METASTABLE COMPLEX VECTOR BUNDLES OVER COMPLEX PROJECTIVE SPACES

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DISSERTATION ABSTRACT

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In the unstable range, topological vector bundles over finite CW complexes are difficult to classify in general. Over complex projective spaces $\mathbb{C}P^n$, such bundles are far from being fully classified, or even enumerated, except for a few small dimensional cases studied in the 1970's [7, 15, 16, 17] using classical tools from homotopy theory, and more recently [14] using the modern tool of chromatic homotopy theory. We apply another modern tool, Weiss calculus, to enumerate topological complex vector bundles over $\mathbb{C}P^n$ with trivial Chern class data, in the first two cases of the metastable range.

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CHAPTER I

INTRODUCTION

As families of vector spaces parametrized by points in a topological space, vector bundles are basic structures studied across topology and geometry. Over reasonable spaces, like manifolds or finite CW complexes, the classification of vector bundles can be phrased in terms of a basic homotopy calculation. That is, vector bundles are representable in the homotopy category. For example, topological complex vector bundles of a fixed rank r over a finite complex X, whose isomorphism classes we denote by $\mathrm{Vect}_r(X)$, are identified with $[X, BU(r)]$ – homotopy classes of maps from X to the classifying space $BU(r)$, for which our standard model is the Grassmannian $\mathrm{Gr}_r(\mathbb{C}^\infty)$ of r-planes in $\mathbb{C}^\infty$. However, calculating homotopy classes of maps between two spaces is notoriously difficult, even when one or both of the spaces is a sphere. It is therefore both surprising and not-so-surprising that even topological bundles over complex projective spaces $\mathbb{C}P^n$ are far from being fully classified, or even enumerated, except for a few low dimensional cases studied in the 1970s [7, 15, 16, 17] and more recently [14].

We applied a modern tool, Weiss calculus [20], to obtain some enumeration results for bundles over $\mathbb{C}P^n$. Here and in what follows, a *bundle* shall stand for a topological complex vector bundle (or an isomorphism class of such, depending on the context) unless otherwise stated. To state our main results, first note that vector bundles can be measured through cohomology invariants, of which first examples include Chern classes, which are remarkably effective but not complete invariants. Since classifying spaces $BU(r)$ are rationally formal, the set of bundles with fixed Chern data is finite. The following notations are fixed throughout:

$\mathrm{Vect}_r(X) \cong [X, BU(r)]$ denotes the pointed set of isomorphism classes of rank r bundles over a finite dimensional CW complex X, and $\mathrm{Vect}^0_r(X)$ denotes the subset of rank r bundles over X whose Chern classes all vanish.

1.1. Main Results

We enumerate certain unstable topological complex vector bundles over complex projective spaces.

Theorem 1.1 (The first unstable case)**.** *Let $l > 2$ be an integer, and let $\psi(l)$ denote the cardinality of $Vect^0_{l-1}(\mathbb{C}P^l)$. Then $\psi(l) = 2$ if l is odd, and $\psi(l) = 1$ if l is even.*

Theorem 1.2 (The second unstable case)**.** *Let $l > 3$ be an integer, and let $\phi(l)$ denote the cardinality of $Vect^0_{l-2}(\mathbb{C}P^l)$. The numbers $\phi(l)$ exhibit the following 24-fold periodic behavior.*

l mod 24	0	1	2	3	4	5	6	7	8	9	10	11
$\psi(l)$	1	1	12	2	1	3	2	4	3	1	4	6

l mod 24	12	13	14	15	16	17	18	19	20	21	22	23
$\phi(l)$	1	1	6	4	1	3	4	2	3	1	2	12

TABLE 1. The enumeration of rank $l-2$ bundles over $\mathbb{C}P^l$, whose Chern classes all vanish.

Broadly speaking, there are two steps to proving these results. First, in Theorem 2.1 we use Weiss calculus to identify stably trivial vector bundles over some d-dimensional complex X with $\{X, \Sigma\mathbb{C}P^\infty_r\}$, when a bundle has rank r with $\frac{d}{4} \leq r \leq \frac{d-1}{2}$, which we call the *metastable range*. Here $\{X, Y\}$ as usual denotes stable homotopy classes of maps, which is the direct limit $\varinjlim[\Sigma^n X, \Sigma^n Y]$, and $\mathbb{C}P^\infty_r$ is the stunted projective space $\mathbb{C}P^\infty/\mathbb{C}P^{r-1}$.

Weiss calculus is a framework which applies to some spaces given by evaluation of functors on the category of vector spaces, resolving them by a tower of fibrations with infinite loop spaces as fibers. Resolution by infinite loop spaces, which are essentially abelian group objects, is a time-honored technique in homotopy theory, with the Postnikov tower and unstable Adams resolutions being standard examples. The machinery of Weiss calculus gives a custom-made resolution of $BU(r)$ for any r, which in the metastable range translates to the stable mapping set above.

Stable mapping sets are amenable to standard tools such as Adams and Atiyah-Hirzebruch spectral sequences, and the second part of our analysis is to employ such tools to make this calculation when X is a complex projective space, with needed incorporation of delicate calculations by Mosher [13], Toda [19], and Matsunaga [11, 12]. Since stable mapping sets are abelian groups, the identification of Theorem 2.1 equips $\mathrm{Vect}_r^0(X)$ with an abelian group structure in the metastable range. We calculate these groups, which are cyclic of the orders given. We prefer to present our main results in terms of cardinality, since we don't have an intrinsic description of a group structure on $\mathrm{Vect}_r^0(X)$.

We also calculate action of vector bundles on spheres in these sets. This action is defined through the collapse map $\mathbb{C}P^l \to \mathbb{C}P^l \vee S^{2l}$, or equivalently by taking as vector bundle on $\mathbb{C}P^l$, finding an isomorphic representative which is trivial on some Euclidean neighborhood, and replacing that trivial bundle on the neighborhood by the corresponding bundle on the sphere.

Proposition 1.3. *The action of $Vect^0_{l-1}(S^{2l})$ on $Vect^0_{l-1}(\mathbb{C}P^l)$ is transitive and free when l is odd, trivial when l is even. The action of $Vect^0_{l-2}(S^{2l})$ on $Vect^0_{l-2}(\mathbb{C}P^l)$ is trivial when $l = 0, 1, 4, 9, 12, 13, 16, 21$ mod 24; transitive and free when $l = 3, 5, 7, 8, 11, 15, 17, 19, 20, 23$ mod 24; transitive but not free in all other cases.*

We also calculate the restriction map between the sets of vector bundles in question.

Proposition 1.4. *The map* $\mathit{Vect}^0_{l-2}(\mathbb{C}P^l) \to \mathit{Vect}^0_{l-2}(\mathbb{C}P^{l-1})$ *induced by the standard inclusion* $\mathbb{C}P^{l-1} \to \mathbb{C}P^l$ *is trivial, for all* $l > 3$*.*

In addition to the question of natural group structure on these sets, open questions invited by our work include finding representatives for these isomorphism classes of bundles, in particular finding whether these have holomorphic representatives, as well as finding invariants distinguishing these bundles. For some of these questions, it might be helpful to have a more direct comparison between $\mathrm{Vect}^0_r(X)$ and stable maps to stunted projective spaces.

For perspective on our techniques, recall that rank r bundles over projective spaces with trivial Chern classes are measured by $[\mathbb{C}P^l, U/U(r)]$, since vanishing of Chern classes in this case implies stable triviality, and the homogeneous space $U/U(r)$ is the homotopy fiber of the standard map $BU(r) \to BU$. So whenever having a rank r bundle over $\mathbb{C}P^l$ that is stably trivial, one obtains, on the face of it, a map from $\mathbb{C}P^l$ to the finite dimensional complex Stiefel manifold $U(l)/U(r)$. However, these classifying maps cannot be added in the naive way that directly resembles bundle addition: although the direct sum of two rank r bundle over $\mathbb{C}P^l$ that are stably trivial is again stably trivial, the geometric dimension of the direct sum can often be strictly greater than r. So the calculation of maps from $\mathbb{C}P^l$ to $U(l)/U(r)$ is unstable in nature, and for example the unstable Adams spectral sequences computing them are not accessible. Using Weiss calculus enables us to replace the Stiefel manifolds $U(l)/U(r)$ by the infinite loop spaces $Q\Sigma\mathbb{C}P^\infty_r$ (where $Q(-) = \Omega^\infty\Sigma^\infty(-)$), so that the loop sum provides an implicit way of adding stably trivial rank r bundles. One is therefore able to replace the calculation of unstable mapping sets with stable ones, and hence to extract the desired unstable information from stable calculations.

1.2. Background and History

We first fix notations. The cohomology ring of $\mathbb{C}P^l$ is isomorphic to the truncated polynomial algebra $\mathbb{Z}[x]/(x^{l+1})$, where x in degree two is identified with the first Chern class of the dual of the tautological line bundle. The Chern classes of any bundle ξ over $\mathbb{C}P^l$ are integer multiples of the powers of x, and will therefore be treated sometimes as integers.

Classification of topological complex vector bundles is typically organized around K-theory and Chern classes, which in special cases give complete information. When $2r \geq \dim X$, bundles are stable – that is, isomorphism is equivalent to stable isomorphism – and can hence be studied through K-theory. When $r = 1$, line bundles are determined by the first Chern class c_1. However, when $1 < r < \frac{1}{2}\dim X$, rank r bundles over X are much harder to compute and detect. For example, the calculation

$$\mathrm{Vect}_2(S^6) \cong [S^6, BU(2)] \cong [S^5, SU(2)] \cong \pi_5(S^3) \cong \mathbb{Z}/2$$

implies that there is a nontrivial rank 2 bundle over the 6-sphere, but such a bundle can only have vanishing Chern classes.

Since the classifying spaces $BU(r)$ are rationally formal, there will only be finitely many rank r bundles with a given set of Chern classes. We call the counting of this set *Chern enumeration.* One key special case of Chern enumeration is finding the cardinality of $\mathrm{Vect}^0_r(X)$, which we call *vanishing Chern enumeration.*

Our main results, Theorems 1.1 and 1.2, resolve the vanishing Chern enumeration question for complex projective spaces, in some of the cases where the first layer of the Weiss tower of $BU(-)$ determines the entire story – see Theorem 2.1.

Another key case of Chern enumeration is determining when the number of vector bundles with fixed Chern classes is non-zero, which we call the *Chern realization question*. Such results are given by arithmetic conditions on the Chern classes. For example, Alan Thomas [18] proves that integer pairs (c_1, c_2) can serve as the Chern classes for some stable bundle over $\mathbb{C}P^l$ with $c_i = 0$ for $i \geq 3$ precisely when the Schwarzenberger condition [10] is satisfied.

Starting with Chern realization results and then applying K-theory techniques, Atiyah and Rees [7] obtain the following Chern enumeration for rank two bundles over $\mathbb{C}P^l$ for $l = 3$ and $l = 4$.

Theorem 1.5 (Atiyah-Rees '76 [7]). *Let $\xi \downarrow \mathbb{C}P^3$ be a stable bundle with $c_i(\xi) = 0$, $i \geq 3$.*

1. *If $c_1(\xi)$ is even, then ξ has exactly two rank 2 representatives.*
2. *If $c_1(\xi)$ is odd, then ξ has a unique rank 2 representative.*

Theorem 1.6 (Atiyah-Rees '76 [7]). *Every stable bundle $\xi \downarrow \mathbb{C}P^4$ with $c_i(\xi) = 0$, $i \geq 3$, contains a unique rank 2 representative.*

As special cases of these theorems, one obtains vanishing Chern enumeration results. We will give alternate proofs of these in Section 3, as illustrative first cases of our more general results.

Corollary 1.7. *Over $\mathbb{C}P^3$ there is a unique nontrivial rank 2 bundle with vanishing Chern classes.*

Corollary 1.8. *Over $\mathbb{C}P^4$ there is no nontrivial rank 2 bundle with vanishing Chern classes.*

It is later proved, both by Rees [15] and by Smith [16], that for every $l \geq 5$ there exists some nontrivial rank 2 bundle over $\mathbb{C}P^l$ with vainishing Chern classes, so that vanishing Chern enumeration is nontrivial. Using obstruction-theoretic techniques, Switzer [17] gives alternative proofs of the Atiyah-Rees theorems and goes further to resolve Chern enumeration for rank two bundles over $\mathbb{C}P^5$ and $\mathbb{C}P^6$.

Recently, progress in this subject has been made by Opie [14], who uses the theory of topological modular forms – a modern tool from chromatic homotopy theory, to obtain a full classification of rank 3 bundles over $\mathbb{C}P^5$. In particular, the Chern enumeration for such bundles is completely determined, and invariants have been constructed to distinguish bundles having the same Chern classes. However, Chern enumeration for complex projective spaces remains a mystery in general.

1.3. Outline

In Chapter 2 we give a brief account of Weiss calculus. The classifying space $BU(r)$ is the value of the functor $BU(-) : V \mapsto BU(V)$ at the standard vector space $\mathbb{C}^r$. To detect rank r bundles over $\mathbb{C}P^l$ we map $\mathbb{C}P^l$ to the Weiss tower of $BU(-)$ evaluated at $V = \mathbb{C}^r$. The Weiss tower of $BU(-)$ is studied in detail by Arone [3]. We shall make use of the description of the first layer and a connectivity estimate of the higher layers to prove our identification

$$\mathrm{Vect}_r^0(\mathbb{C}P^l) \cong \{\mathbb{C}P^l, \Sigma\mathbb{C}P_r^\infty\}$$

in the metastable range.

For those less familiar with stable homotopy techniques, in Chapter 3 we give illustrative examples of our main results, presenting new proofs of Corollaries 1.7 and 1.8, which enumerate $\mathrm{Vect}_2^0(\mathbb{C}P^3)$ and $\mathrm{Vect}_2^0(\mathbb{C}P^4)$. After our stable map identification theorem, these results are proved by showing that $\{\mathbb{C}P^3, \Sigma\mathbb{C}P_2^\infty\} \cong \mathbb{Z}/2$ and $\{\mathbb{C}P^4, \Sigma\mathbb{C}P_2^\infty\} = 0$. To do so we apply the 2-primary Adams spectral sequence to compute some first stable homotopy groups of $\Sigma\mathbb{C}P_2^\infty$, and consider these as coefficients of the cohomology theory represented by the infinite loop space $Q\Sigma\mathbb{C}P_2^\infty$. The computation is then finished with an Atiyah-Hirzerbruch spectral sequence argument.

Chapter 4 is dedicated to the proof of our main results, Theorems 1.1 and 1.2, following the strategy of Section 3. Rank $(l-1)$ and $(l-2)$ bundles over $\mathbb{C}P^l$ are in the metastable range, and the associated Weiss tower yields identifications

$$\mathrm{Vect}_{l-1}^0(\mathbb{C}P^l) \cong \{\mathbb{C}P^l, \Sigma\mathbb{C}P_{l-1}^\infty\} \quad \text{and} \quad \mathrm{Vect}_{l-2}^0(\mathbb{C}P^l) \cong \{\mathbb{C}P^l, \Sigma\mathbb{C}P_{l-2}^\infty\}.$$

In the first case, Theorem 1.1, we apply the 2-primary Adams spectral sequence to compute some first few stable homotopy groups of $\Sigma\mathbb{C}P_{l-1}^\infty$ (with details presented in Appendix A), and then organize the computations of $\{\mathbb{C}P^l, \Sigma\mathbb{C}P_{l-1}^\infty\}$ with the Atiyah-Hirzebruch spectral sequence. The proof of Theorem 1.2 has similar ingredients, but with some added complexity for two reasons. First, both the prime 2 and the prime 3 are involved. Secondly, more detailed study of 3-cell stunted projective spaces is required in order to determine a crucial d_4-differential in the Atiyah-Hirzebruch spectral sequence, where we make use of results of Mosher [13]. Finally, after establishing the main calculations, we prove Propositions 1.3 and 1.4.

CHAPTER II

WEISS CALCULUS

We apply Weiss calculus to equate the vanishing Chern enumeration problem over complex projective spaces with the calculation of a stable homotopy mapping set, in the metastable range. The following simplification of the Weiss tower in the case at hand is key to our metastable computations.

Theorem 2.1. *Let $l > 2$ be an integer. Then the map*

$$Map_*\big(\mathbb{C}P^l, BU(V)\big) \longrightarrow Map_*\big(\mathbb{C}P^l, T_1BU(V)\big)$$

induces a bijection on π_0 after evaluating at $V = \mathbb{C}^r$, provided that $\frac{l}{2} \leq r \leq l-1$. In this case, $\pi_0 Map_*\big(\mathbb{C}P^l, BU(V)\big) = [\mathbb{C}P^l, BU(r)]$ fits into the exact sequence

$$0 \longrightarrow \{\mathbb{C}P^l, \Sigma\mathbb{C}P_r^\infty\} \longrightarrow [\mathbb{C}P^l, BU(r)] \longrightarrow [\mathbb{C}P^l, BU],$$

which we call the metastable exact sequence. It then follows that

$$Vect_r^0(\mathbb{C}P^l) \cong \{\mathbb{C}P^l, \Sigma\mathbb{C}P_r^\infty\}.$$

To prove Theorem 2.1, we begin with a brief account of Weiss calculus.

2.1. Fundamentals of Weiss Calculus

Weiss initiated the study of his calculus, inspired by Goodwillie calculus, in [20]. There he focuses on orthogonal calculus, but we apply unitary calculus here. Let $\mathcal{J}$ be the category whose objects are finite dimensional complex vector spaces with positive definite inner product (all in a fixed universe $\mathbb{C}^\infty$), and whose morphisms are linear isometric inclusions. We consider $\mathcal{J}$ as a topological category since its morphism sets are Stiefel manifolds. Weiss calculus studies continuous functors from $\mathcal{J}$ to pointed spaces.

For each $n \geq 0$ there is a distinguished class of *n-polynomial* functors, and the idea of calculus is to approximate a general functor by these polynomial ones – analogous to the philosophy of classical calculus. Each continuous functor $F : \mathcal{J} \to \mathrm{Top}_*$ is equipped with a tower of fibrations

$$\cdots \longrightarrow T_n F \longrightarrow T_{n-1} F \longrightarrow \cdots \longrightarrow T_1 F \longrightarrow T_0 F$$

where $T_n F$ is n-polynomial, called the Weiss tower of F. For each n there is a comparison map $F \to T_n F$ compatible with the tower, regarded as the universal approximation of F by an n-polynomial functor. We say the tower converges to F if $F \to \mathrm{holim}_n T_n F$ is a weak equivalence. With appropriate connectivity conditions, some towers terminate after finitely many steps.

The homotopy fiber $L_n F := \mathrm{hofib}\,(T_n F \to T_{n-1} F)$ is an *n-homogeneous* functor, which we call the *n-th Weiss layer* of F. A fundamental theorem of Weiss calculus is that n-homogeneous functors are classified by $U(n)$-spectra, and in particular $L_n F$ is of the form

$$V \longmapsto \Omega^\infty (\Theta \wedge S^{nV})_{hU(n)}$$

where Θ is some $U(n)$-spectrum, $nV = \mathbb{C}^n \otimes_{\mathbb{C}} V$ with $U(n)$ acting on the left, and S^{nV} denotes the one point compactification. The classifying spectrum Θ is called the *n-th Weiss derivative* of F.

The Weiss tower, like the Goodwillie tower, can be viewed as a tool which resolves unstable structures by stable ones. While the values of these functors can be viewed as unstable homotopy types, the layers are infinite loop spaces by the above classification theorem, and thus stable. Moreover, the bottom T_0F of the tower is by definition $(T_0F)(V) = \mathrm{hocolim}_k F(V \oplus \mathbb{C}^k)$, which is manifestly a stabilization. For $F(V) := BU(V)$ the bottom layer T_0F is the constant functor $V \mapsto BU$, the classifying space for stable bundles.

2.2. Identification of Derivatives

We build on the seminal work of Arone [3] on the derivatives of the functor $F(V) := BU(V)$, whose n-th derivative is denoted by Θ_n, $n \geq 1$. We denote by $\mathbb{L}_n$ the unreduced suspension of the realization of the category of non-trivial direct-sum decompositions of $\mathbb{C}^n$, and by Ad_n the adjoint representation of $U(n)$. The following result provides a closed-form description of Θ_n.

Theorem 2.2 (Arone '02 [3]). *For every $n \geq 1$, Θ_n is equivalent to $Map_*(\mathbb{L}_n, \Sigma^\infty S^{Ad_n})$.*

It follows immediately that the n-th layer of F is of the form

$$(L_nF)(V) \simeq \Omega^\infty \mathrm{Map}_*(\mathbb{L}_n, \Sigma^\infty S^{\mathrm{Ad}_n} \wedge S^{nV})_{hU(n)}.$$

For a general n, the n-th derivative $\mathrm{Map}_*(\mathbb{L}_n, \Sigma^\infty S^{\mathrm{Ad}_n})$ need not have a homotopy type we can readily describe. However, the first derivative, and hence the first layer, can be made explicit in a way which is of fundamental importance to this paper.

Proposition 2.3. *The first Weiss layer $(L_1F)(V)$ of F is equivalent to $Q(\Sigma S^V)_{hU(1)}$, which in turn is equivalent to $Q\Sigma\mathbb{C}P_r^\infty$ when $V = \mathbb{C}^r$.*

Proof. By Theorem 2.2, $(L_1F)(V)$ is equivalent to $\Omega^\infty \mathrm{Map}_*(\mathbb{L}_1, \Sigma^\infty S^{\mathrm{Ad}_1} \wedge S^V)_{hU(1)}$. Since the space $\mathbb{L}_1$ is just S^0, and S^{Ad_1} is S^1 with trivial $U(1)$-action, $(L_1F)(V)$ is identified with $Q(\Sigma S^V)_{hU(1)}$. To establish the second equivalence, we first observe that when $V = \mathbb{C}^r$ the space $S^V_{hU(1)} = S^{2r}_{hU(1)}$ is the Thom space of the vector bundle $\gamma^{\oplus r}$ over $BU(1) = \mathbb{C}P^\infty$. Indeed, the action of $U(1)$ on $S^{2r} = S^V|_{V=\mathbb{C}^r}$ restricts to scalar multiplication of $U(1)$ on $\mathbb{C}^r$. Therefore the unreduced homotopy orbit

$$uS^{2r}_{hU(1)} := EU(1) \times_{U(1)} S^{2r} = \big(EU(1) \times_{U(1)} S^V\big)|_{V=\mathbb{C}^r}$$

is the fiberwise one-point compactification of the vector bundle $\gamma^{\oplus r}$ over $\mathbb{C}P^\infty$. The reduced homotopy orbit $S^{2r}_{hU(1)}$ is obtained from $uS^{2r}_{hU(1)}$ by collapsing the section of $\mathbb{C}P^\infty$ at infinity, and is therefore the desired Thom space. On the other hand, we recall (say from Proposition 4.3 of [6]) that $\mathbb{C}P^{r+N}/\mathbb{C}P^{r-1}$ is the Thom space of $\gamma_{1,N}^{\oplus r}$, where $\gamma_{1,N}$ denotes the canonical line bundle over $\mathbb{C}P^N$. Letting $N \to \infty$ identifies the Thom space of $\gamma^{\oplus r}$ over $\mathbb{C}P^\infty$ with $\mathbb{C}P_r^\infty = \mathbb{C}P^\infty/\mathbb{C}P^{r-1}$. We conclude that $S^{2r}_{hU(1)} \simeq \mathbb{C}P_r^\infty$, and hence that

$$(L_1F)(\mathbb{C}^r) \simeq Q\Sigma S^{2r}_{hU(1)} \simeq Q\Sigma\mathbb{C}P_r^\infty.$$

□

For each integer $l > 2$, we define a functor F^l so that

$$F^l(V) = \mathrm{Map}_*\big(\mathbb{C}P^l, BU(V)\big).$$

Then $[\mathbb{C}P^l, BU(r)] = \pi_0 F^l(\mathbb{C}^r)$, and the Weiss tower of F^l can be obtained as follows.

Proposition 2.4. *The following are equivalences:*

$$(T_nF^l)(V) \simeq Map_*\big(\mathbb{C}P^l, (T_nF)(V)\big), \textit{ and } (L_nF^l)(V) \simeq Map_*\big(\mathbb{C}P^l, (L_nF)(V)\big).$$

Proof. By the construction of Weiss calculus (see Section 5 of [20]), T_nF is a direct homotopy colimit of some homotopy limits. For every finite complex X the second-variable mapping functor $\mathrm{Map}_*(X, -)$ preserves arbitrary homotopy limits and filtered homotopy colimits. $\square$

Thus the Weiss tower of F^l can be presented by the following diagram.

$$\begin{tikzcd}
 & \vdots \arrow[d] & \\
 & \mathrm{Map}_*\big(\mathbb{C}P^l, (T_2F)(V)\big) \arrow[d] & \mathrm{Map}_*\big(\mathbb{C}P^l, (L_2F)(V)\big) \arrow[l] \\
 & \mathrm{Map}_*\big(\mathbb{C}P^l, (T_1F)(V)\big) \arrow[d] & \mathrm{Map}_*\big(\mathbb{C}P^l, (L_1F)(V)\big) \arrow[l] \\
\mathrm{Map}_*\big(\mathbb{C}P^l, BU(V)\big) \arrow[r] \arrow[ur, bend left] \arrow[uur, bend left] & \mathrm{Map}_*(\mathbb{C}P^l, BU) &
\end{tikzcd}$$

Convergence still holds because the original tower becomes more highly connected in each layers (see Section 2.3). The base $\mathrm{Map}_*(\mathbb{C}P^l, BU)$ of the Weiss tower is generally not a connected space. However, in this paper we work over the base point component, namely the component of the stable trivial bundle.

2.3. Cohomology Descriptions of Layers

An insight of Arone [3] is that the spectra $\Theta_n \simeq \mathrm{Map}_*(\mathbb{L}_n, \Sigma^\infty S^{\mathrm{Ad}_n})$, appearing here as Weiss derivatives of the functor $F(-) = BU(-)$, are closely related to the Goodwillie derivatives of the identity functor. Building on prior work including that of Arone [2], Arone-Dwyer [4] and Arone-Mahoword [5], the following cohomology description of Θ_n, and hence that of the layers L_nF, is established in [3]. In what follows, $\mathcal{A}_{k-1}$ denotes the subalgebra of the mod p Steenrod algebra $\mathcal{A}$, generated by elements $\mathrm{Sq}^1, \mathrm{Sq}^2, \cdots, \mathrm{Sq}^{2^{k-1}}$ if $p = 2$ and by elements $\beta, \mathcal{P}^1, \mathcal{P}^p, \cdots, \mathcal{P}^{p^{k-2}}$ if p is odd.

Theorem 2.5 (Arone '02 [3])**.** *The spectrum Θ_n is rationally contractible for $n > 1$. Integrally, it is contractible unless n is a prime power. If $n = p^k > 1$ then the homology of Θ_{p^k} is all p-torsion, and the mod p cohomology of Θ_{p^k} is free over $\mathcal{A}_{k-1}$.*

The rational contractibility alternatively follows from the fact that the spaces $BU(n)$ are rationally formal, so Chern classes, which are pulled back from the bottom layer of the tower BU, determine vector bundles. The main conclusion we shall draw from Arone's work is the connectivity of the layers. In fact, since $\mathbb{L}_n$ is a CW complex of dimension $n^2 - 1$, and since Ad_n has dimension n^2, the spectrum $\Theta_n = \mathrm{Map}_*(\mathbb{L}_n, \Sigma^\infty S^{\mathrm{Ad}_n})$ is 0-connected. It follows from the dimension estimate that, when n equals the prime power p^k, the lowest nontrivial reduced cohomology of $(L_{p^k}F)(V)$ appears in degree no less than $1 + 2p^k \cdot \dim_{\mathbb{C}} V$. (If n is not a prime power then $(L_nF)(V)$ is infinitely connected, by Theorem 2.5.) The corollary below then follows.

Corollary 2.6. *Let r be the dimension of V. Then $(L_2F)(V)$ is $4r$-connected. The higher layers $(L_iF)(V), i \geq 3$ are more than $4r$-connected.*

2.4. Proof of Theorem 2.1

Let $l > 2$, and r be such that $\frac{l}{2} \leq r \leq l - 1$. Consider the Weiss tower of $F^l(V) = \mathrm{Map}_*\big(\mathbb{C}P^l, BU(V)\big)$ at $V = \mathbb{C}^r$, which by Proposition 2.4 is obtained by mapping $\mathbb{C}P^l$ to the Weiss tower of $F(V) = BU(V)$. According to the connectivity estimate of Corollary 2.6, the layers $L_n F(\mathbb{C}^r)$ of the tower are at least $4r$-connected for all $n \geq 2$. Since $\mathbb{C}P^l$ is $2l$-dimensional and $2l \leq 4r$, spaces

$$(L_n F^l)(\mathbb{C}^r) \simeq \mathrm{Map}_*\big(\mathbb{C}P^l, L_n F(\mathbb{C}^r)\big)$$

are connected for $n \geq 2$ and hence the connected components of the Weiss tower of F^l at $V = \mathbb{C}^r$ stabilize after the first stage. Recall from Proposition 2.3 that $(L_1 F)(V) \simeq Q(\Sigma S^V)_{hU(1)}$, which is $QS^{2r+1}_{hU(1)} \simeq Q\Sigma\mathbb{C}P^\infty_r$ when evaluated at $V = \mathbb{C}^r$. So we conclude that the first layer $L_1 F^l$ of F^l at $V = \mathbb{C}^r$ is of the form

$$(L_1 F^l)(\mathbb{C}^r) \simeq \mathrm{Map}_*\big(\mathbb{C}P^l, QS^{2r+1}_{hU(1)}\big) \simeq \mathrm{Map}_*(\mathbb{C}P^l, Q\Sigma\mathbb{C}P^\infty_r).$$

To sum up, the first stage of the Weiss tower of F^l at $V = \mathbb{C}^r$ consists of the fibration

$$\mathrm{Map}_*\big(\mathbb{C}P^l, Q\Sigma\mathbb{C}P^\infty_r\big) \longrightarrow \mathrm{Map}_*\big(\mathbb{C}P^l, T_1 BU(\mathbb{C}^r)\big) \longrightarrow \mathrm{Map}_*\big(\mathbb{C}P^l, BU\big),$$

and $\mathrm{Map}_*\big(\mathbb{C}P^l, BU(r)\big) \to \mathrm{Map}_*\big(\mathbb{C}P^l, T_1 BU(\mathbb{C}^r)\big)$ induces a bijection on π_0. Given that

$$\pi_1 \mathrm{Map}_*\big(\mathbb{C}P^l, BU\big) \cong [\mathbb{C}P^l, U] = \tilde{K}_U^{-1}(\mathbb{C}P^l) = 0,$$

one obtains the following exact sequence from the homotopy long exact sequence associated with the above fibration:

$$0 \longrightarrow [\mathbb{C}P^l, Q\Sigma\mathbb{C}P^\infty_r] \longrightarrow [\mathbb{C}P^l, BU(r)] \longrightarrow [\mathbb{C}P^l, BU].$$

This exactness implies that the subset of rank r bundles over $\mathbb{C}P^l$ which stabilize to the trivial bundle, is identified with the abelian group $[\mathbb{C}P^l, Q\Sigma\mathbb{C}P_r^\infty] = \{\mathbb{C}P^l, \Sigma\mathbb{C}P_r^\infty\}$. To set up the identification $\mathrm{Vect}_r^0(\mathbb{C}P^l) \cong \{\mathbb{C}P^l, \Sigma\mathbb{C}P_r^\infty\}$, it remains to show that stably trivial bundles over $\mathbb{C}P^l$ are precisely those having trivial Chern classes. Recall that for any finite CW complex X, the Chern character $\mathrm{ch} : \tilde{K}_U^0(X) \to \tilde{H}^{\mathrm{even}}(X;\mathbb{Q})$ is an isomorphism after tensoring with $\mathbb{Q}$. Since $\tilde{K}_U^0(\mathbb{C}P^l)$ is torsion-free, the Chern character is in fact injective on $\mathbb{C}P^l$. Let's also recall the following Chern character formula for $\xi \in \tilde{K}_U^0(\mathbb{C}P^l)$

$$\mathrm{ch}(\xi) = c_1(\xi) + \frac{c_1^2(\xi) - 2c_2(\xi)}{2!} + \cdots + \frac{Q_l(c_1(\xi), \cdots, c_l(\xi))}{l!},$$

where Q_l is the polynomial over $\mathbb{Z}$ characterized by the property that $Q_l(\sigma_1, \cdots, \sigma_l) = x_1^l + \cdots + x_n^l$, where σ_i is the i-th elementary symmetric polynomial in the variables $x_1, \cdots, x_n$. It follows immediately from the above formula that $\mathrm{ch}(\xi) = \mathrm{ch}(\eta)$ for $\xi, \eta \in \tilde{K}_U^0(\mathbb{C}P^l)$ (i.e., for stable bundles ξ, η over $\mathbb{C}P^l$) precisely when $(c_1(\xi), \cdots, c_l(\xi)) = (c_1(\eta), \cdots, c_l(\eta))$. Thus for bundles over $\mathbb{C}P^l$, stable triviality is equivalent to the triviality of their Chern data. Theorem 2.1 is now proved.

CHAPTER III

FIRST CASES: RANK TWO BUNDLES OVER $\mathbb{C}P^3$ AND $\mathbb{C}P^4$

The goal of this chapter is to make calculations with the Weiss tower to recover the two classical examples, Corollaries 1.7 and 1.8. Namely, the vanishing Chern enumeration of rank 2 bundles over $\mathbb{C}P^3$, and that of rank 2 bundles over $\mathbb{C}P^4$. By Theorem 2.1 we have the identifications

$$\mathrm{Vect}_2^0(\mathbb{C}P^3) \cong \{\mathbb{C}P^3, \Sigma\mathbb{C}P_2^\infty\} \quad \text{and} \quad \mathrm{Vect}_2^0(\mathbb{C}P^4) \cong \{\mathbb{C}P^4, \Sigma\mathbb{C}P_2^\infty\}.$$

We prove Corollaries 1.7 and 1.8 by showing that

$$\{\mathbb{C}P^3, \Sigma\mathbb{C}P_2^\infty\} \cong \mathbb{Z}/2 \quad \text{and} \quad \{\mathbb{C}P^4, \Sigma\mathbb{C}P_2^\infty\} = 0.$$

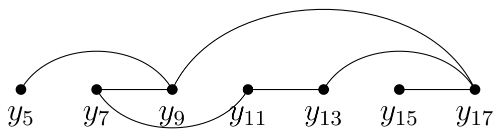
We shall regard these as generalized cohomology computations and apply the Atiyah-Hirzebruch spectral sequence. To learn the coefficient ring, namely the stable homotopy groups of $\Sigma\mathbb{C}P_2^\infty$, we apply the Adams spectral sequence. Note that it suffices to work 2-locally in these two cases. Indeed, if p is an odd prime then the stable homotopy of $\Sigma\mathbb{C}P_2^\infty$ does not have nontrivial p-primary torsion in degrees not exceeding 8 (see part (3) of Lemma 4.5). For those unfamiliar with the Adams spectral sequence, the recent expository paper of Beaudry-Campbell [8] provides a wonderful introduction. We start by describing the action of the Steenrod squares on the cohomology of $\Sigma\mathbb{C}P_2^\infty$, and then construct an explicit minimal resolution to compute the Adams E_2-page through a range. Calculations are 2-local unless otherwise stated.

3.1. First Stable Homotopy Groups of $\Sigma\mathbb{C}P_2^\infty$

We compute $\pi_*(Q\Sigma\mathbb{C}P_2^\infty) = \pi_*^s(\Sigma\mathbb{C}P_2^\infty)$ through a range. The mod two cohomology of $\Sigma\mathbb{C}P_2^\infty$

$$H^*(\Sigma\mathbb{C}P_2^\infty;\mathbb{Z}/2) \cong \mathbb{Z}/2\cdot\{y_5, y_7, \cdots, y_{2n+1}, \cdots\}$$

has a single $\mathbb{Z}/2$-generator in every odd degree $i \geq 5$, which we denote by the y_i. There are no nontrivial cup products. The class y_{2n+1} can be identified with the suspension of the class x^n in the cohomology of $\mathbb{C}P^\infty$, and the action of the Steenrod squares is then identified with that on $\mathbb{C}P^\infty$. The diagram below exhibits the action of the Steenrod squares on elements of $H^*(\Sigma\mathbb{C}P_2^\infty;\mathbb{Z}/2)$ in low degrees. The straight line segments represent a nontrivial action of Sq^2, and the curved ones represent that of Sq^4 or Sq^8.



We now apply the Adams spectral sequence to compute the (2-local part of) the stable homotopy of $\Sigma\mathbb{C}P_2^\infty$, which has

$$E_2^{s,t} = \mathrm{Ext}_{\mathcal{A}}^{s,t}\big(H^*(\Sigma\mathbb{C}P_2^\infty;\mathbb{Z}/2), \mathbb{Z}/2\big) \Longrightarrow \pi_{t-s}^s(\Sigma\mathbb{C}P_2^\infty).$$

In the Appendix we present an explicit minimal $\mathcal{A}$-resolution of $H^*(\Sigma\mathbb{C}P_2^\infty;\mathbb{Z}/2)$ to compute these Ext groups. We summarize the result as follows.

As usual, the horizontal axis is $t - s$ and the vertical axis is s. Each small circled dot represents a $\mathbb{Z}/2$. Bigger circled dots with question marks inside denote

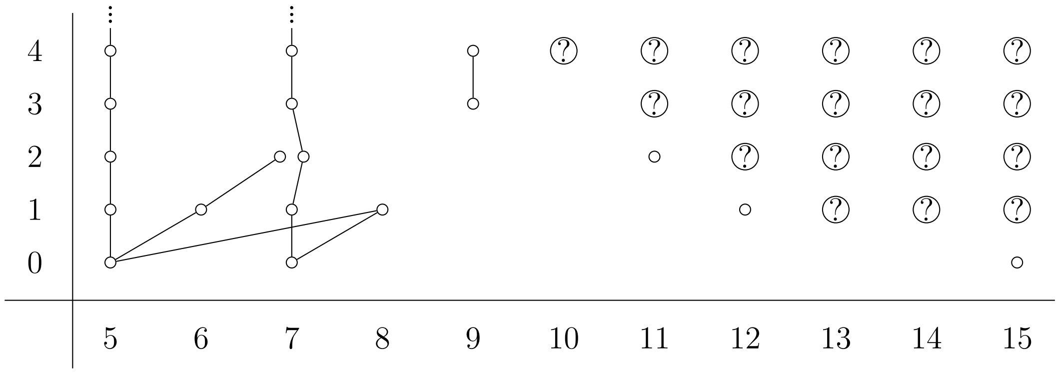


FIGURE 1. The 2-primary Adams E_2-page for $\pi_*^s(\Sigma\mathbb{C}P_2^\infty)$.

unknown groups. Places without circled dots are all zero. As usual, vertical line segments represent multiplication by h_0, line segments of slope one represent multiplication by h_1, and line segments of slope 1/3 represent multiplication by h_2, etc. There is a single h_0 tower along colunms $t-s=5$ and $t-s=7$. There is nothing in the chart whenever $t-s \leq 4$, as $\Sigma\mathbb{C}P_2^\infty$ is 4-connected.

Any d_r-differential starting from or arriving at columns with $t-s \leq 8$, $\forall r \geq 2$ must be trivial. One can then read off some first few stable homotopy groups of $\Sigma\mathbb{C}P_2^\infty$, which we summarize as follows.

Lemma 3.1. *The stable homotopy groups $\pi_i^s(\Sigma\mathbb{C}P_2^\infty)$ for $i \leq 8$ are as follows.*

i	≤ 4	5	6	7	8
$\pi_i^s(\Sigma\mathbb{C}P_2^\infty)$	0	$\mathbb{Z}_{(2)}$	$\mathbb{Z}/2$	$\mathbb{Z}_{(2)} \oplus \mathbb{Z}/2$	$\mathbb{Z}/2$

TABLE 2. The stable homotopy groups $\pi_i^s(\Sigma\mathbb{C}P_2^\infty)$ for $i \leq 8$.

This information of stable homotopy groups of $\Sigma\mathbb{C}P_2^\infty$ is crucial as we compute $\{\mathbb{C}P^3, \Sigma\mathbb{C}P_2^\infty\}$ and $\{\mathbb{C}P^3, \Sigma\mathbb{C}P_2^\infty\}$ in the following sections.

3.2. Recovery of Corollary 1.7

We now make use of Lemma 3.1 to compute $\{\mathbb{C}P^3, \Sigma\mathbb{C}P_2^\infty\}$. To start, consider the following long exact sequence

$$\cdots \to \{S^4, \Sigma\mathbb{C}P_2^\infty\} \longrightarrow \{S^5, \Sigma\mathbb{C}P_2^\infty\} \longrightarrow \{\Sigma\mathbb{C}P^2, \Sigma\mathbb{C}P_2^\infty\} \longrightarrow \{S^3, \Sigma\mathbb{C}P_2^\infty\} \to \cdots .$$

associated to the cofiber sequence $S^3 \xrightarrow{\eta} S^2 \to \mathbb{C}P^2$ (where η denotes the Hopf map). According to Lemma 3.1 the groups $\{S^3, \Sigma\mathbb{C}P_2^\infty\}$ and $\{S^4, \Sigma\mathbb{C}P_2^\infty\}$ both vanish. Therefore the middle two groups in the above sequence are isomorphic, and by Lemma 3.1 we obtain that

$$\{\Sigma\mathbb{C}P^2, \Sigma\mathbb{C}P_2^\infty\} \cong \{S^5, \Sigma\mathbb{C}P_2^\infty\} = \pi_5^s(\Sigma\mathbb{C}P_2^\infty) \cong \mathbb{Z}_{(2)}.$$

Next we consider the cofiber sequence $S^5 \xrightarrow{\eta_3} \mathbb{C}P^2 \to \mathbb{C}P^3$, where η_3 denotes the attaching map of the top cell of $\mathbb{C}P^3$. In the following diagram

$$\begin{tikzcd}
\cdots \arrow[r] & \{\Sigma\mathbb{C}P^2, \Sigma\mathbb{C}P_2^\infty\} \arrow[r, "(\Sigma\eta_3)^*"] & \{S^6, \Sigma\mathbb{C}P_2^\infty\} \arrow[r] & \{\mathbb{C}P^3, \Sigma\mathbb{C}P_2^\infty\} \arrow[r] & 0 \\
& \{S^5, \Sigma\mathbb{C}P_2^\infty\} \cong \mathbb{Z}_{(2)} \arrow[u, "\cong"] \arrow[ur] & & &
\end{tikzcd}$$

the top row is part of the long exact sequence associated with a cofiber sequence, and the vertical map in the triangle is the isomorphism we just analyzed. Let $q_i : \mathbb{C}P^i \to S^{2i}$ be the quotient map by the $(2i-1)$-skeleton. By Lemma 3.1 we have $\{S^6, \Sigma\mathbb{C}P_2^\infty\} \cong \pi_6^s(\Sigma\mathbb{C}P_2^\infty) \cong \mathbb{Z}/2$. Thus

$$(\Sigma\eta_3)^* : \{\Sigma\mathbb{C}P^2, \Sigma\mathbb{C}P_2^\infty\} \longrightarrow \{S^6, \Sigma\mathbb{C}P_2^\infty\}$$

is seen to be a homomorphism $\mathbb{Z}_{(2)} \to \mathbb{Z}/2$, and $\{\mathbb{C}P^3, \Sigma\mathbb{C}P_2^\infty\}$ is the cokernel of this homomorphism. We claim that $(\Sigma\eta_3)^* : \mathbb{Z}_{(2)} \to \mathbb{Z}/2$ is the zero homomorphism. Note that the homomorphism $\{S^5, \Sigma\mathbb{C}P_2^\infty\} \to \{S^6, \Sigma\mathbb{C}P_2^\infty\}$ in the above diagram is induced by the composite

$$\big(S^6 \xrightarrow{\Sigma\eta_3} \Sigma\mathbb{C}P^2 \xrightarrow{\Sigma q_2} S^5\big) = \Sigma\big(S^5 \xrightarrow{\eta_3} \mathbb{C}P^2 \xrightarrow{q_2} S^4\big).$$

Here we recall a general fact. Let $\eta_l : S^{2l-1} \to \mathbb{C}P^{l-1}$ denote the attaching map of the top cell of $\mathbb{C}P^l$. Then the composite $S^{2l-1} \xrightarrow{\eta_l} \mathbb{C}P^{l-1} \xrightarrow{q_{l-1}} S^{2l-2}$ is the suspension of the Hopf map $S^3 \xrightarrow{\eta} S^2$ if l is even, and is null when l is odd. (This fact can be seen as a description of the structure of the stunted projective spaces $\mathbb{C}P_{l-1}^l = \mathbb{C}P^l/\mathbb{C}P^{l-2}$, $l \geq 3$. Indeed, it is detected by the action of Sq^2 that this two-cell complex is equivalent to $S^{2l} \vee S^{2l-2}$ if l is odd, and is equivalent to $\Sigma^{2l-4}\mathbb{C}P^2$ when l is even.) It follows that the composite

$$\mathbb{Z}_{(2)} \cong \{S^5, \Sigma\mathbb{C}P_2^\infty\} \xrightarrow{\cong} \{\Sigma\mathbb{C}P^2, \Sigma\mathbb{C}P_2^\infty\} \xrightarrow{(\Sigma\eta_3)^*} \{S^6, \Sigma\mathbb{C}P_2^\infty\} \cong \mathbb{Z}/2$$

is induced by a null map, and is hence the zero homomorphism. This completes the proof that $\{\mathbb{C}P^3, \Sigma\mathbb{C}P_2^\infty\} \cong \mathbb{Z}/2$, and hence that of Corollary 1.7.

3.3. Recovery of Corollary 1.8

We now show that $\{\mathbb{C}P^4, \Sigma\mathbb{C}P_2^\infty\} = 0$, which recovers Corollary 1.8. Rather than using long exact sequences associated to cofibration sequences, we organize the calculation with the Atiyah-Hirzebruch spectral sequence. Recall that $\{X, \Sigma^* Y\}$ can be regarded as a generalized cohomology theory of X. Filtering by skeleta leads to the Atiyah-Hirzebruch Spectral Sequence (AHSS) for these stable maps with

$E_2^{p,q} = H^p\big(X; \pi_{-q}(Y)\big)$, which converges to $\{X, \Sigma^{p+q}Y\}$ if X is a finite complex. The AHSS computing $\{\mathbb{C}P^4, \Sigma\mathbb{C}P_2^\infty\}$ thus has

$$E_2^{p,q} = H^p\big(\mathbb{C}P^4; \pi^s_{-q}(\Sigma\mathbb{C}P_2^\infty)\big) \Longrightarrow \{\mathbb{C}P^4, \Sigma^{p+q+1}\mathbb{C}P_2^\infty\}.$$

We show the E_2 term through a range below. As $E_2^{p,q}$ vanishes for all $p > 8$ and for all $q > -5$, only the circled groups can contribute in total degree zero.

	4	5	6	7	8	≥ 9
≥ -4	0	0	0	0	0	0
-5	$\mathbb{Z}_{(2)}$	0	$\mathbb{Z}_{(2)}$	0	$\mathbb{Z}_{(2)}$	0
-6	$\mathbb{Z}/2$	0	$\mathbb{Z}/2$	0	$\mathbb{Z}/2$	0
-7	$\mathbb{Z}_{(2)} \oplus \mathbb{Z}/2$	0	$\mathbb{Z}_{(2)} \oplus \mathbb{Z}/2$	0	$\mathbb{Z}_{(2)} \oplus \mathbb{Z}/2$	0
-8	$\mathbb{Z}/2$	0	$\mathbb{Z}/2$	0	$\mathbb{Z}/2$	0

FIGURE 2. The E_2-page of the AHSS for $\{\mathbb{C}P^4, \Sigma\mathbb{C}P_2^\infty\}$.

We analyze the relevant d_2 differentials. By the construction of the AHSS, the differential $d_2 : E_2^{2a,-b} \to E_2^{2a+2,-b-1}$ is induced by the connecting map $\mathbb{C}P^{a+1}/\mathbb{C}P^a \to \Sigma\mathbb{C}P^a/\mathbb{C}P^{a-1}$ in the cofiber sequence

$$\mathbb{C}P^a/\mathbb{C}P^{a-1} \to \mathbb{C}P^{a+1}/\mathbb{C}P^{a-1} \to \mathbb{C}P^{a+1}/\mathbb{C}P^a \to \Sigma\mathbb{C}P^a/\mathbb{C}P^{a-1}.$$

This connecting map, $S^{2a+2} \to S^{2a+1}$, both reflects and is determined by the structure of the two-cell stunted projective space $\mathbb{C}P^{a+1}/\mathbb{C}P^{a-1}$, and is detected by Sq^2. As is discussed in Section 3.2, when a is odd this connecting map is a suspension of $\eta : S^3 \to S^2$, and when a is even it is null. Furthermore,

$$E_2^{2a,-b} = \pi_b^s(\Sigma\mathbb{C}P_2^\infty), \text{ and } E_2^{2a+2,-b-1} = \pi_{b+1}^s(\Sigma\mathbb{C}P_2^\infty).$$

In summary, $d_2 : E_2^{2a,-b} \to E_2^{2a+2,-b-1}$ is a homomorphism $\pi_b^s(\Sigma\mathbb{C}P_2^\infty) \to \pi_{b+1}^s(\Sigma\mathbb{C}P_2^\infty)$, which is multiplication by η when a is odd, and is zero when a is even.

For example, taking $a = 3$ and $b = 6$, one concludes that $E_2^{6,-6} \to E_2^{8,-7}$ is the homomorphism $\mathbb{Z}/2 \to \mathbb{Z}_{(2)} \oplus \mathbb{Z}/2$ induced by η. This homomorphism is onto the second summand of $\mathbb{Z} \oplus \mathbb{Z}/2$, due to the fact that on the Adams E_2-page the dot at $(6,1)$ is connected with that at $(7,2)$ via multiplication by h_1. Similarly, one deduces that $E_2^{6,-7} \to E_2^{8,-8}$ is the homomorphism $\mathbb{Z}_{(2)} \oplus \mathbb{Z}/2 \to \mathbb{Z}/2$ which is the surjection when restricted to the first summand and is zero when restricted to the second.

It follows that $E_r^{p,-p} = 0$ for all p and all $r \geq 3$. So $\{\mathbb{C}P^4, \Sigma\mathbb{C}P_2^\infty\} = 0$, and Corollary 1.8 is proved.

CHAPTER IV

PROOF OF MAIN RESULTS

In this chapter we prove our main results, Theorems 1.1 and 1.2, together with Propositions 1.3 and 1.4. Recall that Theorems 1.1 and 1.2 give, respectively, the vanishing Chern enumeration for bundles of rank $(l-1)$ and $(l-2)$ over $\mathbb{C}P^l$. Both these cases are in the metastable range, so by Theorem 2.1 there are identifications

$$\mathrm{Vect}^0_{l-1}(\mathbb{C}P^l) \cong \{\mathbb{C}P^l, \Sigma\mathbb{C}P^\infty_{l-1}\} \quad \text{and} \quad \mathrm{Vect}^0_{l-2}(\mathbb{C}P^l) \cong \{\mathbb{C}P^l, \Sigma\mathbb{C}P^\infty_{l-2}\}.$$

We perform these stable calculations, following the strategy of Section 3.3 to regard them as generalized cohomology computations.

4.1. First Stable Homotopy Groups of $\Sigma\mathbb{C}P^\infty_n$

Both the calculations of $\{\mathbb{C}P^l, \Sigma\mathbb{C}P^\infty_{l-1}\}$ and $\{\mathbb{C}P^l, \Sigma\mathbb{C}P^\infty_{l-2}\}$ rely on the knowledge of some first few 2-primary stable homotopy groups of $\Sigma\mathbb{C}P^\infty_n$. The latter also requires knowledge of the 3-primary groups. This section is dedicated to presenting the results of these calculations, with details postponed to Appendix A.

The cohomology of $\Sigma\mathbb{C}P^\infty_n$ has a single $\mathbb{Z}/2$-generator $y_{2n+2k+1}$ in every odd degree greater than or equal to $2n+1$. The class $y_{2n+2k+1}$ can be identified with the suspension of the class x^{n+k} in the cohomology of $\mathbb{C}P^\infty$, to compute the action of the Steenrod squares. The stable homotopy groups $\pi_i^s(\Sigma\mathbb{C}P^\infty_n)$ within the range $i \leq 2n+4$ require only the information of the Steenrod square actions on the finite skeleton $\Sigma\mathbb{C}P_n^{n+4}$ of $\Sigma\mathbb{C}P_n^{\infty}$. These actions, and hence the Adams E_2-pages, exhibit an 8-fold periodic behavior. We obtain the following result, whose detailed proof is contained in the Appendix.

Lemma 4.1. *The 2-primary stable homotopy groups $\pi_i^s(\Sigma\mathbb{C}P_n^{\infty})$ for $i \leq 2n+4$ can be described as follows.*

1. $\pi_i^s(\Sigma\mathbb{C}P_n^{\infty}) = 0$ *whenever* $i \leq 2n$, *and* $\pi_{2n+1}^s(\Sigma\mathbb{C}P_n^{\infty}) = \mathbb{Z}_{(2)}$.

2. $\pi_{2n+2}^s(\Sigma\mathbb{C}P_n^{\infty}) = 0$ *if* n *is odd, and* $\pi_{2n+2}^s(\Sigma\mathbb{C}P_n^{\infty}) = \mathbb{Z}/2$ *if* n *is even.*

3. $\pi_{2n+3}^s(\Sigma\mathbb{C}P_n^{\infty}) = \mathbb{Z}_{(2)}$ *if* n *is odd, and* $\pi_{2n+3}^s(\Sigma\mathbb{C}P_n^{\infty}) = \mathbb{Z}_{(2)} \oplus \mathbb{Z}/2$ *if* n *is even.*

4. $\pi_{2n+4}^s(\Sigma\mathbb{C}P_n^{\infty})$ *exhibits the following 8-fold periodicity.*

$n \mod 8$	0	1	2	3	4	5	6	7
$\pi_{2n+4}^s(\Sigma\mathbb{C}P_n^{\infty})$	$\mathbb{Z}/8$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	0	$\mathbb{Z}/4$	$\mathbb{Z}/4$	$\mathbb{Z}/2$	0

TABLE 3. The 8-fold periodic behavior of $\pi_{2n+4}^s(\Sigma\mathbb{C}P_n^{\infty})$.

4.2. The First Unstable Case

We compute $\{\mathbb{C}P^l, \Sigma\mathbb{C}P_{l-1}^{\infty}\}$ through analysis of the AHSS, which has

$$E_2^{p,q} = H^p\big(\mathbb{C}P^l; \pi_{-q}^s(\Sigma\mathbb{C}P_{l-1}^{\infty})\big) \Longrightarrow \{\mathbb{C}P^l, \Sigma^{p+q+1}\mathbb{C}P_{l-1}^{\infty}\}.$$

When l is even, $\pi_{2l-1}^s(\Sigma\mathbb{C}P_{l-1}^{\infty}) = \mathbb{Z}_{(2)}$ and $\pi_{2l}^s(\Sigma\mathbb{C}P_{l-1}^{\infty}) = 0$ by Lemma 4.1, and therefore the E_2-page has the form as described in Figure 3.

The terms $E_2^{p,q}$ vanish for all $p > 2l$ and for all $q > -(2l-1)$. Since all groups are zero along the diagonal $p+q = 0$, one concludes immediately that $\{\mathbb{C}P^l, \Sigma\mathbb{C}P_{l-1}^{\infty}\} = 0$.

	2	3	$\cdots$	$2l-2$	$2l-1$	$2l$	$\geq 2l+1$
$\geq -(2l-2)$	0	0	$\cdots$	0	0	0	0
$-(2l-1)$	$\mathbb{Z}_{(2)}$	0	$\cdots$	$\mathbb{Z}_{(2)}$	0	$\mathbb{Z}_{(2)}$	0
$-2l$	0	0	$\cdots$	0	0	0	0

FIGURE 3. The AHSS E_2-page for $\{\mathbb{C}P^l, \Sigma\mathbb{C}P^\infty_{l-1}\}$, when l is even.

When l is odd, $\pi^s_{2l-1}(\Sigma\mathbb{C}P^\infty_{l-1}) = \mathbb{Z}_{(2)}$ and $\pi^s_{2l}(\Sigma\mathbb{C}P^\infty_{l-1}) = \mathbb{Z}/2$ by Lemma 4.1, and therefore the E_2-page has the form as described in Figure 4.

	2	3	$\cdots$	$2l-2$	$2l-1$	$2l$	$\geq 2l+1$
$\geq -(2l-2)$	0	0	$\cdots$	0	0	0	0
$-(2l-1)$	$\mathbb{Z}_{(2)}$	0	$\cdots$	$\mathbb{Z}_{(2)}$	0	$\mathbb{Z}_{(2)}$	0
$-2l$	$\mathbb{Z}/2$	0	$\cdots$	$\mathbb{Z}/2$	0	$\mathbb{Z}/2$	0

FIGURE 4. The AHSS E_2-page for $\{\mathbb{C}P^l, \Sigma\mathbb{C}P^\infty_{l-1}\}$, when l is odd.

We now determine the differential $d_2 : \mathbb{Z}_{(2)} = E_2^{2l-2,-(2l-1)} \to E_2^{2l,-2l} = \mathbb{Z}/2$. As is previously analyzed, this is a homomorphism $\pi^s_{2l-1}(\Sigma\mathbb{C}P^\infty_{l-1}) \to \pi^s_{2l}(\Sigma\mathbb{C}P^\infty_{l-1})$ induced by a map $S^{2l} \to S^{2l-1}$, which is a suspension of $\eta : S^3 \to S^2$ when l is even, and null when l is odd. Thus the above differential is the zero as l is odd, and the circled group $\mathbb{Z}/2$ survives to the infinity page. Since this is the only nonzero group along the diagonal $p+q=0$ on that page, we conclude that $\{\mathbb{C}P^l, \Sigma\mathbb{C}P^\infty_{l-1}\} = \mathbb{Z}/2$.

We have proved that $\{\mathbb{C}P^l, \Sigma\mathbb{C}P^\infty_{l-1}\} = 0$ when l is even, and $\{\mathbb{C}P^l, \Sigma\mathbb{C}P^\infty_{l-1}\} = \mathbb{Z}/2$ when l is odd, completing the proof of Theorem 1.1.

4.3. The Second Unstable Case

We now prove Theorem 1.2, by calculating $\{\mathbb{C}P^l, \Sigma\mathbb{C}P^\infty_{l-2}\}$. In this case both the prime 2 and the prime 3 are involved. We carry out the 2-local calculation in the next subsection, and the 3-local computation in the second next. The results are as follows.

Theorem 4.2. *Let $l \geq 4$ be an integer. Then $\{\mathbb{C}P^l, \Sigma\mathbb{C}P^\infty_{l-2}\}_{(2)}$ exhibits the following 8-fold periodic behavior.*

$l \mod 8$	0	1	2	3	4	5	6	7
$\{\mathbb{C}P^l, \Sigma\mathbb{C}P^\infty_{l-2}\}_{(2)}$	0	0	$\mathbb{Z}/4$	$\mathbb{Z}/2$	0	0	$\mathbb{Z}/2$	$\mathbb{Z}/4$

TABLE 4. The 8-fold periodic behavior of $\{\mathbb{C}P^l, \Sigma\mathbb{C}P^\infty_{l-2}\}_{(2)}$.

Theorem 4.3. *Let $l \geq 4$ be an integer. Then $\{\mathbb{C}P^l, \Sigma\mathbb{C}P^\infty_{l-2}\}_{(3)}$ exhibits the following 3-fold periodic behavior.*

1. *The group vanishes whenever l is 0 or 1 mod 3.*

2. *The group is isomorphic to $\mathbb{Z}/3$ when l is 2 mod 3.*

Moreover, $\{\mathbb{C}P^l, \Sigma\mathbb{C}P^\infty_{l-2}\}$ has no p-torsion for $p \geq 5$. Combining Theorems 4.2 and 4.3, one obtains immediately the enumerations in Theorem 1.2.

Calculations at the prime 2

We prove Theorem 4.2, with 2-local computations throughout. We regard $\{\mathbb{C}P^l, \Sigma\mathbb{C}P^\infty_{l-2}\}$ again as a cohomology calculation and apply the AHSS, which has

$$E_2^{p,q} = H^p\big(\mathbb{C}P^l; \pi^s_{-q}(\Sigma\mathbb{C}P^\infty_{l-2})\big) \Longrightarrow \{\mathbb{C}P^l, \Sigma^{p+q+1}\mathbb{C}P^\infty_{l-2}\}.$$

By Lemma 4.1, stable homotopy groups of $\Sigma\mathbb{C}P^{\infty}_{l-2}$ can be summarized as follows.

	$\pi^s_{2l-3}(\Sigma\mathbb{C}P^{\infty}_{l-2})$	$\pi^s_{2l-2}(\Sigma\mathbb{C}P^{\infty}_{l-2})$	$\pi^s_{2l-1}(\Sigma\mathbb{C}P^{\infty}_{l-2})$	$\pi^s_{2l}(\Sigma\mathbb{C}P^{\infty}_{l-2})$
$l = 0 \bmod 8$	$\mathbb{Z}_{(2)}$	$\mathbb{Z}/2$	$\mathbb{Z}_{(2)} \oplus \mathbb{Z}/2$	$\mathbb{Z}/2$
$l = 1 \bmod 8$	$\mathbb{Z}_{(2)}$	0	$\mathbb{Z}_{(2)}$	0
$l = 2 \bmod 8$	$\mathbb{Z}_{(2)}$	$\mathbb{Z}/2$	$\mathbb{Z}_{(2)} \oplus \mathbb{Z}/2$	$\mathbb{Z}/8$
$l = 3 \bmod 8$	$\mathbb{Z}_{(2)}$	0	$\mathbb{Z}_{(2)}$	$\mathbb{Z}/2$
$l = 4 \bmod 8$	$\mathbb{Z}_{(2)}$	$\mathbb{Z}/2$	$\mathbb{Z}_{(2)} \oplus \mathbb{Z}/2$	$\mathbb{Z}/2$
$l = 5 \bmod 8$	$\mathbb{Z}_{(2)}$	0	$\mathbb{Z}_{(2)}$	0
$l = 6 \bmod 8$	$\mathbb{Z}_{(2)}$	$\mathbb{Z}/2$	$\mathbb{Z}_{(2)} \oplus \mathbb{Z}/2$	$\mathbb{Z}/4$
$l = 7 \bmod 8$	$\mathbb{Z}_{(2)}$	0	$\mathbb{Z}_{(2)}$	$\mathbb{Z}/4$

TABLE 5. First few nontrivial 2-primary stable homotopy groups of $\Sigma\mathbb{C}P^{\infty}_{l-2}$.

Therefore the AHSS also exhibits an 8-fold periodic behavior. Since the proofs for these eight cases are similar, we shall first present a proof for one case in detail, and then sketch the proofs for the remaining seven cases. Let us consider the case $l \equiv 2 \bmod 8$. In this case our goal is to show that, 2-locally, $\{\mathbb{C}P^l, \Sigma\mathbb{C}P^{\infty}_{l-2}\} \cong \mathbb{Z}/4$.

Part of the E_2-page of the spectral sequence is presented in Figure 5. The groups $E_2^{p,q}$ vanish for all $p > 2l$ and for all $q > -(2l-3)$, so only the circled groups can contribute in total degree zero.

We first analyze the relevant d_2-differentials. As is discussed in Section 3.3, they are determined by the structures of certain two-cell stunted projective spaces, where the attaching maps are detected by Sq^2. For example, for each q the differential $d_2^{2l-2,q} : E_2^{2l-2,q} \longrightarrow E_2^{2l,q-1}$ is a homomorphism induced by the map $\mathbb{C}P^l/\mathbb{C}P^{l-1} \to \Sigma\mathbb{C}P^{l-1}/\mathbb{C}P^{l-2}$, which is part of the cofiber sequence defining $\mathbb{C}P^l/\mathbb{C}P^{l-2}$. This map $S^{2l} \to S^{2l-1}$ is a suspension of $\eta : S^3 \to S^2$ since our l is even. Similarly, by the structure of $\mathbb{C}P^{l-1}/\mathbb{C}P^{l-3}$, the differentials

	$2l-4$	$2l-3$	$2l-2$	$2l-1$	$2l$
$\geq -(2l-4)$	0	0	0	0	0
$-(2l-3)$	$\mathbb{Z}_{(2)}$	0	$\mathbb{Z}_{(2)}$	0	$\mathbb{Z}_{(2)}$
$-(2l-2)$	$\mathbb{Z}/2$	0	$\mathbb{Z}/2$	0	$\mathbb{Z}/2$
$-(2l-1)$	$\mathbb{Z}_{(2)} \oplus \mathbb{Z}/2$	0	$\mathbb{Z}_{(2)} \oplus \mathbb{Z}/2$	0	$\mathbb{Z}_{(2)} \oplus \mathbb{Z}/2$
$-2l$	$\mathbb{Z}/8$	0	$\mathbb{Z}/8$	0	$\mathbb{Z}/8$

FIGURE 5. The 2-primary AHSS E_2-page for $\{\mathbb{C}P^l, \Sigma^{p+q+1}\mathbb{C}P^{\infty}_{l-2}\}$, $l \equiv 2(8)$.

$d_2^{2l-4,q} : E_2^{2l-4,q} \longrightarrow E_2^{2l-2,q-1}$ are all induced by the null map, and are therefore all zero.

The Adams spectral sequence for the stable homotopy of $\Sigma\mathbb{C}P^{\infty}_{l-2}$ has the following form. (See the Appendix for details.)

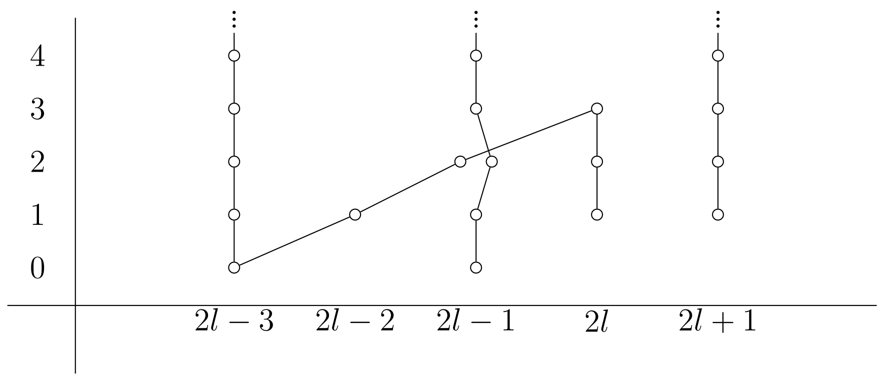


FIGURE 6. The 2-primary Adams E_2-page for $\pi_*^s(\Sigma\mathbb{C}P^{\infty}_{l-2})$, $l \equiv 2(8)$.

When $l = 2 \bmod 8$, we use similar analysis as previous to conclude the following about the d_2 differentials:

1. $d_2^{2l-2,-(2l-3)}$ is the surjection $\mathbb{Z}_{(2)} \to \mathbb{Z}/2$;

2. $d_2^{2l-2,-(2l-2)} : \mathbb{Z}/2 \to \mathbb{Z}_{(2)} \oplus \mathbb{Z}/2$ is the inclusion into the second summand;

3. $d_2^{2l-2,-(2l-1)} : \mathbb{Z}_{(2)} \oplus \mathbb{Z}/2 \to \mathbb{Z}/8$ is given by $(0, 4)$;

4. $d_2^{2l-4,q} = 0$ for all q.

For example, to determine $d_2^{2l-2,-(2l-1)} : \mathbb{Z}_{(2)} \oplus \mathbb{Z}/2 \to \mathbb{Z}/8$ in the third case above, we first note that it is a homomorphism $\pi_{2l-1}^s(\Sigma\mathbb{C}P_{l-2}^\infty) \to \pi_{2l}^s(\Sigma\mathbb{C}P_{l-2}^\infty)$ given by multiplication by η, where η now denotes the stable element. We then examine the Adams chart above to notice that the dot at $(2l-1, 2)$, which yields the $\mathbb{Z}/2$-summand of the domain, is connected with the dot at $(2l, 3)$ by h_1, which represents the element 4 in the target group. This implies that $d_2^{2l-2,-(2l-1)}$ restricted to the $\mathbb{Z}/2$-summand is multiplication by 4. Similarly, we conclude that $d_2^{2l-2,-(2l-1)}$ restricted to the $\mathbb{Z}_{(2)}$-summand must be the zero homomorphism, since the dots at $(2l-1, 0)$ and $(2l, 1)$ are not connected by h_1.

For degree reasons there cannot be any d_3-differential in the AHSS, and the E_4-page is partly depicted in Figure 7. Here along the diagonal line $p + q = 0$ there is a single group of $E_4^{2l,-2l} = \mathbb{Z}/4$, and it follows immediately that $\{\mathbb{C}P^l, \Sigma\mathbb{C}P_{l-2}^\infty\}$ is a quotient of $\mathbb{Z}/4$, by the image of the possibly nontrivial d_4 differential $d_4^{2l-4,-(2l-3)} : E_4^{2l-4,-(2l-3)} \to E_4^{2l,-2l}$.

By the construction of the AHSS, the d_4 differentials is determined by the structure of the three-cell complex $\mathbb{C}P^l/\mathbb{C}P^{l-3}$. The differential $d_4^{2l-4,-(2l-3)} : E_4^{2l-4,-(2l-3)} \to E_4^{2l,-2l}$ is induced by a map $\lambda_l : S^{2l} = \mathbb{C}P^l/\mathbb{C}P^{l-1} \to \Sigma\mathbb{C}P^{l-2}/\mathbb{C}P^{l-3} = S^{2l-3}$, which belongs to the (2-local) third stable stem $\pi_3^s(S^0) \cong \mathbb{Z}/8$. More precisely, consider the following commutative diagram. Here both the rows and the middle two columns are part of cofiber sequences, and δ_{n-1} and δ_{n-2} are connecting maps in those cofiber sequences.

	$2l-4$	$2l-3$	$2l-2$	$2l-1$	$2l$	$>2l$
$\geq -(2l-4)$	0	0	0	0	0	0
$-(2l-3)$	$\mathbb{Z}_{(2)}$	0	$2\mathbb{Z}_{(2)}$	0	$\mathbb{Z}_{(2)}$	0
$-(2l-2)$	$*$	0	0	0	0	0
$-(2l-1)$	$*$	0	$\mathbb{Z}_{(2)}$	0	$\mathbb{Z}_{(2)}$	0
$-2l$	$*$	0	$*$	0	$\mathbb{Z}/4$	0

FIGURE 7. The 2-primary AHSS E_4-page for $\{\mathbb{C}P^l, \Sigma^{p+q+1}\mathbb{C}P^{\infty}_{l-2}\}$, $l \equiv 2(8)$.

$$\begin{tikzcd}
\mathbb{C}P^{l-1}/\mathbb{C}P^{l-2} \arrow[r] & \Sigma\mathbb{C}P^{l-2}/\mathbb{C}P^{l-3} \arrow[r] & \Sigma\mathbb{C}P^{l-1}/\mathbb{C}P^{l-3} \arrow[r] & \Sigma\mathbb{C}P^{l-1}/\mathbb{C}P^{l-2} \\
\mathbb{C}P^{l-1}/\mathbb{C}P^{l-2} \arrow[r] \arrow[u, "="] & \mathbb{C}P^{l}/\mathbb{C}P^{l-2} \arrow[r] \arrow[u, "\delta_{l-2}"] & \mathbb{C}P^{l}/\mathbb{C}P^{l-1} \arrow[r] \arrow[u, "\delta_{l-1}"] & \Sigma\mathbb{C}P^{l-1}/\mathbb{C}P^{l-2} \arrow[u, "="]
\end{tikzcd}$$

When l is even, $\mathbb{C}P^{l-1}/\mathbb{C}P^{l-3}$ splits as $S^{2l-2} \vee S^{2l-4}$. Denote by p the quotient map

$$S^{2l-1} \vee S^{2l-3} = \Sigma\mathbb{C}P^{l-1}/\mathbb{C}P^{l-3} \to \Sigma\mathbb{C}P^{l-2}/\mathbb{C}P^{l-3} = S^{2l-3},$$

which sections the inclusion $\Sigma\mathbb{C}P^{l-2}/\mathbb{C}P^{l-3} \to \Sigma\mathbb{C}P^{l-1}/\mathbb{C}P^{l-3}$. In this case λ_l is the composite $p \circ \delta_{l-1}$. When l is odd, $\mathbb{C}P^{l}/\mathbb{C}P^{l-2}$ splits as $S^{2l} \vee S^{2l-2}$. Denote by j the standard inclusion

$$S^{2l} = \mathbb{C}P^{l}/\mathbb{C}P^{l-1} \to \mathbb{C}P^{l}/\mathbb{C}P^{l-2} = S^{2l} \vee S^{2l-2},$$

which sections the quotient map $\mathbb{C}P^{l}/\mathbb{C}P^{l-2} \to \mathbb{C}P^{l}/\mathbb{C}P^{l-1}$. In this case λ_l is the composite $\delta_{l-2} \circ j$.

$l \mod 8$	0	1	2	3	4	5	6	7
homotopy class of λ_l	ν	ν	0	2ν	ν	ν	2ν	0

TABLE 6. The 8-fold periodic behavior of λ_l.

In [13] Mosher determines the homotopy classes of the λ_l's, which exhibit an 8-fold periodic behavior which we present as follows. Write ν for the Hopf map $S^7 \to S^4$ generating the 2-local third stable stem.

Lemma 4.4 (Mosher [13], Proposition 5.2)**.** *Let $l \geq 4$ be an integer. The homotopy class of λ_l satisfies an 8-fold periodicity as follows.*

Theorem 4.2 follows immediately from Lemma 4.4. When $l = 2 \mod 8$, Lemma 4.4 suggests that the differential $d_4^{2l-4,-(2l-3)}$ we study is induced by the null map. It follows that the cokernel of $d_4^{2l-4,-(2l-3)}$ is $\mathbb{Z}/4$. For degree reasons there can be no further nontrivial differentials starting from or arriving at the diagonal $p+q=0$. Thus we conclude that $\{\mathbb{C}P^l, \Sigma\mathbb{C}P^\infty_{l-2}\} \cong \mathbb{Z}/4$ when for $l = 2 \mod 8$.

In all other cases, determining $d_4^{2l-4,-(2l-3)}$ comes down to learning the homotopy class of λ_l, which by Lemma 4.4 is some multiple of ν. Applying analysis of $\mathbb{C}P^l/\mathbb{C}P^{l-3}$ as above and using Lemma 4.4 proves Theorem 4.2 in all these cases. We present sketched proofs of each case in the following subsection.

Full proof of Theorem 4.2

In this subsection we sketch the proofs of Theorem 4.2 in the remaining seven cases. We compute $\{\mathbb{C}P^l, \Sigma\mathbb{C}P^\infty_{l-2}\}$ at the prime 2, applying the AHSS which has

$$E_2^{p,q} = H^p\big(\mathbb{C}P^l; \pi^s_{-q}(\Sigma\mathbb{C}P^\infty_{l-2})\big) \Longrightarrow \{\mathbb{C}P^l, \Sigma^{p+q+1}\mathbb{C}P^\infty_{l-2}\}.$$

In all following AHSS pages, we shall omit writing out columns labeled by $p \geq 2l+1$, since each such column consists purely of zero groups.

The case $l \equiv 0 \mod 8$

The E_2-page of the AHSS is presented below.

	$2l-4$	$2l-3$	$2l-2$	$2l-1$	$2l$
$\geq -(2l-4)$	0	0	0	0	0
$-(2l-3)$	$\mathbb{Z}_{(2)}$	0	$\mathbb{Z}_{(2)}$	0	$\mathbb{Z}_{(2)}$
$-(2l-2)$	$\mathbb{Z}/2$	0	$\mathbb{Z}/2$	0	$\mathbb{Z}/2$
$-(2l-1)$	$\mathbb{Z}_{(2)} \oplus \mathbb{Z}/2$	0	$\mathbb{Z}_{(2)} \oplus \mathbb{Z}/2$	0	$\mathbb{Z}_{(2)} \oplus \mathbb{Z}/2$
$-2l$	$\mathbb{Z}/2$	0	$\mathbb{Z}/2$	0	$\mathbb{Z}/2$

In this case l is even, so the d_2 differentials $d_2^{2l-2,*}$ from the $p = 2l-2$ column is induced by multiplication by η. Since $l-2 \equiv 6 \mod 8$, it follows from the corresponding Adams Spectral Sequence for $\Sigma\mathbb{C}P^{\infty}_{l-2}$ that

1. $d_2^{2l-2,-(2l-3)}$ is the surjection $\mathbb{Z}_{(2)} \to \mathbb{Z}/2$;

2. $d_2^{2l-2,-(2l-2)} : \mathbb{Z}/2 \to \mathbb{Z}_{(2)} \oplus \mathbb{Z}/2$ is the inclusion into the second summand;

3. $d_2^{2l-2,-(2l-1)} : \mathbb{Z}_{(2)} \oplus \mathbb{Z}/2 \to \mathbb{Z}/2$ is given by $(0,1)$;

Therefore, the two circled groups along the diagonal $p+q=0$ both disappear in the E_3-page. So in this case we conclude that

$$\{\mathbb{C}P^l, \Sigma\mathbb{C}P^{\infty}_{l-2}\} = 0.$$

The case $l \equiv 1 \mod 8$

The E_2-page of the AHSS is presented below.

	$\cdots$	$2l-4$	$2l-3$	$2l-2$	$2l-1$	$2l$
$\geq -(2l-4)$	$\cdots$	0	0	0	0	0
$-(2l-3)$	$\cdots$	$\mathbb{Z}_{(2)}$	0	$\mathbb{Z}_{(2)}$	0	$\mathbb{Z}_{(2)}$
$-(2l-2)$	$\cdots$	0	0	0	0	0
$-(2l-1)$	$\cdots$	$\mathbb{Z}_{(2)}$	0	$\mathbb{Z}_{(2)}$	0	$\mathbb{Z}_{(2)}$
$-2l$	$\cdots$	0	0	0	0	0

In this case we see that all groups along the $p+q = 0$ diagonal are zero groups. It follows that

$$\{\mathbb{C}P^l, \Sigma\mathbb{C}P^\infty_{l-2}\} = 0.$$

The case $l \equiv 3 \mod 8$

The E_2-page of the AHSS is presented below. In this case the only nontrivial group along the $p+q=0$ diagonal is $E_2^{2l,-2l} = \mathbb{Z}/2$. Since l is odd, the d_2 differential $d_2^{2l-2,-(2l-1)}$ is induced by the null map. So this $\mathbb{Z}/2$ survives to the E_4-page.

	$\cdots$	$2l-4$	$2l-3$	$2l-2$	$2l-1$	$2l$
$\geq -(2l-4)$	$\cdots$	0	0	0	0	0
$-(2l-3)$	$\cdots$	$\mathbb{Z}_{(2)}$	0	$\mathbb{Z}_{(2)}$	0	$\mathbb{Z}_{(2)}$
$-(2l-2)$	$\cdots$	0	0	0	0	0
$-(2l-1)$	$\cdots$	$\mathbb{Z}_{(2)}$	0	$\mathbb{Z}_{(2)}$	0	$\mathbb{Z}_{(2)}$
$-2l$	$\cdots$	$\mathbb{Z}/2$	0	$\mathbb{Z}/2$	0	$\mathbb{Z}/2$

It remains to analyze the d_4-differential $d_4^{2l-4,-(2l-3)}$, a homomorphism $\mathbb{Z}_{(2)} \to \mathbb{Z}/2$. By Lemma 4.4, this differential is induced by 2ν. Since $l-2 \equiv 1 \mod 8$, it follows from the corresponding Adams Spectral Sequence for $\Sigma\mathbb{C}P^\infty_{l-2}$ that $d_4^{2l-4,-(2l-3)} = 0$. We conclude that

$$\{\mathbb{C}P^l, \Sigma\mathbb{C}P^\infty_{l-2}\} \cong \mathbb{Z}/2.$$

The case $l \equiv 4 \mod 8$

The E_2-page of the AHSS in this case is the same as that in the case $l \equiv 0 \mod 8$. Similar arguments show that

$$\{\mathbb{C}P^l, \Sigma\mathbb{C}P^\infty_{l-2}\} = 0.$$

The case $l \equiv 5 \mod 8$

The E_2-page of the AHSS in this case is the same as that in the case $l \equiv 1 \mod 8$. The exact same argument shows that

$$\{\mathbb{C}P^l, \Sigma\mathbb{C}P^\infty_{l-2}\} = 0.$$

The case $l \equiv 6 \mod 8$

The E_2-page of the AHSS is presented below.

	$2l-4$	$2l-3$	$2l-2$	$2l-1$	$2l$
$\geq -(2l-4)$	0	0	0	0	0
$-(2l-3)$	$\mathbb{Z}_{(2)}$	0	$\mathbb{Z}_{(2)}$	0	$\mathbb{Z}_{(2)}$
$-(2l-2)$	$\mathbb{Z}/2$	0	$\mathbb{Z}/2$	0	$\mathbb{Z}/2$
$-(2l-1)$	$\mathbb{Z}_{(2)} \oplus \mathbb{Z}/2$	0	$\mathbb{Z}_{(2)} \oplus \mathbb{Z}/2$	0	$\mathbb{Z}_{(2)} \oplus \mathbb{Z}/2$
$-2l$	$\mathbb{Z}/4$	0	$\mathbb{Z}/4$	0	$\mathbb{Z}/4$

In this case l is even, so the d_2 differentials $d_2^{2l-2,*}$ from the $p = 2l-2$ column is induced by multiplication by η. Since $l-2 \equiv 4 \mod 8$, it follows from the corresponding Adams Spectral Sequence for $\Sigma\mathbb{C}P^\infty_{l-2}$ that

1. $d_2^{2l-2,-(2l-3)}$ is the surjection $\mathbb{Z}_{(2)} \to \mathbb{Z}/2$;

2. $d_2^{2l-2,-(2l-2)} : \mathbb{Z}/2 \to \mathbb{Z}_{(2)} \oplus \mathbb{Z}/2$ is the inclusion into the second summand;

3. $d_2^{2l-2,-(2l-1)} : \mathbb{Z}_{(2)} \oplus \mathbb{Z}/2 \to \mathbb{Z}/4$ is the zero homomorphism;

Therefore, the circled group $\mathbb{Z}/2$ along the diagonal $p+q = 0$ both disappears in the E_3-page, while the $\mathbb{Z}/4$ survives to E_4. It remains to analyze the d_4-differential $d_4^{2l-4,-(2l-3)}$, a homomorphism $\mathbb{Z}_{(2)} \to \mathbb{Z}/4$. By Lemma 4.4, this differential is induced by 2ν. Since $l-2 \equiv 4 \mod 8$, it follows from the corresponding Adams Spectral Sequence for $\Sigma\mathbb{C}P^{\infty}_{l-2}$ that $d_4^{2l-4,-(2l-3)}(1) = 2$. The cokernel of $d_4^{2l-4,-(2l-3)}$ is therefore $\mathbb{Z}/2$, which survives to the infinity page. We conclude that

$$\{\mathbb{C}P^l, \Sigma\mathbb{C}P^{\infty}_{l-2}\} \cong \mathbb{Z}/2.$$

The case $l \equiv 7 \mod 8$

The E_2-page of the AHSS is presented below.

	$2l-4$	$2l-3$	$2l-2$	$2l-1$	$2l$
$\geq -(2l-4)$	0	0	0	0	0
$-(2l-3)$	$\mathbb{Z}_{(2)}$	0	$\mathbb{Z}_{(2)}$	0	$\mathbb{Z}_{(2)}$
$-(2l-2)$	0	0	0	0	0
$-(2l-1)$	$\mathbb{Z}_{(2)}$	0	$\mathbb{Z}_{(2)}$	0	$\mathbb{Z}_{(2)}$
$-2l$	$\mathbb{Z}/4$	0	$\mathbb{Z}/4$	0	$\mathbb{Z}/4$

In this case the only nontrivial group along the $p+q = 0$ diagonal is $E_2^{2l,-2l} = \mathbb{Z}/4$. Since l is odd, the d_2 differential $d_2^{2l-2,-(2l-1)}$ is induced by the null map. So this $\mathbb{Z}/4$ survives to the E_4-page. It remains to analyze the d_4-differential $d_4^{2l-4,-(2l-3)}$, a homomorphism $\mathbb{Z}_{(2)} \to \mathbb{Z}/4$. By Lemma 4.4, this differential is induced by the null map, hence must vanish. So the $\mathbb{Z}/4$ along the $p + q = 0$ diagonal survives to the infinity page, and we conclude that

$$\{\mathbb{C}P^l, \Sigma\mathbb{C}P^\infty_{l-2}\} \cong \mathbb{Z}/4.$$

The proof of Theorem 4.2 is now complete.

Calculations at the prime 3

In this subsection we prove Theorem 4.3, and we work 3-locally throughout. At the prime 3, there is one possibly nonzero d_4-differential in the AHSS to be determined. This differential reflects the structure of $\mathbb{C}P^l/\mathbb{C}P^{l-3}$, and in some cases the bottom cell and the top cell in this stunted projective space are related by the 3-primary Steenrod operation $\mathcal{P}^1$, which detects the generator of the third stable stem at the prime 3.

Our strategy for the 3-local calculation is exactly the same as in the 2-local case. First we calculate some 3-local stable homotopy groups of $\Sigma\mathbb{C}P^\infty_{l-2}$. In the Appendix we shall prove the following 3-primary analogue of Lemma 4.1.

Lemma 4.5. *The 3-primary stable homotopy groups $\pi^s_i(\Sigma\mathbb{C}P^\infty_n)$ for $i \leq 2n+4$ are as follows.*

1. $\pi^s_i(\Sigma\mathbb{C}P^\infty_n) = 0$ *for* $i \leq 2n$.
2. $\pi^s_{2n+1}(\Sigma\mathbb{C}P^\infty_n) = \mathbb{Z}_{(3)}$, $\pi^s_{2n+2}(\Sigma\mathbb{C}P^\infty_n) = 0$, *and* $\pi^s_{2n+3}(\Sigma\mathbb{C}P^\infty_n) = \mathbb{Z}_{(3)}$.
3. $\pi^s_{2n+4}(\Sigma\mathbb{C}P^\infty_n) = 0$ *if* $n = 1, 2 \mod 3$, *and* $\pi^s_{2n+4}(\Sigma\mathbb{C}P^\infty_n) = \mathbb{Z}/3$ *if* $n = 0 \mod 3$.

	$2l-4$	$2l-3$	$2l-2$	$2l-1$	$2l$	$>2l$
$\geq -(2l-4)$	0	0	0	0	0	0
$-(2l-3)$	$\mathbb{Z}_{(3)}$	0	$\mathbb{Z}_{(3)}$	0	$\mathbb{Z}_{(3)}$	0
$-(2l-2)$	0	0	0	0	0	0
$-(2l-1)$	$\mathbb{Z}_{(3)}$	0	$\mathbb{Z}_{(3)}$	0	$\mathbb{Z}_{(3)}$	0
$-2l$	0	0	0	0	0	0

FIGURE 8. The 3-primary AHSS E_2-page for $\{\mathbb{C}P^l, \Sigma^{p+q+1}\mathbb{C}P^\infty_{l-2}\}$, $l \equiv 0, 1(3)$.

We can now compute $\{\mathbb{C}P^l, \Sigma\mathbb{C}P^\infty_{l-2}\}$ via the AHSS which has

$$E_2^{p,q} = H^q\big(\mathbb{C}P^l; \pi^s_{-q}(\Sigma\mathbb{C}P^\infty_{l-2})\big) \Longrightarrow \{\mathbb{C}P^l, \Sigma^{p+q+1}\mathbb{C}P^\infty_{l-2}\}.$$

When $l \equiv 0, 1 \mod 3$, the E_2-page of the spectral sequence is presented in Figure 8.

Note that $E_2^{p,q}$ vanish for all $p > 2l$ and for all $q > -(2l-3)$. In particular, all groups are zero along the diagonal $p+q=0$. Thus $\{\mathbb{C}P^l, \Sigma\mathbb{C}P^\infty_{l-2}\} = 0$.

When $l \equiv 2 \mod 3$, the E_2-page of the spectral sequence is presented in Figure 9.

In this case the only nontrivial group along the diagonal $p+q=0$ is the circled $E_2^{2l,-2l} \cong \mathbb{Z}/3$, and the only possible d_2-differential hitting this group is $d_2^{2l-2,-(2l-1)} : E_2^{2l-2,-(2l-1)} \to E_2^{2l,-2l}$. This differential is either induced by the null map or some suspension of $\eta : S^3 \to S^2$, but since η is 3-locally null the differential must vanish. So this copy of $\mathbb{Z}/3$ survives to the E_4-page.

	$2l-4$	$2l-3$	$2l-2$	$2l-1$	$2l$	$>2l$
$\geq -(2l-4)$	0	0	0	0	0	0
$-(2l-3)$	$\mathbb{Z}_{(3)}$	0	$\mathbb{Z}_{(3)}$	0	$\mathbb{Z}_{(3)}$	0
$-(2l-2)$	0	0	0	0	0	0
$-(2l-1)$	$\mathbb{Z}_{(3)}$	0	$\mathbb{Z}_{(3)}$	0	$\mathbb{Z}_{(3)}$	0
$-2l$	$\mathbb{Z}/3$	0	$\mathbb{Z}/3$	0	$\mathbb{Z}/3$	0

FIGURE 9. The 3-primary AHSS E_2-page for $\{\mathbb{C}P^l, \Sigma^{p+q+1}\mathbb{C}P^\infty_{l-2}\}$, $l \equiv 2(3)$.

There is a possibly nonzero differential $d_4^{2l-4,-(2l-3)}$ hitting $E_4^{2l,-2l} \cong \mathbb{Z}/3$. By the construction of the spectral sequence, this homomorphism $\mathbb{Z}_{(3)} \to \mathbb{Z}/3$ is induced by a map $S^{2l} = \mathbb{C}P^l/\mathbb{C}P^{l-1} \to \Sigma\mathbb{C}P^{l-2}/\mathbb{C}P^{l-3} = S^{2l-3}$, which belongs to the (3-local) third stable stem $\pi_3^s(S^0) \cong \mathbb{Z}/3$. We claim that this map must be null. Indeed, if it was essential then it must be detected by the 3-primary Steenrod operation $\mathcal{P}^1$, but $\mathcal{P}^1(x^{l-2}) = 0$ when $l = 2 \mod 3$. So we conclude that the differential $d_4^{2l-4,-(2l-3)}$ is zero when $l = 2 \mod 3$, and hence that $\{\mathbb{C}P^l, \Sigma\mathbb{C}P^\infty_{l-2}\} = \mathbb{Z}/3$ in this case.

We have proved that, 3-locally, $\{\mathbb{C}P^l, \Sigma\mathbb{C}P^\infty_{l-2}\}$ is zero if $l = 0, 1 \mod 3$, and is isomorphic to $\mathbb{Z}/3$ when $l = 2 \mod 3$. Theorem 4.3 then follows.

4.4. Proof of Propositions 1.3 and 1.4

Finally, we prove Propositions 1.3 and 1.4. The former deals with the action of the top cell, and the latter concerns extending a given rank $l-2$ bundle over $\mathbb{C}P^{l-1}$ to $\mathbb{C}P^{l}$.

Given a rank r bundle over $\mathbb{C}P^l$, let $\mathbb{C}P^l \to \mathbb{C}P^l \vee S^{2l}$ be the pinch map with which we define the action of the $\mathrm{Vect}_r(S^{2l})$ on $\mathrm{Vect}_r(\mathbb{C}P^l)$ in the Puppe sequence

$$\cdots \to [S^{2l}, BU(r)] \to [\mathbb{C}P^l, BU(r)] \to [\mathbb{C}P^{l-1}, BU(r)].$$

Restricting to bundles with trivial Chern data, one obtains the action of $\mathrm{Vect}^0_r(S^{2l})$ on $\mathrm{Vect}^0_r(\mathbb{C}P^l)$ in the Puppe sequence

$$\cdots \to \{S^{2l}, \Sigma\mathbb{C}P^\infty_r\} \to \{\mathbb{C}P^l, \Sigma\mathbb{C}P^\infty_r\} \to \{\mathbb{C}P^{l-1}, \Sigma\mathbb{C}P^\infty_r\}.$$

Let's first consider the case $r = l-1$. By Theorem 1.1, the action can be nontrivial only when l is odd. In Section 4.2 we proved the isomorphism $\{S^{2l}, \Sigma\mathbb{C}P^\infty_r\} \cong \{\mathbb{C}P^l, \Sigma\mathbb{C}P^\infty_{l-1}\} \cong \mathbb{Z}/2$ when l is odd. This implies that $\mathrm{Vect}^0_{l-1}(S^{2l})$ acts transitively and freely on $\mathrm{Vect}^0_{l-1}(\mathbb{C}P^l)$ in this case.

We then investigate the case $r = l-2$. At the prime 2, groups $\mathrm{Vect}^0_{l-2}(S^{2l})$, $\mathrm{Vect}^0_{l-2}(\mathbb{C}P^l)$, and $\mathrm{Vect}^0_{l-2}(\mathbb{C}P^l)$ can be summarized as in Table 7, according to Lemma 4.1, Theorem 4.2, and Theorem 1.1, respectively.

$l \mod 8$	0	1	2	3	4	5	6	7
$\{S^{2l}, \Sigma\mathbb{C}P^\infty_{l-2}\}$	$\mathbb{Z}/2$	0	$\mathbb{Z}/8$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	0	$\mathbb{Z}/4$	$\mathbb{Z}/4$
$\{\mathbb{C}P^l, \Sigma\mathbb{C}P^\infty_{l-2}\}$	0	0	$\mathbb{Z}/4$	$\mathbb{Z}/2$	0	0	$\mathbb{Z}/2$	$\mathbb{Z}/4$
$\{\mathbb{C}P^{l-1}, \Sigma\mathbb{C}P^\infty_{l-2}\}$	$\mathbb{Z}/2$	0	$\mathbb{Z}/2$	0	$\mathbb{Z}/2$	0	$\mathbb{Z}/2$	0

TABLE 7. The 2-primary $\mathrm{Vect}^0_{l-2}(S^{2l})$, $\mathrm{Vect}^0_{l-2}(\mathbb{C}P^l)$, and $\mathrm{Vect}^0_{l-2}(\mathbb{C}P^l)$.

So the action of $\mathrm{Vect}^0_{l-2}(S^{2l})$ on $\mathrm{Vect}^0_{l-2}(\mathbb{C}P^l)$ can only be nontrivial when l equals $2, 3, 6$, or 7 mod 8. Our analysis on the AHSS in Section 4.3.1 implies that when $l = 2$ mod 8, the homomorphism $\{\mathbb{C}P^l, \Sigma\mathbb{C}P^\infty_{l-2}\} \to \{\mathbb{C}P^{l-1}, \Sigma\mathbb{C}P^\infty_{l-2}\}$ is the zero homomorphism $\mathbb{Z}/4 \to \mathbb{Z}/2$. Similarly, one can prove that $\{\mathbb{C}P^l, \Sigma\mathbb{C}P^\infty_{l-2}\} \to \{\mathbb{C}P^{l-1}, \Sigma\mathbb{C}P^\infty_{l-2}\}$ is a zero homomorphism in all other cases. One now concludes that $\{S^{2l}, \Sigma\mathbb{C}P^\infty_{l-2}\} \to \{\mathbb{C}P^l, \Sigma\mathbb{C}P^\infty_{l-2}\}$ is surjective when $l = 2, 6$ mod 8, and is an isomorphism when $l = 3, 7$ mod 8. This implies that the action of $\mathrm{Vect}^0_{l-2}(S^{2l})$ on $\mathrm{Vect}^0_{l-2}(\mathbb{C}P^l)$ is transitive and free when $l = 3, 7$ mod 8, and is transitive but not free when $l = 2, 6$ mod 8. At the prime 3, similar analysis shows that the action of $\mathrm{Vect}^0_{l-2}(S^{2l})$ on $\mathrm{Vect}^0_{l-2}(\mathbb{C}P^l)$ is transitive and free when $l = 2$ mod 3, and is trivial in all other cases. Combining the information at primes 2 and 3 yields Proposition 1.3.

The above discussion also proves Proposition 1.4, since $\{\mathbb{C}P^l, \Sigma\mathbb{C}P^\infty_{l-2}\} \to \{\mathbb{C}P^{l-1}, \Sigma\mathbb{C}P^\infty_{l-2}\}$ is always the zero homomorphism.

APPENDIX

SOME RELATED STABLE HOMOTOPY CALCULATIONS

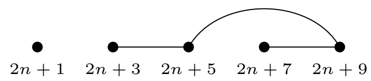
Here we present the details of the calculation of the first few stable homotopy groups of $\Sigma\mathbb{C}P_n^\infty$. Namely, we complete the proof of Lemma 4.1 and Lemma 4.5, which compute, respectively, some first 2-local and 3-local stable homotopy groups of $\Sigma\mathbb{C}P_n^\infty$.

First note that $\Sigma\mathbb{C}P_n^\infty$ is $2n$-connected, and hence $\pi_i^s(\Sigma\mathbb{C}P_n^\infty) = 0$ for $i \leq 2n$. Secondly, $\pi_i^s(\Sigma\mathbb{C}P_n^\infty) = 0$ for $2n+1 \leq i \leq 2n+4$ is controlled by the structure of the stunted projective space $\mathbb{C}P_n^{n+4} = \mathbb{C}P^{n+4}/\mathbb{C}P^{n-1}$.

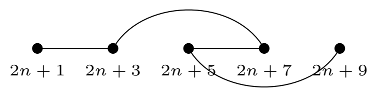
A.1. Proof of Lemma 4.1

We start with the 2-primary calculations. The action of the mod 2 Steenrod algebra on the mod 2 cohomology of $\Sigma\mathbb{C}P_n^{n+4}$ exhibits the following 8-fold periodic behavior. We present these actions in terms of diagrams as follows.

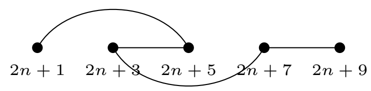
$n = 0 \mod 8$



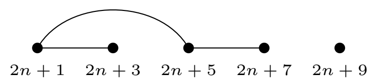
$n = 1 \mod 8$



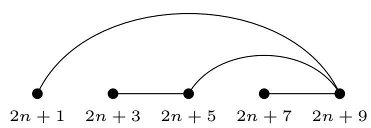
$n = 2 \mod 8$



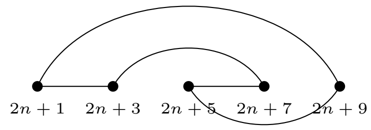
$n = 3 \mod 8$



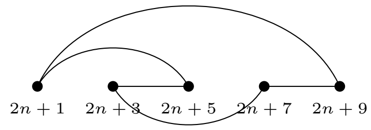
$n = 4 \mod 8$



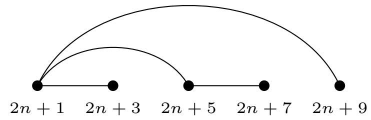
$n = 5 \mod 8$



$n = 6 \mod 8$



$n = 7 \mod 8$

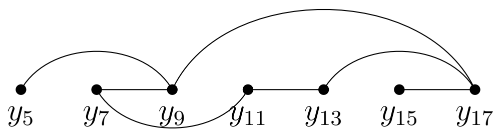


We have constructed explicit minimal $\mathcal{A}$-resolutions

$$\cdots \to P_s \xrightarrow{\partial_s} \cdots P_2 \xrightarrow{\partial_2} P_1 \xrightarrow{\partial_1} P_0 \xrightarrow{\epsilon} H^*(\mathbb{C}P_n^{n+4}) \to 0$$

in each case to compute Adams E_2 pages. We present details in only one example, namely the case $n = 2$, with other cases being similar. In all cases, our hand calculations were kindly verified by Robert Bruner using his computer code [9].

Focusing on the case of $\Sigma\mathbb{C}P_2^\infty$, which is needed to prove Lemma 3.1, we recall that $H^*(\Sigma\mathbb{C}P_2^\infty)$ has the following action of $\mathcal{A}$ in low degrees.



<u>Filtration $s = 0$</u>. To define P_0 which surjects onto $H^*(\Sigma\mathbb{C}P_2^\infty)$, we introduce a free generator $e_{0,5}$ in degree 5 to kill y_5, and a free generator $e_{0,7}$ in degree 7 to kill y_7. That is,

$$P_0 := \mathcal{A}e_{0,5} \oplus \mathcal{A}e_{0,7} \oplus \cdots, \quad \epsilon(e_{0,5}) = y_5, \epsilon(e_{0,7}) = y_7, \cdots$$

The next free generator to introduce would be $e_{0,15}$ in degree 15 so that $\epsilon(e_{0,15}) = y_{15}$, so P_0 has no summands generated in degree t for $7 < t < 15$.

A basis for $\mathrm{Ker}(\epsilon)$ in degrees $t \leq 13$ is presented below.

Deg 6	$\mathrm{Sq}^1 e_{0,5}$			
Deg 7	$\mathrm{Sq}^2 e_{0,5}$			
Deg 8	$\mathrm{Sq}^3 e_{0,5}$	$\mathrm{Sq}^2\mathrm{Sq}^1 e_{0,5}$	$\mathrm{Sq}^1 e_{0,7}$	
Deg 9		$\mathrm{Sq}^3\mathrm{Sq}^1 e_{0,5}$	$\mathrm{Sq}^4 e_{0,5} + \mathrm{Sq}^2 e_{0,7}$	
Deg 10	$\mathrm{Sq}^5 e_{0,5}$	$\mathrm{Sq}^4\mathrm{Sq}^1 e_{0,5}$	$\mathrm{Sq}^3 e_{0,7}$	$\mathrm{Sq}^2\mathrm{Sq}^1 e_{0,7}$
Deg 11	$\mathrm{Sq}^6 e_{0,5}$	$\mathrm{Sq}^5\mathrm{Sq}^1 e_{0,5}$	$\mathrm{Sq}^4\mathrm{Sq}^2 e_{0,5}$	$\mathrm{Sq}^3\mathrm{Sq}^1 e_{0,7}$
Deg 12	$\mathrm{Sq}^7 e_{0,5}$	$\mathrm{Sq}^6\mathrm{Sq}^1 e_{0,5}$	$\mathrm{Sq}^5\mathrm{Sq}^2 e_{0,5}$	$\mathrm{Sq}^4\mathrm{Sq}^2\mathrm{Sq}^1 e_{0,5}$
			$\mathrm{Sq}^5 e_{0,7}$	$\mathrm{Sq}^4\mathrm{Sq}^1 e_{0,7}$
Deg 13	$\mathrm{Sq}^8 e_{0,5}$	$\mathrm{Sq}^7\mathrm{Sq}^1 e_{0,5}$	$\mathrm{Sq}^6\mathrm{Sq}^2 e_{0,5}$	$\mathrm{Sq}^4\mathrm{Sq}^2\mathrm{Sq}^1 e_{0,5}$
			$\mathrm{Sq}^5\mathrm{Sq}^1 e_{0,7}$	$\mathrm{Sq}^4\mathrm{Sq}^2 e_{0,7}$

<u>Filtration $s = 1$</u>. With the kernel of the surjection ϵ from P_0 to the cohomology of $\Sigma\mathbb{C}P_2^\infty$ in hand, we construct P_1 together with a surjection onto $\mathrm{Ker}(\epsilon)$. We define

$$P_1 := \mathcal{A}e_{1,6} \oplus \mathcal{A}e_{1,7} \oplus \mathcal{A}e_{1,8} \oplus \mathcal{A}e_{1,9} \oplus \cdots ,$$

$$\partial_1 e_{1,6} = \mathrm{Sq}^1 e_{0,5}, \partial_1 e_{1,7} = \mathrm{Sq}^2 e_{0,5}, \partial_1 e_{1,8} = \mathrm{Sq}^1 e_{0,7}, \partial_1 e_{1,9} = \mathrm{Sq}^4 e_{0,5} + \mathrm{Sq}^2 e_{0,7}, \cdots .$$

Note that the first element of $\mathrm{Ker}(\epsilon)$ that is not yet in $\partial_1(\mathcal{A}\rceil_{\infty,/}\oplus \mathcal{A}\rceil_{\infty,'} \oplus \mathcal{A}\rceil_{\infty,\forall} \oplus \mathcal{A}\rceil_{\infty,\exists})$ is $\mathrm{Sq}^8 e_{0,5}$ in degree 13. So the next free generator to introduce to P_1 is $e_{1,13}$ so that $\partial_1 e_{1,13} = \mathrm{Sq}^8 e_{0,5}$. In particular, we see that P_1 has no components of degree t for $9 < t < 13$. Furthermore, we note that $e_{1,6}$ is connected with $e_{0,5}$ by h_0,

$e_{1,7}$ is connected with $e_{0,5}$ by h_1, $e_{1,7}$ is connected with $e_{0,7}$ by h_0, $e_{1,9}$ is connected with $e_{0,5}$ by h_2 and with $e_{0,7}$ by h_1.

A basis for $\mathrm{Ker}(\partial_1)$ in degrees $t \leq 13$ is presented below.

Deg 7	$\mathrm{Sq}^1 e_{1,6}$			
Deg 8				
Deg 9	$\mathrm{Sq}^2\mathrm{Sq}^1 e_{1,6}$	$\mathrm{Sq}^3 e_{1,6} + \mathrm{Sq}^2 e_{1,7}$	$\mathrm{Sq}^1 e_{1,8}$	
Deg 10	$\mathrm{Sq}^3\mathrm{Sq}^1 e_{1,6}$	$\mathrm{Sq}^3 e_{1,7}$		
Deg 11	$\mathrm{Sq}^4\mathrm{Sq}^1 e_{1,6}$	$\mathrm{Sq}^5 e_{1,6} + \mathrm{Sq}^3\mathrm{Sq}^1 e_{1,7}$	$\mathrm{Sq}^2\mathrm{Sq}^1 e_{1,8}$	
Deg 12	$\mathrm{Sq}^5\mathrm{Sq}^1 e_{1,6}$	$\mathrm{Sq}^5 e_{1,7} + \mathrm{Sq}^4\mathrm{Sq}^1 e_{1,7}$	$\mathrm{Sq}^3\mathrm{Sq}^1 e_{1,8}$	
Deg 13	$\mathrm{Sq}^6\mathrm{Sq}^1 e_{1,6}$	$\mathrm{Sq}^5\mathrm{Sq}^2 e_{1,6} + \mathrm{Sq}^4\mathrm{Sq}^2 e_{1,7}$	$\mathrm{Sq}^4\mathrm{Sq}^1 e_{1,8}$	
		$\mathrm{Sq}^7 e_{1,6} + \mathrm{Sq}^5 e_{1,8} + \mathrm{Sq}^3\mathrm{Sq}^1 e_{1,9}$	$\mathrm{Sq}^4\mathrm{Sq}^2\mathrm{Sq}^1 e_{1,6}$	$\mathrm{Sq}^5\mathrm{Sq}^1 e_{1,7}$

<u>Filtration $s = 2$</u>. We construct P_2 together with a surjection onto $\mathrm{Ker}(\partial_1)$. We define

$$P_2 := \mathcal{A}e_{2,7} \oplus \mathcal{A}e_{2,9} \oplus \mathcal{A}e'_{2,9} \oplus \cdots ,$$

$$\partial_2 e_{2,7} = \mathrm{Sq}^1 e_{1,6}, \partial_2 e_{2,9} = \mathrm{Sq}^1 e_{1,8}, \partial_2 e'_{2,9} = \mathrm{Sq}^3 e_{1,6} + \mathrm{Sq}^2 e_{1,7}, \cdots .$$

Note that the first element of $\mathrm{Ker}(\epsilon)$ that is not yet in $\partial_2(\mathcal{A}\rceil_{\in,\prime}\oplus\mathcal{A}\rceil_{\in,\exists}\oplus\mathcal{A}\rceil'_{\in,\exists})$ is $\mathrm{Sq}^7 e_{1,6} + \mathrm{Sq}^5 e_{1,8} + \mathrm{Sq}^3\mathrm{Sq}^1 e_{1,9}$ in degree 13. So the next free generator to introduce to P_2 is $e_{2,13}$ so that $\partial_2 e_{2,13} = \mathrm{Sq}^7 e_{1,6} + \mathrm{Sq}^5 e_{1,8} + \mathrm{Sq}^3\mathrm{Sq}^1 e_{1,9}$. In particular, we see that P_2 has no summands of degree t for $9 < t < 13$. Furthermore, $e_{2,7}$ is connected with $e_{1,6}$ by h_0, $e_{2,9}$ is connected with $e_{1,8}$ by h_0, and $e'_{2,9}$ is connected with $e_{1,7}$ by h_1.

A basis for $\mathrm{Ker}(\partial_2)$ in degrees $t \leq 13$ is presented below.

Deg 8	$\mathrm{Sq}^1 e_{2,7}$		
Deg 9			
Deg 10	$\mathrm{Sq}^2\mathrm{Sq}^1 e_{2,7}$	$\mathrm{Sq}^1 e_{2,9}$	
Deg 11	$\mathrm{Sq}^3\mathrm{Sq}^1 e_{2,7}$		
Deg 12	$\mathrm{Sq}^4\mathrm{Sq}^1 e_{2,7}$	$\mathrm{Sq}^2\mathrm{Sq}^1 e_{2,9}$	$\mathrm{Sq}^5 e_{2,7} + \mathrm{Sq}^3 e_{2,9}$
Deg 13	$\mathrm{Sq}^5\mathrm{Sq}^1 e_{2,7}$	$\mathrm{Sq}^3\mathrm{Sq}^1 e_{2,9}$	

Filtration $s = 3$. We construct P_3 together with a surjection onto $\mathrm{Ker}(\partial_2)$. We define

$$P_3 := \mathcal{A}e_{3,8} \oplus \mathcal{A}e_{3,10} \oplus \mathcal{A}e_{3,12} \oplus \cdots ,$$

$$\partial_3 e_{3,8} = \mathrm{Sq}^1 e_{2,7}, \partial_3 e_{3,10} = \mathrm{Sq}^1 e_{2,9}, \partial_3 e_{3,12} = \mathrm{Sq}^5 e_{2,7} + \mathrm{Sq}^3 e_{2,9}, \cdots .$$

Every element of $\mathrm{Ker}(\partial_2)$ in the range given above is contained in $\partial_3(\mathcal{A}\rceil_{\ni,\forall} \oplus \mathcal{A}\rceil_{\ni,\infty\prime} \oplus \mathcal{A}\rceil_{\ni,\infty\in})$. Furthermore, $e_{3,8}$ is connected with $e_{2,7}$ by h_0, and $e_{3,10}$ is connected with $e_{2,9}$ by h_0.

A basis for $\mathrm{Ker}(\partial_3)$ in degrees $t \leq 13$ is presented below.

Deg 9	$\mathrm{Sq}^1 e_{3,8}$	
Deg 10		
Deg 11	$\mathrm{Sq}^2\mathrm{Sq}^1 e_{3,8}$	$\mathrm{Sq}^1 e_{3,10}$
Deg 12	$\mathrm{Sq}^3\mathrm{Sq}^1 e_{3,8}$	
Deg 13	$\mathrm{Sq}^4\mathrm{Sq}^1 e_{3,8}$	$\mathrm{Sq}^2\mathrm{Sq}^1 e_{3,10}$

Filtration $s = 4$. We construct P_4 together with a surjection onto $\mathrm{Ker}(\partial_3)$. We define

$$P_3 := \mathcal{A}e_{4,9} \oplus \mathcal{A}e_{4,11} \oplus \cdots ,$$

$$\partial_4 e_{4,9} = \mathrm{Sq}^1 e_{3,8}, \partial_4 e_{4,11} = \mathrm{Sq}^1 e_{3,10}, \cdots .$$

Every element of $\mathrm{Ker}(\partial_3)$ in the above range is contained in $\partial_4(\mathcal{A}\rceil_{\triangle,\exists} \oplus \mathcal{A}\rceil_{\triangle,\infty\infty})$. Furthermore, $e_{4,9}$ is connected with $e_{3,8}$ by h_0, and $e_{4,11}$ is connected with $e_{3,10}$ by h_0.

<u>Filtration $s > 4$</u>. Inductively, suppose that the basis for $\mathrm{Ker}(\partial_s)$ $(s \geq 4)$ in degrees $t \leq s + 8$ can be taken as follows.

Deg	s+6	$\mathrm{Sq}^1 e_{s,s+5}$	
Deg	s+7		
Deg	s+8	$\mathrm{Sq}^2\mathrm{Sq}^1 e_{s,s+5}$	$\mathrm{Sq}^1 e_{s,s+7}$
Deg	s+9	$\mathrm{Sq}^3\mathrm{Sq}^1 e_{s,s+5}$	
Deg	s+10	$\mathrm{Sq}^4\mathrm{Sq}^1 e_{s,s+5}$	$\mathrm{Sq}^2\mathrm{Sq}^1 e_{s,s+7}$

Then to build P_{s+1} together with a surjection onto $\mathrm{Ker}(\partial_s)$, we let

$$P_{s+1} := \mathcal{A}e_{s+1,s+6} \oplus \mathcal{A}e_{s+1,s+8} \oplus \cdots ,$$

$$\partial_{s+1}e_{s+1,s+6} = \mathrm{Sq}^1 e_{s,s+5}, \partial_{s+1}e_{s+1,s+8} = \mathrm{Sq}^1 e_{s,s+7}.$$

Thus every element of $\mathrm{Ker}(\partial_s)$ in the above range is contained in $\partial_{s+1}(\mathcal{A}\rceil_{\int+\infty,\int+/}\oplus \mathcal{A}\rceil_{\int+\infty,\int+\forall})$. Furthermore, $e_{s+1,s+6}$ is connected with $e_{s,s+5}$ by h_0, and $e_{s+1,s+8}$ is connected with $e_{s,s+7}$ by h_0. It also follows that a basis for $\mathrm{Ker}(\partial_{s+1})$ in degrees $t \leq s + 9$ can be taken as follows.

Deg	s+7	$\mathrm{Sq}^1 e_{s+1,s+6}$	
Deg	s+8		
Deg	s+9	$\mathrm{Sq}^2\mathrm{Sq}^1 e_{s+1,s+6}$	$\mathrm{Sq}^1 e_{s+1,s+8}$
Deg	s+10	$\mathrm{Sq}^3\mathrm{Sq}^1 e_{s+1,s+6}$	
Deg	s+11	$\mathrm{Sq}^4\mathrm{Sq}^1 e_{s+1,s+6}$	$\mathrm{Sq}^2\mathrm{Sq}^1 e_{s+1,s+8}$

Thus we have established the following result, which is summarized in Figure 1

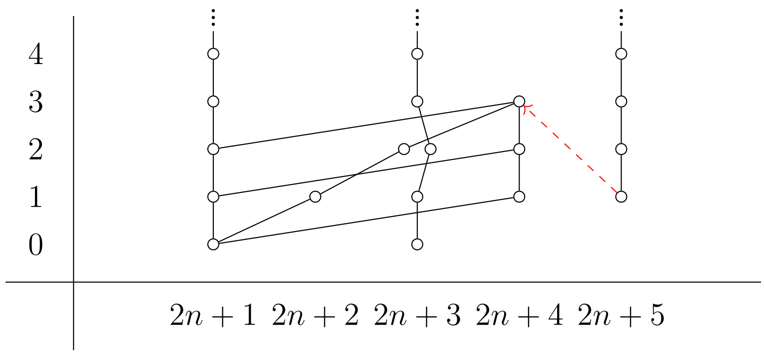
Proposition A.1. *The Adams E_2-page for $\Sigma\mathbb{C}P_2^\infty$ satisfies:*

1. *The groups vanish whenever $t - s \leq 4$.*
2. *There is a single h_0-tower starting at $(s, t-s) = (0, 5)$, and there is nothing else along $t - s = 5$.*
3. *There is a single $\mathbb{Z}/2$ at $(s, t-s) = (1, 6)$, and there is nothing else along $t - s = 6$.*
4. *There is a single h_0-tower starting at $(s, t-s) = (0, 7)$, a single $\mathbb{Z}/2$ at $(s, t-s) = (2, 7)$, and there is nothing else along $t - s = 7$.*
5. *There is a single $\mathbb{Z}/2$ at $(s, t-s) = (1, 8)$, and there is nothing else along $t - s = 8$.*

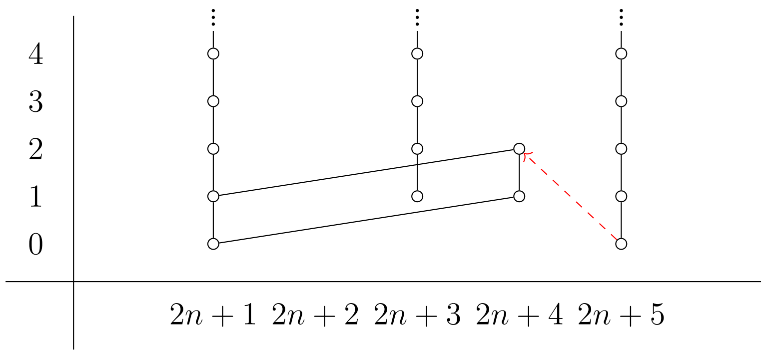
Any differential in the range $t - s \leq 8$ must be trivial, so this is also the Adams E_∞ and thus stable homotopy in this range. More generally if $n = 2$ mod 8, this argument shows that $\Sigma\mathbb{C}P_n^\infty$ has the following stable homotopy groups: $\pi^s_{2n+1}(\Sigma\mathbb{C}P_n^\infty) = \mathbb{Z}_{(2)}$, $\pi^s_{2n+2}(\Sigma\mathbb{C}P_n^\infty) = \mathbb{Z}/2$, $\pi^s_{2n+3}(\Sigma\mathbb{C}P_n^\infty) = \mathbb{Z}_{(2)} \oplus \mathbb{Z}/2$, and $\pi^s_{2n+4}(\Sigma\mathbb{C}P_n^\infty) = \mathbb{Z}/2$.

The other seven cases can be proved in the exact same way. We obtain the following conclusions.

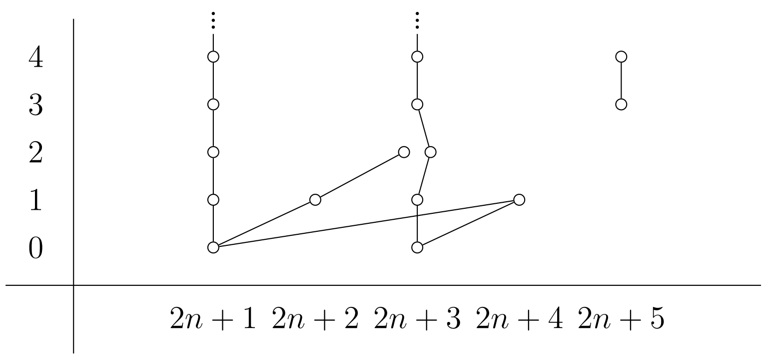
When $n = 0, 4$ mod 8, the Adams E_2-page for $\Sigma\mathbb{C}P_n^\infty$ begins as follows.



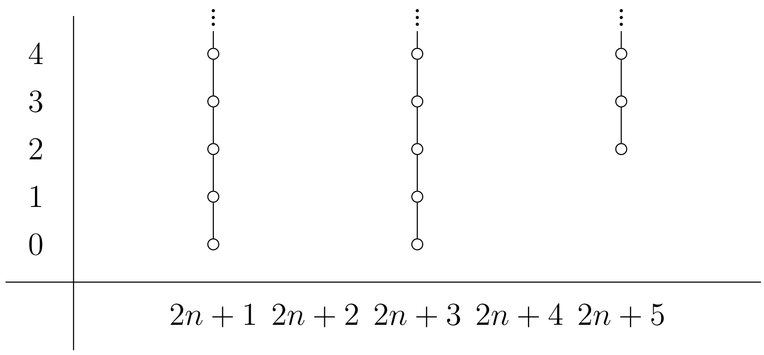
When $n = 1, 5 \bmod 8$, the Adams E_2-page for $\Sigma\mathbb{C}P_n^\infty$ begins as follows.



When $n = 2, 6 \bmod 8$, the Adams E_2-page for $\Sigma\mathbb{C}P_n^\infty$ begins as follows.



When $n = 3, 7 \bmod 8$, the Adams E_2-page for $\Sigma\mathbb{C}P_n^\infty$ begins as follows.



In the last two Adams charts there is no room for any nontrivial differential d_r affecting the region $t - s \leq 2n+4$, for all $r \geq 2$. The following can therefore be concluded immediately.

When $n = 2, 6 \mod 8$, we have that $\pi^s_{2n+1}(\Sigma\mathbb{C}P^\infty_n) = \mathbb{Z}_{(2)}$, that $\pi^s_{2n+2}(\Sigma\mathbb{C}P^\infty_n) = \mathbb{Z}/2$, that $\pi^s_{2n+3}(\Sigma\mathbb{C}P^\infty_n) = \mathbb{Z}_{(2)} \oplus \mathbb{Z}/2$, and that $\pi^s_{2n+4}(\Sigma\mathbb{C}P^\infty_n) = \mathbb{Z}/2$.

When $n = 3, 7 \mod 8$, we have that $\pi^s_{2n+1}(\Sigma\mathbb{C}P^\infty_n) = \mathbb{Z}_{(2)}$, that $\pi^s_{2n+2}(\Sigma\mathbb{C}P^\infty_n) = 0$, that $\pi^s_{2n+3}(\Sigma\mathbb{C}P^\infty_n) = \mathbb{Z}_{(2)}$, and that $\pi^s_{2n+4}(\Sigma\mathbb{C}P^\infty_n) = 0$.

However, in each of the first two Adams charts (i.e., when $n = 0, 4$ or $1, 5 \mod 8$) there is a possible d_2 differential (which is presented as the red dashed arrow in the chart), which has to do with the determination of $\pi^s_{2n+4}(\Sigma\mathbb{C}P^\infty_n)$. To determine these differentials, we recall some classical results.

First a result of Toda [19] relates the stable homotopy groups of $\mathbb{C}P^\infty_n$ to the metastable homotopy groups of unitary groups.

Theorem A.2 (Toda [19]). *Let $0 \leq t < n$. Then $\pi^s_{2n+2t+1}(\mathbb{C}P^\infty_n) = \pi_{2n+2t+1}U(n)$.*

Secondly, the relevant homotopy groups were computed by Matsunaga [11].

Theorem A.3 (Matsunaga [11] [12]). *Two-locally, metastable homotopy groups of $U(n)$ are given as follows*

1. $\pi_{2n+3}U(n) = \mathbb{Z}/8$ *when* $n = 0 \mod 8$.

2. $\pi_{2n+3}U(n) = \mathbb{Z}/4$ *when* $n = 4 \mod 8$.

3. $\pi_{2n+3}U(n) = \mathbb{Z}/2$ *when* $n = 1 \mod 8$.

4. $\pi_{2n+3}U(n) = \mathbb{Z}/4$ *when* $n = 5 \mod 8$.

So by Theorem A.2, $\pi^s_{2n+4}(\Sigma\mathbb{C}P^\infty_n)$ is given by the list of A.3. It follows that the d_2 differentials of interest must be zero when $n = 0, 5 \mod 8$, and must be an isomorphism when $n = 1, 4 \mod 8$. We can now conclude the followings.

1. When $n = 0 \mod 8$, we have that $\pi^s_{2n+1}(\Sigma\mathbb{C}P^\infty_n) = \mathbb{Z}_{(2)}$, that $\pi^s_{2n+2}(\Sigma\mathbb{C}P^\infty_n) = \mathbb{Z}/2$, that $\pi^s_{2n+3}(\Sigma\mathbb{C}P^\infty_n) = \mathbb{Z}_{(2)} \oplus \mathbb{Z}/2$, and that $\pi^s_{2n+4}(\Sigma\mathbb{C}P^\infty_n) = \mathbb{Z}/8$.

2. When $n = 4 \mod 8$, we have that $\pi^s_{2n+1}(\Sigma\mathbb{C}P^\infty_n) = \mathbb{Z}_{(2)}$, that $\pi^s_{2n+2}(\Sigma\mathbb{C}P^\infty_n) = \mathbb{Z}/2$, that $\pi^s_{2n+3}(\Sigma\mathbb{C}P^\infty_n) = \mathbb{Z}_{(2)} \oplus \mathbb{Z}/2$, and that $\pi^s_{2n+4}(\Sigma\mathbb{C}P^\infty_n) = \mathbb{Z}/4$.

3. When $n = 1 \mod 8$, we have that $\pi^s_{2n+1}(\Sigma\mathbb{C}P^\infty_n) = \mathbb{Z}_{(2)}$, that $\pi^s_{2n+2}(\Sigma\mathbb{C}P^\infty_n) = 0$, that $\pi^s_{2n+3}(\Sigma\mathbb{C}P^\infty_n) = \mathbb{Z}_{(2)}$, that $\pi^s_{2n+4}(\Sigma\mathbb{C}P^\infty_n) = \mathbb{Z}/2$.

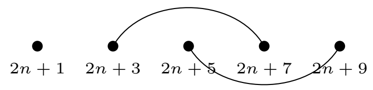
4. When $n = 5 \mod 8$, we have that $\pi^s_{2n+1}(\Sigma\mathbb{C}P^\infty_n) = \mathbb{Z}_{(2)}$, that $\pi^s_{2n+2}(\Sigma\mathbb{C}P^\infty_n) = 0$, that $\pi^s_{2n+3}(\Sigma\mathbb{C}P^\infty_n) = \mathbb{Z}_{(2)}$, and that $\pi^s_{2n+4}(\Sigma\mathbb{C}P^\infty_n) = \mathbb{Z}/4$.

This completes the 2-local computation and proves Lemma 4.1.

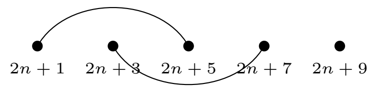
A.2. Proof of Lemma 4.5

We now work 3-locally to prove Lemma 4.5. The strategy is similar. The action of the mod 3 Steenrod algebra on the mod 3 cohomology of $\mathbb{C}P^{n+4}_n$ exhibits the following 3-fold periodicity. We present these actions in terms of diagrams. They correspond, respectively, to cases $n = 0, 1, 2 \bmod 3$. Here each curved segment indicates a nontrivial action of $\mathcal{P}^\infty$.

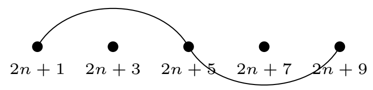
$n = 0 \mod 3$



$n = 1 \mod 3$

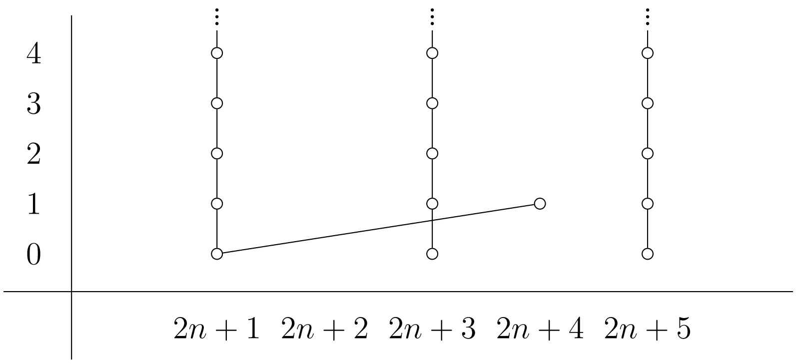


$n = 2 \mod 3$

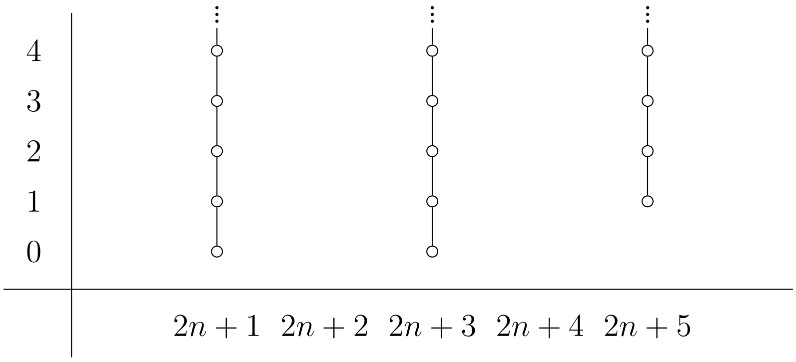


One can then construct explicit minimal resolutions to compute the Adams E_2 page in each case. Note that when $n = 1 \bmod 3$, the resolution can be taken as a degree $2n - 1$ shift of a resolution of $\mathbb{C}P^\infty$, which can be learned from Aikawa [1]. When $n = 0 \bmod 3$, the resolution can be taken as a direct sum of a resolution of $\mathbb{Z}/3$ and that of $\mathbb{C}P^\infty$ followed by a degree shift of $2n + 1$. We omit the details of constructing resolutions and simply provide the Adams charts.

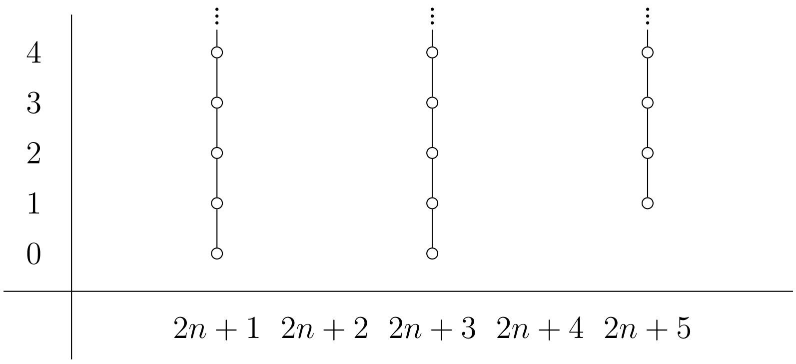
When $n = 0 \bmod 3$, the Adams E_2-page for $\Sigma\mathbb{C}P_n^\infty$ begins as follows.



When $n = 1 \bmod 3$, the beginnings of the Adams E_2-page for $\Sigma\mathbb{C}P_n^\infty$ is as follows.



When $n = 2 \bmod 3$, the beginnings of the Adams E_2-page for $\Sigma\mathbb{C}P_n^\infty$ is as follows.



In all the cases above, there is no room for any differential d_r, for all $r \geq 2$. One can therefore read off the desired 3-local stable homotopy groups immediately. Namely,

1. $\pi^s_{2n+1}(\Sigma\mathbb{C}P_n^\infty) = \mathbb{Z}_{(3)}$, $\pi^s_{2n+2}(\Sigma\mathbb{C}P_n^\infty) = 0$, and $\pi^s_{2n+3}(\Sigma\mathbb{C}P_n^\infty) = \mathbb{Z}_{(3)}$.

2. $\pi^s_{2n+4}(\Sigma\mathbb{C}P_n^\infty)$ exhibits the following 3-fold periodic behavior. It is zero when $n = 0, 1 \bmod 3$, and is $\mathbb{Z}/3$ when $n = 2 \bmod 3$.

This completes the 3-local computations, and proves Lemma 4.5.

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