this case there are no blocks of size 2. The same map which shows that Π_n is isomorphic to the intersection lattice of the braid arrangement, shows that $\Pi_{n,k}$ is isomorphic to the intersection lattice of the k-equal subspace arrangement. The k-equal subspace arrangement is the collection of subspaces of $\mathbb{R}^n$ defined by $x_{i_1} = x_{i_2} = \cdots = x_{i_k}$ for $1 \leq i_1 < i_2 < \cdots < i_k \leq n$. The poset topology of the k-equal partition lattice was used in Björner et al. (1992) and Björner and Lovász (1994) to prove complexity theory lower bound.

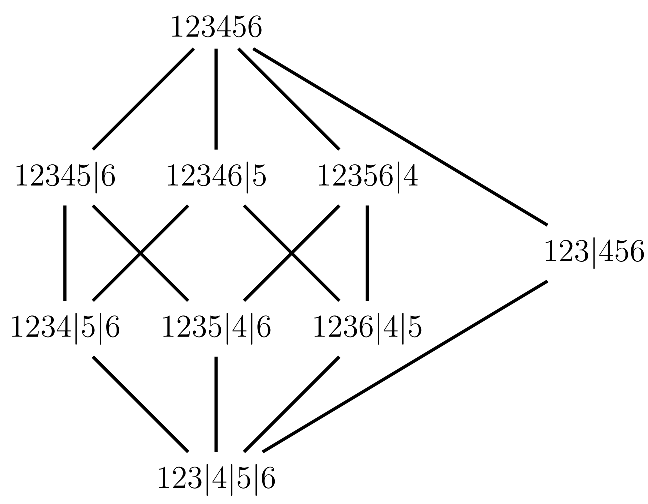


Figure 23. An interval in $\Pi_{6,3}$.

There are three kinds of cover relations in $\Pi_{n,k}$. In Björner and Wachs (1996), the authors give them the following edge labelings from the totally ordered set $\overline{1} < \overline{2} < \cdots < \overline{n} < 1 < 2 < \cdots < n$. The covering $\pi \lessdot \sigma$ is described on the left and the labeling $\lambda = \lambda(\pi \lessdot \sigma)$ is described on the right.

(i) A new block B of size k is created from singletons. $\lambda = \max B$

(ii) A nonsingleton block B is merged with singleton block $\{a\}$. $\lambda = a$

(iii) Two nonsingleton blocks B_1 and B_2 are merged. $\lambda = \overline{\max(B_1 \cup B_2)}$

Theorem 4.1.22 (Theorem 6.1 Björner and Wachs (1996))**.** *The labeling λ of $\Pi_{n,k}$ is an EL-labeling.*

In order to show that $\Pi_{n,k}$ with EL-labeling λ has the property that every maximal chain increase gives a cover relation in the corresponding maximal chain descent order, we need to carefully consider the different types of intervals in $\Pi_{n,k}$ that have at least one maximal chain of length two. These are the chains giving the polygons for maximal chain increases since the decreasing side of the polygons will always have length two. There are seven such types of intervals $[\pi, \tau]$ with a saturated chain of the form $\pi \lessdot \sigma \lessdot \tau$.

(1) In this type, π contains nonsingleton blocks B_1, B_2, B_3 two of which are merged to form σ and then they are merged with the third to form τ. There are three such saturated chains in $[\pi, \tau]$. These are the only saturated chains in the interval. These chains have label sequences

$$\overline{\max(B_1 \cup B_2)}, \overline{\max(B_1 \cup B_2 \cup B_3)}$$

$$\overline{\max(B_1 \cup B_3)}, \overline{\max(B_1 \cup B_2 \cup B_3)}$$

$$\overline{\max(B_2 \cup B_3)}, \overline{\max(B_1 \cup B_2 \cup B_3)}$$

We note that the last label in the the label sequence of type (1) intervals is the same for all three chains. Thus, λ is not a polygon strong EL-labeling, particularly the condition fails for type (1) intervals.

(2) In this type, π has nonsingleton block B_1 and singleton blocks $\{a_1\}, \ldots, \{a_k\}$. We may assume $a_1 < \cdots < a_k$. The singletons are merged to block B_2 to form σ, and then B_1 and B_2 are merged to form τ. This is the only length two saturated chain in $[\pi, \tau]$. The other saturated chains are formed by merging the singletons with B_1 one at a time in all $k!$ possible orders. These additional chains each have length k. The label sequences have the form

$$a_k, \overline{\max(B_1 \cup \{a\})}$$

$$a_{i_1}, \ldots, a_{i_k}$$

for a permutation $i_1, \ldots, i_k$ of $[k]$. We note that the length two saturated chain is the only decreasing saturated chain in $[\pi, \tau]$, and the last label of the length two chain is strictly less than the last label of any other saturated chain, particularly the increasing chain given by the identity permutation. Thus, λ is polygon strong in type (2) intervals.

(3) In this type, π has nonsingleton block B_1 and singleton blocks $\{a\}, \{b\}$ with $a < b$. B_1 is merged with one of a or b to form σ, then the other singleton is merged to form τ. Swapping a and b gives the two saturated chains in $[\pi, \tau]$ both of which are length two. They have label sequences

$$a, b$$

$$b, a$$

The increasing chain has larger top label than the decreasing chain, so λ is polygon strong in type (3) intervals.

(4) B_1 and B_2 are both nonsingleton blocks of π and $\{a\}$ is a singleton block of π. Two of the three blocks are merged to form σ and then merged with the third block to form τ. Each of these saturated chains is length two. They are only saturated chains in the interval. The label sequences are

$$\overline{(B_1 \cup B_2)}, a$$

$$a, \overline{(B_1 \cup \{a\} \cup B_2)}$$

$$a, \overline{(B_1 \cup B_2 \cup \{a\})}$$

The first listed label sequence is from the increasing chain. We again note that the top label of the increasing chain is larger than the top labels of any other saturated chains in the interval. Thus, λ is polygon strong in type (4) intervals.

(5) B_1, B_2, B_3, B_4 are non singleton blocks of π and σ is formed by merging B_1 and B_2, then τ is formed by merging B_3 and B_4. There is one other saturated chain in $[\pi, \tau]$ which is obtained by merging B_3 and B_4, then merging B_1 and B_2. These are the only saturated chains in the interval, and they have label sequences

$$\overline{\max(B_1 \cup B_2)}, \overline{\max(B_3 \cup B_4)}$$

$$\overline{\max(B_3 \cup B_{)}4}, \overline{\max(B_1 \cup B_2)}$$

We thus have that type for (5) intervals, λ is polygon strong.

(6) B_1, B_2 are non singleton blocks of π and $\{a_1\}, \ldots, \{a_k\}$ are singleton blocks of π with $a_1 < \cdots < a_k$. We form σ by merging B_1 and B_2, then form τ by

merging the k singletons. The only other saturated chain in $[\pi, \tau]$ is formed by merging the k singletons first and then merging B_1 and B_2. Both chains have length two, and their label sequences are

$$\overline{\max(B_1 \cup B_2)}, a_k$$

$$a_k, \overline{\max(B_1 \cup B_2)}$$

The first chain is increasing and has larger top label than the top label of the decreasing chain, so λ is polygon strong in type (6) intervals.

(7) π has singleton blocks $\{a_1\}, \ldots, \{a_k\}$ and $\{b_1\}, \ldots, \{b_k\}$ with $a_1 < \cdots < a_k$ and $b_1 < \cdots < b_k$ where the set of a's is disjoint from the set of b's. We form σ by merging the a's into a block of size k and then form τ by merging the b's into a block of size k. The only other saturated chain in the interval is formed by first merging the b's and then merging the a's. Both are length two, and have label sequences

$$a_k, b_k$$

$$b_k, a_k$$

We thus have that for type (7) intervals, λ is polygon strong.

We observe that in all polygons corresponding to maximal chain increases of $\Pi_{n,k}$, the top labels of the relevant label sequences weakly decrease. In particular, the top labels strictly decrease except in intervals of type (1). This allows us to prove all maximal chain increases in $\Pi_{n,k}$ give cover relations in $\Pi_{n,k}$ via Lemma 4.1.20.

Lemma 4.1.23. *The EL-labeling λ of $\Pi_{n,k}$ is polygon complete.*

Proof. First, we suppose $y \in \Pi_{n,k}$ and $z_1, \dots, z_n \in \Pi_{n,k}$ and $x_1, \dots, x_n \in \Pi_{n,k}$ are elements satisfying condition (i) of Lemma 4.1.17 and Lemma 4.1.18. We are thus dealing with at least two intervals $[z_i, y]$ of some type (1)-(7). If any type besides (1) occurs as $[z_i, y]$, then we are in the situation of Lemma 4.1.20 by our observations. Thus, no such collection of elements could exist in $\Pi_{n,k}$. Hence, the only possible case is that each $[z_i, y]$ is of type (1).

Let B_1, B_2, B_3 be the nonsingleton blocks of z_1 which are merged to form the maximal chains of $[z_1, y]$. Without loss of generality, assume $b \in B_3$ is the maximum element of $B_1 \cup B_2 \cup B_3$. Thus, the increasing maximal chain in $[z_1, y]$ comes from merging B_1 and B_2 and then merging the result with B_3. Thus, x_1 is the result of merging B_1 and B_2. Now we also have $x_1 \in [z_n, y]$, but x_1 is in a decreasing chain of $[z_n, y]$. However, in order for $[z_n, y]$ to be of type (1) and contain x_1 and y, z_n must have nonsingleton blocks B_1', B_2', B_3 with $B_1' \cup B_2' = B_1 \cup B_2$. Then the label sequence of $z_n \lessdot x_1 \lessdot y$ is

$$\overline{\max(B_1' \cup B_2')} = \overline{\max(B_1 \cup B_2)}, \overline{\max(B_1' \cup B_2' \cup B_3)} = \overline{\max(B_1 \cup B_2 \cup B_3)}.$$

But this label sequence is the exactly the same as the label sequence from the saturated chain containing x_1 in $[z_1, y]$ and is increasing. Therefore, $y \in \Pi_{n,k}$ and $z_1, \dots, z_n \in \Pi_{n,k}$ and $x_1, \dots, x_n \in \Pi_{n,k}$ satisfying condition (i) of Lemma 4.1.17 and Lemma 4.1.18 cannot arise using only intervals of type (1).

Hence, no such elements $y \in \Pi_{n,k}$ and $z_1, \dots, z_n \in \Pi_{n,k}$ and $x_1, \dots, x_n \in \Pi_{n,k}$ exist. Therefore, all increases between maximal chains in $\Pi_{n,k}$ give cover relations in the maximal chain descent order. $\square$

4.2 A Sufficient Condition for Polygon Completeness via Inversions in Label Sequences

In this section, we introduce a generalization of the notion of inversions of permutations. We speak more generally of inversions of maximal chains with respect to a CL-labeling. The usual notion of inversions of permutations arises from the standard EL-labeling of a Boolean lattice discussed in Section 3.2. We then formulate a condition on inversions of maximal chains of poset P with respect to a CL-labeling λ which implies that λ is polygon complete and that $P_{\boldsymbol{\lambda}}(2)$ possesses some other strong properties.

Definition 4.2.1. *Let P be a finite, bounded poset with a CL-labeling λ. Let m : $\hat{0} = x_0 \lessdot x_1 \lessdot \cdots \lessdot x_{r-1} \lessdot x_r = \hat{1}$ be a maximal chain of P. We say that the pair $(\lambda(x_{i-1}, x_i), \lambda(x_{j-1}, x_j))$ is an **inversion of*** $\mathbf{m}$ ***with respect to*** $\boldsymbol{\lambda}$ *if $1 \leq i < j \leq n$ and $\lambda(x_{i-1}, x_i) \nleq \lambda(x_{j-1}, x_j)$ in Λ. We denote the set of inversions of m with respect to λ by* $\boldsymbol{inv_{\lambda}}(\mathbf{m})$.

Remark 4.2.2. There is a slight abuse of notation in Definition 4.2.1. Technically, the label of a cover relation $x \lessdot y$ contained in maximal chain m from a CL-labeling λ should be written $\lambda(m, x, y)$. However, inversions are only considered with reference to a particular maximal chain, so we use the notation $\lambda(x, y)$ for both EL-labelings and CL-labelings in Definition 4.2.1 to avoid unnecessary clutter.

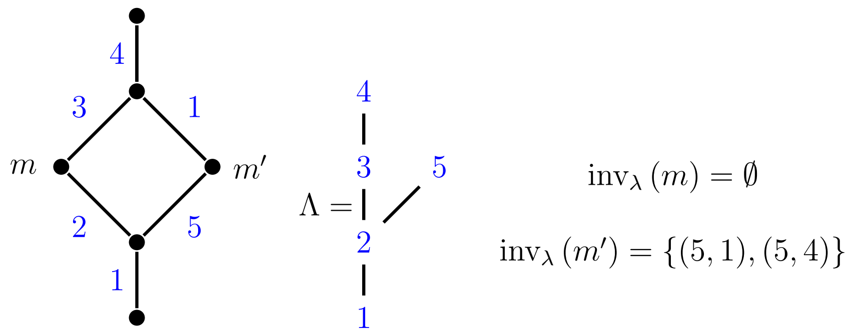


Figure 24. Inversions with respect to an EL-labeling with labels from poset Λ.

The same notion of inversions was considered in McNamara (2003), but only in the context of EL-labelings in which all label sequences are permutations of $[n]$ (known there as S_n EL-labelings). McNamara used inversions in that setting to make induction arguments about supersolvability and 0-Hecke algebra actions on maximal chains of lattices with an S_n EL-labeling.

Next we define a natural condition on inversions which guarantees polygon completeness.

Definition 4.2.3. *Let P be a finite, ranked, bounded poset which admits an EL-labeling or a CL-labeling λ. We say that λ is **inversion ranked** if $m \to m'$ implies $|inv_\lambda(m')| = |inv_\lambda(m)| + 1$.*

Remark 4.2.4. The EL-labeling in the example from Fig. 24 is not inversion ranked since $m \to m'$ while $|\mathrm{inv}_\lambda(m)| = 0$ and $|\mathrm{inv}_\lambda(m')| = 2$.

Example 4.2.5. For any S_n EL-labeling λ, the inversions with respect to λ are the usual inversions of the label sequences as permutations. In this case, λ is always inversion ranked by Proposition 4.1.9.

Remark 4.2.6. In contrast to Example 4.2.5, a CL-labeling in which the label sequence of every maximal chain is some permutation of $[n]$ need not be inversion ranked. We observe this in Example 3.3.8. In that example, the chain labeled 123, which has no inversions, increases by a polygon move to the chain labeled 321, which has three inversions.

Example 4.2.7. Fig. 25 shows two maximal chains which could occur in an inversion ranked EL-labeling λ. In this example, the label set is [4] with its standard total order. For brevity's sake, we set $\lambda_i = \lambda(x_{i-1}, x_i)$ for $1 \leq i \leq 5$ and $\lambda'_3 = \lambda(x_2, x'_3)$ and $\lambda'_4 = \lambda(x'_3, x_4)$. We have

$$\mathrm{inv}_\lambda(m) = \{(\lambda_1, \lambda_2), (\lambda_1, \lambda_3), (\lambda_1, \lambda_4), (\lambda_1, \lambda_5), (\lambda_4, \lambda_5)\}$$

and

$$\mathrm{inv}_\lambda(m') = \{(\lambda_1, \lambda_2), (\lambda_1, \lambda_4), (\lambda_1, \lambda_5), (\lambda'_3, \lambda'_4), (\lambda'_3, \lambda_5), (\lambda'_4, \lambda_5)\},$$

so λ could be inversion ranked.

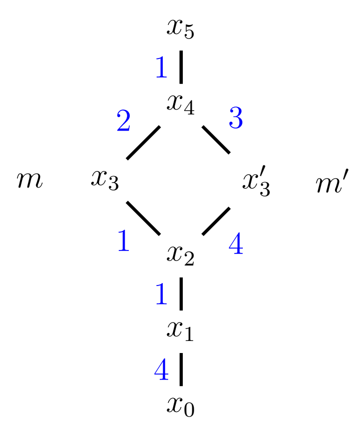


Figure 25. Labeled chains which could occur in an inversion ranked EL-labeling.

Next we observe that the notions of polygon strong and inversion ranked are generally distinct.

Proposition 4.2.8. *The notions of an inversion ranked EL-labeling and a polygon strong EL-labeling are distinct, that is, neither notion implies the other.*

Proof. Let P be the poset with elements $\{a, b, c, d\}$ with maximal chains $a \lessdot b \lessdot d$ and $a \lessdot c \lessdot d$. Let λ be the EL-labeling of P given by $\lambda(a, b) = 1$, $\lambda(b, d) = 2$, $\lambda(a, c) = 4$, and $\lambda(c, d) = 3$. Then λ is inversion ranked, but not polygon strong. On the other hand, minimal labelings of geometric lattices are polygon strong by Theorem 4.1.11 while none of the minimal labelings of the partition lattice Π_4 are inversion ranked, in particular the minimal labeling shown in Fig. 19 is not inversion ranked. □

The following theorem shows that inversion ranked implies that the resulting maximal chain descent order has several nice properties.

Theorem 4.2.9. *Let P be a finite, ranked, bounded poset of rank n. Suppose P admits a CL-labeling λ which is inversion ranked. Then λ is polygon complete. Moreover, $P_\lambda(2)$ is ranked with rank function $|inv_\lambda(\cdot)|$ and $m \in \mathcal{M}(\overline{P})$ is a homology facet of the shellings of $\Delta(\overline{P})$ induced by any linear extension of $P_\lambda(2)$ if and only if $\hat{0} * m * \hat{1}$ has rank $\binom{n}{2}$ in $P_\lambda(2)$.*

Proof. Seeking a contradiction, suppose $m \to m'$ and $m \not\lessdot_\lambda m'$ in $P_\lambda(2)$ for maximal chains $m, m' \in \mathcal{M}(P)$. This implies that there are maximal chains $m_1, \ldots, m_k \in \mathcal{M}(P)$ with $k \geq 1$ such that $m \to m_1 \to \cdots \to m_k \to m'$. Since λ is inversion ranked and since $k \geq 1$, $|\mathrm{inv}_\lambda(m')| = |\mathrm{inv}_\lambda(m)| + k + 1 \geq |\mathrm{inv}_\lambda(m)| + 2$. However, this contradicts the fact that $|\mathrm{inv}_\lambda(m')| = |\mathrm{inv}_\lambda(m)| + 1$ which holds because λ is inversion ranked and $m \to m'$. Therefore, λ is polygon complete.

It follows directly from Proposition 3.3.1 that $P_\lambda(2)$ is ranked with rank function $|\mathrm{inv}_\lambda(\cdot)|$ since the unique ascending maximal chain m_0 of P with respect to λ has $\mathrm{inv}_\lambda(m_0) = \emptyset$. By Lemma 3.4.1 any linear extension of $P_\lambda(2)$ induces a shelling of $\Delta(\overline{P})$ and a maximal chain $m \in \mathcal{M}(\overline{P})$ is a homology facet with respect to such a shelling if and only if $\hat{0} * m * \hat{1}$ is descending with respect to λ. Clearly a maximal chain $m \in \mathcal{M}(\overline{P})$ has $\hat{0} * m * \hat{1}$ descending with respect to λ if and only if $\left|\mathrm{inv}_\lambda\left(\hat{0} * m * \hat{1}\right)\right| = \binom{n}{2}$. Thus, $m \in \mathcal{M}(\overline{P})$ is a homology facet of $\Delta(\overline{P})$ if and only if $\hat{0} * m * \hat{1}$ has rank $\binom{n}{2}$ in $P_\lambda(2)$. $\square$

The previous theorem applies to S_n EL-labelings by Example 4.2.5. In the next chapter, we will prove an even stronger result for S_n EL-labelings.

Corollary 4.2.10. *If P is a finite poset with an S_n EL-labeling λ, then λ is polygon complete, $P_\lambda(2)$ is ranked with rank function $|inv_\lambda(\cdot)|$, and $m \in \mathcal{M}(\overline{P})$ is a homology facet of the shellings of $\Delta(\overline{P})$ given by any linear extension of $P_\lambda(2)$ if and only if $\hat{0} * m * \hat{1}$ has rank $\binom{n}{2}$ in $P_\lambda(2)$.*

CHAPTER V

STRUCTURE THEOREMS AND FURTHER EXAMPLES

In Section 5.1, we give a structure theorem for lower intervals with a maximal chain of length two in maximal chain descent orders. Then in Section 5.2, we discuss in depth several other examples of maximal chain descent orders. Section 5.2 is largely independent of the previous sections, and so can be read first if desired.

5.1 Structure Theorem for Lower Intervals in $P_\lambda(2)$ with a Maximal Chain of Length Two

In this section, we prove that any lower interval in a maximal chain descent order which has some maximal chain of length two has at most two coatoms. This appears as Theorem 5.1.10. As an application, we show that not all posets of regions of hyperplane arrangements are isomorphic to maximal chain descent orders. One might have wondered whether all posets of regions were isomorphic to maximal chain descent orders since the weak order of a finite Coxeter group, in particular in types A and B, is the poset of regions of the corresponding Coxeter arrangement.

Theorem 5.1.10 also implies that such an open lower interval has order complex that is either contractible or has the homotopy type of a 0-sphere. Lastly, much of the work in this section illustrates the qualitative point that working downwards from a maximal chain by considering its descents (as in Björner and Wachs' original proofs of lexicographic shellability) is often more productive than working upwards using its ascents because of the constraint imposed by uniqueness in Proposition 3.1.5.

We first collect some necessary propositions about maximal chain descent orders. Then we prove three technical lemmas which encompass the different ways of producing a lower interval with some maximal chain of length two in a maximal chain descent order. Combined, these three lemmas prove Theorem 5.1.10.

Proposition 5.1.1. *Let P be a finite bounded poset with an EL-labeling λ. Suppose $m, c, n \in \mathcal{M}(P)$ are maximal chains of P such that $m \to c \to n$. Then $n \setminus m = \{x_1, x_2\}$ (see Definition 2.1.1) where $m \setminus c = \{x_1\}$ and $n \setminus c = \{x_2\}$ with $x_1 \neq x_2$.*

Proof. By definition of $m \to c$, $c \setminus m = \{x_1\}$ for a unique element $x_1 \in P$. Similarly, $n \setminus m = \{x_2\}$ for a unique $x_2 \in P$. Also, by definition of $m \to c$, c contains a descent at x_1 with respect to λ. Thus, the polygon corresponding to $m \to n$ does not contain x_1 except as possibly the top or bottom element. Hence, $x_1 \neq x_2$ and $x_1 \in n$. So, we have $n \setminus m = \{x_1, x_2\}$. □

Proposition 5.1.2. *Let P be a finite, bounded poset with EL-labeling λ. Let m_0 be the unique ascending maximal chain of P with respect to λ. Then a maximal chain increase $m_0 \to m$ for a maximal chain $m \in \mathcal{M}(P)$ gives a cover relation $m_0 \prec_\lambda m$ in $P_\lambda(2)$ if and only if m has exactly one descent and, specifically, this is the descent corresponding to $m_0 \to m$.*

Proof. Since $m_0 \to m$, $m \setminus m_0 = \{x\}$ and m has a descent at x.

For the forward direction, suppose seeking contradiction that m has a descent at $y \in m$ for some $y \neq x$. Then by Proposition 3.1.5 there is a unique maximal chain m' such that $m' \to m$ and $m \setminus m' = \{y\}$. Since $y \neq x$, $x \in m'$. Thus, $m' \neq m_0$, so $m_0 \prec_\lambda m'$ by Proposition 3.3.1. However, we then have $m_0 \prec_\lambda m' \to m$ which contradicts the fact that $m_0 \prec_\lambda m$. Therefore, m has a single descent which occurs at x and corresponds to $m_0 \to m$.

For the backward direction, assume m has single descent which occurs at x. Then m_0 is the only maximal chain of P such that $m_0 \to m$. Hence, $m_0 \prec_\lambda m$. $\square$

Now we are ready to prove the three main technical lemmas of this section.

Lemma 5.1.3. *Let P be a finite, bounded poset with an EL-labeling λ. Let $m_0, c, m \in \mathcal{M}(P)$ be maximal chains of P such that $m_0 \prec_\lambda c \prec_\lambda m$ with m_0 the unique ascending chain of P with respect to λ. Let $m \setminus m_0 = \{w, z\}$ (as in Proposition 5.1.1) with $w < z$. Let $b \in m$ satisfy $w \lessdot b$ and let $f \in m$ satisfy $f \lessdot z$. Suppose b is less than f in P. Then*

(i) m has descents at w and z,

(ii) the interval $[m_0, m]_\lambda$ in $P_\lambda(2)$ has exactly two coatoms, one for each of the descents of m at w and z, and

(iii) $[m_0, m]_\lambda$ has exactly four elements.

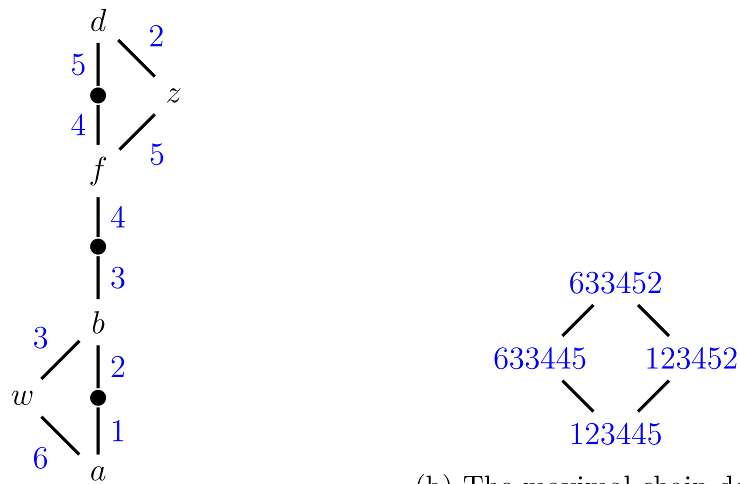
Proof. It may be helpful to follow the proof on the example in Fig. 26. Since $b < f$, $b, f \in m_0$. Let a be the element of m covered by w and let d be the element of m which covers z. Then $a, d \in m_0$ as well. Further, m clearly has descents at w and z. We will show by contradiction that a, b, f and d are not descents of m. Proposition 3.1.5 then ensures the descents of m at w and z provide the two potential coatoms of $[m_0, m]_\lambda$. Last we check that these two potential coatoms are actually covered by n.

We observe that $m_0 \prec_\lambda m_0^f * f * z * d * m_{0d}$ by Corollary 4.1.3. Then by Proposition 5.1.2 $m_0^f * f * z * d * m_{0d}$ has exactly one descent which occurs at z. Thus, f and d are not descents of $m_0^f * f * z * d * m_0^d$. Now seeking contradiction, suppose m has a descent at f or d, then $m_0^f * f * z * d * m_{0d}$ has a descent at f or d, respectively. This contradicts the fact that f and d are not descents of $m_0^f * f * z * d * m_{0d}$.

Next we show that m does not have a descent at a or b. To do this, we first show that $m_0^f \prec_\lambda m_0^a * w * b * m_{0b}^f$ in $[\hat{0}, f]_\lambda(2)$. If $m \setminus c = \{w\}$, then $c^f \to m^f$ and $c^f = m_0^f$ and $m^f = m_0^a * w * b * m_{0b}^f$. Suppose, by way of contradiction, that $m_0^f \not\prec_\lambda m_0^a * w * b * m_{0b}^f$. However, then $c \not\prec_\lambda m$ by Lemma 3.3.3 which contradicts the fact that $c \prec_\lambda m$. If $m \setminus c \neq \{w\}$, then $c \setminus m_0 = \{w\}$. Thus, $c^f = m_0^a * w * b * m_{0b}^f$ and $m_0^f \prec_\lambda m_0^a * w * b * m_{0b}^f$. Then by Proposition 5.1.2, $m_0^a * w * b * m_{0b}^f$ has exactly one descent which occurs at w. So, $m_0^a * w * b * m_{0b}^f$ does not have a descent at a or at b. Suppose seeking contradiction that m has descent at a or b. Then $m_0^a * w * b * m_{0b}^f$ has a descent at a or b, respectively. This contradicts the fact that $m_0^a * w * b * m_{0b}^f$ does not have a descent at a or at b.

Now without loss of generality, assume $m \setminus c = \{w\}$. Since m has exactly two descents, there are exactly two distinct maximal chains of P which increase to m with respect to λ by Proposition 3.1.5. These chains are c and $m_0^f * z * m_{0d}$. By assumption $m_0 \prec_\lambda c$. Now by construction $m_0 \to m_0^f * z * m_{0d}$ and $m_0^f * z * m_{0d}$ has a single descent which occurs at z. Thus, $m_0 \prec_\lambda m_0^f * z * m_{0d}$ by Proposition 5.1.2. This implies c and $m_0^f * z * m_{0d}$ are incomparable in $P_\lambda(2)$. Hence, $m_0^f * z * m_{0d} \prec_\lambda m$, so $[m_0, n]_\lambda$ has two coatoms and four elements in total. $\square$

Example 5.1.4. The poset with EL-labeling λ and the induced maximal chain descent order $P_\lambda(2)$ exhibited in Fig. 26 illustrate Lemma 5.1.3. In $P_\lambda(2)$, the elements are labeled by the label sequence of the maximal chain since the label sequences are all distinct. The elements of P which are labeled a, b, d, f, w, z are those elements referred to in Lemma 5.1.3 and its proof. We see that $P_\lambda(2)$ has two coatoms and has exactly four elements total.



(a) Poset P with EL-labeling λ. (b) The maximal chain descent order $P_\lambda(2)$.

Figure 26. A poset P with EL-labeling λ and its maximal chain descent order $P_\lambda(2)$ illustrating Lemma 5.1.3.

Lemma 5.1.5. *Let P be a finite, bounded poset which admits an EL-labeling λ. Let $m_0, c, m \in \mathcal{M}(P)$ be maximal chains of P such that $m_0 \prec_\lambda c \prec_\lambda m$ with m_0 the unique ascending chain of P. Let $n \setminus m_0 = \{w, z\}$ (as in Proposition 5.1.1) with $w < z$. Let $b \in m$ satisfy $w \lessdot b$ and let $f \in m$ satisfy $f \lessdot z$. Suppose $b = f$. Then m has descents at w and z and the interval $[m_0, m]_\lambda$ in $P_\lambda(2)$ has exactly two coatoms, one for each of the two descents of m.*

Proof. It may be helpful to follow the proof on the example in Fig. 27. In this case, we have $f \in m_0$. Let a be the element of m covered by w and let d be the element of m which covers z. Then $a, d \in m_0$ as well. It is again clear that m has descents at w and z in this case. We will show that a, f, and d are not descents of m. And by Proposition 3.1.5, the descents of m at w and z give the two possible coatoms of $[m_0, n]_\lambda$.

Again by Corollary 4.1.3, $m_0 \prec_\lambda m_0^f * z * d * m_{0d}$. Then by Proposition 5.1.2, $m_0^f * z * d * m_{0d}$ has exactly one descent which occurs at z, so d is not a descent of

$m_0^f * z * d * m_{0d}$. If m had a descent at d, then $m_0^f * z * d * m_{0d}$ would have a descent at d contradicting the fact the d is not a descent of $m_0^f * z * d * m_{0d}$. Thus, m does not have a descent at d.

We now prove that m does not have a descent at a. To accomplish this, we first show that $m_0^f \prec_\lambda m_0^a * w * f$ in $[\hat{0}, f]_\lambda(2)$. If $m \setminus c = \{w\}$, then $c^f \to m^f$ and $c^f = m_0^f$ and $m^f = m_0^a * w * f$. Suppose, looking for a contradiction, that $m_0^f \not\prec_\lambda m_0^a * w * f$. However, then $c \not\prec_\lambda m$ by Lemma 3.3.3 which contradicts the fact that $c \prec_\lambda m$. If $m \setminus c \neq \{w\}$, then $c \setminus m_0 = \{w\}$. Thus, $c^f = m_0^a * w * f$ and $m_0^f \prec_\lambda m_0^a * w * f$. Then by Proposition 5.1.2 $m_0^a * w * b * m_{0b}^f$ has exactly one descent which occurs at w. So, $m_0^a * w * f$ does not have a descent at a. If m had a descent at a, then $m_0^a * w * f$ would have a descent at a contradicting the fact that $m_0^a * w * f$ does not have a descent at a.

Now we turn to showing m does not have a descent at f. We will again proceed by contradiction. For this, we must consider the closed interval $[\hat{0}, z]$ and its maximal chain descent order $[\hat{0}, z]_\lambda(2)$. Let c_0^z be the unique ascending maximal chain of $[\hat{0}, z]$ with respect to λ. Let x be the atom of P contained in m_0, so $x < f \lessdot z$. Then $\lambda(\hat{0}, x)$ is strictly smaller than $\lambda(\hat{0}, e)$ for any atom e of P with $e \neq x$. Thus, $x \in c_0^z$ since c_0^z is lexicographically first among maximal chains of $[\hat{0}, z]$. There are two possibilities for the ascending chain c_0^z. Either $c_0^z = m_0^f * z$ or $c_0^z = \hat{0} * x * c_{0x}^z$ and $f \notin c_{0x}^z$. We show that m does not have a descent at f in the first case. And we show that the second case itself leads to a contradiction, and so cannot even occur.

Assume $c_0^z = m_0^f * z$. There are two cases we must consider: either $c \setminus m_0 = \{w\}$ or $m \setminus c = \{w\}$. Assume first that $c \setminus m_0 = \{w\}$. Let y be the element of m_0 covering f, so $w \lessdot f \lessdot y$ is a subchain of c. Now c does not have a descent at

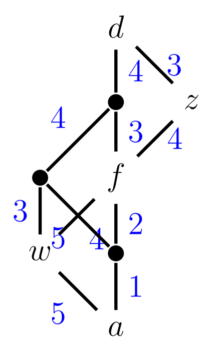
f by Proposition 5.1.2. Thus, $\lambda(w, f) \leq \lambda(f, y)$. Now since $f \lessdot y$ is a subchain of m_0 and $d \in m_0$, $f \lessdot y$ is the first step in the unique ascending chain of $[f, d]$. This implies $\lambda(f, y) < \lambda(f, z)$ because z is an atom of $[f, d]$ distinct from y. Therefore, $\lambda(w, f) \leq \lambda(f, y) < \lambda(f, z)$. Hence, m does not have a descent at f because $w \lessdot f \lessdot z$ is a subchain of m.

Next assume that $m \setminus c = \{w\}$. Then $c \setminus m_0 = \{z\}$, so $c^z = c_0^z$ (recalling that we are in the case that $c_0^z = m_0^f * z$). Thus, $c_0^z \prec_\lambda m^z$ in $[\hat{0}, z]_\lambda(2)$ by Lemma 3.3.3. Now by Proposition 5.1.2 m^z has exactly one descent and it occurs at w, so f is not a descent of m^z. Suppose seeking contradiction that m has has descent at f. Then m_z also has a descent at f which is a contradiction.

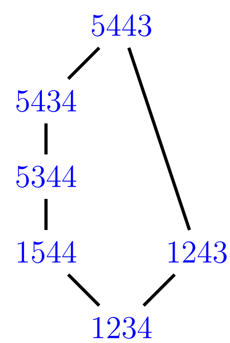
Suppose $c_0^z = \hat{0} * x * c_{0x}^z$ and $f \notin c_{0x}^z$. We prove this gives rise to a contradiction. Again by Corollary 4.1.3 we have $m_0 \prec_\lambda m_0^f * z * m_{0d}$. By Proposition 3.3.1 we also have $c_0^z \prec_\lambda m_0^f * z$ in $[\hat{0}, z]_\lambda(2)$ with strict inequality since $f \in m_0^f * z$ while $f \notin c_0^z$. Thus, by Lemma 3.3.3 $c_0^z * m_{0d} \prec_\lambda m_0^f * z * m_{0d}$ in $P_\lambda(2)$. However, Proposition 3.3.1 then implies $m_0 \prec_\lambda c_0^z * m_{0d} \prec_\lambda m_0^f * z * m_{0d}$ which contradicts that $m_0 \prec_\lambda m_0^f * z * m_{0d}$.

Since m has exactly two descents, there are exactly two distinct maximal chains of P which increase to m with respect to λ by Proposition 3.1.5. One of these chains is c and the other depends on whether $m \setminus c = \{w\}$ or $m \setminus c = \{z\}$. Assume $m \setminus c = \{w\}$. Then the maximal chain corresponding to the descent of m at z is $m_0^a * w * m_{0f}$. By Corollary 4.1.3 $m_0^a * w * m_{0f} \prec_\lambda m$. Thus, $[m_0, m]_\lambda$ has exactly two coatoms c and $m_0^a * w * m_{0f}$. Assume $m \setminus c = \{z\}$. Then the maximal chain corresponding to the descent of m at w is $m_0^f * z * m_{0d}$. We have $m_0 \prec_\lambda m_0^f * z * m_{0d}$ by Corollary 4.1.3. Since we also have $m_0 \prec_\lambda c$, c and $m_0^f * z * m_{0d}$ are incomparable in $P_\lambda(2)$. This implies $m_0^f * z * m_{0d} \prec_\lambda m$. Hence, in this case, $[m_0, m]_\lambda$ has exactly two coatoms c and $m_0^f * z * m_{0d}$. (However, there may be more than four elements in $[m_0, m]_\lambda$ in this case as is exhibited in the example in Example 5.1.6.) $\square$

Example 5.1.6. The poset with EL-labeling λ and the induced maximal chain descent order $P_\lambda(2)$ exhibited in Fig. 27 illustrate Lemma 5.1.5. In $P_\lambda(2)$, the elements are labeled by the label sequence of the maximal chain since the label sequences are all distinct. The elements of P which are labeled a, f, d, w, z, d are those elements referred to in Lemma 5.1.7 and its proof. We see that $P_\lambda(2)$ has two coatoms, but has more than four elements total in contrast to Lemma 5.1.3.



(a) Poset P with EL-labeling λ.



(b) The maximal chain descent order $P_\lambda(2)$.

Figure 27. A poset P with EL-labeling λ and its maximal chain descent order $P_\lambda(2)$ illustrating Lemma 5.1.5.

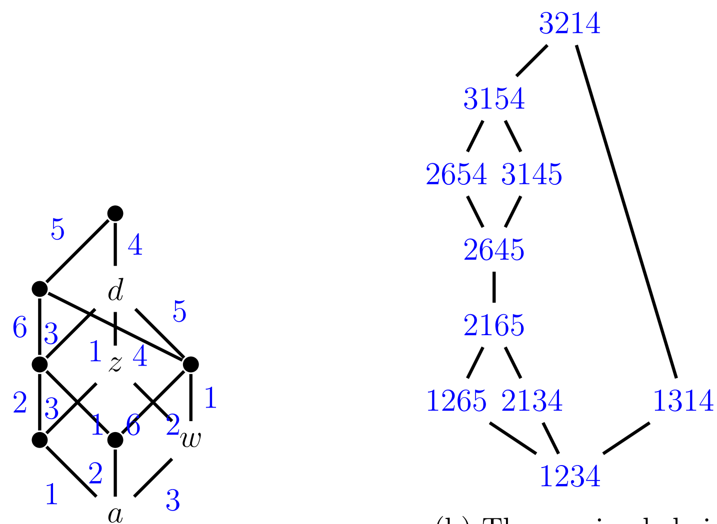
Lemma 5.1.7. *Let P be a finite, bounded poset which admits an EL-labeling λ. Let $m_0, c, m \in \mathcal{M}(P)$ be maximal chains of P such that $m_0 \prec\!\!\cdot_\lambda c \prec\!\!\cdot_\lambda m$ with m_0 the unique ascending chain of P. Let $n \setminus m_0 = \{w, z\}$ (as in Proposition 5.1.1) with $w < z$. Let $b \in m$ satisfy $w \lessdot b$ and let $f \in m$ satisfy $f \lessdot z$. Suppose $f = w$ and $b = z$. Then m has at most two descents, implying that the interval $[m_0, m]_\lambda$ in $P_\lambda(2)$ has at most two coatoms.*

Proof. It may be helpful to follow the proof on the example in Fig. 28. Assume $f = w$ and $b = z$. Let a be the element of m covered by w and let d be the element of m which covers z. Thus, we only have to show that m does not have a descent at a nor at d. We note that $a, d \in m_0$. We again must consider the two cases that either $c \setminus m_0 = \{w\}$ or $m \setminus c = \{w\}$.

Assume $c \setminus m_0 = \{w\}$, so $m \setminus c = \{z\}$. Since $m_0 \prec_\lambda c$ and $c \setminus m_0 = \{w\}$, c has exactly on descent which occurs at w by Proposition 5.1.2. Thus, c does not have a descent at a. Then since $c^w = m^w$, m does not have a descent at a. Next we make some observations necessary to prove that m does not have a descent at d. We have $c_w \prec_\lambda m_w$ in $[w, \hat{1}]_\lambda(2)$ with $m_w \setminus c_w = \{z\}$. We also have that c_w^d is the unique ascending maximal chain of $[w, d]$. Let y be the element of c which covers w and let e the element of c which covers d (m cannot have a descent at d if $d = \hat{1}$). Then $c_w^e \prec_\lambda m_w^e$ in $[w, e]_\lambda(2)$, otherwise Lemma 3.3.3 would imply a contradiction with the fact that $c_w \prec_\lambda m_w$. We also have that $y < d \lessdot e$ is a subchain of m_0 and $c_y^e = {m_0}_y^e$. Then, since c_w^d is the ascending maximal chain of $[w, d]$, c_w^e is the unique ascending maximal chain of $[w, e]$. Proposition 5.1.2 implies m_w^e has exactly one descent which occurs at z since c_w^e is ascending and $c_w^e \prec_\lambda m_w^e$ in $[w, e]_\lambda(2)$. Thus, m_w^e, and consequently m, does not have a descent at d.

Assume $m \setminus c = \{w\}$, so $c \setminus m_0 = \{z\}$. In this case, $c_z = m_z$. Since $m_0 \prec_\lambda c$, c has exactly one descent which occurs at z by Proposition 5.1.2. Thus, d is not a descent of c. So, d is not a descent of m since $c_z = m_z$. Next we show a is not a descent of m. Let g be the element of c covered by z, so $g \in m_0$ and $a < g$. We observe that c_a^z is the unique ascending maximal chain of $[a, z]$. Also, $c^g = m_0^g$, and so is ascending. Thus, c^z is ascending. We also observe that $c^z \prec_\lambda m^z$ otherwise Lemma 3.3.3 would imply a contradiction with the fact that $c \prec_\lambda m$. Then by Proposition 5.1.2 m^z has exactly one descent which occurs at w. Thus, m^z, and so m, does not have a descent at a. This completes the proof. $\square$

Example 5.1.8. Fig. 28 below contains an EL-labeling which illustrates Lemma 5.1.7. The figure shows the poset P with an EL-labeling λ and the induced maximal chain descent order $P_\lambda(2)$. In $P_\lambda(2)$, the elements are labeled by the label sequence of the maximal chain since the label sequences are all distinct. The elements of P labeled a, d, w, z are those elements referred to in Lemma 5.1.7 and its proof. We see that $P_\lambda(2)$ has two coatoms. This example illustrates that, in this case, the lower interval in question may have more than two maximal chains and more than two atoms, despite having at most two coatoms.



(a) Poset P with EL-labeling λ. (b) The maximal chain descent order $P_\lambda(2)$.

Figure 28. A poset P with EL-labeling λ and its maximal chain descent order $P_\lambda(2)$ illustrating Lemma 5.1.7.

Example 5.1.9. The poset and EL-labeling in Fig. 2 gives another example to which Lemma 5.1.7 applies, one in which there is only one coatom in the relevant interval.

The previous three technical lemmas combine to prove the main theorem of this section.

Theorem 5.1.10. *Let P be a finite bounded poset which admits an EL-labeling λ. Let $m_0 \in \mathcal{M}(P)$ be the unique ascending maximal chain of P with respect to λ. Suppose $c, m \in \mathcal{M}(P)$ are maximal chains of P such that $m_0 \prec_\lambda c \prec_\lambda m$. Then the closed interval $[m_0, m]_\lambda$ in $P_\lambda(2)$ has at most two coatoms. Moreover, the only elements of m at which m possibly has descents (which may give rise to coatoms of $[m_0, m]_\lambda$) are the two elements of $m \setminus m_0$.*

Proof. Since $m_0 \prec_\lambda c \prec_\lambda m$, $c \setminus m_0 = \{w, z\}$ with $w < z$. Let $b \in m$ satisfy $w \lessdot b$ and let $f \in m$ satisfy $f \lessdot z$. Then there are three cases: $b < f$, $b = f$, or $f = w$ and $b = z$. The case when $b < f$ is precisely Lemma 5.1.3. The case when $b = f$ is Lemma 5.1.5. Lastly, the case when $f = w$ and $b = z$ is Lemma 5.1.7. □

Theorem 5.1.10 has as a corollary that open lower intervals with some maximal chain of length two in a maximal chain descent order are contractible or homotopy equivalent to a 0-sphere.

Corollary 5.1.11. *Let P be a finite, bounded poset with an EL-labeling λ. Let $m_0 \in \mathcal{M}(P)$ be the unique ascending chain of P with respect to λ. Suppose $m \in \mathcal{M}(P)$ is maximal chains of P such that the interval $[m_0, m]_\lambda$ in $P_\lambda(2)$ has some maximal chain of length two. Then the order complex $\Delta((m_0, m)_\lambda)$ is either contractible or homotopy equivalent to a 0-sphere.*

Proof. By Theorem 5.1.10, $[m_0, m]_\lambda$ has at most two coatoms. One of them is the middle element of the maximal chain of length two in $[m_0, m]_\lambda$. Thus, if $[m_0, m]_\lambda$ has one coatom, then $\Delta((m_0, m)_\lambda)$ is a point. If $[m_0, m]_\lambda$ has two coatoms, then $\Delta((m_0, m)_\lambda)$ has two connected components. One component is the point corresponding to the coatom which is the middle element of the maximal chain of length two in $[m_0, m]_\lambda$. The other component corresponds to the second coatom, and that coatom is a cone point of the component. Hence, $\Delta((m_0, m)_\lambda) \simeq S^0$. $\square$

An application of Theorem 5.1.10 is that not all posets of regions of hyperplane arrangements can be realizes as maximal chain descent orders as exhibited in the following example.

Example 5.1.12. Consider the hyperplane arrangement $\mathcal{A}$ in $\mathbb{R}^3$ consisting of the hyperplanes defined by $x_i = \pm x_j$ for $1 \leq i < j \leq 3$. Then the poset of regions $P_{\mathcal{A}}$ is independent of choice of base region, and is shown in Fig. 29. The poset $P_{\mathcal{A}}$ has rank two and has four coatoms. Thus, $P_{\mathcal{A}}$ is not isomorphic to any maximal chain descent order by Theorem 5.1.10.

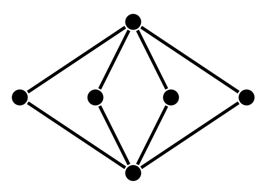


Figure 29. The poset of regions of hyperplane arrangement $\mathcal{A}$.

5.2 Maximal Chain Descent Orders Derived from Well-Known EL-shellable Posets

Our first example sets us up to discuss some related examples later which are endowed with especially rich structure.

5.2.1 $\mathbf{S_n}$ EL-labelings of Finite Supersolvable Lattices.

Here we characterize the intervals in maximal chain descent orders induced by Stanley's M-chain EL-labelings of any finite supersolvable lattice (see Stanley (1972)). For background on supersolvable lattices and M-chain EL-labelings, see Section 2.2.2.

Theorem 5.2.1. *Let P be a finite supersolvable lattice of rank n with an M-chain EL-labeling λ as in Stanley (1972). Let $P_\lambda(2)$ be the maximal chain descent order* *induced by λ. Then any interval in $P_\lambda(2)$ is isomorphic to some interval of weak order on S_n via the map assigning to each maximal chain its label sequence.*

Proof. We show that for any maximal chain $m \in \mathcal{M}(P)$, the lower interval of $P_\lambda(2)$ generated by m is isomorphic to the lower interval of weak order on S_n generated by $\lambda(m)$ via taking label sequences. Then the statement for all closed intervals and open intervals follows immediately. Again the central fact is that the label sequences of maximal chains are permutations of $[n]$.

By Corollary 4.2.10, λ is inversion ranked, and so λ is polygon complete and $P_\lambda(2)$ is ranked by $|\mathrm{inv}_\lambda(\cdot)|$. (Alternatively, we could apply Theorem 4.1.7 to see that λ is polygon complete, but we use the conclusion that P is ranked here as well.) Let m_0 be the unique ascending chain of P with respec to λ, so m_0 is the $\hat{0}$ of $P_\lambda(2)$ and $\lambda(m_0)$ is the identity permutation. We first show that the set of label sequences of the elements in the interval $[m_0, m]_\lambda$ is the set of permutations in the lower interval of weak order $[\lambda(m_0), \lambda(m)]_{wk}$. Then we show that the map $c \mapsto \lambda(c)$ is an isomorphism from $[m_0, m]_\lambda$ to $[\lambda(m_0), \lambda(m)]_{wk}$. For both, we induct on the rank of m in $P_\lambda(2)$.

For each descent of m, the unique chain m' with $m' \rightarrow m$ from Proposition 3.1.5 has label sequence $\lambda(m')$ given by transposing the corresponding descent of $\lambda(m)$ by Proposition 4.1.9. Thus, by induction on the rank of m, the set of label sequences of the elements in $[m_0, m]_\lambda$ is the set of permutations in $[\lambda(m_0), \lambda(m)]_{wk}$. Thus, $c \mapsto \lambda(c)$ is surjective from $[m_0, m]_\lambda$ to $[\lambda(m_0), \lambda(m)]_{wk}$. The fact that $m' \rightarrow m$ implies the label sequence $\lambda(m')$ is obtained from $\lambda(m)$ by transposing a unique descent of $\lambda(m)$ also means that the map $c \mapsto \lambda(c)$ from $[m_0, m]_\lambda$ to $[\lambda(m_0), \lambda(m)]_{wk}$ is order preserving.

Next we show that $c \mapsto \lambda(c)$ from $[m_0, m]_\lambda$ to $[\lambda(m_0), \lambda(m)]_{wk}$ is injective. We again proceed by induction on the rank of m. If the rank of m is zero, then $m = m_0$ by Proposition 3.3.1. Since m_0 is the unique chain of P whose label sequence is the identity permutation, this gives the base case. Now assume the rank of m is greater than zero. Suppose $c, c' \in [m_0, m]_\lambda$ with $\lambda(c) = \lambda(c')$. Observe that m is the only element of $[m_0, m]_\lambda$ with label sequence $\lambda(m)$ by Corollary 3.1.9. Thus, we may assume the rank of c and c' is strictly less than the rank of m.

Let c_1 and c_2 be elements of $[m_0, m']_\lambda$ such that $c \preceq_\lambda c_1 \prec\!\!\cdot_\lambda m$ and $c' \preceq_\lambda c_2 \prec\!\!\cdot_\lambda m$. We have $c_1 \to m$ and $c_2 \to m$. By Proposition 3.1.5, c_1 and c_2 are uniquely determined by the descents of m to which they correspond. Also, the rank of c_1 and c_2 is one less than the rank of m. If $c_1 = c_2$, then $c = c'$ by induction. If $c_1 \neq c_2$, then by induction c is the only element of $[m_0, c_1]_\lambda$ with label sequence $\lambda(c)$ and c' is the only element of $[m_0, c_2]_\lambda$ with label sequence $\lambda(c')$. Thus, it suffices to show that $c' \preceq_\lambda c_1$.

Let $m : \hat{0} = x_0 \lessdot x_1 \lessdot \cdots \lessdot x_n = \hat{1}$ and $c_1 : \hat{0} = x_0 \lessdot x_1 \lessdot \ldots x_{i-1} \lessdot x_i' \lessdot x_{i+1} \lessdot \cdots \lessdot x_n = \hat{1}$ and $c_2 : \hat{0} = x_0 \lessdot x_1 \lessdot \ldots x_{j-1} \lessdot x_j' \lessdot x_{j+1} \lessdot \cdots \lessdot x_n = \hat{1}$. We must have $i \neq j$ since $c_1 \neq c_2$. We have the following two cases: (i) $|i - j| \geq 2$ and (ii) $|i - j| = 1$.

(i) If $|i - j| \geq 2$, then the descents of m corresponding to c_1 and c_2 share no common elements. We may assume without loss of generality that $i < j$. Let c_3 be the maximal chain of P given by $c_3 : \hat{0} = x_0 \lessdot x_1 \lessdot \ldots x_{i-1} \lessdot x_i' \lessdot x_{i+1} \lessdot \cdots \lessdot x_{j-1} \lessdot x_j' \lessdot x_{j+1} \lessdot \cdots \lessdot x_n = \hat{1}$. We have $c_3 \prec\!\!\cdot_\lambda c_1, c_2$ since we could have transposed the descents of m at x_i and x_j in either order to reach c_3 (here we are using Proposition 3.1.5). The label sequence $\lambda(c_3)$ is the meet of the label sequences $\lambda(c_1)$ and $\lambda(c_2)$ in weak order. Thus, $\lambda(c')$ is less than $\lambda(c_3)$ in weak order. We previously showed the the label sequences of elements in $[m_0, c_3]_\lambda$ are the permutations in the weak order interval $[\lambda(m_0), \lambda(c_3)]_\lambda$. Hence, there is some element $c'' \in [m_0, c_3]_\lambda$ with label sequence $\lambda(c')$. Since c'' is also in $[m_0, c_2]_\lambda$ and c' is the unique such element with label sequence $\lambda(c')$, $c'' = c'$. Therefore, $c' \preceq_\lambda c_3 \prec_\lambda c_1$.

(ii) If $|i - j| = 1$, the descents of m corresponding to c_1 and c_2 share a common element. Without loss of generality, we may assume $j = i + 1$. Thus, the saturated subchain of m given by $x_{i-1} \lessdot x_i \lessdot x_{i+1} \lessdot x_{i+2}$ is a descending chain with respect to λ. Let d be the unique ascending saturated chain from x_{i-1} to x_{i+2} and let $c_3 = m^{x_{i-1}} * d * m_{x_{i+2}}$. Then $c_3 \prec_\lambda c_1, c_2$ by Proposition 3.3.1 and Lemma 3.3.3. Further, the label sequence $\lambda(c_3)$ is the meet of $\lambda(c_1)$ and $\lambda(c_2)$ in weak order. Then, by the same argument as in case (i), we have $c' \preceq_\lambda c_3$. Hence, $c' \preceq_\lambda c_3 \prec_\lambda c_1$. This completes the proof of injectivity.

Lastly, Proposition 3.1.5 implies that if $\lambda(c) \lessdot_{wk} \lambda(c')$ in weak order on S_n for $c, c' \in [m_0, m]_\lambda$, then $c \prec\!\!\cdot_\lambda c'$. To see this we suppose $c, c' \in [m_0, m]_\lambda$ and $\lambda(c) \lessdot_{wk} \lambda(c')$. By Proposition 3.1.5 there is a unique maximal chain c'' such that $c'' \to c'$ corresponding to the descent of c' giving rise to $\lambda(c) \lessdot_{wk} \lambda(c')$. Thus, $c'' \in [m_0, m]_\lambda$ and $\lambda(c'') = \lambda(c)$. Then since the map to label sequences is injective on $[m_0, m]_\lambda$, $c'' = c$. Hence, by Corollary 4.2.10 (or by Theorem 4.1.7) $c \prec\!\!\cdot_\lambda c'$ if and only if $\lambda(c) \lessdot_{wk} \lambda(c')$. Therefore, $c \mapsto \lambda(c)$ is an isomorphism from $[m_0, m]_\lambda$ to $[\lambda(m_0), \lambda(m)]_{wk}$. □

Remark 5.2.2. As was noted, finite distributive lattices are especially nice examples of finite supersolvable lattices. They are convenient examples to work with because the M-chain EL-labelings are especially easy to describe and are well controlled. It is not difficult to prove that for any of the M-chain EL-labelings λ_e (described in Remark 2.2.3) of a finite distributive lattice $J(P)$, the corresponding maximal chain descent order is isomorphic to some order ideal in weak order on S_n via the map assigning to each maximal chain its label sequence. We already know in this case from Theorem 5.2.1 that intervals in $J(P)_{\lambda_e}(2)$ are isomorphic to intervals in the weak order on $S_{|P|}$. It is then straight forward to prove the stronger statement that $J(P)_{\lambda_e}(2)$ is isomorphic to the order ideal $\mathcal{L}(P, e)$ of the weak order on $S_{|P|}$. (The notation $\mathcal{L}(P, e)$ is defined in Remark 2.2.3.)

A particularly nice class of distributive lattices are intervals in Young's Lattice, discussed next.

5.2.2 Intervals in Young's Lattice. See Section 2.3 for background on Young's lattice and Young tableaux including the notation used here. In the following proposition, we observe that for tableau $Q \in ST_\alpha$, the label sequence $\lambda_T(m_Q)$ can be read off from the tableaux alone.

Proposition 5.2.3. *Let $T, Q \in ST_\alpha$ be standard tableaux of shape α. Then the label sequence $\lambda_T(m_Q)$ of the maximal chain m_Q of $\mathcal{Y}(\alpha)$ is $\lambda_T(m_Q) = (T(Q^1), T(Q^2), \ldots, T(Q^n))$. Moreover, each label sequence occurs for exactly one maximal chain in $\mathcal{Y}(\alpha)$.*

Proof. The box Q^i is the box added to obtain the rank i element of m_Q from the rank $i-1$ element of m_Q. Now $T(Q^i)$ is the value of the box Q^i under the linear extension defined by T, and so the label of the ith cover relation in m_Q. The uniqueness of label sequences is the same as for the linear extension EL-labelings of any finite distributive lattice. □

Remark 5.2.4. Each choice of standard tableau $T \in ST_\alpha$ defines a maximal chain descent order $\mathcal{Y}(\alpha)_{\lambda_T}(2)$. We may realize these maximal chain descent orders as partial orders on ST_α. By Theorem 4.1.7, λ_T is polygon complete. Proposition 5.2.3

then implies that the cover relations in the maximal chain descent orders can be described by an operation on the tableaux themselves.

Definition 5.2.5. *Let $T \in ST_\alpha$ be a standard tableau of shape α with n boxes. For $1 \le i < j \le n$, let $(i,j)T$ be the filling of α that is the same as T except the entries i and j are switched. If $(i,j)T$ is also a standard tableau, then we call $(i,j)T$ the **ij tableau swap of T**. Further, if $j = i+1$ above, we call $(i,i+1)T$ the **ith tableau swap of T**.*

Lemma 5.2.6. *Suppose $Q, R, T \in ST_\alpha$ are standard tableaux of shape α with n boxes. Then $m_Q \prec_{\lambda_T} m_R$ in $\mathcal{Y}(\alpha)_{\lambda_T}(2)$ if and only if R is the ith tableau swap of Q and $T(Q^i) < T(Q^{i+1})$ for some $1 \le i \le n-1$.*

Proof. By Remark 5.2.2 (which used Theorem 5.2.1), $m_Q \prec_{\lambda_T} m_R$ if and only if $\lambda_T(m_R)$ is obtained from $\lambda_T(m_Q)$ by transposing an ascent of $\lambda_T(m_Q)$. Thus, by Proposition 5.2.3, $m_Q \prec_{\lambda_T} m_R$ if and only if

$$\lambda_T(m_Q) = (T(Q^1), T(Q^2), \dots, T(Q^i), T(Q^{i+1}), \dots, T(Q^n))$$

$$\lambda_T(m_R) = (T(Q^1), T(Q^2), \dots, T(Q^{i+1}), T(Q^i), \dots, T(Q^n))$$

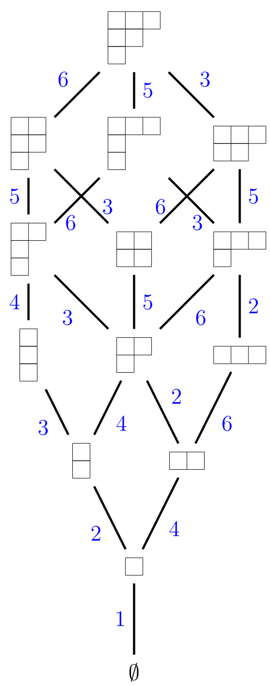
with $T(Q^i) < T(Q^{i+1})$. Hence, $m_Q \prec_{\lambda_T} m_R$ if and only if $R^j = Q^j$ for $j \neq i, i+1$, $R^i = Q^{i+1}$, and $R^{i+1} = Q^i$ which says that R is the ith tableau swap of Q since R is a standard tableau. $\square$

Thus, we have that $\mathcal{Y}(\alpha)_{\lambda_T}(2)$ is isomorphic to the poset on Young tableaux defined as the transitive closure of certain ith tableau swaps.

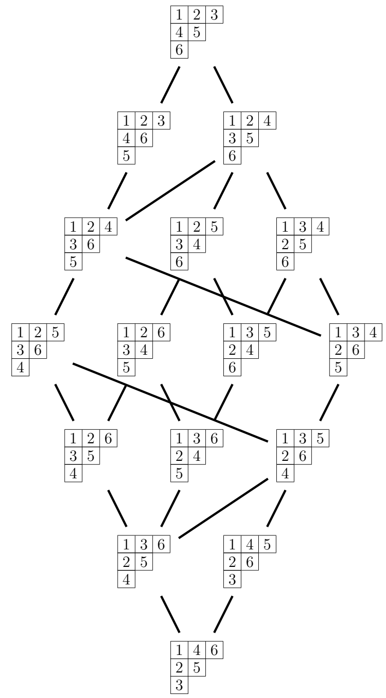
Theorem 5.2.7. *For a standard tableau $T \in ST(\alpha)$ of shape α, $\le_T$ be the partial order on $ST(\alpha)$ defined as the reflexive and transitive closure of $Q \to (i,i+1)Q$ for any $Q \in ST(\alpha)$ such that the ith tableau swap $(i,i+1)Q$ from Definition 5.2.5* *is in $ST(\alpha)$ and $T(Q^i) < T(Q^{i+1})$. Then the map defined by $m \mapsto T_m$ is a poset isomorphism from $\mathcal{Y}(\alpha)_{\lambda_T}(2)$ to $(ST(\alpha), \leq_T)$.*

Proof. We observed that $m \mapsto T_m$ is a bijection from $\mathcal{Y}(\alpha)_{\lambda_T}(2)$ to $(ST(\alpha), \leq_T)$. Then since λ_T is polygon complete by Theorem 4.1.7 (or by Corollary 4.2.10), Lemma 5.2.6 implies that $m \rightarrow m'$ if and only if $T_m \rightarrow T'_m$. This proves the theorem. $\square$

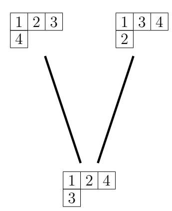
Example 5.2.8. Fig. 30 (a) shows the EL-labeling of an interval in Young's lattice induced by the standard Young tableau $((1, 4, 6), (2, 5), (3))$. Fig. 30 (b) contains the corresponding maximal chain descent order. Fig. 30 (c) shows the maximal chain descent order corresponding induced by the EL-labeling from the standard tableau $((1, 2, 4), (3))$. The maximal chain descent orders are show as the partial orders on standard Young tableaux from Theorem 5.2.7.



(a) The principle order ideal in Young's lattice generated by $(3,2,1)$ with EL-labeling from the Young tableau $((1,4,6),(2,5),(3))$.



(b) The maximal chain descent order induced by the EL-labeling from the Young tableau $((1,4,6),(2,5),(3))$.



(c) The maximal chain descent order induced by the EL-labeling from the Young tableau $((1,4,6),(2,5),(3))$.

Figure 30. The maximal chain descent order induced by the EL-labeling from the Young tableau $((1,2,4),(3))$.

In the next proposition, we recall a special tableau called the row tableau. Fig. 10 shows the row tableau of shape $(3,2,1)$. We mention this tableau because it provides a connection between maximal chain descent orders and the generalized quotients of the weak order on the symmetric group introduced in Björner and Wachs (1988).

Proposition 5.2.9. *Let α be a Young diagram with n boxes and let α_i be the length of the ith row of α. Let R_α be the standard tableau of shape α obtained by labeling the first row of α by $1, 2, \ldots, \alpha_1$ increasing from left to right, labeling the second row by $\alpha_1 + 1, \alpha_1 + 2, \ldots, \alpha_1 + \alpha_2$, and so on. Then for any tableau of shape α, the row word of T is $T(R_\alpha^1), T(R_\alpha^2), \ldots, T(R_\alpha^n)$.*

*The tableau R_α is called **the row tableau** of shape α.*

Proof. This is clear from Proposition 5.2.3. □

In Björner and Wachs (1988), the authors introduce generalized quotients of Coxeter groups. In particular, they study the partial orders induced on these quotients by weak order and Bruhat order on the original Coxeter group. These quotients generalize the notion of quotients of Coxeter groups by parabolic subgroups which are particular choices of coset representatives of a parabolic subgroup. We follow Björner and Wachs' notation and definitions which agree with the notation in Section 2.2.1 in type A. See Björner and Brenti (2010) for general Coxeter groups.

Let (W, S) be a Coxeter system. Let l be the Coxeter length function for (W, S). Subgroups of W generated by a subset $J \subseteq S$, denoted W_J, are called parabolic subgroups. For $J \subseteq S$, **ordinary quotients** are the sets $W^J = \left\{ w \in W \;\middle|\; l(ws) = l(w) + 1 \quad \forall s \in J \right\}$. The ordinary quotient W^J intersects the left cosets of W_J in their minimum length element. This is generalized in Björner and Wachs (1988) as follows:

Definition 5.2.10 (Section 1, Björner and Wachs (1988)). *For any subset $V \subseteq W$, let*

$$W/V = \left\{ w \in W \;\middle|\; l(wv) = l(w) + l(v) \quad \forall v \in V \right\}.$$

*The set W/V is called a **generalized quotient**.*

Restricting the (left) weak order on W to the generalized quotient W/V gives a partial order on W/V which will be referred to as (left) **weak order**.

In Björner and Wachs (1988) Section 7, they introduce a partial order on ST_α called **Left order**. Left order is defined as the reflexive, transitive closure of the order relations given by $Q < T$ whenever T is the ith tableau swap of Q and i appears in a row above $i + 1$ in Q. Björner and Wachs show that Left order is isomorphic to a generalized quotient of the symmetric group.

Theorem 5.2.11 (Björner and Wachs (1988) Theorem 7.2). *Let α have n boxes and let $w(ST_\alpha)$ be the set of row words of all standard tableau of shape α. Then $w(ST_\alpha)$ is a generalized quotient of S_n. Moreover, the map $T \mapsto w(T)$ is a poset isomorphism from Left order on ST_α to weak order on the generalized quotient $w(ST_\alpha)$.*

Choosing the row tableau of a given shape, the induced maximal chain descent order is isomorphic to the Left order on ST_α, and thus isomorphic to a generalized quotient of the symmetric group.

Theorem 5.2.12. *For α with n boxes, the maximal chain descent order $\mathcal{Y}(\alpha)_{\lambda_{R_\alpha}}(2)$ with R_α the row tableau of shape α is isomorphic to Left order on ST_α. Hence, $\mathcal{Y}(\alpha)_{\lambda_{R_\alpha}}(2)$ is also isomorphic to weak order on the generalized quotient $w(ST_\alpha)$ of the symmetric group S_n.*

Proof. By Proposition 5.2.3 and Lemma 5.2.6, the cover relations of $\mathcal{Y}(\alpha)_{\lambda_{R_\alpha}}(2)$ and Left order both correspond to ith tableau swaps $(i, i+1)T$ where i appears in a row above $i+1$ in T. Alternatively, by Proposition 5.2.3 and the definition of $\mathcal{Y}(\alpha)_{\lambda_{R_\alpha}}(2)$, $\mathcal{Y}(\alpha)_{\lambda_{R_\alpha}}(2)$ and Left order are both the reflexive, transitive closure of the ith tableau swaps on ST_α. Then by Theorem 5.2.11, $\mathcal{Y}(\alpha)_{\lambda_{R_\alpha}}(2)$ is isomorphic to weak order on the generalized quotient $w(ST_\alpha)$. □

Remark 5.2.13. We make two observations. First, we note that the label sequences with respect to λ_{R_α} in Theorem 5.2.12 are not the row words of the tableaux corresponding to the maximal chains. Thus, the isomorphism to the generalized quotient in Theorem 5.2.12 which is induced by Theorem 5.2.11 is not the same isomorphism given by Mentioned in Remark 5.2.2. Second, we observe that Theorem 7.6 in Björner and Wachs (1988) can easily be used to extend Theorem 5.2.12 and show which linear extensions of a finite poset P induce EL-labelings of $J(P)$ that give maximal chain descent orders isomorphic to left order on generalized quotients of the symmetric group.

5.2.3 The Partition Lattice.

Another well known M-chain EL-labeling of a supersolvable lattice is the following EL-labeling of the partition lattice Π_{n+1}. Let Π_{n+1} be the collection of set partitions of $[n+1]$ ordered by refinement. The EL-labeling λ due to Gessel and appearing in Björner (1980) is given as follows: if $x \lessdot y$ in Π_{n+1}, then y is obtained from x by merging exactly two blocks B_1 and B_2 of x and $\lambda(x, y) = \max(\{\min B_1, \min B_2\})$. We call λ the **max-min EL-labeling**.

We prove that a class of labeled binary trees given in Definition 5.2.14 are in bijection with the maximal chains of the partition lattice. We then prove that the maximal chain descent order $\Pi_{n+1\lambda}(2)$ is isomorphic to a naturally described poset on these trees. We also note that they are distinct from the trees used in Wachs (1998) to study the (co)homology of the partition lattice.

Definition 5.2.14. *For a positive integer n, let* $\mathbf{PT(n)}$ *denote the set of rooted, unordered, decreasing, full binary trees with $2n$ edges, vertices labeled by $\{1, 2, \ldots, 2n, 2n+1\}$, and leaf set $\{1, 2, \ldots, n, n+1\}$. Rooted means there is a distinguished vertex. Full binary means that each non-leaf vertex has exactly two children. Decreasing means that all of the descendants of a vertex have smaller labels than their ancestor. Unordered means we do not distinguish between the two possible orders of the children of an internal vertex, that is, we may assume that when drawn in the plane, the smaller of the two children of an internal vertex is drawn on the left and the larger is drawn on the right.*

For each integer $0 \leq k \leq n$, let $\mathbf{FPT(n, k)}$ *denote the set of forests of rooted, unordered, decreasing, full binary trees with vertices labeled by $\{1, \ldots, n, n+1, \ldots, n+1+k\}$, leaf set*

$\{1, 2, \ldots, n, n+1\}$, $n+1-k$ components, and $2n-2(n-k) = 2k$ total edges.

For a labeled tree T, let $\mathbf{L(T)}$ *be its leaf set, i.e. the set of labels of leaves of T. Denote the full subtree of T rooted at the vertex labeled i by $\mathbf{T^i}$. If i is the label of an internal vertex of T, let $\mathbf{T^i_1}$ and $\mathbf{T^i_2}$ denote the full subtrees of T rooted at the two children of i.*

Remark 5.2.15. Notice that when $k = n$, $FPT(n, n) = PT(n)$.

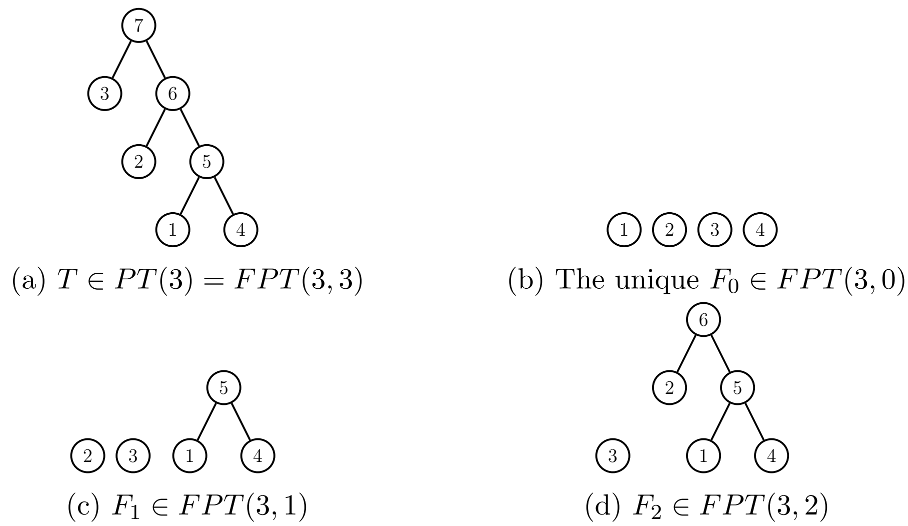
Remark 5.2.16. The trees of Definition 5.2.14 are distinct from the trees used in Wachs (1998) to compute the (co)homology of the partition lattice. The trees in Wachs (1998) are leaf labeled, full binary trees with leaves labeled by $[n+1]$ while internal vertices are not labeled. The same underlying set of leaf labeled, full binary trees appears in Definition 5.2.14, but we also label the internal vertices. In

Wachs (1998), the internal vertices are traversed in post order which describes the maximal chains of Π_{n+1} up to cohomology relations, and so bijects the leaf labeled trees with a cohomology basis. In our trees, the internal vertices are traversed based on the vertex labels which sometimes disagrees with post order. The extra traversals of internal vertices allow trees of $PT(n)$ to biject with maximal chains instead of a cohomology basis.

Definition 5.2.17. *Given a forest $F \in FPT(n,k)$, define a saturated chain* $\mathbf{c}(\mathbf{F})$: $\hat{0} = x_0 \lessdot x_1 \lessdot \cdots \lessdot x_k$ *in the partition lattice* Π_{n+1} *beginning at the unique minimal element* $\hat{0} = 1|2|\dots|n+1$ *as follows: the blocks of x_i are precisely the leaf sets of the components of F restricted to vertices labeled at most $n+1+i$.*

Remark 5.2.18. For $F \in FPT(n,n) = PT(n)$, $c(F)$ is a maximal chain in Π_{n+1}. For any k and any $F \in FTP(n,k)$, the top element of the saturated chain $c(F)$ has blocks which are precisely the leaf sets of the trees in F.

Example 5.2.19. Fig. 31 contains an example of a rooted, unordered, decreasing, full binary tree in $PT(3)$, as well as examples of forests in $FPT(3,k)$ for $0 \leq k \leq 3$. Let F_0 be the forest in Fig. 31 (b), let F_1 be the forest in Fig. 31 (c), let F_2 be the forest in Fig. 31 (d), and let T be the tree in Fig. 31 (a). Then applying Definition 5.2.17 to each of these forests we have $c(F_0) = 1|2|3|4$, $c(F_1) = 1|2|3|4 \lessdot 14|2|3$, $c(F_2) = 1|2|3|4 \lessdot 14|2|3 \lessdot 124|3$, and $c(T) = 1|2|3|4 \lessdot 14|2|3 \lessdot 124|3 \lessdot 1234$.



(a) $T \in PT(3) = FPT(3,3)$ (b) The unique $F_0 \in FPT(3,0)$

(c) $F_1 \in FPT(3,1)$ (d) $F_2 \in FPT(3,2)$

Figure 31. A rooted, unordered, decreasing, full binary tree in $PT(3)$ and forests in $FPT(3,k)$ for $0 \leq k \leq 3$.

Theorem 5.2.20. *The map c of Definition 5.2.17 is a bijection from $FPT(n,k)$ to the saturated chains of Π_{n+1} which begin with the unique minimal element $1|2|\ldots|n+1$ and have length k. In particular, $c : PT(n) \to \mathcal{M}(\Pi_{n+1})$ is a bijection.*

Proof. We will prove Theorem 5.2.20 by induction on k. As a base case we have $k = 0$. The forest F_0 with $n+1$ disconnected vertices labeled $\{1,\ldots,n+1\}$ is the unique forest in $FPT(n,0)$. We have $c(F_0) = 1|2|\ldots|n+1$ which is the unique saturated chain of Π_{n+1} which begins with $1|2|\ldots|n+1$ and has length 0. Thus, c is a bijection when $k = 0$.

Assume c is a bijection when $k = l$ for some $l \geq 0$. Assume $k = l+1$. Observe that for each forest $F \in FTP(n,l+1)$, we obtain a forest $F' \in FTP(n,l)$ by deleting the internal vertex labeled $n+1+l+1$. Further, observe that the saturated chain $c(F')$ is obtained by restricting the saturated chain $c(F)$ to ranks $\{0,1,\ldots,l\}$, i.e. by deleting the element of $c(F)$ at rank $l+1$. We first show that c is injective. Suppose $c(F_1) = c(F_2)$ for forests $F_1, F_2 \in FTP(n,l+1)$. Let $x_l \lessdot x_{l+1}$ be the elements of $c(F_1) = c(F_2)$ of rank l and $l+1$, respectively. So, $c(F_1)^{x_l} =$

$c(F_2)^{x_l}$. Thus, $c(F_1') = c(F_2')$ by our previous observation. Now by the induction hypothesis c is injective on $FTP(n,l)$, so $F_1' = F_2'$. Exactly two blocks of x_l are merged to form x_{l+1}. By Remark 5.2.18 these two blocks are the leaf sets of two of the trees, call them T_1 and T_2, in $F_1' = F_2'$. Then the only way to form a forest $\tilde{F} \in FTP(n, l+1)$ from $F_1' = F_2'$ with x_l as the top element of $c(\tilde{F})$ is to add a vertex labeled $n+1+l+1$ and make its two children the roots of T_1 and T_2. Thus, $F_1 = F_2 = \tilde{F}$, so c is injective on $FTP(n, l+1)$.

Next we show that c is surjective from $FPT(n, l+1)$ to the saturated chains of Π_{n+1} which begin with the unique minimal element $1|2|\dots|n+1$ and have length $l+1$. Let $c_1 : 1|2|\dots|n+1 = y_0 \lessdot y_1 \lessdot \cdots \lessdot y_l \lessdot y_{l+1}$ be a saturated chain in Π_{n+1}. By the induction hypothesis c is surjective from $FTP(n,l)$ to the saturated chains of Π_{n+1} which begin with the unique minimal element $1|2|\dots|n+1$ and have length l. Thus, there is a forest $F \in FTP(n,l)$ such that $c(F) = c_1^{y_l}$. By Remark 5.2.18 there are two trees T_1 and T_2 in F whose leaf sets are the two blocks of y_l which are merged to form y_{l+1}. Let F_1 be the forest formed from F by adding a vertex labeled $n+1+l+1$ and make its two children the roots of T_1 and T_2. It is clear that $F_1 \in FTP(n, l+1)$ since $F \in FTP(n,l)$ and we added the internal vertex $n+1+l+1$ and reduced the number of connected components by exactly one while leaving all other labels the same. By construction the top element of $c(F_1)$ is y_{l+1} while $c(F_1) \setminus \{y_{l+1}\} = c_1^{y_l}$, so $c(F_1) = c_1$. Hence, c is surjective from $FPT(n, l+1)$ to the saturated chains of Π_{n+1} which begin with the unique minimal element $1|2|\dots|n+1$ and have length $l+1$. Therefore, the theorem holds by induction. □

Proposition 5.2.21. *Let T be a tree with $T \in TP(n)$. Let λ be the max-min EL-labeling of the partition lattice Π_{n+1}. Then for each $1 \leq i \leq n$, the ith entry in the* *label sequence* $\lambda(c(T))$ *is the maximum of the minima of the leaf sets of* T_1^{n+1+i} *and* T_2^{n+1+i}*, that is* $\lambda(c(T))_i = \max\left\{\min(L(T_1^{n+1+i})), \min(L(T_2^{n+1+i}))\right\}$.

Proof. By definition of $c(T)$ the two blocks merged to form the rank i element of $c(T)$ from the rank $(i-1)$ element of $c(T)$ are exactly $L(T_1^{n+1+i})$ and $L(T_2^{n+1+i})$. Then the proposition follows by definition of λ. □

Next we see that two maximal chains in the partition lattice differing by a polygon can be described by two simple operations on the corresponding trees, one for each of the two types of rank two intervals in the partition lattice.

Lemma 5.2.22. *Let* $T, S \in TP(n)$ *be trees. Then* $c(T)$ *and* $c(S)$ *differ by a polygon at rank* i *in the partition lattice* Π_{n+1} *if and only if exactly one of the two following conditions holds:*

(i) S *is obtained from* T *by swapping the labels* $n+1+i$ *and* $n+1+i+1$*, or*

(ii) $n+1+i$ *is a child of* $n+1+i+1$ *in* T *and* S *is obtained from* T *by swapping the full subtree of* T *whose root is the child of* $n+1+i+1$ *which is not* $n+1+i$ *and a full subtree of* T *whose root is a child of* $n+1+i$.

Proof. First, we observe that conditions (i) and (ii) are mutually exclusive because if $n+1+i$ is a child of $n+1+i+1$, then swapping the labels $n+1+i$ and $n+1+i+1$ results in a tree which is not decreasing. We now prove the forward direction. We have $c(T) : 1|2|\dots|n+1 = x_0 \lessdot x_1 \lessdot \cdots \lessdot x_{i-1} \lessdot x_i \lessdot x_{i+1} \lessdot \cdots \lessdot x_n = 12\dots n+1$ and $c(S) : 1|2|\dots|n+1 = x_0 \lessdot x_1 \lessdot \cdots \lessdot x_{i-1} \lessdot x_i' \lessdot x_{i+1} \lessdot \cdots \lessdot x_n = 12\dots n+1$ for some $x_i' \neq x_i$. There are two cases: either (a) x_{i-1} contains blocks B_1, B_2, B_3, and B_4; x_i contains blocks $B_1 \cup B_2$, B_3, and B_4; and x_{i+1} contains blocks $B_1 \cup B_2$ and $B_3 \cup B_4$, or (b) x_{i-1} contains blocks B_1, B_2, and B_3; x_i contains blocks $B_1 \cup B_2$ and B_3; and x_{i+1} contains blocks $B_1 \cup B_2 \cup B_3$.

We show that these two cases precisely give rise to conditions (i) and (ii) of the theorem, respectively.

In case (a), we have the that x_i' is obtained from x_{i-1} by merging blocks B_3 and B_4 and x_{i+1} is formed from x_i' by merging blocks B_1 and B_1. Thus, swapping the labels $n+1+i$ and $n+1+i+1$ in T gives a tree $\tilde{S} \in TP(n)$ with $c(\tilde{S}) = c(S)$. Then by Theorem 5.2.20, $\tilde{S} = S$. Thus, condition (i) holds.

Assume we are in case (b). By definition of c, T restricted to labels at most $n+1+i-1$ contains connected components T_1, T_2, and T_3 with leaf sets $L(T_1) = B_1$, $L(T_2) = B_2$, and $L(T_3) = B_3$. Further, T_1, T_2, and T_3 are full subtrees of T. Also, by definition of c, $n+1+i$ is a child of $n+1+i+1$ and the root of T_3 is a child of $n+1+i+1$ since B_1 and B_2 are merged to form x_i from x_{i-1} and B_3 is merged with $B_1 \cup B_2$ to form x_{i+1} from x_i. Now x_i' is formed from x_{i-1} either by merging B_1 and B_3 or by merging B_2 and B_3, then x_{i+1} is formed from x_i' by merging B_2 with $B_1 \cup B_3$ or by merging B_1 with $B_2 \cup B_3$, respectively. Thus, in the first case, swapping the subtrees T_2 and T_3 results in a tree $\tilde{S} \in TP(n)$ with $c(\tilde{S}) = c(S)$. In the second case, swapping the subtrees T_1 and T_3 results in a tree $\tilde{S} \in TP(n)$ with $c(\tilde{S}) = c(S)$. Either way, Theorem 5.2.20 implies $\tilde{S} = S$. Hence, condition (ii) holds which completes the proof of the forward direction.

For the backward direction, if condition (i) is satisified, then $c(T)$ and $c(S)$ differ by a polygon at rank i and the interval $[x_{i-1}, x_{i+1}]$ is of type (a) above. If condition (ii) is satisified, then $c(T)$ and $c(S)$ differ by a polygon at rank i and the interval $[x_{i-1}, x_{i+1}]$ is of type (b). This completes the proof. $\square$

The tree operations of Lemma 5.2.22 give rise to the following partial order on $TP(n)$ which we then show is isomorphic to $\Pi_{n+1\lambda}(2)$.

Definition 5.2.23. *Define a partial order on the trees $PT(n)$ as follows: let $\preceq$ be the reflexive, transitive closure of $T \rightharpoonup S$ if S is obtained from T by either condition (i) or (ii) in Lemma 5.2.22 and* $\max\left\{\min(L(T_1^{n+1+i})), \min(L(T_2^{n+1+i}))\right\} < \max\left\{\min(L(T_1^{n+1+i+1})), \min(L(T_2^{n+1+i+1}))\right\}$.

Theorem 5.2.24. *Let λ be the max-min EL-labeling of Π_{n+1}. Then the map $c : (PT(n), \preceq) \to \Pi_{n+1\lambda}(2)$ is a poset isomorphism. Moreover, the cover relations of $(PT(n), \preceq)$ are precisely given by $T \rightharpoonup S$ for $S, T \in PT(n)$ as given in Definition 5.2.23.*

Proof. The map $c : (PT(n), \preceq) \to \Pi_{n+1\lambda}(2)$ is a bijection by Theorem 5.2.20. By Definition 5.2.23 and Lemma 5.2.22, $m \to m'$ if and only if $c^{-1}(m) \rightharpoonup c^{-1}(m')$. Thus, $c : (PT(n), \preceq) \to \Pi_{n+1\lambda}(2)$ is a poset isomorphism. Lastly, by Theorem 4.1.7 (alternatively, by Corollary 4.2.10) the cover relations of $(PT(n), \preceq)$ are precisely given by $T \rightharpoonup S$ for $S, T \in PT(n)$ since λ is an S_n EL-labeling. $\square$

As a corollary, Theorem 5.2.1 and Theorem 5.2.24 imply that intervals in $(PT(n), \preceq)$ are isomorphic to intervals in the weak order on S_n.

Corollary 5.2.25. *Every interval in the maximal chain descent order $\Pi_{n+1\lambda}(2)$ and every interval in the poset $(PT(n), \preceq)$ from Definition 5.2.23 is isomorphic to some interval in weak order on the symmetric group S_n.*

Example 5.2.26. The partition lattice Π_4 with the max-min EL-labeling is shown in Fig. 32. The induced maximal chain descent order is pictured in Fig. 33 and illustrates Theorem 5.2.24 and Corollary 5.2.25.

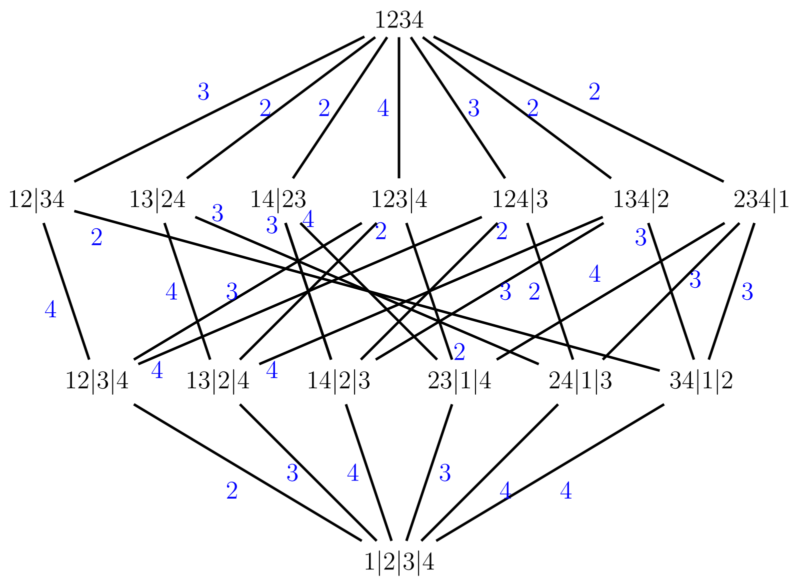


Figure 32. The partition lattice Π_4 with the max-min EL-labeling.

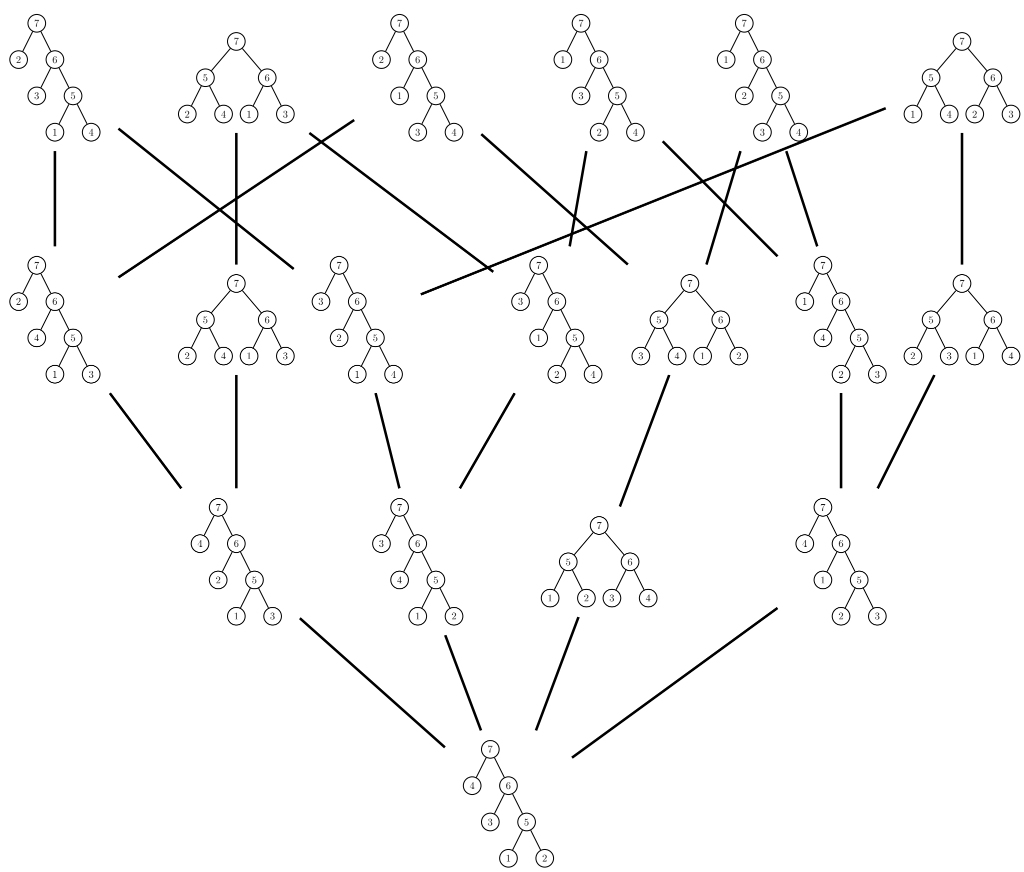


Figure 33. $\Pi_{4\lambda}(2)$ induced by the max-min EL-labeling λ of Π_4.

Remark 5.2.27. We can restrict $PT(n)$ to those trees with "non-crossing" leaf sets and obtain a subposet of $(PT(n), \preceq)$ which is isomorphic to the maximal chain descent order of the max-min EL-labeling λ restricted to the non-crossing partition lattice NC_{n+1}. Alternatively, we could also similarly construct a different poset isomorphic to $NC_{n+1\lambda}(2)$ from the rooted k-ary trees which Edelman and Simion used to study chains in NC_{n+1} in Edelman and Simion (1994) and Edelman (1982). A similar simple operation on the trees describes the cover relations in that case.

5.2.4 Weak Order of Type B as a Maximal Chain Descent Order. In this section, we the type B analog of Theorem 3.2.1 which showed that the weak order on the symmetric group is the maximal chain descent order from the standard EL-labeling of the Boolean lattice. We show that the weak order of the Type B Coxeter group can be realized as a maximal chain descent order. This is motivated by the fact that S_n is the symmetry group of the n-simplex Δ_n and B_n is the face poset Δ_n. We similarly construct the weak order on the type B Coxeter group as the maximal chain descent order of an EL-labeling of the face poset of the n-cube C_n whose symmetry group is the type B Coxeter group. For background on the type B Coxeter groups, in particular the combinatorial representation we use here, see Section 2.2.1.

We will construct a dual EL-labeling of the face poset of the n-cube, that is, an EL-labeling of the face poset of the n-dimensional cross polytope, whose maximal chain descent order is isomorphic to the weak order on S_n^B. For a thorough review of polytopes, their face posets, and n-cubes in particular, see Ziegler (1995), for instance. This EL-labeling of the face lattice of the cube was certainly known previously. Björner and Wachs (1983) (Theorem 4.3 and Corollary 4.4) showed that the face lattice of any polytope admits a CL-labeling though their proof does not explicitly construct the labeling, so it was not clear when it would produce an EL-labeling. In Björner (1980), it was shown the face lattices of cubes and their duals (cross polytopes) admit EL-labelings although the EL-labelings there are distinct from the EL-labelings presented here. Then in Björner, Garsia, and Stanley (1982) (Figure 13), an EL-labeling of the face lattice of the square is exhibited which is an example of the EL-labeling presented here.

We recall that the n-dimensional cube is the convex hull

$$\mathrm{conv}\left(\left\{(\epsilon_1, \epsilon_2, \ldots, \epsilon_n) \;\middle|\; \epsilon_i = \pm 1\right\}\right).$$

The k-dimensional faces for $k \geq 0$ are exactly the convex hulls of those sets of vertices defined by fixing $n-k$ coordinates, that is, a k-face is determined by a $n-k$-subset $\{i_1, \ldots, i_{n-k}\} \subset [n]$ and a sequence $(\epsilon'_{i_1}, \ldots, \epsilon'_{i_{n-k}}) \in \{\pm 1\}^{n-k}$ as

$$\mathrm{conv}\left(\left\{(\epsilon_1, \epsilon_2, \ldots, \epsilon_n) \;\middle|\; \epsilon_i = \pm 1, \epsilon_{i_j} = \epsilon'_{i_j} 1 \leq j \leq n-k\right\}\right).$$

We also have the unique -1-dimensional empty face $\emptyset$. We recall a useful fact which we will use in an induction argument; each proper face of an n-cube is isomorphic to a lower dimensional cube. We also recall that face posets of polytopes are always graded by dimension of the face plus one.

Thus, cover relations in $F(C_n)$, the face poset of the n-cube, are of the form $f \lessdot f'$ where f is a k-face of C_n for $k \geq 0$ defined by $\{i_1, \ldots, i_{n-k}\} \subset [n]$ and $(\epsilon'_{i_1}, \ldots, \epsilon'_{i_{n-k}}) \in \{\pm 1\}^{n-k}$ while f' is a $(k+1)$-dimensional face defined by $\left\{i_1, \ldots, \hat{i_j}, \ldots, i_{n-k}\right\}$ and $(\epsilon'_{i_1}, \ldots, \hat{\epsilon_{i_j}}, \ldots, \epsilon'_{i_{n-k}}) \in \{\pm 1\}^{n-k-1}$ for some $1 \leq j \leq n-k$ where $\hat{\cdot}$ denotes deleting that element from the set or sequence. Or $f = \emptyset$ and f' is 0-face, a vertex. We define an edge labeling λ of $F(C_n)$ by the integers $\Lambda = [-n, n]$ as $\lambda(f \lessdot f') = \epsilon'_{i_j} i_j$ if f is of dimension at least 0 and $\lambda(f \lessdot f') = 0$ if $f = \emptyset$. We totally order Λ by $n, n-1, \ldots, 2, 1, 0, -1, -2, \ldots, -n+1, -n$, that is the reverse of the normal order on integers. We show this edge labeling is a dual EL-labeling and that the label sequences of maximal chains, when read from bottom to top are given by the signed permutations in one line notation with a 0 appended at the beginning.

Theorem 5.2.28. *The edge labeling λ of the face poset of the n-cube $F(C_n)$ is dual EL-labeling. Moreover, the label sequences of maximal chains of $F(C_n)$ when read from bottom to top are given by the signed permutations of n in one line notation with a* 0 *appended at the beginning.*

Proof. First, it is clear that when read from bottom to top the label sequences are precisely the signed permutations of n with a 0 appended at the beginning, and that each signed permutation appears as the label sequence of exactly one maximal chain of $F(C_n)$. We will now read the label sequences from top to bottom since we claim this is a dual EL-labeling. So, when referring to an increasing chain in an interval $[f, f']$ in $F(C_n)$, we mean a chain whose label sequence is increasing when read downward from f' to f. (Later, when showing the corresponding maximal chain descent order is isomorphic to the type B weak order we will return to reading the label sequences from bottom to top.)

Next, we observe that if $f < f'$ in $F(C_n)$ and $f \neq \emptyset$, then f is defined by $\{i_1, \ldots, i_{n-k}\} \subset [n]$ and $(\epsilon'_{i_1}, \ldots, \epsilon'_{i_{n-k}}) \in \{\pm 1\}^{n-k}$ while f is defined by $\left\{i_1, \ldots, \hat{i_{s_1}}, \ldots, \hat{i_{s_r}}, \ldots, i_{n-k}\right\} \subset [n]$ and $(\epsilon'_{i_1}, \ldots, \epsilon' \hat{i}_{s_1}, \ldots, \epsilon' \hat{i}_{s_r}, \ldots, \epsilon'_{i_{n-k}}) \in \{\pm 1\}^{n-k}$. Thus, the label sequences in the interval $[f, f']$ of $F(C_n)$ are precisely the permutations of the set $\left\{\epsilon'_{i_{s_1}} i_{s_1}, \ldots, \epsilon'_{i_{s_r}} i_{s_r}\right\}$ which is a totally orderd subset of Λ. We note these are the permutations of the set $\left\{\epsilon'_{i_{s_1}} i_{s_1}, \ldots, \epsilon'_{i_{s_r}} i_{s_r}\right\}$ in the usual sense, not signed permutations even though the elements being permuted may have signs. Hence, every interval $[f, f']$ with $f \neq \emptyset$ has a unique increasing maximal chain which lexicographically precedes all other maximal chains in the interval.

Thus, it only remains to consider intervals $[f, f']$ in which $f = \emptyset$.

Suppose f' has dimension $k \geq 1$ and is defined by $\{i_1, \ldots, i_{n-k}\} \subset [n]$ and $(\epsilon'_{i_1}, \ldots, \epsilon'_{i_{n-k}}) \in \{\pm 1\}^{n-k}$. Then the maximal chains of $[f, f']$ are determined by fixing the remainder of the coordinates in any order as either plus or minus one, that is, a choice of permutation of $[n] \setminus \{i_1, \ldots, i_{n-k}\}$ and a choice of sign for each element of $[n] \setminus \{i_1, \ldots, i_{n-k}\}$. Thus, the label sequences of maximal chains of $[f, f']$ are precisely the signed permutations in one line notation of $[n] \setminus \{i_1, \ldots, i_{n-k}\}$ with 0 appended at the end of each permutation. We will show that maximal chain m_0 of $[f, f']$ corresponding to the label sequence $j_k, j_{k-1}, \ldots, j_2, j_1, 0$ for $\{j_1 < j_2 < \cdots < j_k\} = [n] \setminus \{i_1, \ldots, i_{n-k}\}$ is the unique increasing chain of $[f, f']$. It is clear that m_0 is increasing, and it is clear m_0 lexicographically precedes all other maximal chains of $[f, f']$ since $j_k, j_{k-1}, \ldots, j_2, j_1$ lexicographically precedes all other signed permutations of $[n] \setminus \{i_1, \ldots, i_{n-k}\}$ in one line notation with respect to Λ.

If the label sequence of a maximal chain m of $[f, f']$ has a negative sign for any element of $j \in [n] \setminus \{i_1, \ldots, i_{n-k}\}$, then $\lambda(m)$ contains a descent by the discrete intermediate value theorem since $\lambda(m)$ ends in 0 and $-i > 0$ in Λ. If the label sequence of a maximal chain m of $[f, f']$ has positive signs for all $j \in [n] \setminus \{i_1, \ldots, i_{n-k}\}$, but begins with some permutation other than $j_k, j_{k-1}, \ldots, j_2, j_1$ when read from top to bottom, then $\lambda(m)$ contains a descent since $j_k, j_{k-1}, \ldots, j_2, j_1$ is the only permutation of $\{j_k, j_{k-1}, \ldots, j_2, j_1\}$ without any descents with respect to Λ. Then since each label sequence in $[f, f']$ is unique, m_0 is the unique increasing maximal chain in $[f, f']$ and lexicographically precedes all other maximal chains in $[f, f']$.

Therefore, λ is a dual EL-labeling of $F(C_n)$. $\square$

This brings us to our theorem that the type B weak order is isomorphic to a maximal chain descent order.

Theorem 5.2.29. *Let λ be the dual EL-labeling of the face poset $F(C_n)$ from Theorem 5.2.28. Then the weak order on S_n^B is isomorphic to the maximal chain descent order $F(C_n)_\lambda(2)$.*

Proof. We claim the map $\varphi : F(C_n)_\lambda(2) \to S_n^B$ given by reading the label sequence of a maximal chain from bottom to top and deleting the 0 from the beginning is a poset isomorphism between $F(C_n)_\lambda(2)$ and S_n^B. By our initial observations in the proof of Theorem 5.2.28, φ is a bijection.

We now recall that face posets of polytopes are thin, that is, each interval of length two contains precisely four elements. Thus, for the increasing chain in each length two interval of $F(C_n)$, there is an unique descending chain. And we previously observed that face posets of polytopes are graded, so the polygon moves induced by λ only depend on rank two intervals of $F(C_n)$.

We thus need a short anyalsis of the length two intervals of $F(C_n)$. Let $[f, f']$ be an interval of length two with $f \neq \emptyset$. In the proof of Theorem 5.2.28, we observed that the label sequences of the two maximal chains in $[f, f']$ will have the form $\epsilon_i i$, $\epsilon_j j$ and $\epsilon_j j$, $\epsilon_i i$ for some $i, j \in [n]$. Thus, the label sequence of the unique descending chain in $[f, f']$ is obtained by transposing the label sequence of the unique increasing chain of $[f, f']$.

Now let $[f, f']$ be an interval of length two with $f = \emptyset$. Thus, f' is 1-dimensional, so an edge. The two vertices of f' must have the form $v_1 = (\epsilon'_1, \dots, \epsilon'_{i-1}, 1, \epsilon'_{i+1}, \dots \epsilon'_n)$ and $v_2 = (\epsilon'_1, \dots, \epsilon'_{i-1}, -1, \epsilon'_{i+1}, \dots \epsilon'_n)$ for some $1 \leq i \leq n$.

Hence, the two maximal chains of $[f, f']$ are $\emptyset \lessdot v_1 \lessdot f'$ and $\emptyset \lessdot v_2 \lessdot f'$. These chains have label sequences $\lambda(\emptyset \lessdot v_1) = 0, \lambda(v_1 \lessdot f') = i$ and $\lambda(\emptyset \lessdot v_2) = 0, \lambda(v_2 \lessdot f') = -i$, respectively. Thus, the label sequence of the unique descending chain in $[f, f']$ is obtained by switching the sign of the first label after the 0, when read from bottom to top. Further, the intervals of length two of the form $[\emptyset, f']$, the increasing chain always has positive first label while the descending chain always has negative first label. Thus, if the set of labels appearing in the label sequences ever changes due to a polygon move with respect to λ, it must occur because the chains differed by some length two interval $[\emptyset, f']$ and the new set of labels is strictly lexicographically larger than the old set of labels with respect to Λ.

Hence, the two possible types of polygon moves with respect to the dual EL-labeling λ of $F(C_n)$ precisely correspond to the simple reflections giving cover relations in $wk(S_n^B)$. Here we are critically using that φ reads the label sequences from bottom to top deleting 0 and that $\Lambda : n, n-1, \ldots, 2, 1, 0, -1, -2, \ldots, -n+1, -n$.

Thus, it only remains to show that each polygon move with respect to λ actually gives a cover relation in $F(C_n)_\lambda(2)$, that is λ is polygon complete. We observe that λ is nearly a polygon strong EL-labeling, but length two intervals of the form $[\emptyset, f']$ have all there maximal chains ending in 0 and so are not polygon strong. So, we will argue directly that every maximal chain increase gives a cover relation in $F(C_n)$. We might try using Lemma 4.1.18 or Theorem 4.2.9. But that approach essentially requires the direct proof below.

Suppose seeking contradiction that $m \rightarrow m'$ for maximal chains m, m' of $F(C_n)$, but $m \not\lessdot_\lambda m'$ in $F(C_n)_\lambda(2)$. Let i be the height of the polygon on which m and m' differ. Then there is some maximal chain m'' such that $m \prec_\lambda m'' \prec_\lambda m'$.

By Lemma 4.1.2, the polgyon on which m and m'' differ is at height j for some $j < i$ (recalling that λ is a dual EL-labeling). Thus, $1 \leq i$. so $\lambda(m)$ and $\lambda(m')$ use precisely the same set of labels. Thus, $\lambda(m)$ and $\lambda(m')$ are permutations (again unsigned though some elements may have signs) of the same set $\{\epsilon_1, \epsilon_2 2, \ldots, \epsilon_n n\}$ with 0 at the beginning. Then by the fact that deleting 0 from $\lambda(m)$ covered by $\lambda(m)'$ in weak order on the permutations of $\{\epsilon_1, \epsilon_2 2, \ldots, \epsilon_n n\}$, at some maximal chain m_c with $m'' \preceq_\lambda m_c \preceq_\lambda m'$ in $F(C_n)_\lambda(2)$ the set of labels appearing in the label sequence must change. That is, $\lambda(m_c)$ has a distinct subset of labels from Λ appearing than do $\lambda(m)$ or $\lambda(m')$. However, then our previous observation that if a set of appearing labels changes because of a polygon move implies the set of appearing labels strictly lexicographically increases. This contradicts that $\lambda(m)$ and $\lambda(m')$ have the same set of labels appearing while $m \prec_\lambda m_c \preceq_\lambda m'$.

Therefore, λ is polygon complete which completes the proof that φ is a poset isomorphism. □

Remark 5.2.30. Theorem 3.4.6 and Theorem 5.2.29 together recover the type B case of the result in Björner (1984) that the linear extensions of the weak order on a Coxeter group induce shellings of the corresponding Coxeter complex just as was noted for type A in Remark 3.4.3.

CHAPTER VI

SB-LABELING OF S-WEAK ORDER AND THE S-TAMARI LATTICE

In this chapter, we largely follow the notation and definitions of Ceballos and Pons (2019) which can be found in Section 2.2.4 and Section 2.2.5. The work in this chapter appears in Lacina (2022).

6.1 Constructing an SB-labeling of s-weak order

In this section, we prove a series of lemmas on s-decreasing trees and multi-inversion sets which we then use to prove that the following edge labeling of s-weak order is an SB-labeling as Theorem 6.1.20. We label a cover relation in s-weak order by taking the unique tree ascent (pair of distinct labeled vertices) corresponding to the cover relation by Theorem 2.2.22 and use the smaller of the two elements of the tree ascent as the label, that is we label cover relations by the label of the root vertex of the subtree moved to achieve the cover relation. Fig. 6 includes two examples of our labeling of s-weak order.

Definition 6.1.1. *Let $T \prec Z$ be a cover relation in s-weak order. Let $T \xrightarrow{(a,b)} Z$ be the s-tree rotation of T along the unique tree ascent (a,b) associated to $T \prec Z$ by Theorem 2.2.22. Define λ to be the edge labeling of s-weak order given by $\lambda(T,Z) = a$.*

The notion of tree ascent is defined in Definition 2.2.16. The notation $T \xrightarrow{(a,b)} Z$ and corresponding notion of s-tree rotation are given in Definition 2.2.21 and Remark 2.2.24.

Remark 6.1.2. In the case $s = (1,\dots,1)$, the SB-labeling of Definition 6.1.1 gives an SB-labeling of weak order on $S_{l(s)}$. Our labeling is distinct from the one given for symmetric groups in Hersh and Mészáros (2017).

The main thing to do in our proof that Definition 6.1.1 gives an SB-labeling of s-weak order is showing that for any $T \prec Z, Q$, the Hasse diagram of the interval $[T, Z \vee Q]$ is a diamond, a pentagon, or a hexagon. Examples of all three types of such intervals can be seen in Fig. 6. In particular, $[T, Z \vee Q]$ has precisely two maximal chains. Then we will verify that, in any case, the labeling on the two maximal chains of $[T, Z \vee Q]$ satisfies Definition 2.1.8. Many of our proofs are easier with Fig. 7 and Remark 2.2.24 in mind so it is worth a few moments to internalize those.

The following proposition restricts the possible tree ascents of an s-decreasing tree. In particular, if (a, b) is a tree ascent of some s-decreasing tree T with $s(a) > 0$, then no labeled vertices of T^a besides a can form a tree ascent with b. For instance, $(5, 9)$ is a tree ascent of the s-decreasing tree in Fig. 4, but no vertex below 5 forms a tree ascent with 9 because the rightmost child of 5 must be a leaf. We use this to characterize the multi-inversion set of $Z \vee Q$ for any $T \prec Z, Q$ and to restrict the chains that can appear in $[T, Z \vee Q]$.

Proposition 6.1.3. *Let T be an s-decreasing tree and let $1 \leq a < b \leq n$ be such that (a, b) is a tree ascent of T with $s(a) > 0$. Then no pair of the form (e, b) such that $e \in T^a$ and $e < a$ is a tree ascent of T.*

Proof. Let (a, b) be a tree ascent of T. Assume (e, b) is also a tree ascent of T with $e \in T^a$ and $e < a$. Then $e \in T^a_{s(a)}$ by (ii) of Definition 2.2.16 of (e, b) being a tree ascent of T because $e < a < b$. Thus, $T^a_{s(a)}$ is not a leaf. However, since $s(a) > 0$, this contradicts (iii) of Definition 2.2.16 of (a, b) being a tree ascent of T. Thus, such a pair (e, b) is not a tree ascent of T. $\square$

Remark 6.1.4. If $s(a) = 0$, it is possible that (a, b) and (e, b) for some $e \in T^a$ with $e < a$ are both tree ascents of T.

The situation precluded by Proposition 6.1.3 may occur if $s(a) = 0$.

We use the following two definitions to describe $Z \vee Q$ for any $T \prec Z, Q$ in terms of tree inversion sets.

Definition 6.1.5. *Let T be a s-decreasing tree and let $1 \leq a < b \leq n$ be such that (a,b) is a tree ascent of T. Let Z be the s-decreasing tree obtained by $T \xrightarrow{(a,b)} Z$. Define* ***the set of inversions added by the s-tree rotation along (a,b)****, denoted $\boldsymbol{A_T(a,b)}$, by*

$$A_T(a,b) = \left\{ (f,e) \;\middle|\; \#_Z(f,e) > \#_T(f,e) \right\}.$$

Definition 6.1.6. *Let T be an s-decreasing tree. Let (a,b) and (c,d) be tree ascents of T with $a < c$. We note that b and d are determined by Remark 2.2.19 once we know a and c are each the smaller element of a tree ascent. Define the following set valued function:*

$$F_T(a,c) = \begin{cases} \left\{ (d,e) \;\middle|\; e \in T^a \setminus 0 \right\} & \text{if } b = c \text{ and } a \in T_0^c \\ \\ \emptyset & \text{otherwise} \end{cases}$$

Example 6.1.7. Let T be the s-decreasing tree in Fig. 4. As we saw in Example 2.2.17, $(5,9)$ and $(4,5)$ are both tree ascents of T. Also, $4 \in T_0^5$. If we perform the s-tree rotation of T along $(5,9)$ using Remark 2.2.24, we observe that $A_T(5,9) = \{(9,5)\}$ and $A_T(4,5) = \{(5,1),(5,2),(5,4)\}$. Also, by definition $F_T(4,5) = \{(9,1),(9,2),(9,4)\}$.

In the next proposition, we explicitly compute the tree inversions added by an s-tree rotation, that is, $A_T(a,b)$ from Definition 6.1.5. The proposition can be verified on Example 6.1.7 above.

Proposition 6.1.8. *Let T be an s-decreasing tree and let $1 \leq a < b \leq n$ be such that (a, b) is a tree ascent of T. Suppose $T \xrightarrow{(a,b)} Z$. Then for $1 \leq e < f \leq n$, $(f, e) \in A_T(a, b)$ if and only if $f = b$ and $e \in T^a \setminus 0$ in which case*

$$\#_Z(f, e) = \#_T(f, e) + 1.$$

The notation $\#_Z(f, e)$ and the corresponding notion of cardinality are given in Definition 2.2.5.

Proof. First, we note that if $e \in T^a \setminus 0$ and $e < a$, then $s(a) > 0$. Thus, $T^a_{s(a)}$ is a leaf by condition (iii) of Definition 2.2.16 of (a, b) being a tree ascent of T. Hence, for any $e \in T^a \setminus 0$, $e \notin T^a_{s(a)}$. Then both parts of the statement follow from Remark 2.2.24 by considering the only subtrees that change in an s-tree rotation (see Fig. 7). □

A particularly simple case of Proposition 6.1.8 is when the smaller element of a tree ascent has only a single child.

Corollary 6.1.9. *If (a, b) is a tree ascent of T with $s(a) = 0$ and $T \xrightarrow{(a,b)} Z$, then $A_T(a, b) = \{(b, a)\}$.*

The subsequent lemma essentially shows that the sets of inversions added by s-tree rotations along distinct tree ascents are disjoint. This is illustrated by Example 6.1.7 where the particular sets of inversions added are pairwise disjoint. We use this lemma in the proof of one of two different upcoming characterizations of $Z \vee Q$ for any $T \prec Z, Q$.

Lemma 6.1.10. *Let T be an s-decreasing tree. Let $1 \leq a < b \leq n$ and $1 \leq c < d \leq n$ be such that (a, b) and (c, d) are tree ascents of T with $a < c$. Then $A_T(a, b)$, $A_T(c, d)$, and $F_T(a, c)$ are pairwise disjoint.*

The notation $A_T(a,b)$ and $F_T(a,c)$ are given in Definition 6.1.5 and Definition 6.1.6, respectively.

Proof. We assume seeking contradiction that there is some $(f,e) \in A_T(a,b) \cap A_T(c,d)$. Then by Proposition 6.1.8, $f = b = d$ and $e \in T^a, T^c$. Now by Definition 2.2.16 of (a,b) and (c,d) being tree ascents of T, $a,c \in T^b$. Then, by the fact that e is only below one child of b in T and by Remark 2.2.8, $a,c \in T_i^b$. Then since (a,b) and (c,b) are both tree ascents of T, $a,c \in {}_RT_i^b$ by Remark 2.2.20. Now by definition of ${}_RT_i^b$, $a \in T^c$. If $s(c) > 0$, then (a,b) and (c,b) both being tree ascents of T contradicts Proposition 6.1.3. Thus, $s(c) = 0$. Then by Corollary 6.1.9, $A_T(c,d) = \{(d,c)\}$ so $(f,e) = (d,c)$. But that contradicts Proposition 6.1.8 because $a < c$ so $c \notin T^a \setminus 0$.

If $F_T(a,c) \neq \emptyset$, then $b = c \neq d$ and $a \in T_0^c$ by Definition 6.1.6. Thus, $F_T(a,c)$ is disjoint from $A_T(a,b)$ by Proposition 6.1.8 since $b \neq d$. Also in this case, $F_T(a,c)$ is disjoint from $A_T(c,d)$ by Proposition 6.1.8 because each $e \in T^a \setminus 0$ is also in T_0^c. □

Now we have the first of two different descriptions of $Z \vee Q$ for any $T \prec Z, Q$. Intuitively, this lemma says we can reach $Z \vee Q$ by first performing the s-tree rotation of T along the tree ascent associated with Z and then the s-tree rotation of Z along the tree ascent associated with Q or vice versa. In reality, we run into situations where the tree ascent of T associated with Q is not actually a tree ascent of Z or vice versa. So this intuitive picture is not always defined. We address those situations with later lemmas. We use this description to establish the desired saturated chains in $[T, Z \vee Q]$, while we use the second description in the proofs that there are no other saturated chains to such a join.

Lemma 6.1.11. *Let T be an s-decreasing tree and let $1 \leq a < b \leq n$ and $1 \leq c < d \leq n$ be such that (a,b) and (c,d) are distinct tree ascents of T. Suppose $T \xrightarrow{(a,b)} Z$ and $T \xrightarrow{(c,d)} Q$. Then $inv(Z \vee Q) = ((inv(T) + (b,a))^{tc} + (d,c))^{tc}$. Moreover, the order of the pairs in this equality of multi-inversion sets can be reversed.*

The notation $(\cdot)^{tc}$ and the corresponding notion of transitive closure are given just prior to Theorem 2.2.14. The notion of containment of multi-inversion sets is given in Definition 2.2.9. The notation $I + J$ and associated idea of the sum of multi-inversion sets are given just after Example 2.2.12.

Proof. First, by Theorem 2.2.14, $\mathrm{inv}(Z \vee Q) = (\mathrm{inv}(Z) \cup \mathrm{inv}(Q))^{tc}$. Let $I = \mathrm{inv}(Z) \cup \mathrm{inv}(Q)$. By definition of transitive closure, to show

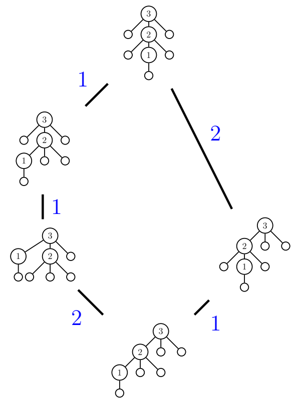
$$((\mathrm{inv}(T) + (b,a))^{tc} + (d,c))^{tc} = I^{tc}$$

it suffices to show that $(\mathrm{inv}(T) + (b,a))^{tc} + (d,c) \subseteq I$ and $\mathrm{inv}(Z), \mathrm{inv}(T) + (d,c) \subseteq (\mathrm{inv}(T) + (b,a))^{tc} + (d,c)$. We will show the inclusions in that order.

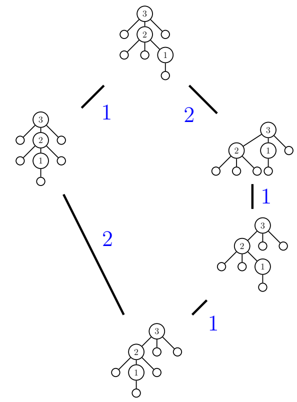
We recall by Definition 2.2.21 that $\mathrm{inv}(Z) = (\mathrm{inv}(T)+(b,a))^{tc}$ and $\mathrm{inv}(Q) = (\mathrm{inv}(T)+(c,d))^{tc}$. By Proposition 6.1.8 and Lemma 6.1.10, $\#_Z(d,c) = \#_T(d,c)$ and $\#_Q(d,c) = \#_T(d,c) + 1$ so $\#_I(d,c) = \#_T(d,c) + 1$. Thus, $(\mathrm{inv}(T) + (b,a))^{tc} + (d,c) \subset I$. On the other hand, $\mathrm{inv}(T) + (d,c) \subset (\mathrm{inv}(T) + (b,a))^{tc} + (d,c)$ since $\mathrm{inv}(T) \subset (\mathrm{inv}(T) + (b,a))^{tc}$. Thus, $\mathrm{inv}(T) + (d,c), \mathrm{inv}(Z) \subset (\mathrm{inv}(T) + (b,a))^{tc} + (d,c)$. Therefore, $\mathrm{inv}(Z \vee Q) = ((\mathrm{inv}(T) + (b,a))^{tc} + (d,c))^{tc}$. Similarly, the tree ascents may appear in the other order, that is $\mathrm{inv}(Z \vee Q) = ((\mathrm{inv}(T) + (d,c))^{tc} + (b,a))^{tc}$. □

In the next lemma, we begin with distinct tree ascents (a,b) and (c,d) of an s-decreasing tree T and let Z and Q be the s-tree rotations of T along those tree ascents, respectively. We characterize when one pair ceases to be a tree ascent of the s-tree rotation along the other pair. The four possibilities turn out to correspond to different relationships between (a, b) and (c, d) in T. These four possibilities end up characterizing the intervals $[T, Z \vee Q]$ which have Hasse diagrams that are diamonds, pentagons, and hexagons. The nature of this interval characterization is essentially contained in the following examples in Fig. 34 of intervals from the s-weak orders in Fig. 6. These intervals act as guides for the remainder of this section.

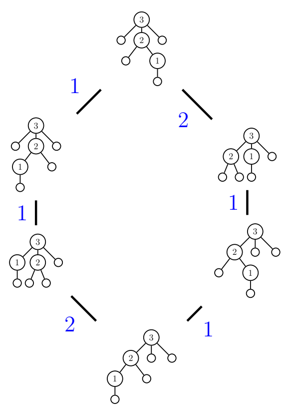
In later lemmas, we will show that in particular, when (a, b) is a tree ascent of Q and (c, d) is a tree ascent of Z, $[T, Z \vee Q]$ has Hasse diagram that is a diamond. When exactly one of (a, b) is not a tree ascent of Q or (c, d) is not a tree ascent of Z, $[T, Z \vee Q]$ has Hasse diagram which is a pentagon. When both (a, b) is not a tree ascent of Q and (c, d) is not a tree ascent of Z, $[T, Z \vee Q]$ has Hasse diagram that is a hexagon. Lemma 6.1.12 below can be illustrated with the s-decreasing tree in Fig. 4. Using Remark 2.2.24, we can perform the s-tree rotations of the s-decreasing tree in Fig. 4 along the following pairs of tree ascents: $(5, 9)$ and $(7, 8)$, $(2, 4)$ and $(3, 4)$, $(3, 4)$ and $(4, 5)$, $(2, 4)$ and $(4, 5)$, $(4, 5)$ and $(5, 9)$. These exemplify the ways a pair of tree ascents can be related and all of the ways one tree ascent can cease to be a tree ascent after the s-tree rotation along another tree ascent.



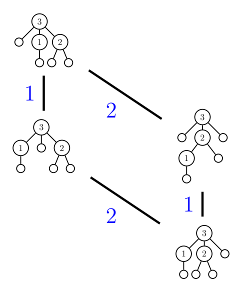
(a) Pentagon in $s = (0, 2, 2)$.



(b) Pentagon in $s = (0, 2, 2)$.



(c) Hexagon in $s = (0, 1, 2)$.



(d) Diamond in $s = (0, 1, 2)$.

Figure 34. Examples of pentagonal, hexagonal, and diamond intervals which arise in s-weak order.

Lemma 6.1.12. *Let T be a s-decreasing tree. Let $1 \leq a < b \leq n$ and $1 \leq c < d \leq n$ be such that (a, b) and (c, d) are tree ascents of T with $a < c$. Let $T \xrightarrow{(a,b)} Z$ and $T \xrightarrow{(c,d)} Q$. If either of (a, b) is not a tree ascent of Q or (c, d) is not a tree ascent of Z, then $b = c$ and $s(c) > 0$. Moreover, if (a, c) is not a tree ascent of Q, then $a \in T_0^c$. If (c, d) is not a tree ascent of Z, then $a \in T_{s(c)-1}^c$.*

Proof. We will argue that there are four cases that we must check in more detail for the way in which one of the tree ascents (a, b) or (c, d) can cease to be a tree ascent after the s-tree rotation along the other. We will check these four cases and show that two of them cannot actually occur and that the other two are precisely the conclusions of the lemma. Suppose that either (a,b) is not a tree ascent of Q or (c,d) is not a tree ascent of Z. Then after the s-tree rotation along one of (a,b) or (c,d), at least one of the three conditions of Definition 2.2.16 must be violated by the other pair.

We begin with three observations with which we show four simpler cases cannot occur leaving us with the four cases mentioned above. First, Remark 2.2.24 and the fact that $a < c$ imply $Q^a = T^a$ since the only vertices which have changes to the subtrees rooted at them in the s-tree rotation from T to Q are c or have label greater than c. Second, since $a < c$, Remark 2.2.24 implies the s-tree rotation along (a,b) does not move vertex c or any vertices above c in T. Third, s-tree rotations never decrease the cardinalities of tree inversions by Proposition 6.1.8.

The first observation shows condition (iii) of Definition 2.2.16 cannot be violated by (a,b) in Q. This is because $Q^a = T^a$, so if $s(a) > 0$, then $Q^a_{s(a)}$ is a leaf since (a,b) is a tree ascent of T. The second observation shows that condition (i) of Definition 2.2.16 cannot be violated by (c,d) in Z because the relative positions of c and d in T are not changed by the s-tree rotation along (a,b). The second and third observations together show that condition (ii) of Definition 2.2.16 cannot be violated by (c,d) in Z. This is because the second observation implies that for any e with $c < e < d$, $c \in T^e$ if and only if $c \in Z^e$. By condition (ii) of Definition 2.2.16 of (c,d) being a tree ascent of T, $\#_T(e,c) = s(e)$. Then by the third observation, $\#_T(e,c) \leq \#_Z(e,c)$ so $\#_Z(e,c) = s(e)$, which is exactly condition (ii) of Definition 2.2.16 of (c,d) being a tree ascent of Z. Lastly, the third observation shows that condition (ii) of Definition 2.2.16 cannot be violated by (a,b) in Q in certain cases, namely by any e such that $a < e < b$, $a \in Q^e$, and

$a \in T^e$. This is again because condition (ii) of Definition 2.2.16 of (a, b) being a tree ascent of T implies $\#_T(e, a) = s(e)$ and the third observation implies $\#_T(e, a) \leq \#_Q(e, a)$ so $\#_Q(e, a) = s(e)$. The case of $a < e < b$ with $a \in Q^e$, but $a \notin T^e$ is covered as case (1) below.

Thus, there are four possible cases for how conditions (i), (ii), or (iii) of Definition 2.2.16 might be violated.

(1) (a, b) is not a tree ascent of Q because (ii) is violated by some $a < e < b$ such that $a \notin T^e$, but $a \in Q_i^e$ and $i < s(e)$.

(2) (a, b) is not a tree ascent of Q because (i) is violated by $\#_Q(b, a) = s(b)$,

(3) (a, b) is not a tree ascent of Q because (i) is violated by $a \notin Q^b$,

(4) (c, d) is not a tree ascent of Z because (iii) is violated by $s(c) > 0$ and $Z_{s(c)}^c$ is not a leaf.

We show cases (1) and (2) cannot occur and that cases (3) and (4) give the conclusions of Lemma 6.1.12.

(1) Assume there is some e such that $a < e < b$, $a \notin T^e$, and $a \in Q_i^e$ with $i < s(e)$. By Remark 2.2.24, there are two ways that a is below vertex in Q which it was not below in T. Either $a \in Q_{s(c)}^c$ or $a \in T^c \setminus 0$. If $a \in Q_{s(c)}^c$, then the only vertex that a is below in Q which it was not below in T is c. Thus, $e = c$, but $\#_Q(c, a) = s(c)$ so (ii) would not be violated. If $a \in T^c \setminus 0$, then $a \neq c$ implies $s(c) > 0$. However, if $c < b$, then $a \in T_{s(c)}^c$ because (a, b) is a tree ascent of T. This contradicts (c, d) being a tree ascent of T because $s(c) > 0$ and $T_{s(c)}^c$ is not a leaf. If $b \leq c$, then $b \in T^c \setminus 0$ by Remark 2.2.8. Then by Remark 2.2.24,

if e has $a \in Q^e$ and $a \notin T^e$, then $c \in Q^e$ also. Thus, $e \geq c \geq b$ contradicting $e < b$. Thus, this case cannot occur.

(2) Assume $\#_Q(b,a) = s(b)$. Since (a,b) is a tree ascent of T, $\#_T(b,a) < s(b)$. Thus, $\#_Q(b,a) = s(b)$ implies $(b,a) \in A_T(c,d)$ by Proposition 6.1.8. However, this contradicts Lemma 6.1.10. Hence, this case cannot occur.

(3) Suppose $a \notin Q^b$. We note that Remark 2.2.24 (Fig. 7) implies that $a \in T^b$ and $a \notin Q^b$ if and only if $b = c$ and $a \in T_0^c$ by considering the subtrees which change with the s-tree rotation. Thus, $b = c$ and $a \in T_0^c$. Then since $b = c$ and (a,b) is a tree ascent of T, $s(c) > 0$ by Remark 2.2.18. This is precisely the first of the two possible conclusions of Lemma 6.1.12.

(4) Suppose $s(c) > 0$ and $Z^c_{s(c)}$ is not a leaf. We note that $T^c_{s(c)}$ is a leaf by (iii) of Definition 2.2.16 of (c,d) being a tree ascent of T since $s(c) > 0$. Now Remark 2.2.24 implies that $T^c_{s(c)}$ is a leaf and $Z^c_{s(c)}$ is not a leaf if and only if $c = b$ and $a \in T^c_{s(c)-1}$ again by considering the subtrees which change with the s-tree rotation. Hence, $b = c$ and $a \in T^c_{s(c)-1}$. This is exactly the second possible conclusion of Lemma 6.1.12.

□

Remark 6.1.13. Assuming the hypotheses of Lemma 6.1.12, if $s(c) = 0$, condition (iii) of Definition 2.2.16 cannot be violated by (c,d) in Z. In this case, (c,d) will be a tree ascent of Z.

In the following lemma, we give the second description of $Z \vee Q$ for any $T \prec Z, Q$. We explicitly find the multi-inversion set difference between $\text{inv}(T)$ and $\text{inv}(Z \vee Q)$, in contrast with Lemma 6.1.11 which was the first description of $Z \vee Q$. Similarly to Lemma 6.1.11 though, $\text{inv}(T)$ and $\text{inv}(Z \vee Q)$ is obtained from inv (T) by adding the tree inversions necessary to reach Z from T and then the tree inversions needed to reach Q from T but with a correction of some additional tree inversions if (a, b) is not a tree ascent of Z. In practice, this lemma shows the possible pairs that may occur as tree ascents corresponding to cover relations in the interval $[T, Z \vee Q]$. We use this lemma to restrict the chains that can occur in $[T, Z \vee Q]$. We can also verify the lemma on the s-decreasing tree in Fig. 4 in the case of the cover relations given by the tree ascents and tree inversions added in Example 6.1.7.

Lemma 6.1.14. *Let T be an s-decreasing tree. Let $1 \leq a < b \leq n$ and $1 \leq c < d \leq n$ be such that (a, b) and (c, d) are tree ascents of T with $a < c$. Suppose $T \xrightarrow{(a,b)} Z$ and $T \xrightarrow{(c,d)} Q$, then $inv\,(Z \vee Q) - inv\,(T) = A_T(a, b) \cup A_T(c, d) \cup F_T(a, c)$.*

The notation $inv\,(\cdot) - inv\,(\cdot)$ and the corresponding notion of multi-inversion set difference are defined in Definition 2.2.9. The notations $A_T(\cdot, \cdot)$ and $F_T(\cdot, \cdot)$ are defined in Definition 6.1.5 and Definition 6.1.6.

Proof. There are two cases. Either (a, b) is a tree ascent of Q or not.

Suppose (a, b) is a tree ascent of Q. Then either $b \neq c$ or $a \notin T_0^c$ by Lemma 6.1.12. Either way, $F_T(a, c) = \emptyset$ by definition. Then by Lemma 6.1.11, $Q \xrightarrow{(a,b)} Z \vee Q$. Thus, by Proposition 6.1.8, inv $(Z \vee Q) -$inv $(T) = A_T(c, d) \cup A_Q(a, b)$. Again using Proposition 6.1.8, $A_Q(a, b) = \left\{(b, e) \;\middle|\; e \in Q^a \setminus 0\right\}$. Now since $a < c$, $c \notin T^a$ and $c \notin Q^a$. Thus, Remark 2.2.24 implies $Q^a \setminus 0 = T^a \setminus 0$. Hence, $A_Q(a, b) = A_T(a, b)$ so inv $(Z \vee Q) -$ inv $(T) = A_T(c, d) \cup A_T(a, b)$.

Next suppose (a, b) is not a tree ascent of Q. Then $b = c$, $a \in T_0^c$, and $s(c) > 0$ by Lemma 6.1.12. We first argue that the multi-inversion set difference between inv $(Z \vee Q)$ and inv (T) contains the stated tree inversions. We then produce an s-decreasing tree P' whose multi-inversion set difference with inv (T)

actually equals the stated tree inversions. Then the lemma holds because the join is the least upper bound, in this context has the smallest multi-inversion set difference with $\mathrm{inv}\,(T)$ by inclusion. We produce P', which is $Z \vee Q$, in the argument by finding a particular saturated chain starting at T.

We first observe that by Proposition 6.1.8 and Lemma 6.1.10, $A_T(a,b) \cup A_T(c,d) \subseteq \mathrm{inv}\,(Z \vee Q) - \mathrm{inv}\,(T)$. Next we show that by transitivity $F_T(a,c) \subset \mathrm{inv}\,(Z \vee Q) - \mathrm{inv}\,(T)$. It suffices to show that $\#_{Z\vee Q}(d,e) \geq \#_T(d,e) + 1$ for all $e \in T^a \setminus 0$. To show this inequality we first note that since $b = c$, $e \in Z_1^c$ for all $e \in T^a \setminus 0$ by Remark 2.2.24. Thus, $\#_{Z\vee Q}(c,e) \geq 1$ for all $e \in T^a \setminus 0$. Now for any such $e \in T^a \setminus 0$, $e < c < d$ so by transitivity $\#_{Z\vee Q}(d,e) \geq \#_{Z\vee Q}(d,c)$. Next we observe that by Proposition 6.1.8 and the fact that $Q \preceq Z \vee Q$, $\#_{Z\vee Q}(d,c) \geq \#_T(d,c) + 1$. Lastly, we note that since $a \in T^c$, $\#_T(d,e) = \#_T(d,c)$ for all $e \in T^a \setminus 0$. Thus, $\#_{Z\vee Q}(d,e) \geq \#_T(d,e) + 1$.

It remains to show that there is an s-decreasing tree P' with $\mathrm{inv}\,(P') - \mathrm{inv}\,(T) = A_T(a,b) \cup A_T(c,d) \cup F_T(a,c)$. We claim there is a saturated chain

$$T \xrightarrow{(c,d)} Q \xrightarrow{(a,d)} P \xrightarrow{(a,c)} P'$$

and that $\mathrm{inv}\,(P') - \mathrm{inv}\,(T) = A_T(a,b) \cup A_T(c,d) \cup F_T(a,c)$. Fig. 35 illustrates this chain and guides the proof.

We first show (a,d) is a tree ascent of Q and then that (a,c) is a tree ascent of the s-decreasing tree P resulting from the s-tree rotation of Q along (a,d). We recall that to show that (a,d) is a tree ascent of Q, it suffices to show that $a \in {}_RT_j^d$ for some $j < s(d)$ and that if $s(a) > 0$, then $T_{s(a)}^a$ is a leaf and similarly for (a,c) in P.

We observe that $c \in {}_RT_j^d$ for some $j < s(d)$ since (c,d) is a tree ascent of T. Also, $a \in {}_RT_0^c$ since (a,c) is a tree ascent of T with $a \in T_0^c$. Then by Remark 2.2.24,

$a \in {}_R Q^d_j$ since $f = c$ was the only $a < f < d$ with $a \in T^f_k$ and $k < s(f)$. Further, Remark 2.2.24 implies $Q^a = T^a$. If $s(a) > 0$, then $T^a_{s(a)}$ is a leaf because (a, c) is a tree ascent of T. So $Q^a_{s(a)}$ would be a leaf also. Hence, (a, d) is a tree ascent of Q.

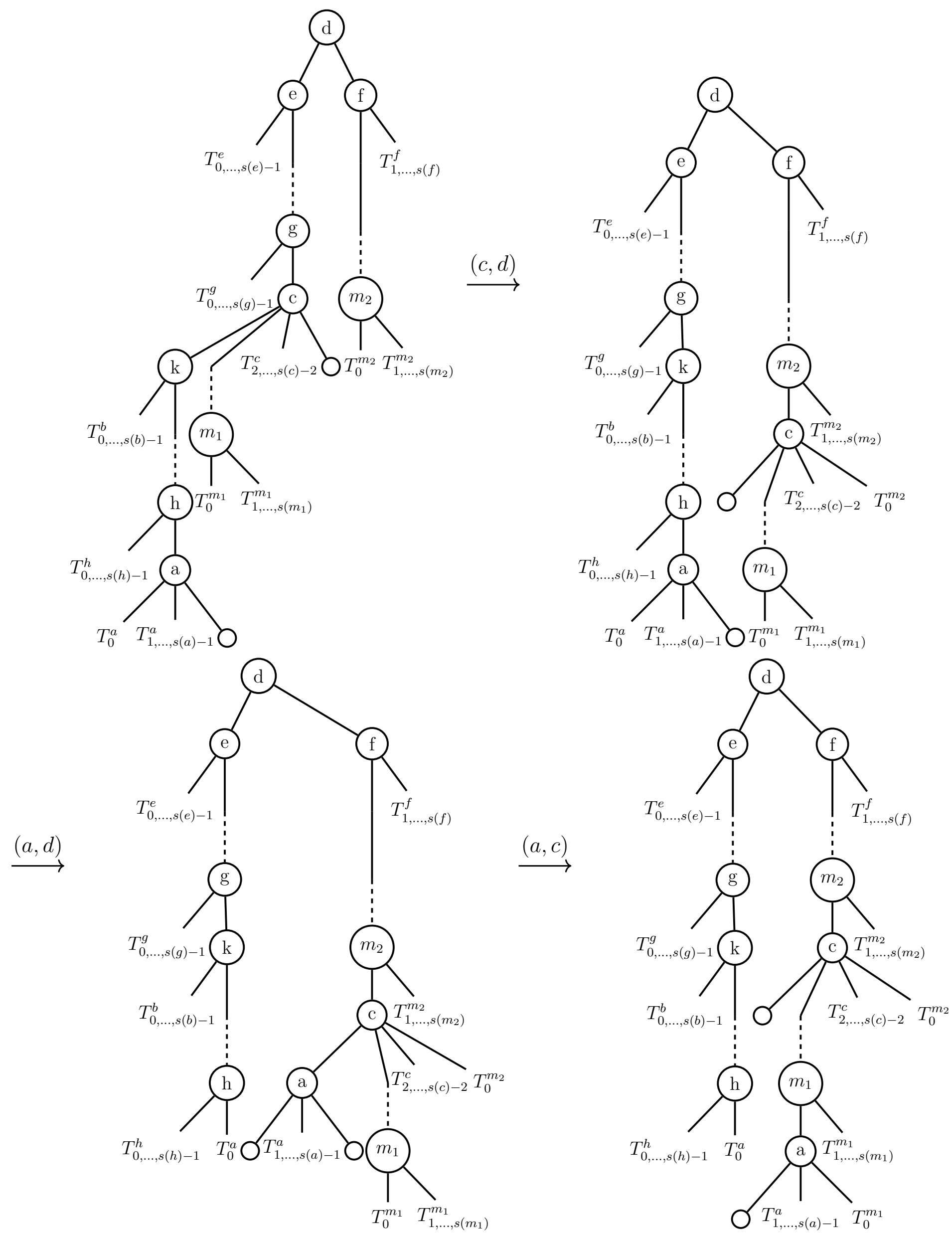


Figure 35. The length three side of an $a \in T^c_0$ pentagon from Lemma 6.1.14. m_1 is the smallest element of ${}_L T^c_1$ that is larger than a and m_2 is the smallest element of ${}_L T^d_{j+1}$ that is larger than c.

Next we observe that Q_0^c is a leaf by Remark 2.2.24 and the fact that $0 < s(c)$ by supposition. Thus, also by Remark 2.2.24, $P_0^c = P^a$. Hence, $a \in {}_RP_0^c$. Again, since Q_0^c is a leaf, $P_{s(a)}^a$ is a leaf by Remark 2.2.24. Hence, (a,c) is a tree ascent of P. Therefore, we have the claimed saturated chain.

Now by Proposition 6.1.8, $\mathrm{inv}\,(P') - \mathrm{inv}\,(T) = A_T(c,d) \cup A_Q(a,d) \cup A_P(a,c)$. But by Remark 2.2.24, we have $Q^a = T^a$ and $P^a = T^a \setminus 0$. Hence, $A_Q(a,d) = F_T(a,c)$. Further, since $b = c$, $A_P(a,c) = A_T(a,b)$. Therefore, $\mathrm{inv}\,(Z \vee Q) - \mathrm{inv}\,(T) = A_T(a,b) \cup A_T(c,d) \cup F_T(a,c)$. $\square$

In the following lemma, we establish that in the interval $[T, Z \vee Q]$ for any $T \prec Z, Q$, the only atoms are Z and Q. We use this in part of the proof that there are only two maximal chains in such an interval.

Lemma 6.1.15. *Let T be an s-decreasing tree. Let $1 \le a < b \le n$ and $1 \le c < d \le n$ be such that (a,b) and (c,d) are tree ascents of T with $a < c$. Suppose $T \xrightarrow{(a,b)} Z$ and $T \xrightarrow{(c,d)} Q$, then Z and Q are the only atoms in $[T, Z \vee Q]$.*

Proof. First, Theorem 2.2.22 implies that atoms of the $[T, Z \vee Q]$ correspond to the tree ascents (e,f) of T such that $(f,e) \in \mathrm{inv}\,(Z \vee Q) - \mathrm{inv}\,(T)$. Thus, by Lemma 6.1.14 the atoms of $[T, Z \vee Q]$ correspond to pairs $(f,e) \in A_T(a,b) \cup A_T(c,d) \cup F_T(a,c)$ such that (e,f) is a tree ascent of T. By Proposition 6.1.8 and Proposition 6.1.3, the only pairs $(f,e) \in A_T(a,b) \cup A_T(c,d)$ such that (e,f) are tree ascents of T are $(f,e) = (b,a), (d,c)$. Further, if $F_T(a,c) \neq \emptyset$ and $(f,e) \in F_T(a,c)$, then $b = c$, $f = d$, and $e \in T^a \setminus 0$ by Definition 6.1.6. For all $e \in T^a$, $e \in T_k^b$ with $k < s(b)$ since (a,b) is a tree ascent of T. Then since $b < d$, (e,d) such that $e \in T^a$ does not satisfy condition (ii) of Definition 2.2.16, and so is not a tree ascent of T. Therefore, the only atoms of $[T, Z \vee Q]$ are Z and Q. $\square$

In the next lemma, we consider the case of $T \xrightarrow{(a,b)} Z$ and $T \xrightarrow{(c,d)} Q$ for tree ascents (a, b) and (c, d) of T, but when (c, d) is not a tree ascent of Z. This is one of the cases from Lemma 6.1.12. We construct a saturated chain from T to $Z \vee Q$. This is similar to the construction of the saturated chain in the proof of Lemma 6.1.14. This new chain is illustrated in Fig. 36 below. As an example, we can construct this chain using the s-decreasing tree in Fig. 4 and its tree ascents $(2, 4)$ and $(4, 5)$.

Lemma 6.1.16. *Let $T \prec Z, Q$ be cover relations in s-weak order corresponding to $T \xrightarrow{(a,b)} Z$ and $T \xrightarrow{(c,d)} Q$ for tree ascents (a, b) and (c, d) of T with $a < c$. Suppose (c, d) is not a tree ascent of Z, then there is a saturated chain of the form $T \xrightarrow{(a,b)} Z \xrightarrow{(a,d)} P \xrightarrow{(c,d)} Z \vee Q$.*

Proof. First, by Lemma 6.1.12, $b = c$, $a \in T^c_{s(c)-1}$, and $s(c) > 0$. So the two tree ascents of interest in T are (a, c) and (c, d). We claim that there is a saturated chain

$$T \xrightarrow{(a,c)} Z \xrightarrow{(a,d)} P \xrightarrow{(c,d)} P'.$$

We first show that (a, d) is a tree ascent of Z, and then that (c, d) is a tree ascent of the s-decreasing tree P resulting from the s-tree rotation of Z along (a, d). Then we show that $P' = Z \vee Q$. This is illustrated in Fig. 36 which also guides the proof.

First, we note that $c \in {}_RT^d_j$ for some $0 \leq j < s(d)$ since (c, d) is a tree ascent of T. Thus, $c \in {}_RZ^d_j$ because the tree rotation of T along (a, b) does not move any vertices above a in T. Also, $T^c_{s(c)}$ is a leaf because (c, d) is a tree ascent of T and $s(c) > 0$. Then by Remark 2.2.24, $Z^c_{s(c)} = T^a \setminus 0$ so a is the $s(c)$th child c in Z. Thus, $a \in {}_RZ^d_j$ since a is the $s(c)$th child of c in Z. Further, $Z^a_{s(a)}$ is a leaf again by Remark 2.2.24 and the fact that $T^c_{s(c)}$ is a leaf. Hence, (a, d) is a tree ascent of Z.

Now, again by Remark 2.2.24, $c \in {}_RP^d_j$ where j is the same j as above so $0 \leq j < s(d)$. Lastly, $P^c_{s(c)} = Z^a_0$ which is a leaf by Remark 2.2.24 and the fact that $T^c_{s(c)}$ is a leaf. Thus, (c, d) is a tree ascent of P.

Now we claim $P' = Z \vee Q$. By Proposition 6.1.8, $\operatorname{inv}(P') - \operatorname{inv}(T) = A_T(a, c) \cup A_Z(a, d) \cup A_P(c, d)$. Thus, by Lemma 6.1.14, it remains to show that $A_T(a, c) \cup A_Z(a, d) \cup A_P(c, d) = A_T(a, b) \cup A_T(c, d) \cup F_T(a, c)$. Since $b = c$, $A_T(a, b) = A_T(a, c)$. To show $A_Z(a, d) \cup A_P(c, d) = A_T(c, d) \cup F_T(a, c)$, there are two cases because $b = c$. Either $a \in T^c_0$ or $a \notin T^c_0$, that is, $F_T(a, c)$ is possibly non-empty or $F_T(a, c) = \emptyset$, respectively, by Definition 6.1.6.

Suppose $a \in T^c_0$. Then, as above, by Remark 2.2.24 and the fact that $T^c_{s(c)}$ is a leaf, $Z^a = T^a \setminus 0$. Thus, by Proposition 6.1.8 and Definition 6.1.6, $A_z(a, d) = F_T(a, c)$. Further, by Remark 2.2.24 along with the fact that $a \in T^c_0$ and our previous observations that $P^c_{s(c)}$ and $T^c_{s(c)}$ are leaves, $P^c \setminus 0 = T^c \setminus 0$. Thus, by Proposition 6.1.8, $A_P(c, d) = A_T(c, d)$.

Now suppose $a \notin T^c_0$ so $F_T(a, c) = \emptyset$. To show that $A_Z(a, d) \cup A_P(c, d) = A_T(c, d)$, we need to show that $T^c \setminus 0 = Z^a \setminus 0 \cup P^c \setminus 0$ as sets of labeled vertices. We previously argued that $Z^c_s(c) = Z^a \setminus 0 = T^a \setminus 0$. Also, as sets of labeled vertices $P^c \setminus 0 = (T^c \setminus 0) \setminus (T^a \setminus 0)$ by Remark 2.2.24. This completes the proof. $\square$

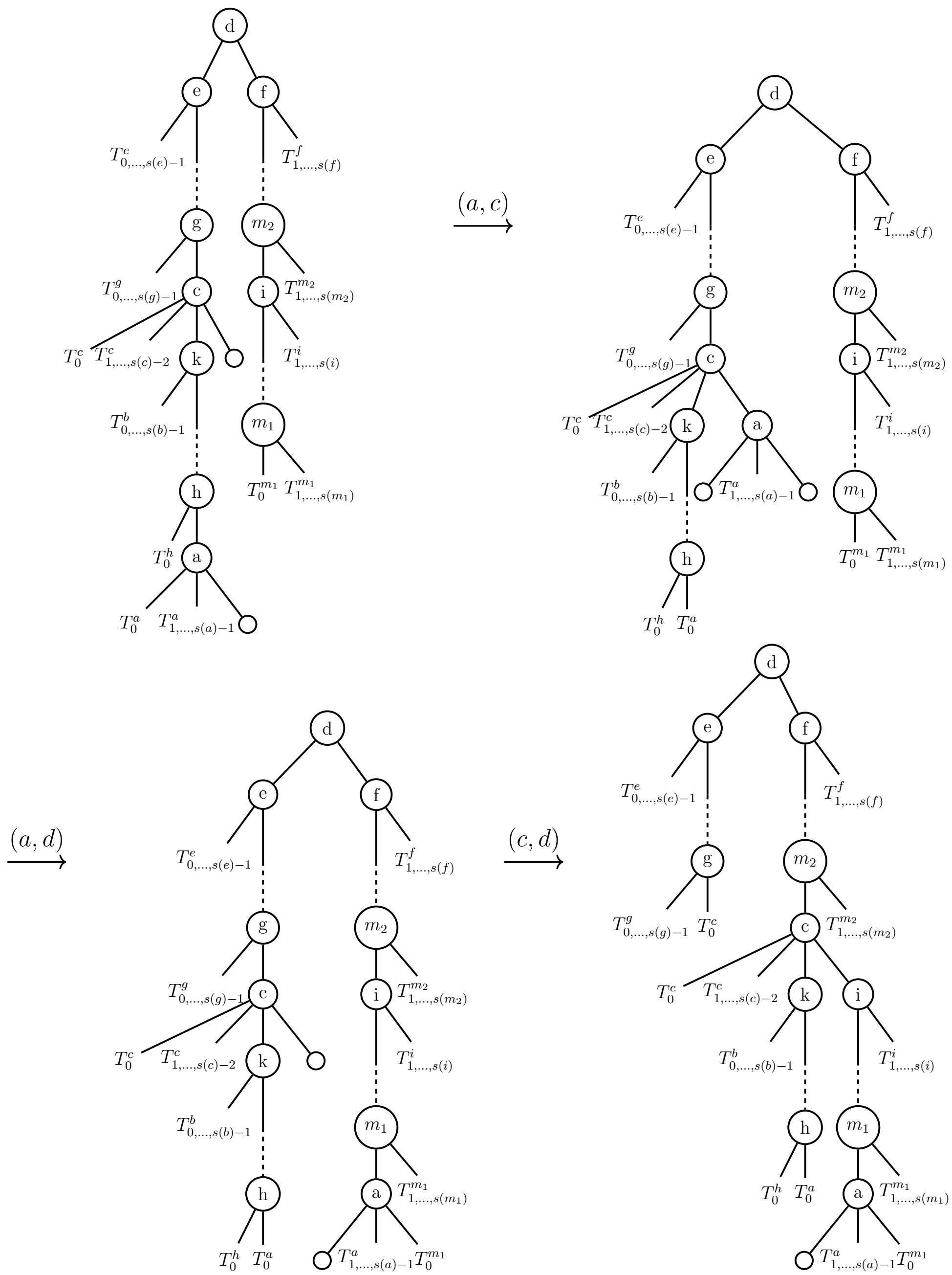


Figure 36. The length three side of an $a \in T^c_{s(c)-1}$ pentagon from Lemma 6.1.16. m_1 is the smallest element of ${}_LT^d_{j+1}$ that is larger than a and m_2 is the smallest element of ${}_LT^d_{j+1}$ that is larger than c.

In the next three lemmas, we begin with $[T, Z \vee Q]$ for $T \prec Z, Q$ having (a, b) and (c, d) the tree ascents of T associated with Z and Q, respectively.

We prove that three of the four relationships given by Lemma 6.1.12 result in $[T, Z \vee Q]$ having Hasse diagram that is a diamond or a pentagon and that, in any of these three cases, our labeling in Definition 6.1.1 satisfies the conditions of an SB-labeling. A pentagon arises in two different ways, once each from two of the relationships. The fourth relationship is when both relationships giving pentagons occur at the same time in T. In that case, $[T, Z \vee Q]$ has Hasse diagram a hexagon. We show the hexagonal case in the proof of Theorem 6.1.20. Theorem 2.2.22 characterizing cover relations in s-weak order and Lemma 6.1.11 along with the chains constructed in Lemma 6.1.14 and Lemma 6.1.16, establish the two maximal chains of $[T, Z \vee Q]$ in these cases. Thus, the bulk of the proofs of the next three lemmas is showing that there are no other maximal chains in $[T, Z \vee Q]$ in the respective cases. We note that our labeling always satisfies the first condition of an SB-labeling by Remark 2.2.19. All three lemmas can again be verified on the appropriate intervals of the examples of s-weak order in (b) and (c) of Fig. 6.

Lemma 6.1.17. *Let $T \prec Z, Q$ be cover relations in s-weak order corresponding to $T \xrightarrow{(a,b)} Z$ and $T \xrightarrow{(c,d)} Q$ for tree ascents (a, b) and (c, d) of T with $a < c$. Suppose (a, b) is a tree ascent of Q and (c, d) is a tree ascent of Z. Then $[T, Z \vee Q]$ has Hasse diagram which is a diamond and the edge labeling of Definition 6.1.1 on its two maximal chains satisfies Definition 2.1.8.*

Proof. By Lemma 6.1.11, $\mathrm{inv}_{=}(Z \vee Q)\,(\mathrm{inv}\,(Z) + (d, c))^{tc} = (\mathrm{inv}\,(Q) + (b, a))^{tc}$. Then since (c, d) is a tree ascent of Z and (a, b) is a tree ascent of Q, $R \xrightarrow{(c,d)} Z \vee Q$ and $Q \xrightarrow{(a,b)} Z \vee Q$. Hence, $R, Q \prec Z \vee Q$ Thus, $T \prec Z \prec Z \vee Q$ and $T \prec Q \prec Z \vee Q$ are two distinct saturated chains from T to $Z \vee Q$. Then to show there is not a third such saturated chain, it suffices to show there is not a third atom in the interval $[T, Z \vee Q]$. We showed this fact as Lemma 6.1.15.

Now we observe that the label sequences of the saturated chains $T \prec Z \prec Z \vee Q$ and $T \prec Q \prec Z \vee Q$ are a, c and c, a, respectively. Therefore, Definition 2.1.8 is satisfied. $\square$

Lemma 6.1.18. *Let $T \prec Z, Q$ be cover relations in s-weak order corresponding to $T \xrightarrow{(a,b)} Z$ and $T \xrightarrow{(c,d)} Q$ for tree ascents (a, b) and (c, d) of T with $a < c$. Suppose (a, b) is a tree ascent of Q and (c, d) is not a tree ascent of Z. Then $[T, Z \vee Q]$ has Hasse diagram which is a pentagon and the edge labeling of Definition 6.1.1 on its two maximal chains satisfies Definition 2.1.8.*

Proof. Fig. 36 illustrates this case and provides a guide for this proof. First, we observe that $Q \prec Z \vee Q$ by Lemma 6.1.11 because (a, b) is a tree ascent of Q. This cover relation is given by the s-tree rotation $Q \xrightarrow{(a.b)} Z \vee Q$. Thus, the label sequence for the saturated chain $T \prec Q \prec Z \vee Q$ is c, a.

Next, by Lemma 6.1.12, $b = c$ and $a \in T^c_{s(c)-1}$ with $s(c) - 1 > 0$. Then by Lemma 6.1.16 there is a saturated chain of the form $T \xrightarrow{(a,c)} Z \xrightarrow{(a,d)} P \xrightarrow{(c,d)} Z \vee Q$.

Thus, it remains to show that there are no other maximal chains in $[T, Z \vee Q]$ in this case. Proposition 6.1.8 shows $Q \not\preceq P$. Thus, it suffices to show there are no other elements in $[T, Z \vee Q]$ besides $T, Z, Q, P, Z \vee Q$.

We note the only atoms in $[T, Z \vee Q]$ are Z and Q by Lemma 6.1.15. Then since $Q \prec Z \vee Q$, the only other possibility of an element in $[T, Z \vee Q]$ besides the five listed above is that there is an atom of $[Z, Z \vee Q]$ distinct from P. Assume there is such an atom, Z'. Then by Theorem 2.2.22 and Proposition 6.1.8, there exists $(f, e) \in A_Z(a, d) \cup A_P(c, d)$ such that (e, f) is a tree ascent of Z with $Z \xrightarrow{(e,f)} Z'$. Now by Proposition 6.1.8 and Proposition 6.1.3, the only pair $(f, e) \in A_Z(a, d)$ such that (e, f) is a tree ascent of Z is $(f, e) = (d, a)$. However, $(f, e) \neq (d, a)$ since $Z' \neq P$. Next we note that any $(f, e) \in A_P(c, d)$ has the form (d, e) for some $e \in P^c \setminus 0$ by

Proposition 6.1.8. We observe that by Remark 2.2.24, $P^c = Z^c \setminus s(c)$. Thus, any such any $e \in P^c \setminus 0$ with $e \neq c$ has $e \in Z^c_i$ with $i \neq s(c)$. Thus, (e, d) does not satisfy (ii) of Definition 2.2.16 of (e, d) being a tree ascent of Z because $e < c < d$. Thus, (c, d) must be the tree ascent of Z corresponding to Z'. However, this contradicts the hypothesis of the lemma that (c, d) is not a tree ascent of Z. Hence, P is the only atom of $[Z, Z \vee Q]$, and there are no other elements of $[T, Z \vee Q]$ besides the five listed earlier.

The two saturated chains have label sequences c, a and a, a, c which satisfy Definition 2.1.8. $\square$

Lemma 6.1.19. *Let $T \prec Z, Q$ be cover relations in s-weak order corresponding to $T \xrightarrow{(a,b)} Z$ and $T \xrightarrow{(c,d)} Q$ for tree ascents (a, b) and (c, d) of T with $a < c$. Suppose (a, b) is not a tree ascent of Q, but (c, d) is a tree ascent of Z. Then $[T, Z \vee Q]$ has Hasse diagram which is a pentagon and the edge labeling of Definition 6.1.1 on its two maximal chains satisfies Definition 2.1.8.*

Proof. In this case, $Z \prec Z \vee Q$ by Lemma 6.1.11. This cover relation is given by the s-tree rotation $Z \xrightarrow{(c,d)} Z \vee Q$. Thus, there is a saturated chain $T \prec R \prec Z \vee Q$ with label sequence a, c.

Since (a, b) is not a tree ascent of Q, $b = c$ and $a \in T^c_0$ with $s(c) > 1$ by Lemma 6.1.12. Then by the proof of Lemma 6.1.14, there is a saturated chain of the form

$$T \xrightarrow{(c,d)} Q \xrightarrow{(a,d)} P \xrightarrow{(a,c)} Z \vee Q.$$

Thus, it remains to show these are the only saturated chains in the interval $[T, Z \vee Q]$. Again Proposition 6.1.8 implies $Z \not\prec P$. Hence, it suffices to show there are no other elements in $[T, Z \vee Q]$ besides $T, Z, Q, P, Z \vee Q$.

Again the only atoms in $[T, Z \vee Q]$ are Z and Q by Lemma 6.1.15. Since $Z \prec Z \vee Q$, the only other possibility is that there is an atom Q' in $[Q, Z \vee Q]$ distinct from P. Assume Q' is such an atom. Then by Theorem 2.2.22 and Proposition 6.1.8, there exists $(f, e) \in A_Q(a, d) \cup A_P(a, c)$ such that (e, f) is a tree ascent of Q and $Q \xrightarrow{(e,f)} Q'$. By Proposition 6.1.3 and Proposition 6.1.8, the only pair $(f, e) \in A_Q(a, d)$ such that (e, f) is a tree ascent of Q is $(f, e) = (d, a)$. But $(f, e) \neq (d, a)$ since $Q' \neq P$. Next we note that any $(f, e) \in A_P(a, c)$, has the form (c, e) for some $e \in P^a \setminus 0$. By Remark 2.2.24, $P^a \setminus 0 = Q^a \setminus 0 = T^a \setminus 0$ since $a \in T_0^c$ and $s(c) > 1$. Also by Remark 2.2.24, no element of T^a is in Q^c since $a \in T_0^c$. Thus, for $e \in P^a \setminus 0$, $e \notin Q^c$. Thus, no $(f, e) \in A_P(a, c)$ has (e, f) a tree ascent of Q. Hence, P is the only atom of $[Q, Z \vee Q]$.

Lastly, the label sequences for these two chains are a, c and c, a, a which satisfy Definition 2.1.8. $\square$

This brings us to the proof that Definition 6.1.1 gives an SB-labeling of s-weak order.

Theorem 6.1.20. *Let $T \prec Z$ be a cover relation in s-weak order. Let $T \xrightarrow{(a,b)} Z$ be the s-tree rotation of T along the unique tree ascent (a, b) associated to $T \prec Z$ by Theorem 2.2.22. Let λ to be the edge labeling $\lambda(T, Z) = a$. Then λ is an SB-labeling of s-weak order.*

Proof. Suppose $T \prec Z, Q$ correspond to $T \xrightarrow{(a,b)} Z$ and $T \xrightarrow{(c,d)} Q$ for tree ascents of (a, b) and (c, d) of T with $a < c$. By Remark 2.2.19, λ satisfies property (i) of Definition 2.1.8. To verify properties (ii) and (iii) of Definition 2.1.8, there are four cases we must check:

(1) (a, b) is a tree ascent of Q and (c, d) is a tree ascent of Z, or

(2) (a, b) is a tree ascent of Q while (c, d) is not a tree ascent of Z, or

(3) (c, d) is a tree ascent of Z while (a, b) is not a tree acsent of Q, or

(4) (a, b) is not a tree ascent of Q and (c, d) is not a tree ascent of Z.

Case (1) is Lemma 6.1.17. Case (2) is Lemma 6.1.18. Case (3) is Lemma 6.1.19. Case (4) results in $[T, Z \vee Q]$ having Hasse diagram which is a hexagon and follows from Lemma 6.1.18 and Lemma 6.1.19 and their proofs as we show now.

In case (4), Lemma 6.1.12 implies $b = c$, but this time $a \in T_0^c$ and $s(c) = 1$ so $a \in T_{s(c)-1}^c$. Then the proofs of Lemma 6.1.18 and Lemma 6.1.19 imply that there are two distinct maximal chains in $[T, Z \vee Q]$. Both maximal chains are of length three and their label sequences are a, a, c and c, a, a. Additionally, the proofs that there are no other maximal chains in the intervals in Lemma 6.1.18 and Lemma 6.1.19 combine to show there are no other maximal chains in $[T, Z \vee Q]$. Thus, (ii) and (iii) of Definition 2.1.8 are satisfied. Therefore, λ is an SB-labeling of s-weak order. □

Thus, we can characterize the homotopy types of open intervals in s-weak order and the Möbius function of s-weak order as follows.

Corollary 6.1.21. *Let $T \preceq Z$ in s-weak order. Then $\Delta(T, Z)$, the order complex of the open interval (T, Z), is homotopy equivalent to a ball or a sphere of some dimension. Moreover, the Möbius function of s-weak order satisfies $\mu(T, Z) \in \{-1, 0, 1\}$.*

Proof. The characterization of homotopy type follows from Theorem 2.1.9 and Theorem 6.1.20. The result on the Möbius function follows from the fact that

$\mu(T, Z) = \tilde{\chi}(\Delta(T, Z))$ along with the fact that the reduced Euler characteristic of a ball is 0 and a d-sphere is $(-1)^d$. $\square$

Lastly, we give an intrinsic characterization of the intervals which are homotopy spheres and the dimension of those spheres.

Lemma 6.1.22. *If $T \prec Z$ in s-weak order, then Z is the join of the atoms in $[T, Z]$ if and only if*

$$inv\,(Z) = (inv\,(T) + A_T(a_1, b_1) + \cdots + A_T(a_l, b_l))^{tc}$$

where $(a_1, b_1), \ldots, (a_l, b_l)$ are the tree ascents of T such that $(b_i, a_i) \in inv\,(Z) - inv\,(T)$. Moreover, the number of atoms in the interval $[T, Z]$ is l regardless of whether or not Z is the join of atoms in the interval.

Proof. Let $T \preceq Z$ in s-weak order. The number of atoms in $[T, Z]$ follows from the characterization of cover relations in s-weak order.

Let $(a_1, b_1), \ldots, (a_l, b_l)$ be all of the tree ascents of T contained in $\mathrm{inv}\,(Z) - \mathrm{inv}\,(T)$. Let $T_1, \ldots, T_l$ be the corresponding atoms of $[T, Z]$, respectively. Then to prove the characterization of the join of atoms, it suffices to show $\mathrm{inv}\left(\bigvee_{i=1}^{l} T_i\right) = (\mathrm{inv}\,(T) + A_T(a_1, b_1) + \cdots + A_T(a_l, b_l))^{tc}$. We note that by induction, $\mathrm{inv}\left(\bigvee_{i=1}^{l} T_i\right) = (\mathrm{inv}\,(T_1) \cup \cdots \cup \mathrm{inv}\,(T_l))^{tc}$. Now by Proposition 6.1.8, $\mathrm{inv}\,(T_i) = \mathrm{inv}\,(T) + A_T(a_i, b_i)$. By Lemma 6.1.10, the sets $A_T(a_i, b_i)$ are pairwise disjoint. Thus,

$$\mathrm{inv}\,(T) + A_T(a_1, b_1) + \cdots + A_T(a_l, b_l) \subset \mathrm{inv}\,(T_1) \cup \cdots \cup \mathrm{inv}\,(T_l)$$

so

$$(\mathrm{inv}\,(T) + A_T(a_1, b_1) + \cdots + A_T(a_l, b_l))^{tc} \subset \mathrm{inv}\left(\bigvee_{i=1}^{l} T_i\right).$$

On the other hand, $\text{inv}(T) + A_T(a_i, b_i) \subset \text{inv}(T) + A_T(a_1, b_1) + \cdots + A_T(a_l, b_l)$ for each $i \in [l]$ so $\text{inv}(T_i) \subset (\text{inv}(T) + A_T(a_1, b_1) + \cdots + A_T(a_l, b_l))^{tc}$ for each $i \in [l]$. Thus, $\text{inv}\left(\bigvee_{i=1}^{l} T_i\right) \subset (\text{inv}(T) + A_T(a_1, b_1) + \cdots + A_T(a_l, b_l))^{tc}$ which gives the result. □

Lemma 6.1.22 and Theorem 2.1.9 combine to give directly the following theorem.

Theorem 6.1.23. *If $T \prec Z$, then $\Delta(T, Z)$ is homotopy equivalent to a sphere if and only if*

$$inv(Z) = (inv(T) + A_T(a_1, b_1) + \cdots + A_T(a_l, b_l))^{tc}$$

where $(a_1, b_1), \ldots, (a_l, b_l)$ are the tree ascents of T such that $(b_i, a_i) \in inv(Z) - inv(T)$. Moreover, in this case the dimension of the sphere is $l - 2$.

6.2 Constructing an SB-labeling of the s-Tamari lattice

In this section, we prove that a quite similar edge labeling of the s-Tamari Lattice is an SB-labeling. The notation and notions for the s-Tamari lattice are defined in Section 2.2.5 and are quite similar to those for s-weak order. We use a subscript of Tam to differentiate between s-weak order and the s-Tamari lattice, for instance $\prec_{Tam}$ instead of $\prec$ for cover relations. For the join however, we still use $\vee$ as in s-weak order because the s-Tamari lattice is a sublattice of s-weak order. We follow a quite similar structure of lemmas as in the proof for s-weak order. The proofs are quite similar to the case of s-weak order with the only major difference being that intervals $[T, Z \vee Q]_{Tam}$ for any $T \prec_{Tam} Z, Q$ have Hasse diagrams which are only diamonds or pentagons. Further, there is only one way that pentagonal intervals arise. There are also some minor differences in the details we must check, but these details are usually simpler than in the case of s-weak order because Tamari tree ascents are always a pair of a parent and child as defined just after Theorem 2.2.26. Because of the similarities, the proofs presented here are more cursory.

Intuitively, we label cover relations in the s-Tamari lattice by the label of the root vertex of the subtree that is moved to obtain the cover relation, that is we label by the smaller element of the Tamari tree ascent associated to the cover relation by Theorem 2.2.28, just as in s-weak order.

Definition 6.2.1. *Let $T \prec_{Tam} Z$ be a cover relation in the s-Tamari lattice. Let $T \xrightarrow{Tam(a,b)} Z$ be the s-Tamari rotation of T along the Tamari tree ascent (a, b) of T associated to $T \prec_{Tam} Z$ by Theorem 2.2.28. Define λ to be the edge labeling $\lambda(T, Z) = a$.*

For $T \prec_{Tam} Z, Q$, we prove that $[T, Z \vee Q]_{Tam}$ has Hasse diagram which is either a diamond or a pentagon, and that the labeling on the two maximal chains satisfies Definition 2.1.8 in either case. In the s-Tamari lattice, there is only one type of pentagonal interval instead of two. Similarly to s-weak order, our first proposition restricts the Tamari tree ascents which can occur in an s-Tamari tree. We use it to characterize when $[T, Z \vee Q]_{Tam}$ has Hasse diagram which is a diamond or which is a pentagon, as well as to describe the atoms in such intervals.

Proposition 6.2.2. *Let T be an s-Tamari tree and let $1 \leq a < b \leq n$ be such that (a, b) is a Tamari tree ascent of T. Then no pair of the form (c, b) such that $c \in T^a$ and $c < a$ is a Tamari tree ascent of T.*

Proof. Since (a, b) is a Tamari tree ascent of T, a is a child of b in T. No other $c \in T^a$ is a child of b in T. $\square$

Just as in the s-weak order case, the next two definitions let us describe $\operatorname{inv}(Z \vee Q)$ when $T \prec_{Tam} Z, Q$. The subsequent proposition explicitly computes the tree inversions added by an s-Tamari rotation along a Tamari tree ascent.

Definition 6.2.3. *Let T be a s-Tamari tree and let $1 \leq a < b \leq n$ be such that (a,b) is a Tamari tree ascent of T. Let Z be the s-Tamari tree obtained by $T \xrightarrow{Tam(a,b)} Z$. Define **the set of inversions added by the s-Tamari rotation along** (a,b), denoted $\boldsymbol{A_T^{Tam}(a,b)}$, by*

$$A_T^{Tam}(a,b) = \left\{(f,e) \;\middle|\; \#_Z(f,e) > \#_T(f,e)\right\}.$$

Definition 6.2.4. *Let T be an s-Tamari tree and let (a,b) and (c,d) be Tamari tree ascents of T with $a < c$. We note that b and d are determined by a and c since they are the parents of a and c, respectively. Define the following set valued function:*

$$F_T^{Tam}(a,c) = \begin{cases} \left\{(d,e) \;\middle|\; e \in T^a \setminus 0\right\} & \text{if } b = c \text{ and } a \in T_0^c \\ \\ \emptyset & \text{otherwise} \end{cases}$$

Proposition 6.2.5. *Let T be an s-Tamari tree and let $1 \leq a < b \leq n$ be such that (a,b) is a Tamari tree ascent of T. Suppose $T \xrightarrow{Tam(a,b)} Z$. Then $(f,e) \in A_T^{Tam}(a,b)$ if and only if $f = b$ and $e \in T^a \setminus 0$ in which case*

$$\#_Z(f,e) = \#_T(f,e) + 1.$$

Proof. This follows from Remark 2.2.29 by keeping track of the only subtrees that change in an s-Tamari rotation. □

Again as in the s-weak order case, we use the following lemma in one of two different characterizations of $Z \vee Q$ for $T \prec_{Tam} Z, Q$.

Lemma 6.2.6. *Let T be an s-Tamari tree. Let $1 \leq a < b \leq n$ and $1 \leq c < d \leq n$ be such that (a,b) and (c,d) are Tamari tree ascents of T with $a < c$. Then $A_T^{Tam}(a,b)$, $A_T^{Tam}(c,d)$, and $F_T^{Tam}(a,c)$ are pairwise disjoint.*

Proof. Assume seeking contradiction that $A_T^{Tam}(a,b) \cap A_T^{Tam}(c,d) \neq \emptyset$. Then by Proposition 6.2.5, $b = d$. Then $a \in T_i^b$ and $c \in T_j^b$ with $i \neq j$ since a and c are distinct children of d. Thus, T^a and T^c are disjoint. However, the intersection being non-empty then contradicts Proposition 6.2.5.

If $F_T^{Tam}(a,c) \neq \emptyset$, then $b = c$ and $a \in T_0^c$ by Definition 6.2.4. Thus, $F_T^{Tam}(a,c)$ is disjoint from $A_T^{Tam}(a,b)$ since $b \neq d$. $F_T^{Tam}(a,c)$ is also disjoint from $A_T^{Tam}(c,d)$ by Proposition 6.2.5 because every $e \in T^a \setminus 0$ is in T_0^c since $a \in T_0^c$. $\square$

In the following lemma, we show the first of two descriptions of $Z \vee Q$ for $T \prec_{Tam} Z, Q$. The second description of $Z \vee Q$ is Lemma 6.2.9 below. Our proof of Lemma 6.2.7 is nearly identical to the proof of Lemma 6.1.11 since the s-Tamari lattice is a sublattice of s-weak order.

Lemma 6.2.7. *Let T be an s-Tamari tree and let $1 \leq a < b \leq n$ and $1 \leq c < d \leq n$ be such that (a,b) and (c,d) are distinct Tamari tree ascents of T. Suppose $T \xrightarrow{Tam(a,b)} Z$ and $T \xrightarrow{Tam(c,d)} Q$, then $inv(Z \vee Q) = ((inv(T) + (b,a))^{tc} + (d,c))^{tc}$.*

Proof. Since the s-Tamari lattice is a sublattice of s-weak order, $Z \vee Q$ is the same s-decreasing tree in the s-Tamari lattice as in s-weak order. Thus, this proof is the same as the proof of Lemma 6.1.11, but with Proposition 6.1.8 and Lemma 6.1.10 replaced by Proposition 6.2.5 and Lemma 6.2.6, respectively. $\square$

In the next lemma, we begin with s-Tamari Tree T with distinct Tamari tree ascents (a,b) and (c,d) with $a < c$. We show (c,d) is always a Tamari tree ascent of the s-Tamari rotation of T along (a,b). We also show that the only way that (a,b)

ceases to be a Tamari tree ascent of the s-Tamari rotation of T along (c, d) is if $b = c$ and a is the 0th child of c in T. In contrast with the four possibilities we say in Lemma 6.1.12 for s-weak order, there are only two possibilities in the s-Tamari lattice. These turn out to characterize the diamond and pentagonal intervals of the s-Tamari lattice.

Lemma 6.2.8. *Let T be a s-Tamari tree. Let $1 \leq a < b \leq n$ and $1 \leq c < d \leq n$ be such that (a, b) and (c, d) are Tamari tree ascents of T with $a < c$. Let $T \overset{Tam(a,b)}{\longrightarrow} Z$ and $T \overset{Tam(c,d)}{\longrightarrow} Q$. If (a, b) is not a Tamari tree ascent of Q, then $b = c$ and a is the 0th child of c. Moreover, (c, d) is a Tamari tree ascent of Z.*

Proof. By Remark 2.2.29, the s-Tamari rotation along (a, b) changes nothing above c in T. Thus, c is still a non-right most child of d in Z so (c, d) is a Tamari tree ascent of Z. Because $a < c$, there are only two ways that (a, b) might not be a Tamari tree ascent of Q. Either (1) $a \in Q^b_{s(b)}$ or (2) a is not a child of b in Q.

For (1), we note that $a \in T^b_j$ for some $j < s(b)$ since (a, b) is a Tamari tree ascent of T. Then by Proposition 6.2.5, $a \in Q^b_{s(b)}$ implies $b = d$ and $a \in T^c$. Then, however, since $a < c$, (a, d) being a Tamari tree ascent of T contradicts Proposition 6.2.2. Thus, (1) cannot occur. For (2), Remark 2.2.29 implies a is a child of b in T, but not a child of b in Q if and only if $b = c$ and a is the 0th child of b in T. This is precisely the conclusion of this lemma. □

Next we give a second description of $Z \vee Q$ for $T \prec_{Tam} Z.Q$, this time in terms of explicit multi-inversion sets instead of using the transitive closure.

Lemma 6.2.9. *Let T be an s-Tamari tree. Let $1 \leq a < b \leq n$ and $1 \leq c < d \leq n$ be such that (a, b) and (c, d) are Tamari tree ascents of T with $a < c$. Suppose*

$T \overset{Tam(a,b)}{\longrightarrow} Z$ *and* $T \overset{Tam(c,d)}{\longrightarrow} Q$. *Then* $inv\,(Z \vee Q) - inv\,(T) = A_T^{Tam}(a,b) \cup A_T^{Tam}(c,d) \cup F_T^{Tam}(a,c)$.

Proof. If (a,b) is a Tamari tree ascent of Q, then a similar argument to that in the proof of Lemma 6.1.14, but with the corresponding lemmas for s-Tamari trees shows that the result holds.

If (a,b) is not a Tamari tree ascent of Q, then $b = c$ and a is the $0th$ child of c by Lemma 6.2.8. A similar argument to that in the proof of Lemma 6.1.14 using transitivity shows that $A_T^{Tam}(a,b) \cup A_T^{Tam}(c,d) \cup F_T^{Tam}(a,c) \subseteq \mathrm{inv}\,(Z \vee Q) - \mathrm{inv}\,(T)$. Thus, it suffices two show there is an s-Tamari tree P' with $\mathrm{inv}\,(P') - \mathrm{inv}\,(T) = A_T^{Tam}(a,b) \cup A_T^{Tam}(c,d) \cup F_T^{Tam}(a,c)$. We claim there is a saturated chain

$$T \overset{Tam(c,d)}{\longrightarrow} Q \overset{Tam(a,d)}{\longrightarrow} P \overset{Tam(a,c)}{\longrightarrow} P'$$

with P' such an s-Tamari tree.

Since a is the $0th$ child of c, a is the $0th$ child of d in Q by Remark 2.2.29. Thus, (a,d) is a tree ascent of Q. Then, again by Remark 2.2.29, a is the $0th$ child of c in P. Hence, (a,c) is a Tamari tree ascent of P. Thus, we have the claimed saturated chain. Now we apply Proposition 6.2.5 at each step of the chain which gives $\mathrm{inv}\,(P') - \mathrm{inv}\,(T) = A_T^{Tam}(c,d) \cup A_Q^{Tam}(a,d) \cup A_P^{Tam}(a,c)$. Now by Remark 2.2.29 we have $A_Q^{Tam}(a,d) = F_T(a,c)$ and $A_P^{Tam}(a,c) = A_T^{Tam}(a,c)$. Thus, $\mathrm{inv}\,(P') - \mathrm{inv}\,(T) = A_T^{Tam}(c,d) \cup A_T^{Tam}(a,c) \cup F_T^{Tam}(a,c)$ and these sets are pairwise disjoint by Lemma 6.2.6. □

In the next lemma, we show that the only atoms in $[T, Z \vee Q]_{Tam}$ with $T \prec_{Tam} Z, Q$ are Z and Q using Lemma 6.2.9.

Lemma 6.2.10. *Let T be an s-decreasing tree. Let $1 \le a < b \le n$ and $1 \le c < d \le n$ be such that (a,b) and (c,d) are Tamari tree ascents of T with $a < c$. Suppose $T \overset{Tam(a,b)}{\longrightarrow} Z$ and $T \overset{Tam(c,d)}{\longrightarrow} Q$, then Z and Q are the only atoms in $[T, Z \vee Q]_{Tam}$.*

Proof. Assume $T' \in [T, Z \vee Q]_{Tam}$ and $T \prec_{Tam} T'$ with $T' \neq Z, Q$. Let (e,f) be the Tamari tree ascent of T corresponding to T'. By Lemma 6.2.9, $(f,e) \in A_T^{Tam}(a,b) \cup A_T^{Tam}(c,d) \cup F_T^{Tam}(a,c)$. $(e,f) \neq (a,b),(c,d)$ since $T' \neq Z, Q$. Any other pair $(f,e) \in A_T^{Tam}(a,b) \cup A_T^{Tam}(c,d) \cup F_T^{Tam}(a,c)$ being a Tamari tree ascent of T contradicts Proposition 6.2.2 because either $f = b$ or $f = d$ and e is below a or c in T and so cannot be a child of f. □

In the subsequent two lemmas, we show the s-Tamari lattice intervals of the form $[T, Z \vee Q]_{Tam}$ where $T \prec_{Tam} Z, Q$ have Hasse diagrams that are either diamonds or pentagons and that the labeling of Definition 6.2.1 satisfies the definition of SB-labeling.

Lemma 6.2.11. *Let $T \prec_{Tam} Z, Q$ be cover relations in the s-Tamari lattice corresponding to $T \overset{Tam(a,b)}{\longrightarrow} Z$ and $T \overset{Tam(c,d)}{\longrightarrow} Q$ for distinct Tamari tree ascents of (a,b) and (c,d) of T. Suppose (a,b) is a Tamari tree ascent of Q. Then $[T, Z \vee Q]_{Tam}$ has Hasse diagram which is a diamond and the edge labeling of Definition 6.2.1 on its two maximal chains satisfies Definition 2.1.8.*

Proof. Similarly to the corresponding proof in s-weak order, we use Lemma 6.2.7 to show $Z \overset{Tam(c,d)}{\longrightarrow} Z \vee Q$ and $Q \overset{Tam(a,b)}{\longrightarrow} Z \vee Q$. Hence, $Z, Q \prec_{Tam} Z \vee Q$ Thus, $T \prec_{Tam} Z \prec_{Tam} Z \vee Q$ and $T \prec_{Tam} Q \prec_{Tam} R \vee Q$ are two distinct saturated chains from T to $Z \vee Q$. To show there is not a third such saturated chain it suffices to show there is not a third atom in the interval $[T, Z \vee Q]_{Tam}$, but this is Lemma 6.2.10. Hence, the above chains are the only two saturated chains from T to $Z \vee Q$.

Now we only need observe that the label sequences of the saturated chains $T \prec_{Tam} Z \prec_{Tam} Z \vee Q$ and $T \prec_{Tam} Q \prec_{Tam} Z \vee Q$ are a, c and c, a, respectively. Therefore, Definition 2.1.8 is satisfied. $\square$

Lemma 6.2.12. *Let $T \prec_{Tam} Z, Q$ be cover relations in the s-Tamari lattice corresponding to $T \xrightarrow{Tam(a,b)} Z$ and $T \xrightarrow{Tam(c,d)} Q$ for Tamari tree ascents (a, b) and (c, d) of T with $a < c$. Suppose (a, b) is not a Tamari tree ascent of Q. Then $[T, Z \vee Q]_{Tam}$ has Hasse diagram which is a pentagon and the edge labeling of Definition 6.2.1 on its two maximal chains satisfies Definition 2.1.8.*

Proof. By Lemma 6.2.8, $b = c$ and a is the 0th child of c. Again by Lemma 6.2.7, we have the saturated chain $T \prec_{Tam} Z \prec_{Tam} Z \vee Q$ given by the s-Tamari rotations $T \xrightarrow{Tam(a,b)} Z$ and $Z \xrightarrow{Tam(c,d)} Z \vee Q$. By the proof of Lemma 6.2.9, we have a saturated chain

$$T \xrightarrow{Tam(c,d)} Q \xrightarrow{Tam(a,d)} P \xrightarrow{Tam(a,c)} Z \vee Q.$$

We note that by Proposition 6.2.5 $Z \preceq_{Tam} P$. Thus, to show the Hasse diagram of $[T, Z \vee Q]_{Tam}$ is a pentagon, it suffices to show there are no other elements in the interval besides $T, Z, Q, P, Z \vee Q$. To show there are no other elements in the interval, it suffices to show there are no other atoms in $[T, Z \vee Q]_{Tam}$ besides Z and Q and that there are no other atoms in $[Q, Z \vee Q]_{Tam}$ besides P. The fact that there are no atoms of $[T, Z \vee Q]_{Tam}$ besides Z and Q is Lemma 6.2.10. Similarly to the proof of Lemma 6.1.19 for s-weak order, Lemma 6.2.9 implies the existence of an atom in $[Q, Z \vee Q]_{Tam}$ besides P would contradict Proposition 6.2.2. Hence, the Hasse diagram of the interval is a pentagon whose only maximal chains are the two already shown.

The label sequences for the maximal chains $T \prec_{Tam} Z \prec_{Tam} R \vee Q$ and $T \prec_{Tam} Q \prec_{Tam} P \prec_{Tam} Z \vee Q$ are a, c and c, a, a, respectively. These label sequences satisfy Definition 2.1.8. $\square$

The previous two lemmas together prove the labeling of Definition 6.2.1 is an SB-labeling.

Theorem 6.2.13. *Let $T \prec_{Tam} Z$ be a cover relation in the s-Tamari lattice. Let $T \xrightarrow{Tam(a,b)} Z$ be the s-Tamari rotation of T along the Tamari tree ascent (a, b) of T associated to $T \prec_{Tam} Z$ by Theorem 2.2.28. Let λ be the edge labeling $\lambda(T, Z) = a$. Then λ is an SB-labeling of the s-Tamari lattice.*

Proof. Condition (i) of Definition 2.1.8 is satisfied by Remark 2.2.19. Lemma 6.2.8, Lemma 6.2.11, and Lemma 6.2.12 together imply conditions (ii) and (iii) of Definition 2.1.8 are satisfied proving the theorem. $\square$

Theorem 6.2.13 and Theorem 2.1.9 prove a characterization of the homotopy type of open intervals in the s-Tamari lattice and so also characterize its Möbius function.

Corollary 6.2.14. *Let $T \preceq_{Tam} Z$ in the s-Tamari lattice. Then $\Delta(T, Z)_{Tam}$, the order complex of the open interval $(T, Z)_{Tam}$, is homotopy equivalent to a ball or a sphere of some dimension. Moreover, the Möbius function of the s-Tamari lattice satisfies $\mu_{Tam}(T, Z) \in \{-1, 0, 1\}$.*

Furthermore, we give the analogous intrinsic description of open s-Tamari intervals whose order complexes are homotopy spheres as for s-weak order.

Lemma 6.2.15. *If $T \prec_{Tam} Z$, then Z is the join of the atoms in $[T, Z]_{Tam}$ if and only if*

$$inv\,(Z) = \left(inv\,(T) + A_T^{Tam}(a_1, b_1) + \cdots + A_T^{Tam}(a_l, b_l)\right)^{tc}$$

where $(a_1, b_1), \ldots, (a_l, b_l)$ *are the Tamari tree ascents of* T *such that* $(b_i, a_i) \in inv\,(Z) - inv\,(T)$*. Moreover, the number of atoms in the interval* $[T, Z]_{Tam}$ *is* l *regardless of whether or not* Z *is the join of atoms in the interval.*

Proof. The number of atoms follows from Theorem 2.2.28, the characterization of cover relations in the s-Tamari lattice. The rest of the statement follows from the same argument as in the proof of Lemma 6.1.22 with the lemmas about s-weak order replaced by the corresponding lemmas for the s-Tamari lattice because the s-Tamari lattice is a sublattice of s-weak order. □

Theorem 6.2.16. *If* $T \prec Z$*, then* $\Delta(T, Z)_{Tam}$ *is homotopy equivalent to a sphere if and only if*

$$inv\,(Z) = \big(inv\,(T) + A_T^{Tam}(a_1, b_1) + \cdots + A_T^{Tam}(a_l, b_l)\big)^{tc}$$

where $(a_1, b_1), \ldots, (a_l, b_l)$ *are the Tamari tree ascents of* T *such that* $(b_i, a_i) \in inv\,(Z) - inv\,(T)$*. Moreover, in this case the dimension of the sphere is* $l - 2$*.*

Proof. This follows from combining Lemma 6.2.15 and Theorem 2.1.9. □

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