COMPOSITION AND COBORDISM MAPS

by

JESSE ASHER COHEN

A DISSERTATION

Presented to the Department of Mathematics
and the Division of Graduate Studies of the University of Oregon
in partial fulfillment of the requirements
for the degree of
Doctor of Philosophy

September 2023

DISSERTATION APPROVAL PAGE

Student: Jesse Asher Cohen

Title: Composition and Cobordism Maps

This dissertation has been accepted and approved in partial fulfillment of the requirements for the Doctor of Philosophy degree in the Department of Mathematics by:

Robert Lipshitz	Chair and Advisor
Nicolas Addington	Core Member
Daniel Dugger	Core Member
Ben Elias	Core Member
Hans Dreyer	Institutional Representative

and

Krista Chronister	Vice Provost of Graduate Studies

Original approval signatures are on file with the University of Oregon Graduate School.

Degree awarded September 2023

DISSERTATION ABSTRACT

Jesse Asher Cohen

Doctor of Philosophy

Department of Mathematics

September 2023

Title: Composition and Cobordism Maps

We study the relationship between the algebra of module homomorphisms under composition and 4-dimensional cobordisms in the context of bordered Heegaard Floer homology. In particular, we prove that composition of module homomorphisms of type-D structures induces the pair of pants cobordism map on Heegaard Floer homology in the morphism spaces formulation of the latter, due to Lipshitz–Ozsváth–Thurston. Along the way, we prove a gluing result for cornered 4-manifolds constructed from bordered Heegaard triples.

As applications, we present a new algorithm for computing arbitrary cobordism maps on Heegaard Floer homology and construct new nontrivial A_∞-deformations of Khovanov's arc algebras. Motivated by this last result and a Künneth theorem for Heegaard Floer complexes of connected sums, we also prove the existence of a tensor product decomposition for arc algebras in characteristic 2 and show that there cannot be such a splitting over $\mathbb{Z}$.

CURRICULUM VITAE

NAME OF AUTHOR: Jesse Asher Cohen

GRADUATE AND UNDERGRADUATE SCHOOLS ATTENDED:

University of Oregon, Eugene, OR
University of California, Riverside, Riverside, CA
University of California, Los Angeles, Los Angeles, CA

DEGREES AWARDED:

Doctor of Philosophy, Mathematics, 2023, University of Oregon
Master of Science, Mathematics, 2016, University of California, Riverside
Bachelor of Science, Mathematics, 2014, University of California, Los Angeles

AREAS OF SPECIAL INTEREST:

Low Dimensional Topology

PROFESSIONAL EXPERIENCE:

Teaching Assistant, University of California, Riverside, 2014–2017

Graduate Employee, University of Oregon, 2017–2023

GRANTS, AWARDS AND HONORS:

PUBLICATIONS:

Cohen, J. (2023). Composition Maps in Heegaard Floer Homology. arXiv:2209.01705 (submitted for publication to Geometry & Topology)

Cohen, J. (2022). An Exceptional Splitting of Khovanov's Arc Algebras in Characteristic 2. arXiv:2209.01705 (submitted for publication to Fundamenta Mathematicae)

ACKNOWLEDGEMENTS

I would like to thank my advisor Robert Lipshitz for his invaluable guidance, support, encouragement, and kindness during my time at the University of Oregon.

Immeasurable thanks are due to my parents — Daria, Joe, Joyce, and Mike — without whose unconditional love and encouragement none of this would have ever been possible, and to my extended family — Aaron, Ari, Armi, Chris, Denise, Eva, Gabriella, Jenny, Karl, Ove, and Sienna. Their unwavering belief in my success has sustained me at every turn. Lastly, though very far from least, I owe a lifetime of love and gratitude to my grandparents Gabriella — who turned 101 during the writing of this thesis — and Karl, Julia, Max, and Yvonne, the four of whom are with me now only in memory. I love you all more than words can express.

I also owe an immense dept of gratitude to my friends — Ally, Cordell, Eli, Hypatia, Kieran, Michael, Tera, Warren, Wendryn, and so many more. I wouldn't have made it this far without you!

I would also like to thank my friends in the mathematics community. They have, every one, made me a better mathematician and a better person. Though I owe a part of my success to far too many to list here, I am especially grateful to Allan, Alonso, Andrew, Andy, Arya, Catherine, Champ, Daniel, Gary, Greg, Halley, Holt, Ivo, Karuna, Katrin, Katrina, Kristen, Lawrence, Leigh, Maggie, Marissa, Miriam, Olivia, Siavash, Stewart, Tim, Tzula, Xavier, and Yang for their friendship, encouragement, and many enlightening conversations.

This research was supported in part by the National Science Foundation under Grant Nos. DMS-2204214 and DMS-1928930.

For my parents

TABLE OF CONTENTS

LIST OF FIGURES

Figure Page

CHAPTER I

INTRODUCTION

Nontechnical Introduction

The overarching topic of this dissertation is *low-dimensional topology.* Roughly speaking, topology is the study of intrinsic properties of shapes which remain unchanged when you bend, twist, rotate, or resize them, or otherwise deform them without cutting, poking holes in, or gluing them together to create new complications. The adjective 'low-dimensional', in this context, means that the types of shapes whose properties we will be concerned with will have dimension at most 4, meaning that the number of independent directions one could travel within such a shape is at most 4. For example, a line without thickness would be a one-dimensional object since one can only travel in a single direction while remaining inside of the line. In contrast, the surface of a donut — a *torus* — has two independent directions of motion out of which any motion inside of it can be built: one can either travel in a circle around the donut hole or in a circle through it, and any other direction of motion in the torus is some combination of the two. A zero-dimensional object is simply some collection of points which are not connected to each other, and so on. In fact, we constrain ourselves slightly further, in order to ensure that the shapes we work with are well-behaved: we will be concerned primarily with *compact, smooth manifolds.* These are shapes which, though we will allow them to have a boundary — like the edge of a disk — or even corners along the boundary — like the points on the edge of a filled-in square — do not otherwise exhibit sudden changes in their dimension, sharp corners in their interiors, or other pathological behavior. For instance, the shape made by a plane together with a line going through it at a 90 degree angle is excluded from consideration since its dimension has a local change from 2 to 1 as one travels inside it from the disk to the line and vice versa. Manifolds of dimension n always locally "look like" the n-dimensional Euclidean space; e.g. a 3-dimensional manifold locally looks like a patch of the sort of space we live in. The requirement that our shapes be 'compact' means that, no matter how one builds it out of local "patches", one can always select out a finite number of those patches which are enough to cover it. Here, 'smooth' means that there is no "roughness" at any scale or of any order in them.

Zero-dimensional manifolds can be characterized by the number of points they contain so they are not particularly interesting on their own from the perspective of topology. Smooth 1-dimensional manifolds are slightly more interesting: up to the sorts of deformations we allow in topology, which for us will be *diffeomorphisms*, there is a single compact 1-dimensional manifold without boundary, namely the circle. If we allow our 1-manifolds to have boundary, there is one more up to diffeomorphism: the compact interval $[0, 1]$, i.e. a line segment with two endpoints. In two dimensions, things become a little more interesting, since compact surfaces can have many holes — in the same sense that a torus has a single "donut hole" — but a complete classification of compact surfaces without boundary has been known since the late 1800s (cf. [Poi07]), and classifying surfaces with boundary and corners is not much harder. Manifolds of dimensions 3 and 4, and the different ways circles can sit inside the former and surfaces can sit inside the latter, on the other hand, are much more difficult to classify. The question of whether two 3-manifolds are the same up to diffeomorphism is often a

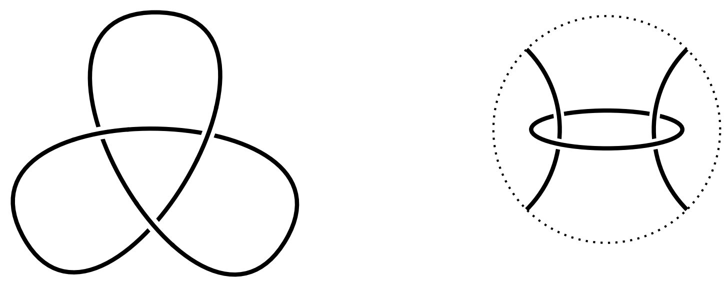


FIGURE 1. Planar diagrams representing the right-handed trefoil knot in Euclidean 3-space (left) and a tangle in the 3-ball (right).

very difficult question to answer. A *knot* in a 3-manifold Y is an *embedding* of a circle inside of Y, i.e. a closed loop in Y which might wrap around itself, or around parts of Y that it can get "caught" on, but which never intersects itself (cf. Figure 1). A *link* in Y is an embedding of some positive number of circles in Y, i.e. a collection of knots in Y which might wrap around each other in interesting ways. Two knots (or links) in Y are *isotopic* to each other if it is possible to "wiggle" one of them around in Y, without passing it through itself, until it becomes a copy of the other one; we think of isotopic knots as being "the same." If Y has non-empty boundary ∂Y, one may also consider *tangles* in Y: embeddings of some positive number of intervals and circles in Y such that the ends of the embedded intervals lie on ∂Y. Two tangles are equivalent if they are isotopic in the above sense, with the caveat that we require isotopies of tangles to fix their endpoints. In similar fashion to the case of 3-manifolds up to diffeomorphism, the question of whether or not two knots or tangles are isotopic is often a very difficult question to answer. Distinguishing 4-manifolds and knotted surfaces inside of them is often even more difficult.

Rather than asking whether two manifolds or knots are the same, then, we instead ask "how can we tell them apart?" The answer to this question comes to us in the form of *invariants*: these are quantities — numbers, polynomials, sets, or more complicated objects like graded vector spaces or functors — which one can assign to a manifold or a knot, or some data representing them, which remain unchanged when we deform the objects in question. Some invariants, e.g. the invariant which assigns the number 0 to every knot, do not tell us any information whatsoever but others contain a lot of information. Two of these invariants, *Heegaard Floer homology* and *Khovanov homology*, contain a plethora of useful data, not just about 3-manifolds and knots, respectively, but also about 4-manifolds and surfaces inside of them, with or without boundary. These invariants, and their variations, also admit interesting algebraic structures when suitably interpreted. In this dissertation, we explore the relationship between these algebraic structures and the underlying topology witnessed by the invariants.

Summary of Results

In this section, we provide a brief overview of the organization of this dissertation. In the remainder of Chapter I, we provide background on A_∞-algebras and their (bi)modules, (bordered) Heegaard Floer homology, Khovanov homology, and Khovanov's arc algebras.

In Chapter II, we discuss the morphism spaces formulation of Heegaard Floer homology [LOT11] and prove the following main theorem, which exhibits a strong relationship between the algebra of module homomorphisms and the 4-dimensional topology witnessed by Floer theory.

Theorem 1.0.1 (Theorem 2.0.1). *Let Y_1, Y_2, and Y_3 be bordered 3-manifolds, all of which have boundaries parametrized by the same surface F, and let $\mathcal{A} = \mathcal{A}(-F)$ be the algebra associated to $-F$. Let $Y_{ij} = -Y_i \cup_\partial Y_j$ and consider the pair of pants* *cobordism* $W : Y_{12} \sqcup Y_{23} \to Y_{13}$ *given by*

$$W = (\triangle \times F) \cup_{e_1 \times F} (e_1 \times Y_1) \cup_{e_2 \times F} (e_2 \times Y_2) \cup_{e_3 \times F} (e_3 \times Y_3), \tag{1.1}$$

where $\triangle$ *is a triangle with edges* e_1*,* e_2*, and* e_3 *in cyclic order. If we define* $\mathrm{Mor}^{\mathcal{A}}(Y_i, Y_j) := \mathrm{Mor}^{\mathcal{A}}(\widehat{CFD}(Y_i), \widehat{CFD}(Y_j))$ *to be the space of left* $\mathcal{A}$*-module homomorphisms* $\widehat{CFD}(Y_i) \to \widehat{CFD}(Y_j)$*, then the composition map* $f \otimes g \mapsto g \circ f$ *fits into a homotopy commutative square of the form*

$$\begin{array}{ccc} \mathrm{Mor}^{\mathcal{A}}(Y_1, Y_2) \otimes \mathrm{Mor}^{\mathcal{A}}(Y_2, Y_3) & \xrightarrow{f \otimes g \mapsto g \circ f} & \mathrm{Mor}^{\mathcal{A}}(Y_1, Y_3) \\ \simeq \downarrow & & \downarrow \simeq \\ \widehat{CF}(Y_{12}) \otimes \widehat{CF}(Y_{23}) & \xrightarrow{\widehat{f}_W} & \widehat{CF}(Y_{13}) \end{array} \tag{1.2}$$

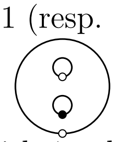
where $\widehat{f}_W$ *is the map induced by* W *and the vertical maps come from the Heegaard Floer pairing theorem for morphism spaces* [LOT11, Theorem 1]*.*

Along the way, we prove several technical lemmas, including a non-existence result for holomorphic disks in a particular bordered Heegaard triple and a gluing result for cornered 4-manifolds constructed from bordered Heegaard triples, both of which are crucial to the above theorem. As a consequence of Theorem 2.0.1, we give a new algorithm for computing cobordism maps on Heegaard Floer homology via composition of morphisms. This algorithm gives an alternative to the approaches of [LMW08] and [MOT20].

In Chapter III, we introduce Heegaard Floer analogues of Khovanov's arc algebras H_n, which take the form of endomorphism rings $\mathfrak{h}_n$ of type-D structures associated to sets of crossingless matchings, and show that, in general, $\mathfrak{h}_n$ is a nontrivial A_∞-deformation of H_n. In [Shu14], Shumakovitch showed that

Khovanov homology $Kh(L)$ of a link L, with coefficients in $\mathbb{F}_2$, decomposes as $Kh(L) \cong \widetilde{Kh}(L) \otimes V$, where $\widetilde{Kh}(L)$ is the *reduced* Khovanov homology of L and $V = \mathbb{F}_2[x]/(x^2)$. Motivated by computations in the preceding section, alongside a Künneth theorem for Heegaard Floer homology of connected sums, we show that the analog of Shumakovitch's result holds for the arc algebras H_n on $2n$ points defined over a ring R of characteristic 2: that there is an isomorphism of algebras $H_n \cong \widetilde{H}_n \otimes_R R[x]/(x^2)$, where $\widetilde{H}_n$ is a reduced version of H_n. We also show that there is no such isomorphism of arc algebras defined over $\mathbb{Z}$ when $n > 1$.

Notation

Throughout this dissertation, we work almost exclusively over the field $\mathbb{F}_2$ with two elements. As such, we will denote this field simply by $\mathbb{F}$. We will often denote commutative algebras over $\mathbb{F}$ by $\Bbbk$ and arbitrary commutative rings by R. In Chapter III, we will work with algebras whose elements are R-linear combinations of configurations of circles in the plane whose components are labeled by elements of the basis $\{1, x\}$ for the commutative ring $R[x]/(x^2)$. To avoid notational clutter, we will indicate that a component is labeled by 1 (resp. x) by placing a hollow dot $\circ$ (resp. solid dot $\bullet$) on it. For example, in , the outermost circle and higher of the two smaller circles are labeled with 1, while the lower of the two smaller circles is labeled with x.

1.1 Background on A_∞-algebras and Related Structures

We recall here several definitions of algebraic structures which are central to bordered Heegaard Floer homology. We refer the reader to [LOT18] and [LOT15] for thorough treatments of these structures and their categories.

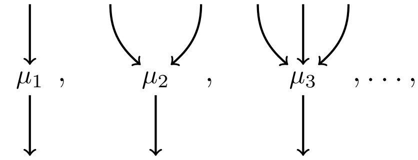
Definition 1. Let $M = \bigoplus_{d\in\mathbb{Z}} M_d$ be a $\mathbb{Z}$-graded module and $n \in \mathbb{Z}$. Define $M[n]$ to be the graded module whose d^{th} summand is defined by $(M[n])_d = M_{d-n}$, which is to say that $M[n]$ is M with all summands shifted *up* in grading by n.

Definition 2. Let $\Bbbk$ be a ring of characteristic two. An A_∞-*algebra* $\mathcal{A}$ over $\Bbbk$ is a $\mathbb{Z}$-graded $\Bbbk$-module A together with $\Bbbk$-linear maps $\mu_i : A^{\otimes_{\Bbbk} i} \to A[2-i]$ for $i \geq 1$ subject to the A_∞-relations[1]

$$\sum_{i+j=n+1} \sum_{k=1}^{n-j+1} \mu_i \circ (\mathrm{id}^{\otimes k-1} \otimes \mu_j \otimes \mathrm{id}^{\otimes n-j-k+1}) = 0 \tag{1.3}$$

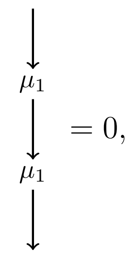
for every $n \geq 1$. Following [LOT18], we will consistently denote the underlying $\Bbbk$-module by A. An A_∞-algebra is called *strictly unital* if there is some $1 \in A$ which acts as a unit for the map μ_2 and $\mu_i(a_1, \ldots, a_i) = 0$ for $i \neq 2$ if $a_j = 1$ for some j.

There is a convenient visual mnemonic for the A_∞ relations: we may represent μ_i by a downwardly-oriented tree with one internal vertex, i input leaves, and one output leaf, i.e.

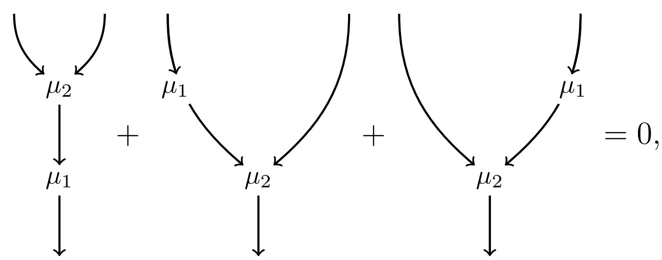


[1]In characteristics other than two, these relations include signs.

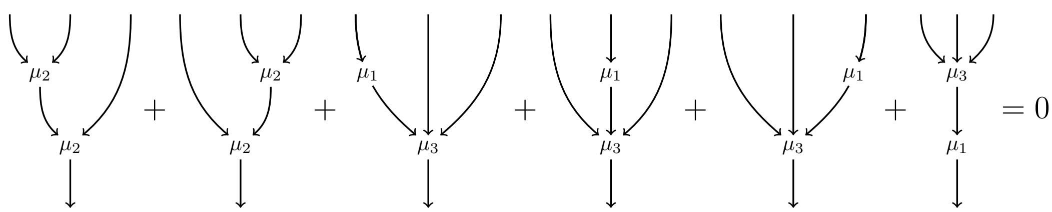
then the A_∞ relation for n tells us that the sum over all downwardly-oriented trees with two interior vertices, n input leaves, and one output leaf is zero. For example, in these terms, the $n = 1$ case becomes



i.e. $\mu_1^2 = 0$, the $n = 2$ case is



i.e. μ_1 satisfies the Leibniz rule with respect to μ_2, and the $n = 3$ case is



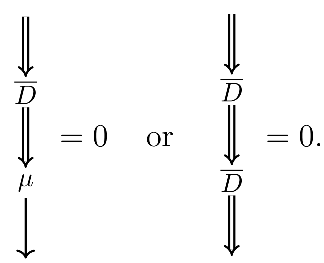
which tells us that μ_2 is associative up to a homotopy μ_3 with respect to the differential μ_1 on A and the induced differential $\mu_1 \otimes \mathrm{id} \otimes \mathrm{id} + \mathrm{id} \otimes \mu_1 \otimes \mathrm{id} + \mathrm{id} \otimes \mathrm{id} \otimes \mu_1$ on $A^{\otimes 3}$. In particular, if $\mu_i = 0$ for $i > 2$, then $\mathcal{A}$ is an ordinary differential graded associative algebra over $\Bbbk$ with multiplication μ_2 and differential μ_1. There are many A_∞-algebras whose higher operations do not vanish but, if there is some n

for which $\mu_i = 0$ for all $i > n$, then one says that $\mathcal{A} = (A, \{\mu_i\}_{i=1}^{\infty})$ is *operationally bounded* or just *bounded*.

It is often convenient to consolidate all of the maps μ_i into a single linear map $\mu : T(A[1]) \to A[2]$, where $T(A[1]) = \bigoplus_{n=0}^{\infty} A^{\otimes_{\Bbbk} n}[n]$ is the tensor algebra of $A[1]$ and we set $\mu_0 = 0$. Defining a map $\overline{D} : T(A[1]) \to T(A[1])$ by

$$\overline{D} = \sum_{j=1}^{n} \sum_{k=1}^{n-j+1} \mathrm{id}^{\otimes k-1} \otimes \mu_j \otimes \mathrm{id}^{\otimes n-j-k+1}, \tag{1.4}$$

then the A_∞-relations are given equivalently by either $\mu \circ \overline{D} = 0$ or $\overline{D} \circ \overline{D} = 0$ or, graphically, as



(1.5)

Definition 3. A (*right*) A_∞-*module* $\mathcal{M}$ over an A_∞-algebra $\mathcal{A}$ is a graded $\Bbbk$-module M together with $\Bbbk$-linear maps $m_i : M \otimes_{\Bbbk} A^{\otimes(i-1)} \to M[2-i]$ for all $i \geq 1$ such that

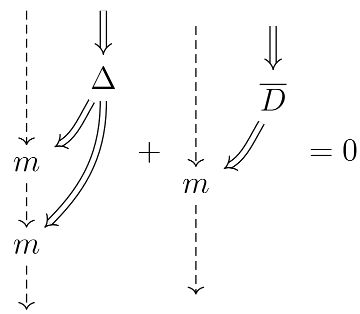
$$\begin{aligned} &\sum_{i+j=n+1} m_i \circ (m_j \circ (\mathrm{id}_M \otimes \mathrm{id}^{\otimes j-1}) \otimes \mathrm{id}^{\otimes n-j}) \\ &+ \sum_{i+j=n+1} \sum_{k=1}^{n-j} m_i \circ (\mathrm{id}_M \otimes \mathrm{id}^{\otimes k-1} \otimes \mu_j \otimes \mathrm{id}^{\otimes n-j-k}) = 0 \end{aligned} \tag{1.6}$$

for all $n \geq 1$. In other words, the m_i are compatible with the A_∞-operations μ_j in the sense that these maps, in concert, satisfy the usual A_∞-relations, except that the first input is a module element rather than an algebra element. As with

A_∞-algebras, an A_∞-module $\mathcal{M}$ is *strictly unital* if there exists $1 \in \mathcal{A}$ such that $m_2(x \otimes 1) = x$ and $m_i(x \otimes a_1 \otimes \cdots \otimes a_{i-1}) = 0$ for $i > 2$ if $a_j = 1$ for some j. We say $\mathcal{M}$ is *bounded* if there exists some n for which $m_i = 0$ for all $i > n$.

In particular an A_∞-algebra $\mathcal{A}$ is an A_∞-module over itself in an obvious way and is strictly unital (resp. bounded) as an A_∞-module if and only if it is strictly unital (resp. bounded) as an A_∞-algebra.

The A_∞-module relation also has a convenient graphical description:



(1.7)

where $\Delta : T(A) \to T(A) \otimes_{\Bbbk} T(A)$ is the canonical comultiplication map defined on pure tensors by

$$\Delta(a_1 \otimes \cdots \otimes a_n) = \sum_{m=0}^{n} (a_1 \otimes \cdots \otimes a_m) \otimes (a_{m+1} \otimes \cdots a_n).$$

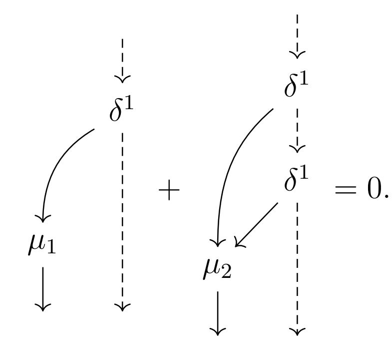
Here, algebra elements are denoted by solid arrows while module elements are denoted by dotted arrows.

Remark. If $\mathcal{M}$ is a right A_∞-module over a differential graded algebra $\mathcal{A}$ and $m_i = 0$ for all $i > 2$, then $\mathcal{M}$ is a differential graded $\mathcal{A}$-module in the usual sense.

Definition 4. Let $\mathcal{A}$ be a differential graded algebra over a ring $\Bbbk$ with differential μ_1 and multiplication map μ_2. A (left) type D structure over $\mathcal{A}$ consists of a graded $\Bbbk$-module N equipped with a $\Bbbk$-linear morphism $\delta^1 : N \to (\mathcal{A} \otimes_{\Bbbk} N)[1]$ satisfying the compatibility condition

$$(\mu_2 \otimes \mathrm{id}_N) \circ (\mathrm{id}_{\mathcal{A}} \otimes \delta^1) \circ \delta^1 + (\mu_1 \otimes \mathrm{id}_N) \circ \delta^1 = 0, \tag{1.8}$$

which can be represented graphically as



(1.9)

A type-D structure homomorphism is a $\Bbbk$-module map $f : N_1 \to \mathcal{A} \otimes N_2$ satisfying the equation

$$(\mu_2 \otimes \mathrm{id}_{N_2}) \circ (\mathrm{id}_{\mathcal{A}} \otimes f) \circ \delta^1_{N_1} + (\mu_2 \otimes \mathrm{id}_{N_2}) \circ (\mathrm{id}_{\mathcal{A}} \otimes \delta^1_{N_2}) \circ f + (\mu_1 \otimes \mathrm{id}_{N_2}) \circ f = 0 \tag{1.10}$$

and a homotopy between type-D structure homomorphisms $f, g : N_1 \to \mathcal{A} \otimes N_2$ is a $\Bbbk$-module homomorphism $h : N_1 \to (\mathcal{A} \otimes N_2)[-1]$ such that

$$(\mu_2 \otimes \mathrm{id}_{N_2}) \circ (\mathrm{id}_{\mathcal{A}} \otimes h) \circ \delta^1_{N_1} + (\mu_2 \otimes \mathrm{id}_{N_2}) \circ (\mathrm{id}_{\mathcal{A}} \otimes \delta^1_{N_2}) \circ h + (\mu_1 \otimes \mathrm{id}_{N_2}) \circ h = f - g.$$

Example 1. Suppose that X is a differential graded $\mathcal{A}$-module which is free with basis $\{x_i\}$ and has differential determined by

$$\partial x_i = \sum_j a_{ij} x_j. \tag{1.11}$$

Let $N = \mathrm{span}_{\Bbbk}\{x_i\}$. Then the map $\delta^1 : N \to (\mathcal{A} \otimes_{\Bbbk} N)[1]$ defined on basis elements by

$$\delta^1(x_i) = \sum_j a_{ij} \otimes x_j \tag{1.12}$$

makes the pair (N, δ^1) into a type D structure. To see this, compute

$$\begin{aligned}
&((\mu_2 \otimes \mathrm{id}_N) \circ (\mathrm{id}_{\mathcal{A}} \otimes \delta^1) \circ \delta^1 + (\mu_1 \otimes \mathrm{id}_N) \circ \delta^1)(x_i) \\
&= \sum_j ((\mu_2 \otimes \mathrm{id}_N) \circ (\mathrm{id}_{\mathcal{A}} \otimes \delta^1) + (\mu_1 \otimes \mathrm{id}_N))(a_{ij} \otimes x_j) \\
&= \sum_{j,k} (\mu_2 \otimes \mathrm{id}_N)(a_{ij} \otimes a_{jk} \otimes x_k) + \sum_j \mu_1(a_{ij}) \otimes x_j \\
&= \sum_{j,k} a_{ij} a_{jk} \otimes x_k + \sum_j \mu_1(a_{ij}) \otimes x_j \\
&= \sum_{j,k} (a_{ij} a_{jk} + \mu(a_{ik})) \otimes x_k.
\end{aligned} \tag{1.13}$$

On the other hand, we have that

$$
\begin{aligned}
\partial^2 x_i &= \partial \sum_j a_{ij} x_j \\
&= \sum_j (\partial a_{ij}) x_j + a_{ij} \partial x_j \\
&= \sum_j \mu_1(a_{ij}) x_j + a_{ij} \left(\sum_k a_{jk} x_k \right) \\
&= \sum_{j,k} a_{ij} a_{jk} x_k + \sum_j \mu_1(a_{ij}) x_j \\
&= \sum_{j,k} a_{ij} a_{jk} x_k + \sum_j \mu_1(a_{ik}) x_k \\
&= \sum_{j,k} (a_{ij} a_{jk} + \mu_1(a_{ik})) x_k
\end{aligned}
\tag{1.14}
$$

but $\{x_i\}$ is an $\mathcal{A}$-basis for X so the fact that $\partial^2 x = 0$ implies $a_{ij}a_{jk} + \mu_1(a_{ik}) = 0$ for all i, j, and k so δ^1 satisfies the compatibility condition. Any dg-module homomorphism $X_1 \to X_2$ induces a corresponding map of type-D structures and the converse is true for type-D structures obtained in this manner. Similarly, homotopies of such maps are equivalent to homotopies of type-D structures.

On the other hand, if (N, δ^1) is a left type-D structure over $\mathcal{A}$, then $\mathcal{A} \otimes_{\Bbbk} N$ is a left differential $\mathcal{A}$-module with differential

$$
m_1 = (\mu_2 \otimes \mathrm{id}_N) \circ (\mathrm{id}_{\mathcal{A}} \otimes \delta^1) + \mu_1 \otimes \mathrm{id}_N \tag{1.15}
$$

and module structure map $m_2 = \mu_2 \otimes \mathrm{id}_N$. As in the above example, type-D homomorphisms and homotopies induce chain homomorphisms and chain homotopies, respectively.

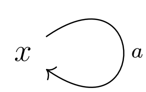
Definition 5. Given a left type D structure (N, δ^1) over a dg-algebra $\mathcal{A}$, there are higher structure maps $\delta^k : N \to (\mathcal{A}^{\otimes k} \otimes_{\Bbbk} N)[k]$ defined recursively by

$$\delta^k = (\mathrm{id}_{\mathcal{A}^{\otimes(k-1)}} \otimes \delta^1) \circ \delta^{k-1}. \tag{1.16}$$

We say (N, δ^1) is *operationally bounded* — or just *bounded* — if $\delta^k = 0$ for all k sufficiently large and *unbounded* otherwise.

Remark. It is frequently useful, in the case that a type D structure (N, δ^1) is constructed from a differential graded $\mathcal{A}$-module with a finite $\mathcal{A}$-basis $\{x_i\}$, to represent it as a directed graph $\Gamma := \Gamma_{(N,\delta^1)}$ with vertices x_i and one edge $x_i \to x_j$ labeled by a_{ij} for each i and j — where, by convention, an unlabeled arrow corresponds to $a_{ij} = 1$. Framed in this way, it is easy to see that (N, δ^1) is operationally bounded if and only if the corresponding graph Γ contains no directed cycles, except possibly those in which there are successive edges $x_i \overset{a_{ij}}{\to} x_j$ and $x_j \overset{a_{jk}}{\to} x_k$ such that $a_{ij} \otimes a_{jk} = 0 \in \mathcal{A} \otimes_{\Bbbk} \mathcal{A}$.

Example 2. Let $\mathcal{A}$ be the associative $\mathbb{F}$-algebra $\mathbb{F}[a]/(a^2)$ with zero differential and consider the free $\mathcal{A}$-module $X = \mathcal{A}\langle x \rangle$ with differential given by $\partial x = ax$. Then the corresponding type D structure (N, δ^1) with $N = \mathbb{F}\langle x \rangle$ is the one whose associated directed graph is

$\Gamma =$ 

i.e. $\delta^k(x) = a \otimes \overset{k}{\cdots} \otimes a \otimes x$. In particular, this an example of an unbounded type D structure.

This graphical interpretation of type-D structures is especially useful for efficiently computing complexes of module homomorphisms between (modulifications of) them: if (N_1, ∂_1) and (N_2, ∂_2) are dg-modules over a differential algebra $\mathcal{A}$, then the space $\mathrm{Mor}^{\mathcal{A}}(N_1, N_2)$ of $\mathcal{A}$-module homomorphisms $N_1 \to N_2$ is naturally a complex of modules over the ground ring $\Bbbk$ when equipped with the differential $\partial f = \partial_2 \circ f + f \circ \partial_1$. If Γ_i, $i = 1, 2$, is the graph for the type-D structure associated to (N_i, ∂_i) and $f : N_1 \to N_2$ is a module homomorphism with

$$f(x_i) = \sum_j f_{ij} y_j, \tag{1.17}$$

we may form a new graph Γ_f from $\Gamma_1 \sqcup \Gamma_2$ by adding a new edge $x_i \overset{f_{ij}}{\to} y_j$ for each nonzero term in $f(x_i)$ for all i. This new graph is the graph for the mapping cone of f and represents a type-D structure if and only if f is a type-D structure homomorphism. The morphism ∂f can then be computed by summing over all length 2 paths in Γ_f which contain one of these edges, in the sense that a path of the form $x_h \overset{a}{\to} x_i \overset{f_{ij}}{\to} y_j$ contributes a summand of $(\partial f)(x_h)$ of the form $a f_{ij} y_j$ and a path of the form $x_i \overset{f_{ij}}{\to} y_j \overset{b}{\to} y_k$ contributes a summand of the form $f_{ij} y_k$.

Example 3. Consider the $\mathbb{F}$-algebra $\mathcal{A}$ with basis

$$\{\iota_0, \iota_1, \rho_1 = \iota_0\rho_1\iota_1, \rho_2 = \iota_1\rho_2\iota_0, \rho_3 = \iota_0\rho_3\iota_1, \rho_{12} = \rho_1\rho_2, \rho_{23} = \rho_2\rho_3, \rho_{123} = \rho_1\rho_2\rho_3\}, \tag{1.18}$$

where ι_0 and ι_1 are orthogonal idempotents, relations $\rho_2\rho_1 = 0$ and $\rho_3\rho_2 = 0$, and trivial differential (we will encounter this algebra again shortly). Let (N, δ^1) be the left type-D structure with $N = \mathbb{F}\langle x, y, z\rangle$, $x = \iota_1 x$, $y = \iota_0 y$, $z = \iota_0 z$, and associated graph

$$\begin{array}{ccc} y & \longrightarrow & z \\ & {\scriptstyle \rho_3}\searrow \quad \nearrow{\scriptstyle \rho_2} & \\ & x & \end{array} \quad . \tag{1.19}$$

Consider the endomorphism $f : N \to N$ given by $f(z) = \rho_{12} y$ and $f(x) = f(y) = 0$. Then

$$\Gamma_f = \begin{array}{ccc} y & \longrightarrow & z \\ & {\scriptstyle \rho_3}\searrow \quad \nearrow{\scriptstyle \rho_2} & \\ & x & \end{array} \xrightarrow{\rho_{12}} \begin{array}{ccc} y & \longrightarrow & z \\ & {\scriptstyle \rho_3}\searrow \quad \nearrow{\scriptstyle \rho_2} & \\ & x & \end{array} \tag{1.20}$$

and we can read off ∂f as

$$\partial f = [y \mapsto \rho_{12} y] + [z \mapsto \rho_{123} x] + [z \mapsto \rho_{12} z]. \tag{1.21}$$

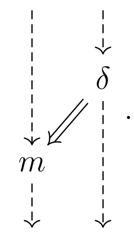
Note that the path $x \xrightarrow{\rho_2} z \xrightarrow{\rho_{12}} y$ does not contribute a nonzero term since $\rho_2 \rho_{12} = 0$.

The box tensor product

Fix a dg-algebra $\mathcal{A}$ over a ring $\Bbbk$. Given a right A_∞-module $\mathcal{M} = (M, \{m_i\})$ and a left type-D structure $\mathcal{N} = (N, \delta^1)$, both over $\mathcal{A}$, such that at least one of $\mathcal{M}$ or $\mathcal{N}$ is operationally bounded, we may pair $\mathcal{M}$ and $\mathcal{N}$ to obtain a chain complex $\mathcal{M} \boxtimes \mathcal{N}$ of $\Bbbk$-modules, called the *box tensor product* of $\mathcal{M}$ and $\mathcal{N}$. The underlying vector space of $\mathcal{M} \boxtimes \mathcal{N}$ is $M \otimes_\Bbbk N$ and the differential $\partial^\boxtimes$ is given on generators $\boldsymbol{x} \otimes \boldsymbol{y}$ by

$$\partial^\boxtimes(\boldsymbol{x} \otimes \boldsymbol{y}) = \sum_{k=0}^{\infty} (m_{k+1} \otimes \mathrm{id}_N)(\boldsymbol{x} \otimes \delta^k(\boldsymbol{y})) \tag{1.22}$$

or, graphically,

$$\partial^{\boxtimes} = $$  . (1.23)

Type-DA bimodules

There are several types of bimodules over A_∞-algebras that arise in Floer homology. We will primarily be concerned with type-DA bimodules, which combine the notions of left type-D structures and right A_∞-modules, so we review their definition here.

Definition 6. Let $\mathcal{A}$ and $\mathcal{B}$ be A_∞-algebras over ground rings $\Bbbk$ and $\mathbb{j}$. A *type-DA bimodule* $\mathcal{N}$ over $(\mathcal{A}, \mathcal{B})$ is a graded $(\Bbbk, \mathbb{j})$-bimodule N equipped with degree zero $(\Bbbk, \mathbb{j})$-bimodule homomorphisms

$$\delta^1_{1+j} : N \otimes_{\mathbb{j}} \mathcal{B}[1]^{\otimes j} \to \mathcal{A}[1] \otimes_{\Bbbk} N \tag{1.24}$$

satisfying the following compatibility relation. Let $\delta^1 = \sum_{j=0}^{\infty} \delta^1_{1+j}$ and recursively define maps $\delta^i : N \otimes_{\mathbb{j}} T^*(\mathcal{B}[1]) \to \mathcal{A}[1]^{\otimes i} \otimes_{\Bbbk} N$ by taking $\delta^0 = \mathrm{id}_N$ and

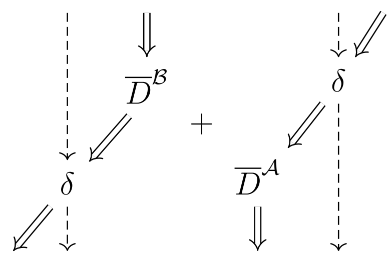
$$\delta^{i+1} = (\mathrm{id}_{A^{\otimes i}} \otimes \delta^1) \circ (\delta^i \otimes \mathrm{id}_{T^*(\mathcal{B}[1])}) \circ (\mathrm{id}_N \otimes \Delta), \tag{1.25}$$

where $\Delta : T^*(\mathcal{B}[1]) \to T^*(\mathcal{B}[1]) \otimes T^*(\mathcal{B}[1])$ is the canonical comultiplication map.

Finally, define $\delta = \sum_{i=0}^{\infty} \delta^i$. Then the compatibility condition for $\mathcal{N}$ is

$$\delta \circ (\mathrm{id}_N \otimes \overline{D}^{\mathcal{B}}) + (\overline{D}^{\mathcal{A}} \otimes \mathrm{id}_N) \circ \delta = 0, \tag{1.26}$$

or graphically



$= 0$ (1.27)

A type-DA bimodule is *strictly unital* if $\delta_2^1(\boldsymbol{x}, 1) = 1 \otimes \boldsymbol{x}$ for any $\boldsymbol{x} \in N$ and $\delta_{1+i}^1(\boldsymbol{x}, b_1, \ldots, b_i) = 0$ if $i > 1$ and any of the b_j is an element of $\mathbb{j}$. The boundedness condition for these structures is more complicated than for A_∞-modules or type-D structures, so we refer the reader to [LOT15] for a rigorous presentation, but one may think of it as follows: each summand of δ may be represented by a directed graph with some number of input and output vertices and $\mathcal{N}$ is bounded if there is some n such that each summand with i input vertices and j output vertices vanishes whenever $i + j > n$.

Like type-D structures, one may represent type-DA bimodules as oriented graphs whose vertices correspond to generators. In this setting, an edge $\boldsymbol{x} \to \boldsymbol{y}$ is labeled by the sum of symbols $a_{\mathrm{out}} \otimes (a_{\mathrm{in}}^1, \ldots, a_{\mathrm{in}}^n)$, one for each summand $a_{\mathrm{out}} \otimes \boldsymbol{y}$ of $\delta_{1+n}^1(\boldsymbol{x}, a_{\mathrm{in}}^1, \ldots, a_{\mathrm{in}}^n)$, letting $a_{\mathrm{in}}^1, \ldots, a_{\mathrm{in}}^n$ range over all sequences of algebra elements.

1.2 Background on Floer Homology

Heegaard Floer homology

Heegaard Floer homology is a suite of invariants of closed, oriented 3-manifolds and cobordisms between them introduced by Peter Ozsváth and Zoltán Szabó in [OS04b]. The particular variant of Heegaard Floer homology we will be concerned with is the so-called 'hat' version. This invariant associates to a closed, oriented 3-manifold Y a graded $\mathbb{F}$-vector space $\widehat{HF}(Y)$ and to each smooth, connected, 4-dimensional cobordism $W : Y_0 \to Y_1$ a map $\widehat{F}_W : \widehat{HF}(Y_0) \to \widehat{HF}(Y_1)$. This assignment is functorial with respect to composition of cobordisms (cf. [OS06, JTZ21, Zem21a]). The vector space $\widehat{HF}(Y)$ is the homology of a complex $\widehat{CF}(Y)$ defined as a variant of the Lagrangian-intersection Floer complex of a pair of Lagrangian tori in a Kähler manifold. We briefly recall this definition here.

Definition 7. A (pointed) *Heegaard diagram* is a quadruple $\mathcal{H} = (\Sigma_g, \boldsymbol{\alpha}, \boldsymbol{\beta}, z)$ consisting of a closed, oriented surface Σ of some genus g, two collections $\boldsymbol{\alpha} = \{\alpha_1, \dots, \alpha_g\}$ and $\boldsymbol{\beta} = \{\beta_1, \dots, \beta_g\}$ of pairwise disjoint embedded circles in Σ, and a basepoint $z \in \Sigma \smallsetminus (\boldsymbol{\alpha} \cup \boldsymbol{\beta})$. Here, we define $\Sigma \smallsetminus \boldsymbol{c} = \Sigma \smallsetminus (c_1 \cup \cdots \cup c_k)$ for any collection $\boldsymbol{c} = \{c_1, \dots, c_k\}$ of embedded circles in Σ. In addition, we require that $\Sigma \smallsetminus \boldsymbol{\alpha}$ and $\Sigma \smallsetminus \boldsymbol{\beta}$ are connected and that any intersection between $\boldsymbol{\alpha}$-circles and $\boldsymbol{\beta}$-circles is transverse.

A Heegaard diagram specifies a closed oriented 3-manifold Y as follows: attach 3-dimensional 2-handles to each $\alpha_i \times \{0\}$ and $\beta_j \times \{1\}$ in $\Sigma \times [0,1]$ and smooth corners. The resulting manifold has two S^2 boundary components and we obtain Y by filling each of these with a copy of the 3-ball.

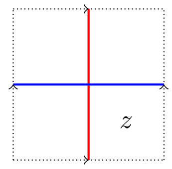


FIGURE 2. The standard genus 1 Heegaard diagram for S^3.

Example 4. The genus 1 Heegaard diagram shown in Figure 2 is a diagram specifying S^3.

Every closed oriented 3-manifold Y admits a Heegaard diagram specifying it in the above manner as follows: choose a self-indexing Morse function $f : Y \to [0,3] \subset \mathbb{R}$ with exactly one critical point of index 0 and one of index 3 and a Riemannian metric g on Y such that (f,g) is Morse–Smale. Since f is self-indexing, $\frac{3}{2}$ is not a critical value so $\Sigma = f^{-1}(\frac{3}{2})$ is a smooth surface by the implicit function theorem and inherits an orientation from Y. Let $\{a_1, \dots, a_g\} = f^{-1}(1)$ and $\{b_1, \dots, b_g\} = f^{-1}(2)$ be the sets of index 1 and index 2 critical points of Y. Given $x \in Y$, let $\gamma_x : \mathbb{R} \to Y$ be the unique solution to the downward gradient flow equation

$$\dot{\gamma}(t) + \nabla f_{\gamma(t)} = 0 \tag{1.28}$$

satisfying the initial condition $\gamma_x(0) = x$. For $i = 1, \dots, g$, let

$$W^s(a_i) = \left\{ x \in Y \,\middle|\, \lim_{t\to\infty} \gamma_x(t) = a_i \right\} \tag{1.29}$$

and

$$W^u(b_i) = \left\{ x \in Y \,\middle|\, \lim_{t\to-\infty} \gamma_x(t) = b_i \right\} \tag{1.30}$$

be the stable and unstable manifolds of a_i and b_i, respectively. Then the embedded circles $\alpha_i = \Sigma \cap W^a(a_i)$ and $\beta_i = \Sigma \cap W^u(b_i)$ specify an unpointed Heegaard diagram $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$ for Y. One may obtain a pointed Heegaard diagram $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, z)$ for Y by choosing a downward gradient flow trajectory from the index 3 critical point to the index 0 critical point and taking z to be its intersection with Σ. Two Heegaard diagrams specify the same closed 3-manifold Y up to diffeomorphism if and only if they can be related by a sequence of *Heegaard moves.* If $\boldsymbol{\gamma}$ is either $\boldsymbol{\alpha}$ or $\boldsymbol{\beta}$, these are:

1. Isotopies: smoothly deforming $\boldsymbol{\gamma}$ inside $\Sigma \setminus z$ in such a way that the curves $\gamma_1, \ldots, \gamma_g$ remain disjoint throughout.

2. Handleslides: replacing $\gamma \in \boldsymbol{\gamma}$ with a curve γ'' with the property that there exists a third curve $\gamma' \in \boldsymbol{\gamma}$ such that γ, γ', and γ'' bound an embedded pair of pants surface.

3. Stabilizations/Destabilizations: taking connected sums with the standard genus 1 Heegaard diagram for S^3 and the inverse of this operation.

Given a Heegaard diagram $\mathcal{H} = (\Sigma_g, \boldsymbol{\alpha}, \boldsymbol{\beta}, z)$, let $\mathrm{Sym}^g(\Sigma) = \Sigma^{\times g}/S_g$ be the g-fold symmetric product of its underlying surface. The space $\mathrm{Sym}^g(\Sigma)$ is a smooth manifold which inherits a complex structure from Σ. The collections $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ determine embedded half-dimensional tori $\mathbb{T}_\alpha = \alpha_1 \times \cdots \times \alpha_g$ and $\mathbb{T}_\beta = \beta_1 \times \cdots \times \beta_g$ which intersect transversely and are disjoint from the subvariety $V_z = \{z\} \times \mathrm{Sym}^{g-1}(\Sigma)$.

Definition 8. For $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$, a *Whitney disk* from $\boldsymbol{x}$ to $\boldsymbol{y}$ is a continuous map $u : D^2 \to \mathrm{Sym}^g(\Sigma)$ such that $u(-i) = \boldsymbol{x}$, $u(i) = \boldsymbol{y}$, and u maps the part of the boundary of D^2 with non-negative real part to $\mathbb{T}_\alpha$ and the part with non-positive real part to $\mathbb{T}_\beta$. Let $\pi_2(\boldsymbol{x}, \boldsymbol{y})$ be the set of homotopy classes of such disks.

Given $\phi \in \pi_2(\boldsymbol{x}, \boldsymbol{y})$ and a path J_s of almost complex structures on $\mathrm{Sym}^g(\Sigma)$, let $\widetilde{\mathcal{M}}_{J_s}(\phi)$ denote the moduli space of J_s-holomorphic representatives of ϕ, i.e. Whitney disks $u : D^2 \to \mathrm{Sym}^g(\Sigma)$ from $\boldsymbol{x}$ to $\boldsymbol{y}$ satisfying the differential equation $J_s \circ du = du \circ i$. For generic choices of path J_s, the space $\widetilde{\mathcal{M}}_{J_s}(\phi)$ is a smooth manifold whose dimension $\mu(\phi)$, the *Maslov index* of ϕ, is given by the index of the $\overline{\partial}$-operator associated to the complex structure on the disk and J_s. Recall that the automorphism group of the disk is $\mathbb{R}$ so there is an $\mathbb{R}$-action on $\widetilde{\mathcal{M}}_{J_s}(\phi)$ by translation. In the case that $\mu(\phi) = 1$, the quotient $\mathcal{M}(\phi) = \widetilde{\mathcal{M}}_{J_s}(\phi)/\mathbb{R}$ is a compact 0-dimensional manifold. We call the elements of this space *rigid holomorphic disks*. Define $n_z(\phi) = \#\phi^{-1}(V_z)$ to be the intersection number of ϕ with V_z.

Definition 9. As an $\mathbb{F}$-vector space, $\widehat{CF}(\mathcal{H})$ is freely generated by $\mathbb{T}_\alpha \cap \mathbb{T}_\beta$. The differential $\partial : \widehat{CF}(\mathcal{H}) \to \widehat{CF}(\mathcal{H})$ is defined on generators by

$$\partial \boldsymbol{x} = \sum_{\boldsymbol{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta} \sum_{\phi \in \pi_2(\boldsymbol{x}, \boldsymbol{y}) \,|\, \mu(\phi)=1,\, n_z(\phi)=0} \#\mathcal{M}(\phi)\boldsymbol{y}, \tag{1.31}$$

i.e. $\partial \boldsymbol{x}$ counts rigid holomorphic Whitney disks from $\boldsymbol{x}$ to $\boldsymbol{y}$ which avoid the subvariety V_z.

The chain homotopy type of $\widehat{CF}(\mathcal{H})$ is invariant under Heegaard moves, hence an invariant of the 3-manifold Y determined by $\mathcal{H}$, so we are justified in writing $\widehat{CF}(Y)$ for $\widehat{CF}(\mathcal{H})$, and its homology $\widehat{HF}(Y)$ is an invariant of Y.

Bordered Floer homology

Bordered Heegaard Floer homology, defined by Lipshitz–Ozsváth–Thurston in [LOT18], is a suite of invariants associated to a 3-manifold Y with parametrized boundary taking the form of homotopy types of A_∞-modules over algebras $\mathcal{A}(\mathcal{Z})$ associated to a combinatorialization $\mathcal{Z}$ of the boundary parametrization. In particular, if Y has one boundary component, the bordered Floer package gives us a left type-D structure $\widehat{CFD}(Y)$ over $\mathcal{A}(-\mathcal{Z}) \cong \mathcal{A}(\mathcal{Z})^{\mathrm{op}}$, which one may think of as a projective left dg-module, and a right A_∞-module $\widehat{CFA}(Y)$ over $\mathcal{A}(\mathcal{Z})$, whose homotopy types are invariants of Y. We briefly recall the construction of this object in Section 2. These modules satisfy pairing theorems as follows: if Y_1 and Y_2 are 3-manifolds with the same connected boundary surface and $Y_{12} = -Y_1 \cup_\partial Y_2$ is the closed 3-manifold obtained by gluing Y_1 and Y_2 along their respective boundary parametrizations, then there are homotopy equivalences $\widehat{CF}(Y_{12}) \simeq \widehat{CFA}(-Y_1) \boxtimes \widehat{CFD}(Y_2) \simeq \mathrm{Mor}^{\mathcal{A}}(\widehat{CFD}(Y_1), \widehat{CFD}(Y_2))$, where $\mathrm{Mor}^{\mathcal{A}}(\widehat{CFD}(Y_1), \widehat{CFD}(Y_2))$ is the chain complex of $\mathcal{A} = \mathcal{A}(-\mathcal{Z})$-module homomorphisms $\widehat{CFD}(Y_1) \to \widehat{CFD}(Y_2)$. In the box tensor pairing, $\widehat{CFD}(Y_2)$ is being regarded as a genuine type-D structure, while in the morphism space pairing, both of the $\widehat{CFD}(Y_i)$ are being thought of as dg-modules.

For a complete treatment of the material in this section, we refer the reader to [LOT18, LOT15].

Definition 10. A *pointed matched circle* is a quadruple $\mathcal{Z} = (Z, \boldsymbol{a}, M, z)$ consisting of an oriented circle Z, $4k$ points $\boldsymbol{a} = \{a_1, \ldots, a_{4k}\}$ in Z, a 2-to-1 function $M : \boldsymbol{a} \to [2k]$ called a *matching*, and a basepoint $z \in Z \smallsetminus \boldsymbol{a}$ such that the result of surgering Z along the matching M is connected, i.e. a single circle.

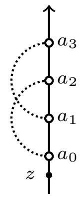


FIGURE 3. The pointed matched circle for the torus.

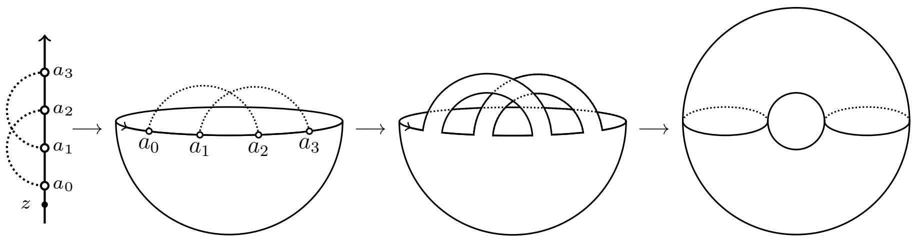


FIGURE 4. Reconstructing the torus from a pointed matched circle. The two 1-handles in the penultimate image correspond to the handles determined by the index 1 critical points of the height function on the vertical torus in the final image, with the handle depicted as crossing under the other corresponding to the lower critical point.

We will regard each pointed matched circle $\mathcal{Z}$ as a contact 1-manifold and refer to intervals $\rho \subset Z$ which have ends on $\boldsymbol{a}$ and do not cross z as *Reeb chords*.

A pointed matched circle specifies an oriented surface $F(\mathcal{Z})$ by filling Z with a disk, adding 2-dimensional 1-handles along each pair of matched points, and then filling the boundary circle of the resulting surface with a disk. For example, the unique pointed matched circle for $\mathbb{T}^2$ is depicted in Figure 3, with matching specified by dotted arcs, and the reconstruction of $\mathbb{T}^2$ from this data is shown in Figure 4.

Definition 11. A *bordered* 3-*manifold* Y is an oriented 3-manifold with boundary together with an orientation-preserving diffeomorphism $\phi : F(\mathcal{Z}) \to \partial Y$ for some pointed matched circle $\mathcal{Z}$. Such data can be specified by a *bordered Heegaard diagram*, which is a quadruple $(\overline{\Sigma}, \boldsymbol{\alpha}, \boldsymbol{\beta}, z)$ where:

– $\overline{\Sigma}$ is a compact, oriented, surface of some genus g,

– $\boldsymbol{\alpha} = \boldsymbol{\alpha}^a \cup \boldsymbol{\alpha}^c = \{\alpha_1^a, \ldots, \alpha_{2k}^a, \alpha_1^c, \ldots, \alpha_{g-k}^c\}$ is a collection of $g + k$ pairwise-disjoint curves in $\overline{\Sigma}$ consisting of $g - k$ embedded circles α_i^c in the interior of $\overline{\Sigma}$ and $2k$ arcs α_j^a with boundary on and transverse to $\partial\overline{\Sigma}$,

– $\boldsymbol{\beta} = \{\beta_1, \ldots, \beta_g\}$ is a collection of g pairwise disjoint embedded circles β_i in the interior of $\overline{\Sigma}$,

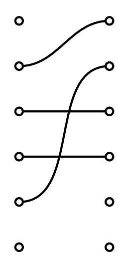
– and z is a point in $\partial\overline{\Sigma} \smallsetminus (\boldsymbol{\alpha} \cap \partial\overline{\Sigma})$

such that $\overline{\Sigma}\smallsetminus\boldsymbol{\alpha}$ and $\overline{\Sigma}\smallsetminus\boldsymbol{\beta}$ are connected and any intersections of α- and β curves is transverse. Moreover, two bordered Heegaard diagrams specify the same bordered 3-manifold Y if and only if they can be related to one another by a finite sequence of Heegaard moves fixing the endpoints of the α-arcs (cf. [LOT18, Chapter 4]).

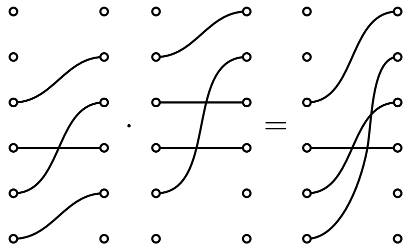
The algebra of a pointed matched circle

Given non-negative integers n and k such that $n \geq k$, let $\mathcal{A}(n, k)$ be the $\mathbb{F}$-vector space generated by non-decreasing *partial permutations* of k elements: triples (S, T, ρ), where S and T are k-element subsets of $[n] = \{1, 2, \ldots, n\}$ and $\rho : S \to T$ is a bijection such that $i \leq \rho(i)$ for all $i \in S$. For a generator $a = (S, T, \rho)$, let $\mathrm{inv}(a)$ be the number of *inversions* of ρ: the number of pairs $i, j \in S$ such that $i < j$ but $\phi(j) < \phi(i)$. We can make $\mathcal{A}(n, k)$ into a graded algebra, which we call the *strands algebra with k strands and n places*, as follows: given generators $a = (S, T, \rho)$ and $b = (T, U, \sigma)$ with $\mathrm{inv}(\sigma \circ \rho) = \mathrm{inv}(\rho) + \mathrm{inv}(\sigma)$, we define the product ab by $ab = (S, U, \sigma \circ \rho)$. If, instead, the domain of σ is not equal to the range of ρ, or if $\mathrm{inv}(\sigma \circ \rho) \neq \mathrm{inv}(\rho) + \mathrm{inv}(\sigma)$, we define $ab = 0$. Generators $a = (S, T, \rho)$ are homogeneous of degree $\mathrm{inv}(a)$. Note that there is an idempotent

$I(S) \in \mathcal{A}(n,k)$ for each k-element subset S of $[n]$ given by $I(S) = (S, S, \mathrm{id}_S)$. This algebra has a graphical presentation in terms of *strands diagrams* with k strands and n places: planar isotopy classes of diagrams in $[0,1] \times [1,n]$ consisting of k non-decreasing smooth curves $x_s : [0,1] \to [1,n]$, where $s \in S$ for some k-element subset S of $[n]$, which we call *strands*. We require that strands have left-boundary on $\{0\} \times [n]$ given by $x_s(0) = s$, right-boundary on $\{1\} \times [n]$, that $x_i \pitchfork x_j$ whenever $i \neq j$, and that no two strands share a common endpoint or intersect more than once. Such a diagram represents a partial permutation $a = (S, T, \rho)$ by taking S as above $T = \{x_s(1) : s \in S\}$, and $\rho(s) = x_s(1)$. For example, the strands diagram

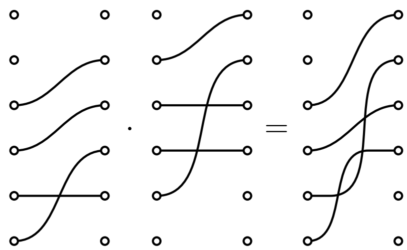


represents the partial permutation $(\{2,3,4,5\}, \{3,4,5,6\}, \rho) \in \mathcal{A}(6,4)$ given by $\rho(2) = 5$, $\rho(5) = 6$, and $\rho(i) = i$ for $i = 3,4$. It is straightforward to show that, conversely, any partial permutation (S, T, ρ) can be represented using a strands diagram. Presented in this way, the product ab is given by horizontal concatenation of diagrams with a on the left and b on the right, subject to the condition that the product is zero if either the right endpoints of the strands of a do not match up with the left endpoints of the strands of b or if any two of the strands cross each other more than once. The latter is the case precisely when the corresponding partial permutations have $\mathrm{inv}(\sigma \circ \rho) \neq \mathrm{inv}(\rho) + \mathrm{inv}(\sigma)$. For example, we have



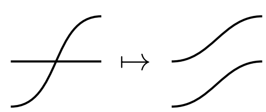
(1.32)

while



$= 0.$ (1.33)

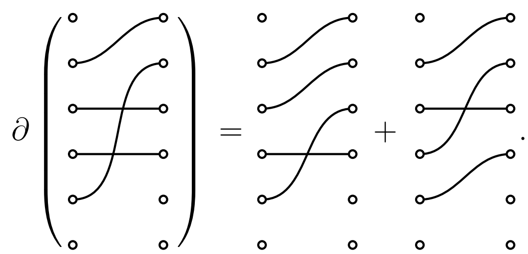
Note that, in this presentation for $\mathcal{A}(n, k)$, we may think of $\operatorname{inv}(a)$ as the number of crossings of two strands in a strands diagram a. Given a crossing in a strands diagram, there is an unique way to resolve it so that the result is again a strands diagram — namely



— and we may use this to define a differential on $\mathcal{A}(n, k)$. Given a strands diagram a, let $\operatorname{Cross}(a)$ be the set of crossings of a. If $c \in \operatorname{Cross}(a)$, let a_c be the strands diagram obtained by resolving a at c and define

$$\partial a = \sum_{\text{crossings } c} a_c. \tag{1.34}$$

For example,


(1.35)

The fact that $\partial^2 = 0$ is automatic from the fact that double crossings occur in pairs $a_{cc'} = a_{c'c}$ and $\mathbb{F}$ has characteristic 2.

Definition 12. The *strands algebra with n places* is $\mathcal{A}(n) = \bigoplus_k \mathcal{A}(n,k)$ with the usual product algebra structure and the differential induced by the differentials on the summands.

Given a pointed matched circle $\mathcal{Z} = (Z, \boldsymbol{a}, M, z)$, we define an algebra $\mathcal{A}(\mathcal{Z})$ as follows. Given a set $\boldsymbol{\rho} = \{\rho_1, \dots, \rho_j\}$ of intervals in $\mathcal{Z} \setminus z$ with ends on $\boldsymbol{a}$, which we call *Reeb chords*, such that no two ρ_i share a common endpoint — in which case we say $\boldsymbol{\rho}$ is *consistent* — we may regard $\boldsymbol{\rho}$ as a strands diagram with $4k$ places by placing two vertical copies of $\overline{Z \setminus z}$ parallel to each other and regarding each ρ_i as a strand connecting the initial endpoint ρ_i^- of ρ_i in the left-hand copy of $\overline{Z \setminus z}$ to the final endpoint ρ_i^+ in the right-hand copy. As a partial permutation, this strands diagram is $(\boldsymbol{\rho}^-, \boldsymbol{\rho}^+, \phi)$, where $\boldsymbol{\rho}^- = \{\rho_1^-, \dots, \rho_j^-\}$, $\boldsymbol{\rho}^+ = \{\rho_1^+, \dots, \rho_j^+\}$, and $\phi : \boldsymbol{\rho}^- \to \boldsymbol{\rho}^+$ is the function defined by $\phi(\rho_i^-) = \rho_i^+$. We then associate a strands algebra element $a_0(\boldsymbol{\rho}) \in \mathcal{A}(n)$ to $\boldsymbol{\rho}$ by taking

$$a_0(\boldsymbol{\rho}) = \sum_{S \,|\, S \cap (\boldsymbol{\rho}^- \cup \boldsymbol{\rho}^+) = \varnothing} (S \cup \boldsymbol{\rho}^-, S \cup \boldsymbol{\rho}^+, \phi_S), \tag{1.36}$$

where $\phi_S : S \cup \boldsymbol{\rho}^- \to S \cup \boldsymbol{\rho}^+$ is the unique extension of ϕ such that $\phi_S|_S = \mathrm{id}_S$. Now, given a subset $\boldsymbol{s} \subset [2k]$, say that $S \subset [4k]$ is a *section* of M over $\boldsymbol{s}$ if $M|_S$ maps S bijectively onto $\boldsymbol{s}$. Define an idempotent $I(\boldsymbol{s}) \in \mathcal{A}(4k)$ by

$$I(\boldsymbol{s}) = \sum_{\text{sections } S \text{ of } M \text{ over } \boldsymbol{s}} I(S). \tag{1.37}$$

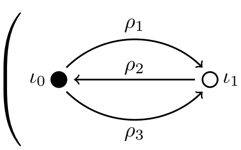
Definition 13. The ring of idempotents $\mathcal{I}(\mathcal{Z})$ is the subalgebra of $\mathcal{A}(4k)$ generated by the idempotents $I(\boldsymbol{s})$. This algebra has unit

$$\mathbf{I} = \sum_{\boldsymbol{s} \subset [2k]} I(\boldsymbol{s}). \tag{1.38}$$

We now define $\mathcal{A}(\mathcal{Z})$ to be the subalgebra of $\mathcal{A}(4k)$ generated by $\mathcal{I}(\mathcal{Z})$ and the elements $\mathbf{I}a_0(\boldsymbol{\rho})\mathbf{I}$, where $\boldsymbol{\rho}$ ranges over all consistent sets of Reeb chords in $\mathcal{Z}$. Define the *weight* i part $\mathcal{A}(\mathcal{Z}, i)$ of $\mathcal{A}(\mathcal{Z})$ by $\mathcal{A}(\mathcal{Z}, i) = \mathcal{A}(\mathcal{Z}) \cap \mathcal{A}(4k, k+i)$.

Example 5. The algebra associated to the torus — whose pointed matched circle is shown in Figure 3 — is isomorphic to the following path algebra quotient:

$\mathcal{A}(\mathbb{T}^2) \cong \mathrm{Path}\,\Bigg($



$\Bigg) \Bigg/ \begin{matrix}\rho_2\rho_1 = 0 \\ \rho_3\rho_2 = 0\end{matrix}\,.$ (1.39)

This algebra has basis $\iota_0, \iota_1, \rho_1, \rho_2, \rho_3, \rho_{12}, \rho_{23}, \rho_{123}$.

$\widehat{CFA}$ and $\widehat{CFD}$

Definition 14. Let $\mathcal{H} = (\overline{\Sigma}, \boldsymbol{\alpha}, \boldsymbol{\beta}, z)$ be a genus g bordered Heegaard diagram for a bordered 3-manifold $(Y, \phi : F(\mathcal{Z}) \to \partial Y)$. A *generator* of $\mathcal{H}$ is an unordered g-tuple $\boldsymbol{x} = \{x_1, \ldots, x_g\}$ of points in Σ such that precisely one x_i lies on each β-

circle, precisely one x_i lies on each α-circle, and at most one x_i lies on each α-arc. We denote the set of generators for $\mathcal{H}$ by $\mathfrak{S}(\mathcal{H})$.

Given a generator $\boldsymbol{x} \in \mathfrak{S}(\mathcal{H})$, let

$$o_{\boldsymbol{x}} = \{i : \boldsymbol{x} \cap \alpha_i^a \neq \varnothing\}, \tag{1.40}$$

i.e. $o_{\boldsymbol{x}} \subset [2k]$ is the set of α-arcs *occupied* by $\boldsymbol{x}$. We then associate idempotents $I_A(\boldsymbol{x}) \in \mathcal{I}(\mathcal{Z})$ and $I_D(\boldsymbol{x}) \in \mathcal{I}(-\mathcal{Z})$ to $\boldsymbol{x}$ by taking $I_A(\boldsymbol{x}) = I(o_{\boldsymbol{x}})$ and $I_D(\boldsymbol{x}) = I([2k] \setminus o_{\boldsymbol{x}})$. These then give us a right-action of $\mathcal{I}(\mathcal{Z})$ and a left-action of $\mathcal{I}(-\mathcal{Z})$ on the vector space $\mathbb{F}\mathfrak{S}(\mathcal{H})$ as follows:

$$\boldsymbol{x} \cdot I(\boldsymbol{s}) = \begin{cases} \boldsymbol{x} & I(\boldsymbol{s}) = I_A(\boldsymbol{x}) \\ 0 & \text{else} \end{cases} \tag{1.41}$$

and

$$I(\boldsymbol{s}) \cdot \boldsymbol{x} = \begin{cases} \boldsymbol{x} & I(\boldsymbol{s}) = I_D(\boldsymbol{x}) \\ 0 & \text{else,} \end{cases} \tag{1.42}$$

respectively. In either case, the weight i summands of $\mathcal{I}(\mathcal{Z})$ act trivially on $\mathbb{F}\mathfrak{S}(\mathcal{H})$.

Definition 15. We now define an A_∞-module $\widehat{CFA}(\mathcal{H})$ over $\mathcal{A}(\mathcal{Z})$. As a right $\mathcal{I}(\mathcal{Z})$-module, $\widehat{CFA}(\mathcal{H})$ is just $\mathbb{F}\mathfrak{S}(\mathcal{H})$. Now define maps

$$m_{1+n} : \widehat{CFA}(\mathcal{H}) \otimes_{\mathcal{I}(\mathcal{Z})} \mathcal{A}(\mathcal{Z}) \otimes_{\mathcal{I}(\mathcal{Z})} \cdots \otimes_{\mathcal{I}(\mathcal{Z})} \mathcal{A}(\mathcal{Z}) \to \widehat{CFA}(\mathcal{H}) \tag{1.43}$$

by

$$m_{1+n}(\boldsymbol{x}, a(\boldsymbol{\rho}_1), \ldots, a(\boldsymbol{\rho}_n)) = \sum_{\boldsymbol{y}\in\mathfrak{S}(\mathcal{H})} \sum_{\substack{B \in \pi_2(\boldsymbol{x},\boldsymbol{y}) \\ \operatorname{ind}(B,\vec{\boldsymbol{\rho}})=1}} \#\mathcal{M}^B(\boldsymbol{x}, \boldsymbol{y}; \boldsymbol{\rho}_1, \ldots, \boldsymbol{\rho}_n)\boldsymbol{y}, \quad (1.44)$$

where $\pi_2(\boldsymbol{x}, \boldsymbol{y})$, $\operatorname{ind}(B, \vec{\boldsymbol{\rho}})$, and $\mathcal{M}^B(\boldsymbol{x}, \boldsymbol{y}; \boldsymbol{\rho}_1, \ldots, \boldsymbol{\rho}_n)$ are as defined in [LOT18], and taking $m_2(\boldsymbol{x}, \mathbf{I}) = \boldsymbol{x}$ and $m_{1+n}(\boldsymbol{x}, \ldots, \mathbf{I}, \ldots) = 0$ if $n > 1$.

Definition 16. We similarly define a left differential module $\widehat{CFD}(\mathcal{H})$ over $\mathcal{A}(-\mathcal{Z})$, which we will think of interchangeably with its corresponding type-D structure. As a left $\mathcal{A}(-\mathcal{Z})$-module, $\widehat{CFD}(\mathcal{H})$ is $\mathcal{A}(-\mathcal{Z}) \otimes_{\mathcal{I}(-\mathcal{Z})} \mathbb{F}\mathfrak{S}(\mathcal{H})$. Now, given a sequence of Reeb chords $\vec{\rho} = (-\rho_1, \ldots, -\rho_n)$ in $-\mathcal{Z} = -\partial\mathcal{H}$, let $a(\vec{\rho}) = a(-\rho_1) \cdots a(-\rho_n)$. Given $\boldsymbol{x}, \boldsymbol{y} \in \mathfrak{S}$ and $B \in \pi_2(\boldsymbol{x}, \boldsymbol{y})$, let

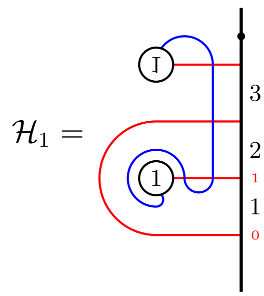
$$a^B_{\boldsymbol{x},\boldsymbol{y}} = \sum_{\vec{\rho}\,|\,\operatorname{ind}(B,\vec{\rho})=1} \#\mathcal{M}^B(\boldsymbol{x}, \boldsymbol{y}; \vec{\rho})a(-\vec{\rho}). \quad (1.45)$$

The differential on $\widehat{CFD}(\mathcal{H})$ is then given by

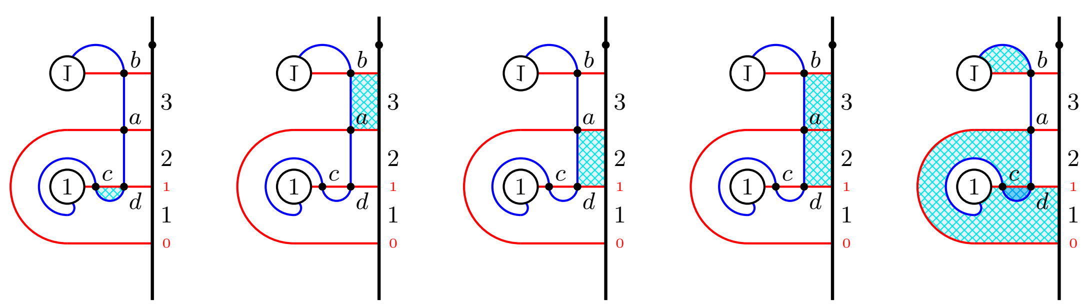
$$\partial(\mathbf{I} \otimes \boldsymbol{x}) = \sum_{\boldsymbol{y}\in\mathfrak{S}(\mathcal{H})} \sum_{B\in\pi_2(\boldsymbol{x},\boldsymbol{y})} a^B_{\boldsymbol{x},\boldsymbol{y}} \otimes \boldsymbol{y}. \quad (1.46)$$

We now give two examples adapted from ones given in [Lev17] and [LOT18] and compute their box tensor product. For additional examples, we refer the reader to Section 3.3.

Example 6. Consider the following planar representation of a bordered Heegaard diagram $\mathcal{H}_1 = (\overline{\Sigma}, \boldsymbol{\alpha}, \boldsymbol{\beta}, z)$.

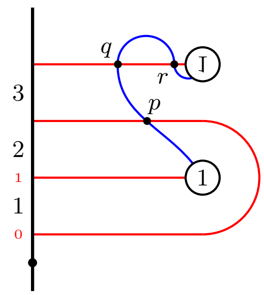


We regard this planar diagram as residing in a disk D whose boundary is the vertical line at right. The two disks labeled 1 represent a handle in Σ, which we can recover from D by deleting the interiors of these two disks and gluing the resulting boundary components along the identity map of S^1. The right A_∞-module for this diagram is given as an $\mathbb{F}$-vector space by $\widehat{CFA}(\mathcal{H}_1) = \mathbb{F}\langle a, b, c, d\rangle$ with idempotents given by $a\iota_0 = a$, $b\iota_1 = b$, $c\iota_1 = c$, and $d\iota_1 = d$. The holomorphic disks supported by $\mathcal{H}_1$ are

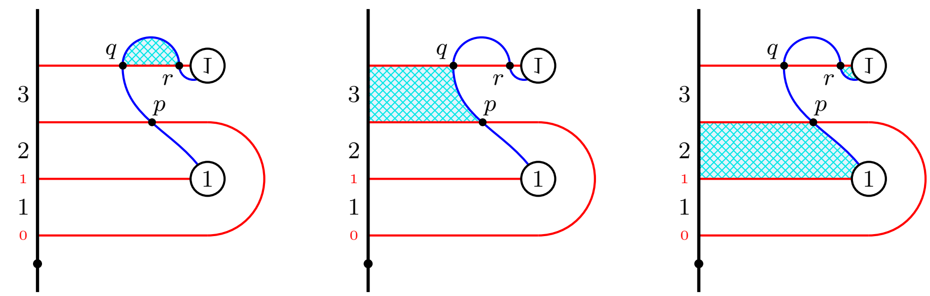


which tell us that $m_1(d) = c$, $m_2(a, \rho_3) = b$, $m_2(d, \rho_2) = a$, $m_2(d, \rho_{23}) = b$, and $m_2(a, \rho_1) = b$ and that all the higher structure maps vanish. Here, we indicate regions in which a disk has multiplicity greater than 1 with a darker color.

Example 7. Now consider the bordered Heegaard diagram

$\mathcal{H}_2 =$ 

with $\widehat{CFD}(\mathcal{H}_2) = \mathbb{F}\langle p, q, r\rangle$ with idempotents given by $\iota_1 p = p$, $\iota_0 q = q$, and $\iota_0 r = r$. The rigid holomorphic disks supported by this diagram are

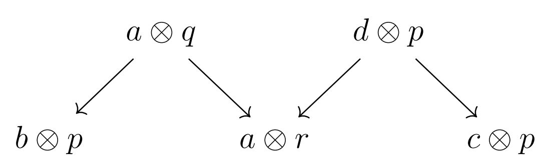


which tells us that δ^1 is given by $\delta^1(q) = 1 \otimes r + \rho_3 \otimes p$ and $\delta^1(p) = \rho_2 \otimes r$. Using this, we get that $\delta^2(q) = \rho_3 \otimes \rho_2 \otimes r$ and all higher δ^k vanish.

Combining these, we get $\widehat{CFA}(\mathcal{H}_1) \boxtimes \widehat{CFD}(\mathcal{H}_2) = \mathbb{F}\langle a\otimes q, a\otimes r, b\otimes p, c\otimes p, d\otimes p\rangle$. It is not hard to see that $\partial^{\boxtimes}(a\otimes r) = \partial^{\boxtimes}(b\otimes p) = \partial^{\boxtimes}(c\otimes p) = 0$ and we can compute the remaining contributions to the box differential as follows:

$$
\begin{aligned}
\partial^{\boxtimes}(a \otimes q) &= (m_2 \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \delta^1)(a \otimes q) \\
&= (m_2 \otimes \mathrm{id})(a \otimes (\iota_0 \otimes r + \rho_3 \otimes p)) \\
&= m_2(a, \iota_0) \otimes r + m_2(a, \rho_3) \otimes p \\
&= a \otimes r + b \otimes p \\
\partial^{\boxtimes}(d \otimes p) &= m_1(d) \otimes p + (m_2 \otimes \mathrm{id})(d \otimes (\rho_2 \otimes r)) \\
&= c \otimes p + m_2(d, \rho_2) \otimes r \\
&= c \otimes p + a \otimes p.
\end{aligned}
$$

In other words, the complex $(\widehat{CFA}(\mathcal{H}_1) \boxtimes \widehat{CFD}(\mathcal{H}_2), \partial^{\boxtimes})$ is equal to



which has 1-dimensional homology. Indeed, the Heegaard diagram $\mathcal{H}_1 \cup \mathcal{H}_2$ is a Heegaard diagram for S^3 and $\dim_{\mathbb{F}} \widehat{HF}(S^3) = 1$.

Bimodule invariants

In order to make full use of the power of bordered Floer homology, one must also consider invariants of 3-manifolds with more than a single boundary component. In the case of manifolds with two boundary components, the bordered Floer package gives us four different types of bimodules. The input data for these invariants consists of a compact 3-manifold with two parameterized boundary components, as one would expect, along with a distinguished disk in each boundary component, a basepoint in the boundary of each disk, and a framed arc connecting the two basepoints. These data may be encoded combinatorially in the form of bordered Heegaard diagrams with two boundary components.

Definition 17. A genus g *arced bordered Heegaard diagram* with two boundary components is a quadruple $\mathcal{H} = (\overline{\Sigma}, \overline{\boldsymbol{\alpha}}, \boldsymbol{\beta}, \boldsymbol{z})$ consisting of:

- a compact genus g surface $\overline{\Sigma}$ with two boundary components $\partial_L\overline{\Sigma}$ and $\partial_R\overline{\Sigma}$
- a g-tuple $\boldsymbol{\beta} = \{\beta_1, \dots, \beta_g\}$ of pairwise disjoint circles in the interior of $\overline{\Sigma}$
- a collection $\overline{\boldsymbol{\alpha}} = \boldsymbol{\alpha}^{a,L} \cup \boldsymbol{\alpha}^c \cup \boldsymbol{\alpha}^{a,R}$, where $\boldsymbol{\alpha}^{a,L} = \{\alpha_1^{a,L}, \dots, \alpha_{2\ell}^{a,L}\}$ are arcs with boundary on $\partial_L\overline{\Sigma}$, $\boldsymbol{\alpha}^{a,R} = \{\alpha_1^{a,R}, \dots, \alpha_{2r}^{a,R}\}$ are arcs with boundary on $\partial_R\overline{\Sigma}$, and $\boldsymbol{\alpha}^c = \{\alpha_1^c, \dots, \alpha_{g-\ell-r}^c\}$ are circles in the interior of $\overline{\Sigma}$, all of which are pairwise disjoint, and
- a path $\boldsymbol{z}$ in $\overline{\Sigma} \smallsetminus (\overline{\boldsymbol{\alpha}} \cup \boldsymbol{\beta})$ between $\partial_L\overline{\Sigma}$ and $\partial_R\overline{\Sigma}$,

such that $\overline{\Sigma} \smallsetminus \overline{\boldsymbol{\alpha}}$ and $\overline{\Sigma} \smallsetminus \boldsymbol{\beta}$ are both connected and $\overline{\boldsymbol{\alpha}}$ and $\boldsymbol{\beta}$ intersect transversely.

Note that an arced bordered Heegaard diagram with two boundary components gives rise to two pointed matched circles $\mathcal{Z}_L$ and $\mathcal{Z}_R$ given by

$$\mathcal{Z}_L = (-\partial_L\overline{\Sigma}, \boldsymbol{\alpha}^{a,L} \cap \partial_L\overline{\Sigma}, m_L, \boldsymbol{z} \cap \partial_L\overline{\Sigma}),$$

and

$$\mathcal{Z}_R = (\partial_R\overline{\Sigma}, \boldsymbol{\alpha}^{a,R} \cap \partial_R\overline{\Sigma}, m_R, \boldsymbol{z} \cap \partial_R\overline{\Sigma}),$$

where m_L and m_R are the matchings induced by the arcs $\boldsymbol{\alpha}^{a,L}$ and $\boldsymbol{\alpha}^{a,R}$, respectively.

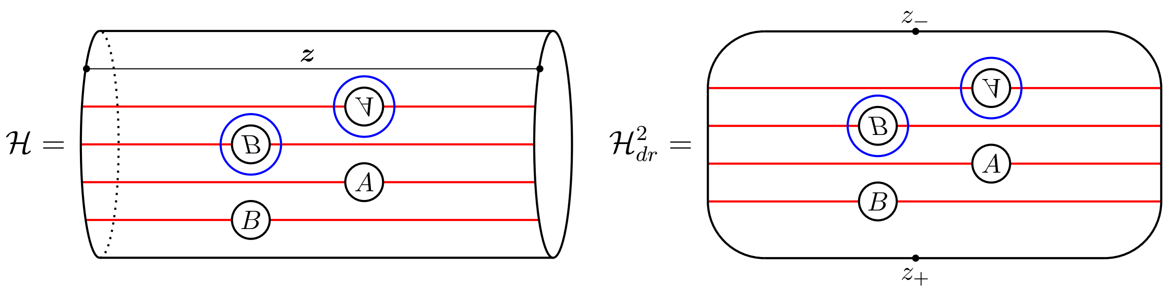


FIGURE 5. An arced bordered Heegaard diagram for the cylinder $T^2 \times [0,1]$ (left) and the corresponding doubly pointed drilled diagram (right).

Definition 18. A *drilling* of an arced bordered Heegaard diagram $\mathcal{H} = (\overline{\Sigma}, \overline{\boldsymbol{\alpha}}, \boldsymbol{\beta}, \boldsymbol{z})$ is the ordinary bordered Heegaard diagram $\mathcal{H}_{dr}$ obtained by deleting a small neighborhood nbd$(\boldsymbol{z})$ of $\boldsymbol{z}$ from $\overline{\Sigma}$, smoothing corners, and then placing a basepoint on any of the boundary components of nbd$(\boldsymbol{z})$ which meets the interior of $\overline{\Sigma}$. In the case of an arced bordered Heegaard diagram with two boundary components, there are two possible choices of basepoint up to isotopy, z_+ and z_-, and we denote the associated doubly pointed diagram by $\mathcal{H}^2_{dr}$. Note that the pointed matched circle determined by $\mathcal{H}_{dr}$ is $\mathcal{Z} = \mathcal{Z}_L \# \mathcal{Z}_R$. In particular, the algebra $\mathcal{A}(\mathcal{Z}_L) \otimes \mathcal{A}(\mathcal{Z}_R)$ sits naturally inside of the algebra $\mathcal{A}(\mathcal{Z})$.

Definition 19. If $\mathcal{H}$ is an arced bordered Heegaard diagram, a *generator* of $\mathcal{H}$ is a generator of $\mathcal{H}_{dr}$. As before, we denote the set of generators of $\mathcal{H}$ by $\mathfrak{S}(\mathcal{H})$.

Definition 20. To an arced bordered Heegaard diagram $\mathcal{H}$, one can associate a type-DA bimodule $\widehat{CFDA}(\mathcal{H})$. By restricting to the subalgebra $\mathcal{I}(\mathcal{Z}_L) \otimes \mathcal{I}(\mathcal{Z}_R)$ of $\mathcal{I}(\mathcal{Z}_L \# \mathcal{Z}_R)$, the vector space $\mathbb{F}\mathfrak{S}(\mathcal{H})$ becomes a left-right $(\mathcal{I}(-\mathcal{Z}_L), \mathcal{I}(\mathcal{Z}_R))$-bimodule. As a left $\mathcal{A}(-\mathcal{Z}_L)$-module, $\widehat{CFDA}(\mathcal{H}) = \mathcal{A}(-\mathcal{Z}_L) \otimes_{\mathcal{I}(-\mathcal{Z}_L)} \mathbb{F}\mathfrak{S}(\mathcal{H})$. The

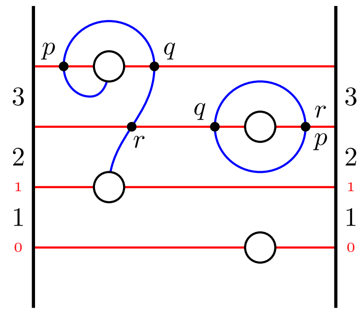


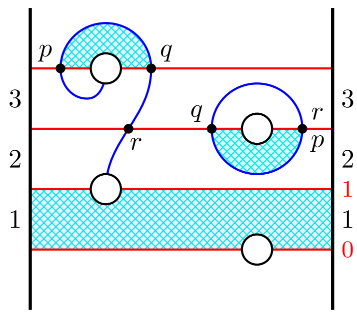
FIGURE 6. An arced bordered Heegaard diagram for the meridional Dehn twist of the torus.

structure maps δ^1_{1+n} are defined by strict unitality and

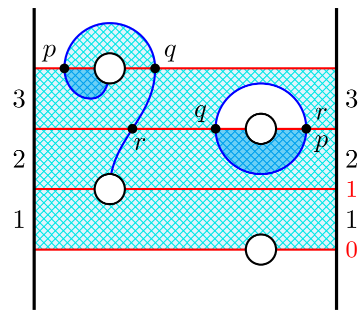
$$\delta^1_{1+n}(\boldsymbol{x}, a(\boldsymbol{\rho}_1), \ldots, a(\boldsymbol{\rho}_n)) = \sum_{\boldsymbol{y} \in \mathfrak{S}(\mathcal{H})} \sum_{\substack{B \in \pi_2(\boldsymbol{x}, \boldsymbol{y}) \\ \operatorname{ind}(B, \vec{\rho}_L, \vec{\boldsymbol{\rho}}_R) = 1}} \#\mathcal{M}^B(\boldsymbol{x}, \boldsymbol{y}; \vec{\rho}_L; \vec{\boldsymbol{\rho}}_R) a(-\vec{\rho}_L) \boldsymbol{y}, \tag{1.47}$$

where $\operatorname{ind}(B, \vec{\rho}_L, \vec{\boldsymbol{\rho}}_R)$ and $\mathcal{M}^B(\boldsymbol{x}, \boldsymbol{y}; \vec{\rho}_L; \vec{\boldsymbol{\rho}}_R)$are as defined in [LOT15].

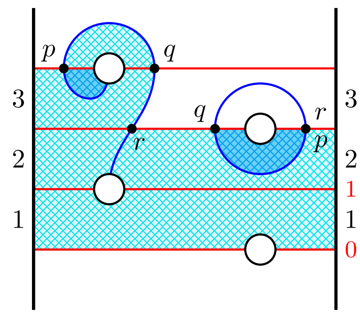
Example 8. Consider the (weight 0) type-DA bimodule $\widehat{CFDA}(\tau_\mu, 0) = \mathbb{F}\langle p, q, r\rangle$, corresponding to the arced bordered Heegaard diagram for the mapping cylinder of the meridional Dehn twist τ_μ of the torus shown in Figure 6. The generators of this diagram are the sets of intersection points p, q, and r determined by the corresponding labels in Figure 6 and these have idempotents given by $\iota_0 p \iota_0 = p$, $\iota_1 q \iota_1 = q$, and $\iota_1 r \iota_0 = r$. One can show that the rigid disks supported by this diagram, and the corresponding terms of the type-DA structure maps, are



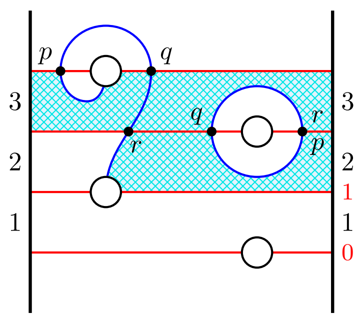
$\delta(p, \rho_1) = \rho_1 \otimes q$



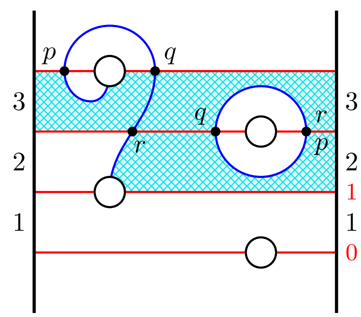
$\delta(p, \rho_{123}) = \rho_{123} \otimes q$



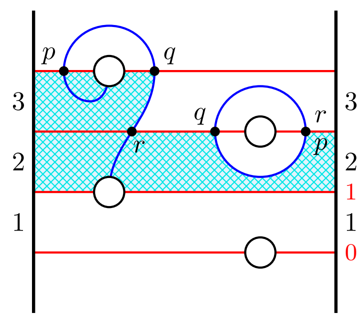
$\delta(p, \rho_{12}) = \rho_{123} \otimes r$



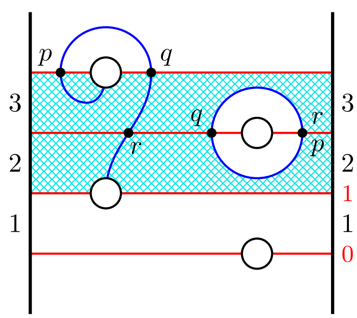
$\delta(p, \rho_3, \rho_2) = \rho_3 \otimes r$



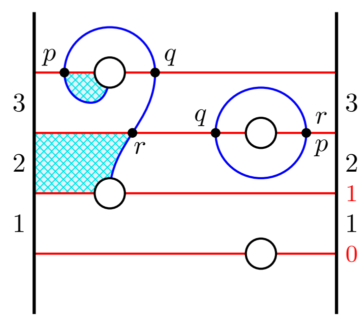
$\delta(p, \rho_3, \rho_{23}) = \rho_3 \otimes q$



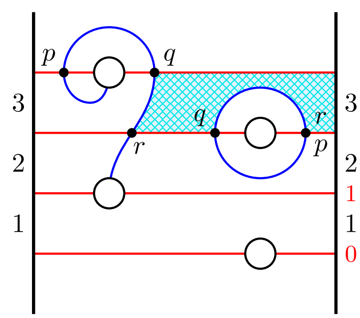
$\delta(q, \rho_2) = \rho_{23} \otimes r$



$\delta(q, \rho_{23}) = \rho_{23} \otimes q$

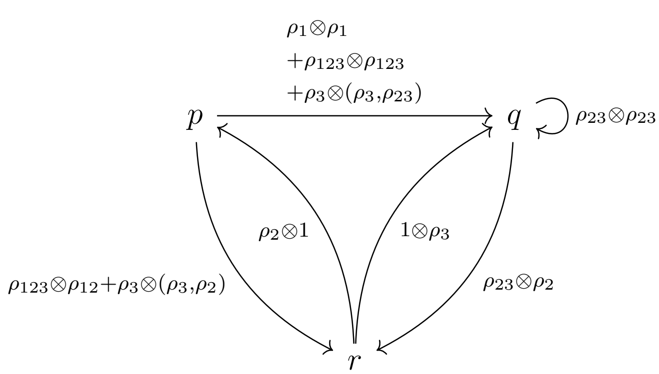


$\delta(r) = \rho_2 \otimes p$



$\delta(r, \rho_3) = q$

so, graphically, this type-DA bimodule is



.

1.3 Background on Khovanov Homology

Topological quantum field theories

Definition 21. Let R be a ring. A *Frobenius algebra* over R is a free R-module V equipped with a multiplication map $m : V \otimes V \to V$, a comultiplication map $\Delta : V \to V \otimes V$, a unit map $\mathbb{1} : R \to V$, and a counit map $\varepsilon : V \to R$ such that $(V, m, \mathbb{1})$ is an associative R-algebra, (V, Δ, ε) is a coassociative coalgebra, and the diagrams

$$\begin{array}{ccc} V \otimes V & \xrightarrow{\Delta\otimes\mathrm{id}} & V \otimes V \otimes V \\ {\scriptstyle m}\downarrow & & \downarrow{\scriptstyle \mathrm{id}\otimes m} \\ V & \xrightarrow[\Delta]{} & V \otimes V \end{array} \quad \text{and} \quad \begin{array}{ccc} V \otimes V & \xrightarrow{\mathrm{id}\otimes\Delta} & V \otimes V \otimes V \\ {\scriptstyle m}\downarrow & & \downarrow{\scriptstyle m\otimes\mathrm{id}} \\ V & \xrightarrow[\Delta]{} & V \otimes V \end{array} \tag{1.48}$$

commute.

It is a classical result, the proof of which uses Cerf theory, that commutative Frobenius algebras over R are in bijective correspondence with 2-*dimensional topological quantum field theories*. The latter are symmetric monoidal functors $\mathscr{F} : \mathcal{C}ob^{1+1} \to {}_R\mathrm{Mod}$, where $\mathcal{C}ob^{1+1}$ is the category of closed 1-dimensional manifolds and compact cobordisms between them with monoidal product given by disjoint union. This correspondence associates the multiplication and comultiplication maps to the pair of pants cobordisms $\bigcirc \sqcup \bigcirc \to \bigcirc$ and $\bigcirc \to \bigcirc \sqcup \bigcirc$, given by merging and splitting two circles, respectively, and the unit and counit maps to the cup and cap cobordisms $\varnothing \to \bigcirc$ and $\bigcirc \to \varnothing$, respectively. Of particular interest to us is the 2-dimensional commutative Frobenius algebra $V = R[x]/(x^2)$, where R is a ring, with counit given by $\varepsilon(1) = 0$ and $\varepsilon(x) = 1$ and comultiplication given by $\Delta(1) = 1 \otimes x + x \otimes 1$ and $\Delta(x) = x \otimes x$. We henceforth refer to V as the *Khovanov TQFT*.

Khovanov homology

In 1985 [Jon97], Vaughan F.R. Jones defined his now famous polynomial invariant $\widehat{J}(L)$ for links using von Neumann algebras. It was quickly realized by Kauffman [Kau87] that the Jones polynomial can be computed combinatorially as

$$\widehat{J}(L) = (-1)^{n_-} q^{n_+ - 2n_-} \langle D \rangle \tag{1.49}$$

where D is any diagram for L, n_+ and n_- are the number of positive and negative crossings in D, and $\langle D \rangle$ is the *Kauffman bracket* uniquely characterized by its value

$$\langle \bigcirc \rangle = q + q^{-1} \tag{1.50}$$

on the unknot and the *Kauffman bracket skein relation*

$$\left\langle \times \right\rangle = \left\langle)\,(\right\rangle - q \left\langle \asymp \right\rangle . \tag{1.51}$$

Not long after its introduction, in the 1990s, the Jones polynomial was used by Kauffman [Kau87], Murasugi [Mur88], Thistlethwaite [Thi87b, Thi87a], and Menasco–Thistlethwaite [MT93] to prove the Tait conjectures, which were first formulated by Peter Guthrie Tait in the 1890s [Tai98]. Later, in the early 2000s [Kho00, Kho03], Mikhail Khovanov introduced invariants for links $L \subset S^3$, the *Khovanov homology* $Kh(L)$ and *reduced Khovanov homology* $\widetilde{Kh}(L)$ respectively, taking values in the category of bigraded abelian groups — or more generally R-modules. These invariants are categorifications of the Jones polynomial in the sense that the graded Euler characteristics $\chi_q(Kh(L))$ and $\chi_q(\widetilde{Kh}(L))$ coincide with the unnormalized and normalized Jones polynomials $\widehat{J}(L)$ and $J(L) = \frac{\widehat{J}(L)}{q+q^{-1}}$, respectively. Here, the graded Euler characteristic of a bigraded abelian group $\mathcal{C} = \bigoplus_{i,j} \mathcal{C}^{i,j}$ is the polynomial

$$\chi_q(\mathcal{C}) = \sum_{i,j} (-1)^i q^j \mathrm{rk}(\mathcal{C}^{i,j}). \tag{1.52}$$

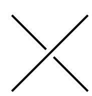
These homology groups are functorial under smooth link cobordisms and have been used to great effect in low-dimensional topology. There are a variety of spectral sequences, many of which are themselves link invariants (cf. [BHL19]), whose E^2-pages are given by either Khovanov homology or its reduced version $\widetilde{Kh}(L)$ (cf. [OS05, Blo11, KM11, BHL19, BS15, Dow18] for some examples). In [Ras10], Rasmussen used the spectral sequence defined by Lee in [Lee02] to define the s-invariant $s(K)$ of a knot K and used this to give a combinatorial reproof of the Milnor conjecture — that the slice genus of the (p, q)-torus knot is $\frac{(p-1)(q-1)}{2}$ — the original proof of which, due to Kronheimer-Mrowka [KM93], relied heavily on gauge theory. Similarly, the s-invariant can be used to give a combinatorial proof of the existence of exotic smooth structures on $\mathbb{R}^4$ (cf. [Ras05]). More recently, the s-invariant was used by Piccirillo in [Pic20] to show that the Conway knot is not smoothly slice and, in a similar vein, Hayden-Sundberg show in [HS21] that the cobordism maps on Khovanov homology can be used to distinguish exotically knotted smooth surfaces in the 4-ball which are topologically but not smoothly isotopic.

In [Kho02], Khovanov defined algebras H_n, the *arc algebras* on $2n$ points, and associated to an $(2m, 2n)$-tangle diagram T a complex of (H_m, H_n)-bimodules $\mathcal{C}_{Kh}(T)$ whose chain homotopy type is an invariant of the underlying tangle in

$D^2 \times I$. These bimodules and their variants can also be used to define invariants of annular links (cf. [BPW19, Lip20, LLS22]) as well as links in $S^2 \times S^1$ (cf. [Roz10, Wil21, MMSW19]).

The construction

Khovanov homology with R-coefficients is defined using the Khovanov TQFT as follows: given a diagram D for a link L with c crossings, numbered from 1 to c, we first produce a commutative cube $\mathbf{2}^c \to \mathcal{C}ob_q^{1+1}$, where $\mathbf{2}^c$ is the cube with vertices $\{0,1\}^c$ and one edge $\boldsymbol{a} \to \boldsymbol{b}$ if and only if we have $\boldsymbol{a} = (a_1, \ldots, a_c)$ and $\boldsymbol{b} = (b_1, \ldots, b_c)$ with $a_i = b_i$ for all $i \neq j$ and $a_j = 0$ while $b_j = 1$. Here, $\mathcal{C}ob_q^{1+1}$ is the category obtained from $\mathcal{C}ob^{1+1}$ by allowing all objects to be formally q-graded; we refer to the q grading as the *quantum grading*. To construct this cube, we replace each crossing of the form



with the local morphism

$$h^{-n_-} q^{n_+ - 2n_-} \left(\underline{)\,(} \longrightarrow q \asymp \right) \tag{1.53}$$

where the map is given by the (minimal) saddle cobordism, extended by the identity away from the crossings. We declare the underlined term of this map to lie in *homological grading* zero, h and q are the homological and quantum grading shift operators, and (n_-, n_+) is either $(1,0)$ or $(0,1)$, depending on whether or not the crossing is positive or negative according to the convention shown in Figure 7.

FIGURE 7. Positive and negative crossings.

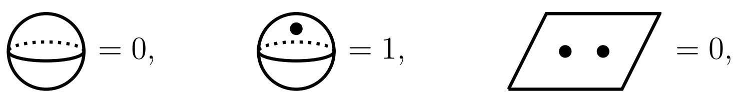
Remark. We use Khovanov's original convention for resolutions of crossings, i.e. $)($ is the 0-resolution of $\times$ while $\asymp$ is the 1-resolution. Much of the literature, including [OS05], uses the opposite convention.

One then obtains the Khovanov cube by applying V, thought of as a topological quantum field theory, to this cube of total resolutions, declaring that x lies in quantum grading -1 and 1 lies in quantum grading 1, and inserting signs on the edges so that each face anticommutes. The Khovanov complex $\mathcal{C}_{Kh}(D;R)$ of D with R-coefficients is then obtained by flattening this cube along homological gradings and the homotopy type of this complex is an isotopy invariant of the underlying link, justifying the notation $Kh(L;R)$ for its homology. The reduced Khovanov complex $\widetilde{\mathcal{C}}_{Kh}(D;R)$ is defined similarly except that one first chooses a basepoint on L whose image in D is not a crossing point and then associates the submodule Rx of V to the marked component of each resolution, associating V to the remaining components as usual[2]. This still gives a well-defined complex and its homology $\widetilde{Kh}(L;R)$ is again an isotopy invariant of L but it depends, in general, on the component of L upon which the basepoint was placed. However, $\widetilde{Kh}(L;\mathbb{F})$ is known to be basepoint-invariant.

[2]One may instead consider the quotient of $\mathcal{C}_{Kh}(D;R)$ by this subcomplex; the resulting homology is the same, a fact which will be important for us later.

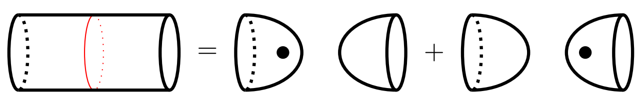
We collect here a few basic facts regarding Bar-Natan's geometric interpretation of Khovanov homology for tangles as they appear in [BN05] and [BN07].

Definition 22. Define[3] $\mathcal{C}ob_\bullet$ to be the category whose objects are formally q-graded direct sums of (possibly empty) compact 1-manifolds and whose morphisms are matrices of dotted cobordisms between them modulo the following local relations:



(1.54)

and

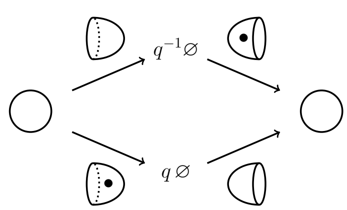


(1.55)

called the *sphere*, *dotted sphere*, *two dot*, and *neck cutting* relations, respectively.

In [BN07], Bar-Natan proves the following theorem.

Theorem 1.3.1 (Delooping)**.** *The maps*

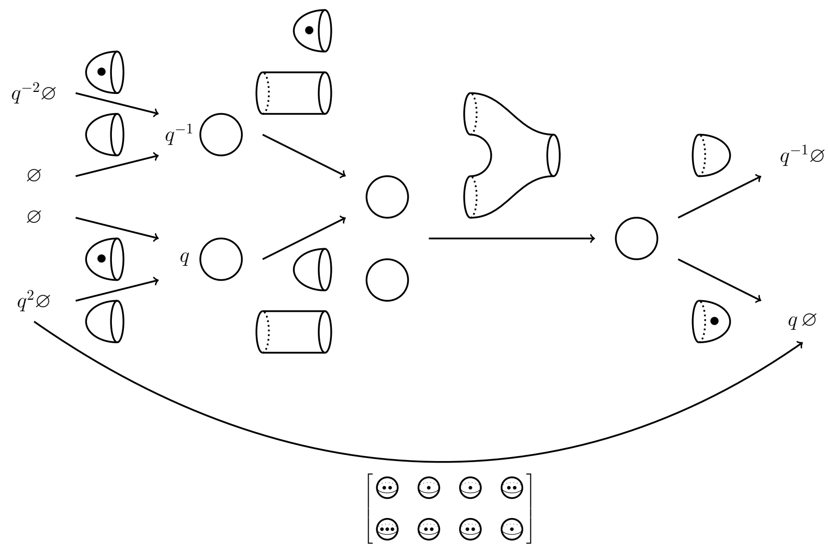


(1.56)

are mutually inverse isomorphisms in $\mathcal{C}ob_\bullet$, where the middle column is regarded as the formal direct sum $q^{-1}\varnothing \oplus q\varnothing$.

[3]What we denote by $\mathcal{C}ob_\bullet$ is actually what Bar-Natan denotes by $\mathrm{Mat}(\mathcal{C}ob^3_{\bullet/\ell})$.

Delooping shows us that $\mathcal{C}ob_\bullet$ encodes the Frobenius algebra $V = R[x]/(x^2)$ as a topological quantum field theory in the sense that, if one replaces each instance of the empty set by the ring R, then the two pairs of pants yield the multiplication and comultiplication maps and the cup and cap cobordisms yield the unit and counit maps after delooping. For example, the diagram



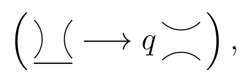
obtained via delooping exhibits a factorization of the multiplication map on $V \cong q^{-1}R \oplus qR$, given in matrix form with respect to the basis $\{x, 1\}$ by

$$\begin{bmatrix} q^{-2}R \\ R \\ R \\ q^2R \end{bmatrix} \xrightarrow{\begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}} \begin{bmatrix} q^{-1}R \\ q\,R \end{bmatrix}, \tag{1.57}$$

as a sequence of elementary cobordism maps in $\mathcal{C}ob_\bullet$.

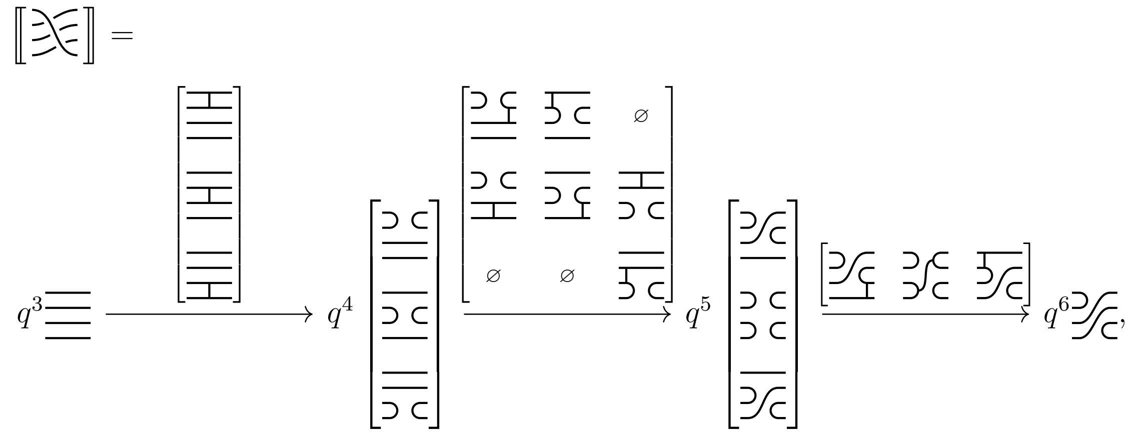
In [BN05], Bar-Natan introduced a variation on $\mathcal{C}ob_\bullet$ associated to a closed disk with an even number of marked points on the boundary — or more generally an "output" disk with some number of "input" disks removed from its interior and an even number of endpoints on each boundary component.

The objects of this category are, instead, flat tangles in the disk with ends on the marked points and morphisms are cobordisms of flat tangles in $D^2 \times [0,1]$ which are cylindrical near $\partial D^2 \times [0,1]$ and transverse to the ends, modulo the same local relations. In particular, delooping also holds (for nullhomotopic loops) in these categories. Let $\mathrm{Kom}(\mathcal{C}ob_\bullet)$ be the category of chain complexes in $\mathcal{C}ob_\bullet$. One can then associate a chain complex $[\![T]\!] \in \mathrm{Kom}(\mathcal{C}ob_\bullet)$ to any oriented tangle T in the disk, as in the construction of Khovanov homology, by replacing each crossing with the two term complex

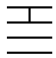
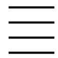
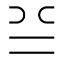
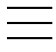
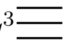
$$h^{-n_-}q^{n_+-2n_-}$$ 

extending each map by the identity cobordism away from the crossings, introducing signs so that each face of the cube anticommutes, and flattening the resulting cubical complex in each homological grading.

Example 9. One may show that the complex associated to the braid $s_1s_2s_3 \in B_4$, where s_i is the i^{th} positive braid generator, is given by



(1.58)

where a planar tangle diagram with an arc connecting two strands denotes the saddle cobordism obtained by merging the two strands along the arc — for example, the morphism  :  $\to$  is the saddle cobordism joining the top two strands of . Here, the q^3 term sits in homological grading zero.

The homotopy type of the complex $[\![T]\!]$ is a tangle invariant and it can be shown that complexes in these categories fit together into the structure of a planar algebra [BN05, Section 5] in such a way that if T is a tangle diagram which decomposes as $T = T' \cup T''$, where T'' is the intersection of T with a small disk, then the complex associated to T is homotopy equivalent to the planar algebraic tensor product of the complexes for T' and T''.

Theorem 1.3.2 (Gaussian elimination [BN07])**.** *Let $\varphi : b_1 \to b_2$ be an isomorphism in an additive category $\mathcal{C}$, then a chain complex in $Mat(\mathcal{C})$ containing a four-term segment of the form*

$$\cdots\ A \xrightarrow{\begin{bmatrix}\alpha\\ \beta\end{bmatrix}} \begin{bmatrix}b_1\\ B\end{bmatrix} \xrightarrow{\begin{bmatrix}\varphi & \delta\\ \gamma & \varepsilon\end{bmatrix}} \begin{bmatrix}b_2\\ C\end{bmatrix} \xrightarrow{\begin{bmatrix}\zeta & \eta\end{bmatrix}} D\ \cdots \tag{1.59}$$

is isomorphic to the complex in which this segment has been replaced with

$$\cdots\ A \xrightarrow{\begin{bmatrix}0\\ \beta\end{bmatrix}} \begin{bmatrix}b_1\\ B\end{bmatrix} \xrightarrow{\begin{bmatrix}\varphi & 0\\ 0 & \varepsilon-\gamma\varphi^{-1}\delta\end{bmatrix}} \begin{bmatrix}b_2\\ C\end{bmatrix} \xrightarrow{\begin{bmatrix}0 & \eta\end{bmatrix}} D\ \cdots \tag{1.60}$$

and both are homotopy equivalent to the simplified complex

$$\cdots\ A \xrightarrow{\beta} B \xrightarrow{\varepsilon-\gamma\varphi^{-1}\delta} C \xrightarrow{\eta} D\ \cdots. \tag{1.61}$$

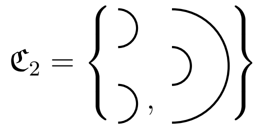


FIGURE 8. The set $\mathfrak{C}_2$ of planar crossingless matchings on 4 points.

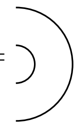
Delooping and Gaussian elimination are particularly useful for computing Khovanov homology in concert with the planar algebraic nature of the complexes via a divide and conquer strategy. In particular, one can decompose a link diagram into sub-tangles, compute and simplify locally using delooping and Gaussian elimination, then glue the simplified complexes back together to obtain a complex homotopy equivalent to the complex for the original link diagram. After replacing each empty set with R and flattening along homological gradings, the resulting complex of R-modules is homotopy equivalent to $\mathcal{C}_{Kh}(L; R)$.

<u>Bimodule invariants of tangles</u>

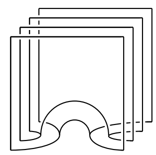
We recall the following definitions and results from [Kho02]. For any n, let $\mathfrak{C}_n$ be the set of planar crossingless matchings on $2n$ points. The *arc algebra* H_n on $2n$ points over R is defined by

$$H_n = q^{-n} \bigoplus_{a,b\in\mathfrak{C}_n} \mathcal{C}_{Kh}\left(a^! \, b \right), \tag{1.62}$$

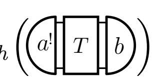
where $a^!$ denotes the diagram a flipped across the vertical axis and $\mathcal{C}_{Kh}(L)$ is the usual Khovanov complex associated to a link by applying the Khovanov TQFT $\mathcal{C}_{Kh}(\bigcirc) = V := R[x]/(x^2)$ to the Bar-Natan complex. The multiplication on this algebra is given by the maps $\mathcal{C}_{Kh}(a^!b \sqcup b^!c) \to \mathcal{C}_{Kh}(a^!c)$ induced by the minimal saddle cobordisms $b \sqcup b^! \to \mathrm{id}$. For example, if

$b =$ 

then the minimal saddle cobordism for b is

$b \sqcup b^! \to \mathrm{id} =$  .

There is a complex of (H_m, H_n)-bimodules $\mathcal{C}_{Kh}(T)$ associated to any $(2m, 2n)$-tangle T, given by

$$\mathcal{C}_{Kh}(T) = q^{-n} \bigoplus_{a \in \mathfrak{C}_m, b \in \mathfrak{C}_n} \mathcal{C}_{Kh}$$  . (1.63)

The homotopy type of $\mathcal{C}_{Kh}(T)$ is an invariant of T as a tangle in $D^2 \times [0,1]$ and these bimodules satisfy the gluing property

$$_{H_\ell}\mathcal{C}_{Kh}(T_1)_{H_m} \otimes_{H_m} \mathcal{C}_{Kh}(T_2)_{H_n} \cong {}_{H_\ell}\mathcal{C}_{Kh}(T_1 T_2)_{H_n}, \tag{1.64}$$

where $T_1 T_2$ is the tangle obtained by gluing the right-endpoints of T_1 to the left-endpoints of T_2, making $\mathcal{C}_{Kh}$ into a projective 2-functor from the 2-category of tangles and tangle cobordisms (cf. [Kho06]) to the 2-category of bimodules over the algebras H_n, where n ranges over all non-negative integers.

CHAPTER II

BORDERED FLOER HOMOLOGY AND COMPOSITION

In this Chapter, we prove the following theorem.

Theorem 2.0.1. *Let Y_1, Y_2, and Y_3 be bordered 3-manifolds, all of which have boundaries parametrized by the same surface F, and let $\mathcal{A} = \mathcal{A}(-F)$ be the algebra associated to $-F$. Let $Y_{ij} = -Y_i \cup_\partial Y_j$ and consider the pair of pants cobordism $W : Y_{12} \sqcup Y_{23} \to Y_{13}$ given by*

$$W = (\triangle \times F) \cup_{e_1 \times F} (e_1 \times Y_1) \cup_{e_2 \times F} (e_2 \times Y_2) \cup_{e_3 \times F} (e_3 \times Y_3), \tag{2.1}$$

where $\triangle$ is a triangle with edges e_1, e_2, and e_3 in cyclic order. If we define $\mathrm{Mor}^{\mathcal{A}}(Y_i, Y_j) := \mathrm{Mor}^{\mathcal{A}}(\widehat{CFD}(Y_i), \widehat{CFD}(Y_j))$ to be the space of left $\mathcal{A}$-module homomorphisms $\widehat{CFD}(Y_i) \to \widehat{CFD}(Y_j)$, then the composition map $f \otimes g \mapsto g \circ f$ fits into a homotopy commutative square of the form

$$\begin{array}{ccc} \mathrm{Mor}^{\mathcal{A}}(Y_1, Y_2) \otimes \mathrm{Mor}^{\mathcal{A}}(Y_2, Y_3) & \xrightarrow{f \otimes g \mapsto g \circ f} & \mathrm{Mor}^{\mathcal{A}}(Y_1, Y_3) \\ \simeq \downarrow & & \downarrow \simeq \\ \widehat{CF}(Y_{12}) \otimes \widehat{CF}(Y_{23}) & \xrightarrow{\quad \widehat{f}_W \quad} & \widehat{CF}(Y_{13}) \end{array} \tag{2.2}$$

where $\widehat{f}_W$ is the map induced by W and the vertical maps come from the pairing theorem [LOT11, Theorem 1].

To show this, we use a bordered Heegaard triple AT, originally defined by Auroux in [Aur10]. In particular, we prove the following.

Theorem 2.0.2. *Let $\mathcal{H}_i$ be bordered Heegaard diagrams for bordered 3-manifolds Y_i for $i = 1, 2, 3$ and let $\mathcal{H}_i^+$ be the bordered Heegaard triple obtained by doubling* *the β-circles in $\mathcal{H}_i$ by a small Hamiltonian isotopy. Then the map*

$$\widehat{G}_{\mathsf{AT}} : \mathrm{Mor}^{\mathcal{A}}(Y_1, Y_2) \otimes \mathrm{Mor}^{\mathcal{A}}(Y_2, Y_3) \to \mathrm{Mor}^{\mathcal{A}}(Y_1, Y_3) \tag{2.3}$$

induced by counting pseudoholomorphic triangles in $\mathsf{AT}_{1,2,3} := \mathsf{AT} \cup \mathcal{H}_1^+ \cup \mathcal{H}_2^+ \cup \mathcal{H}_3^+$*, identifying* $\mathrm{Mor}^{\mathcal{A}}(Y_i, Y_j)$ *with* $\overline{\widehat{CFD}(Y_i)} \boxtimes \mathcal{A} \boxtimes \widehat{CFD}(Y_j)$*, agrees up to homotopy with the composition map* $f \otimes g \mapsto g \circ f$*.*

We then discuss a construction of 4-manifolds with boundary and corners from bordered Heegaard triples and show (Corollary 2.3.2) that the triple $\mathsf{AT}_{1,2,3}$ represents a variant of the pair of pants cobordism described above and use this to prove Theorem 2.0.1 via results of Zemke [Zem21a, Zem21b]. Lastly, as a consequence of Theorem 2.0.1, we give a new algorithm for computing the map $\widehat{HF}(Y_0) \to \widehat{HF}(Y_1)$ associated to a cobordism $X : Y_0 \to Y_1$, at the chain level, via composition of morphisms. This algorithm gives an alternative to the combinatorial approaches of [LMW08] and [MOT20].

2.1 An Interpolating Triple

In [LOT11], Lipshitz–Ozsváth–Thurston show that, for bordered 3-manifolds Y_1 and Y_2 with the same boundary, the chain complex $\mathrm{Mor}^{\mathcal{A}}(\widehat{CFD}(Y_1), \widehat{CFD}(Y_2))$ of $\mathcal{A}$-module maps $\widehat{CFD}(Y_1) \to \widehat{CFD}(Y_2)$ is homotopy equivalent to the Heegaard Floer chain complex $\widehat{CF}(-Y_1 \cup_\partial Y_2)$. There, they considered an (α, β)-bordered Heegaard diagram $\mathsf{AZ}(\mathcal{Z})$, first introduced by Auroux in [Aur10], associated to $\mathcal{Z}$ and show that the bordered Floer bimodule $\widehat{CFAA}(\mathsf{AZ}(\mathcal{Z}))$ is isomorphic, as a left-right $(\mathcal{A}(-\mathcal{Z}), \mathcal{A}(-\mathcal{Z}))$-bimodule, to the regular bimodule ${}_{\mathcal{A}(-\mathcal{Z})}\mathcal{A}(-\mathcal{Z})_{\mathcal{A}(-\mathcal{Z})}$. As a corollary, they then deduce that

$$\begin{aligned}\widehat{CF}(-Y_1 \cup_\partial Y_2) &\simeq \overline{\widehat{CFD}(Y_1)} \boxtimes \widehat{CFAA}(\mathsf{AZ}(\mathcal{Z})) \boxtimes \widehat{CFD}(Y_2) \\ &\cong \mathrm{Mor}^{\mathcal{A}(-\mathcal{Z})}(\widehat{CFD}(Y_1), \widehat{CFD}(Y_2)).\end{aligned} \tag{2.4}$$

The diagram $\mathsf{AZ}(\mathcal{Z})$ is defined as follows: if k is the genus of the surface $F(\mathcal{Z})$ determined by $\mathcal{Z}$, consider the planar triangle $\triangle_k$ bounded by the coordinate axes and the line $x + y = 4k + 1$, which we will call the *diagonal* of $\triangle_k$. Let Σ' be the quotient of $\triangle_k$ by identifying small neighborhoods of the points $(i, 4k + 1 - i)$ and $(j, 4k + 1 - j)$ in the diagonal if i and j are matched in $\mathcal{Z}$ in such a way that the result is an orientable genus k surface with a single boundary component. If i and j are matched in $\mathcal{Z}$, then the disconnected subspace $\triangle_k \cap (\{x = i\} \cup \{x = j\})$ descends to a single arc β_i in Σ' and, similarly, the subspace $\triangle_k \cap (\{y = 4k+1-i\} \cup \{y = 4k + 1 - j\})$ descends to a single arc α_i. Let Σ be the result of attaching a 1-handle to $\partial\Sigma'$ along the 0-sphere $\{(0, 0), (4k+1, 0)\}$ and let $\boldsymbol{z}$ be a neighborhood of the core of this 1-handle. Then $\mathsf{AZ}(\mathcal{Z})$ is the diagram $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{z})$, where $\boldsymbol{\alpha} = \{\alpha_i\}$ and $\boldsymbol{\beta} = \{\beta_i\}$.

We will consider a similarly defined bordered Heegaard triple associated to $\mathcal{Z}$, also due to Auroux, which we call $\mathsf{AT}(\mathcal{Z})$. We construct $\mathsf{AT}(\mathcal{Z})$ as follows: if, as before, k is the genus of $F(\mathcal{Z})$, consider the square $\square_k$ in the plane bounded by the coordinate axes and the lines $x = 4k + 1$ and $y = 4k + 1$ and let Σ' be the quotient of $\square_k$ obtained by identifying small neighborhoods of the points $(i, 4k + 1)$ and $(j, 4k + 1)$ in the segment $\square_k \cap \{y = 4k + 1\}$ if i and j are matched in $\mathcal{Z}$ in such a way that the result is an orientable genus k surface with one boundary component.

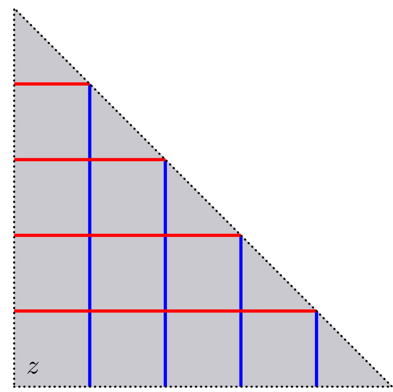


FIGURE 9. The triangle $\triangle_1$ and the arcs which descend to the α- and β-arcs in the interpolating piece $\mathsf{AZ}(\mathcal{Z})$ associated to the unique genus 1 pointed matched circle.

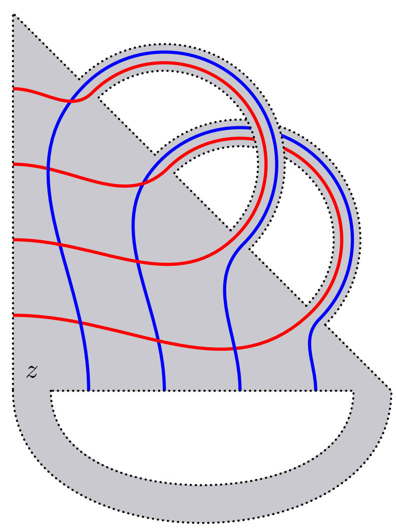


FIGURE 10. The diagram $\mathsf{AZ}(\mathcal{Z})$ associated to the genus 1 pointed matched circle.

Now, if i and j are matched in $\mathcal{Z}$, then the disconnected subspaces

$$\begin{aligned} g_i &= \square_k \cap (\{-x+y=4k+1-i\} \cup \{-x+y=4k+1-j\}) \\ d_i &= \square_k \cap (\{x=i\} \cup \{x=j\}) \\ e_i &= \square_k \cap (\{x+y=4k+1-i\} \cup \{x+y=4k+1-j\}) \end{aligned} \tag{2.5}$$

descend to single arcs γ_i', δ_i', and ε_i', respectively, in Σ'. Now let $\overline{\Sigma}_{\mathsf{AT}}$ be the result of attaching 1-handles to $\partial\Sigma'$ along the 0-spheres $\{(0,0),(4k+1,0)\}$ and $\{(4k+1,0),(4k+1)\}$ and let $\boldsymbol{z}$ be a neighborhood of the core of either handle and take $\mathsf{AT}(\mathcal{Z})$ to be the triple $(\overline{\Sigma}_{\mathsf{AT}}, \boldsymbol{\gamma}, \boldsymbol{\delta}, \boldsymbol{\varepsilon}, \boldsymbol{z})$, where, as before, $\boldsymbol{\gamma} = \{\gamma_i\}$, $\boldsymbol{\delta} = \{\delta_i\}$, and $\boldsymbol{\varepsilon} = \{\varepsilon_i\}$ are given by suitably generic Hamiltonian perturbations of the arcs γ_i', δ_i', and ε_i'. Note that the unperturbed arcs have nongeneric triple intersections so the perturbations are strictly necessary in order for the result to be an admissible diagram in the sense of [LOT18]. We will perturb the triple intersections, in the same manner as given by Auroux in [Aur10], as shown in Figure 11. We also include in AT the data of an embedded trivalent tree $\boldsymbol{z}$ as shown in Figure 12; in the quotient AT, this tree has one leaf on each boundary component.

Since it will be convenient for us to have done so later, we will modify AT slightly by assuming that the spaces g_i and e_i are given by lines of slope $\tan(\frac{\pi}{6})$ and $\tan(\frac{5\pi}{6})$, respectively, instead of 1 and -1. We assume these again meet the top boundary segment of $\square_k$ at the points $(i, 4k+1)$. If we think of these lines as the intersections of lines in $\mathbb{R}^2$ with $\square_k$, then the perturbations of the curves in AT which removes the nongeneric triple points can be realized by translations of the g- and e-lines in the plane as shown in Figure 13. This choice is motivated by the proof of Lemma 2.2.5.

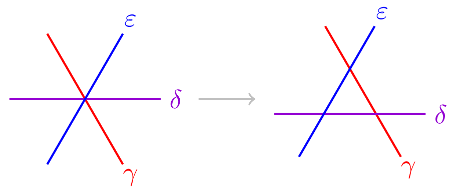


FIGURE 11. Auroux's perturbation convention for triple intersections in $\mathsf{AT}(\mathcal{Z})$.

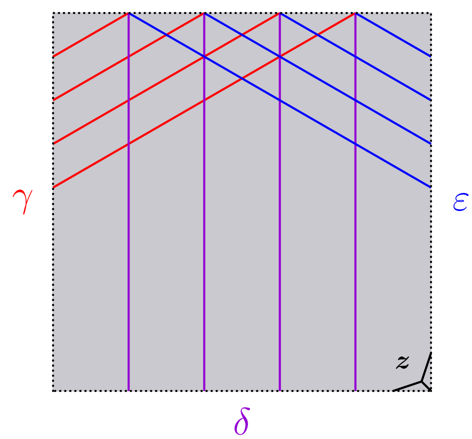


FIGURE 12. The square $\square_1$ and the arcs which descend to the γ-, δ-, and ε-arcs in the interpolating triple $\mathsf{AT}(\mathcal{Z})$ associated to the unique genus 1 pointed matched circle.

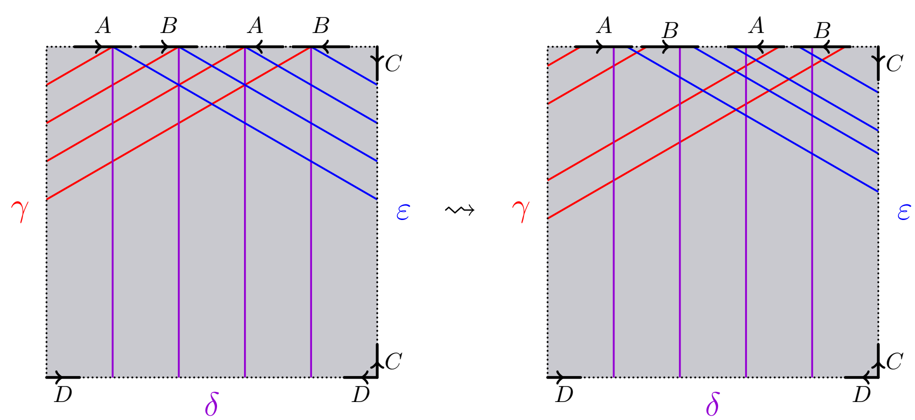


FIGURE 13. Perturbing the diagram using planar translations to obtain the triple $\mathsf{AT}(\mathcal{Z})$ (right) associated to the genus 1 pointed matched circle. Here, we draw the segments of $\partial\square_k$ which are identified in AT as oriented black lines and label the glued pairs of segments with the same letter.

Now let $\boldsymbol{\eta}$ be any one of $\boldsymbol{\gamma}$, $\boldsymbol{\delta}$, or $\boldsymbol{\varepsilon}$ and let $\partial_\eta \mathsf{AT}(\mathcal{Z})$ be the component of $\partial \mathsf{AT}(\mathcal{Z})$ which intersects $\boldsymbol{\eta}$ nontrivially. Note that, by construction, the result of forgetting $\boldsymbol{\eta}$ and gluing a disk to Σ along $\partial_\eta \mathsf{AT}(\mathcal{Z})$ is a copy of $\mathsf{AZ}(\mathcal{Z})$. For $\boldsymbol{\eta}, \boldsymbol{\theta} \in \{\boldsymbol{\gamma}, \boldsymbol{\delta}, \boldsymbol{\varepsilon}\}$, let $\mathsf{AZ}_{\eta\theta}$ be the diagram obtained by deleting the collection of arcs $\boldsymbol{\zeta} \in \{\boldsymbol{\gamma}, \boldsymbol{\delta}, \boldsymbol{\varepsilon}\} \smallsetminus \{\boldsymbol{\eta}, \boldsymbol{\theta}\}$ and let $\mathcal{A}_{\eta\theta} = \widehat{CFAA}(\mathsf{AZ}_{\eta\theta})$. We recall [Aur10, Proposition 4.8] which says that the map $\mathcal{A}_{\delta\varepsilon} \otimes_{\mathbb{F}} \mathcal{A}_{\gamma\delta} \to \mathcal{A}_{\gamma\varepsilon}$ given by counting provincial holomorphic triangles in $\mathsf{AT}(\mathcal{Z})$ coincides with multiplication under the identification of $\mathcal{A}_{\eta\theta}$ with $\mathcal{A}(\mathcal{Z})$.

Proposition 2.1.1 ([LOT11], Proposition 4.1)**.** *The left-right* $(\mathcal{A}(\mathcal{Z}), \mathcal{A}(\mathcal{Z}))$*-bimodule* $\widehat{CFAA}(\mathsf{AZ}_{\eta\theta})$ *is isomorphic to* $\mathcal{A}(\mathcal{Z})$*.*

Sketch. We identify the generating set $\mathfrak{S}(\mathsf{AZ}_{\eta\theta})$ with the usual basis for $\mathcal{A}(\mathcal{Z})$ in terms of strand diagrams. A generator $\boldsymbol{x} \in \mathfrak{S}(\mathsf{AZ}_{\eta\theta})$ is a set of points in $\boldsymbol{\eta} \cap \boldsymbol{\theta}$. To a single intersection point $x \in \boldsymbol{\eta} \cap \boldsymbol{\theta}$, we associate a Reeb chord or smeared horizontal strand in $\mathcal{Z} = (Z, \boldsymbol{a}, M)$ as follows. First, draw $\mathcal{Z}$ above the square, oriented from left to right, with the set of points $\boldsymbol{a}$ identified with the boundary intersection points of $\boldsymbol{\eta}$ and $\boldsymbol{\theta}$. Next, note that there are unique segments e and g in the square passing through x and there is an unique triangular (or empty) region T_x of $\square_k$ bounded by the segments e and g and the line $y = 4k + 1$. If T_x is empty, then x is a boundary intersection point and we associate to it the smeared horizontal strand given by the matching M. Otherwise, we associate to x the Reeb chord ρ_x in $\mathcal{Z}$ determined by the line segment $T_x \cap \{y = 4k + 1\}$. A generator $\boldsymbol{x} \in \mathfrak{S}(\mathsf{AZ}_{\eta\theta})$ may therefore be identified with a set of Reeb chords and smeared horizontal strands and, hence, with a strand diagram. It is straightforward to see that this identification gives a bijection between $\mathfrak{S}(\mathsf{AZ}_{\eta\theta})$ and the usual basis for $\mathcal{A}(\mathcal{Z})$. Note also that we may identify the left- and right-idempotents of a generator $\boldsymbol{x}$

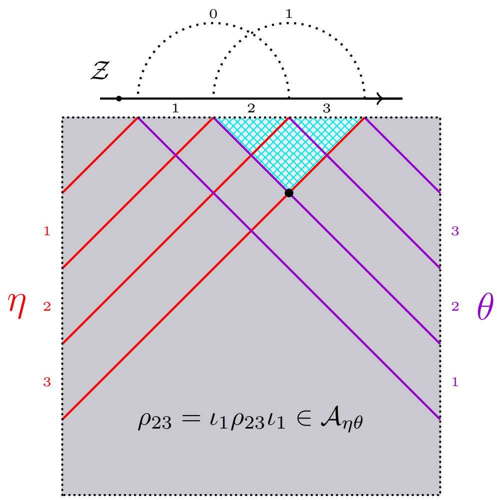


FIGURE 14. Identifying a generator $\boldsymbol{x} \in \mathfrak{S}(\mathsf{AZ}_{\eta\theta})$ with the algebra element $\rho_{23} \in \mathcal{A}(\mathcal{Z})$. In $\mathcal{A}(-\mathcal{Z})$, this same generator is identified with ρ_{12}.

with the collections of left- and right-endpoints of the segments $T_x \cap \{y = 4k+1\}$, respectively. The identification we have given here is equivalent to the one given in [LOT11]. To recover theirs from ours, note that if T_x is nonempty, then there is an unique rectangular domain R_x in $\mathsf{AZ}_{\eta\theta}$ bounded by the leftmost segment of $\partial\square_k$, $\boldsymbol{\eta}$, and $T_x \cap \boldsymbol{\theta}$ with vertices at x and the topmost endpoint of $T_x \cap \boldsymbol{\theta}$. Drawing $\mathcal{Z}$ oriented downward and to the left of $\mathsf{AZ}_{\eta\theta}$ so that $\boldsymbol{a}$ is identified with $\boldsymbol{\eta} \cap \partial\mathsf{AZ}_{\eta\theta}$, one can verify readily that the Reeb chord in $\mathcal{Z}$ determined by $R_x \cap \partial\mathsf{AZ}_{\eta\theta}$ is precisely ρ_x. Lastly, the diagram $\mathsf{AZ}_{\eta\theta}$ is *nice* in the sense of [SW10] so the differential on $\widehat{CFAA}(\mathsf{AZ}_{\eta\theta})$ counts only embedded rectangles, the only nontrivial A_∞-operations are the m_2 maps, and these operations count half-strips — i.e. bigons asymptotic to Reeb chords at the boundary. It is then straightforward to identify the differential and bimodule structures on $\widehat{CFAA}(\mathsf{AZ}_{\eta\theta})$ with those on $\mathcal{A}(\mathcal{Z})$. □

Remark. One way to think about the module actions on $\widehat{CFAA}(\mathsf{AZ}_{\eta\theta})$ is as follows. Suppose $\boldsymbol{x}$ and $\boldsymbol{y}$ are generators such that the collection of right-endpoints of the segments $T_{x_i} \cap \{y = 4k+1\}$ for $x_i \in \boldsymbol{x} = \{x_1, \ldots, x_k\}$ coincides with the collection of left-endpoints of the segments $T_{y_j} \cap \{y = 4k+1\}$ for $y_j \in \boldsymbol{y} = \{y_1, \ldots, y_k\}$. In this case, there is a bijection $f : [k] \to [k]$ with the property that $T_{x_i} \cap T_{y_{f(j)}}$ is precisely the common vertex of the triangles T_{x_i} and $T_{y_{f(i)}}$ when $i = j$ and empty otherwise. Note that there is an unique (possibly empty) rectangular region R_i with the property that $T_{z_i} := T_{x_i} \cup T_{y_{f(i)}} \cup R_i$ is again a triangle. The product $\boldsymbol{x} \cdot \boldsymbol{y}$ is then precisely the collection of intersection points $\boldsymbol{z} = \{z_1, \ldots, z_k\}$. One may verify that this coincides with the usual algebra structure on $\mathcal{A}(\mathcal{Z})$ under the above identification and with the left- and right-module structures under the identification from [LOT11].

We now define the map $m : \mathcal{A}_{\delta\varepsilon} \otimes_{\mathbb{F}} \mathcal{A}_{\gamma\delta} \to \mathcal{A}_{\gamma\varepsilon}$. Let $\triangle$ be a triangle with edges e_γ, e_δ, and e_ε, ordered clockwise, and let $e_{\eta\theta}$ be the unique point in $e_\eta \cap e_\theta$. Now let $W = \mathrm{int}(\mathsf{AT}) \times \triangle$ and fix generators $\rho \in \mathfrak{S}(\mathsf{AZ}_{\delta\varepsilon})$, $\sigma \in \mathfrak{S}(\mathsf{AZ}_{\gamma\delta})$, and $\tau \in \mathfrak{S}(\mathsf{AZ}_{\gamma\varepsilon})$. Denote by $\pi_2(\rho, \sigma, \tau)$ the collection of all homology classes of maps $(S, \partial S) \to (W, \boldsymbol{\gamma} \times e_\gamma \cup \boldsymbol{\delta} \times e_\delta \cup \boldsymbol{\varepsilon} \times e_\varepsilon)$, where S is a Riemann surface with boundary and boundary marked points $s_{\gamma\delta}$, $s_{\delta\varepsilon}$, and $s_{\varepsilon\gamma}$ such that $s_{\gamma\delta} \mapsto \rho$, $s_{\delta\varepsilon} \mapsto \sigma$, and $s_{\varepsilon\gamma} \mapsto \tau$. As in Section 10 of [Lip06], one may pick a sufficiently nice almost complex structure J on W so that, for each $A \in \pi_2(\rho, \sigma, \tau)$, the moduli space $\mathcal{M}^A(\rho, \sigma, \tau)$ of embedded J-holomorphic curves $(S, \partial S) \xrightarrow{u} (W, \boldsymbol{\gamma} \times e_\gamma \cup \boldsymbol{\delta} \times e_\delta \cup \boldsymbol{\varepsilon} \times e_\varepsilon)$ in the homology class A such that $u(s_{\gamma\delta}) = \rho$, $u(s_{\delta\varepsilon}) = \sigma$, and $u(s_{\varepsilon\gamma}) = \tau$ is a smooth manifold whose dimension is given by the Maslov index $\mathrm{ind}(A)$ of A. We then define m on generators by

$$m(\rho \otimes \sigma) = \sum_{\tau \in \mathfrak{S}(\mathsf{AZ}_{\gamma\varepsilon})} \sum_{\mathrm{ind}(A)=0} \#\mathcal{M}^A(\rho, \sigma, \tau)\tau. \tag{2.6}$$

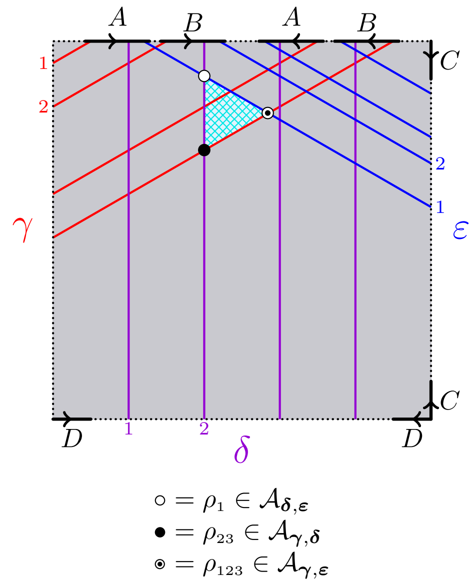


FIGURE 15. An embedded holomorphic triangle in $\mathsf{AT}(\mathcal{Z})$ representing the multiplication $m^{\text{op}}(\rho_{23} \otimes \rho_1) = \rho_{123}$ in the algebra $\mathcal{A}(\mathcal{Z})^{\text{op}}$ or, equivalently, the multiplication $m(\rho_{12} \otimes \rho_3) = \rho_{123}$ in $\mathcal{A}(-\mathcal{Z})$, where $\mathcal{Z}$ is the genus 1 pointed matched circle.

Proposition 2.1.2 ([Aur10, Proposition 4.8])**.** *The map $m : \mathcal{A}_{\delta\varepsilon} \otimes_{\mathbb{F}} \mathcal{A}_{\gamma\delta} \to \mathcal{A}_{\gamma\varepsilon}$ coincides with the multiplication map under the identification of each $\mathcal{A}_{\eta\theta}$ with $\mathcal{A}(\mathcal{Z})$.*

As noted in the introduction, we will be working over the algebra $\mathcal{A}(-\mathcal{Z})$. However, it is a standard fact that this algebra is isomorphic to $\mathcal{A}(\mathcal{Z})^{\text{op}}$. Indeed, one can identify the generators $\mathfrak{S}(\mathsf{AZ}_{\eta\theta})$ with the usual generators for $\mathcal{A}(-\mathcal{Z})$ in precisely the same way as we did for $\mathcal{A}(\mathcal{Z})$ with the sole exception that we draw $\mathcal{Z}$ above $\mathsf{AZ}_{\eta\theta}$ oriented from right to left, rather than from left to right.

Corollary 2.1.3. *m coincides with the multiplication map $\mathcal{A}_{\gamma\delta} \otimes_{\mathbb{F}} \mathcal{A}_{\delta\varepsilon} \to \mathcal{A}_{\gamma\varepsilon}$ under the identification of $\mathcal{A}_{\eta\theta}$ with $\mathcal{A}(-\mathcal{Z})$.*

Remark. By construction, the map m counts only pseudoholomorphic triangles which do not meet the boundary of AT. One could instead count all rigid triangles in AT, in which case one would expect to see additional terms in m. However, Lemma 2.2.5 below tells us that these maps coincide. See [LOT16] for further details on pseudoholomorphic polygon maps in bordered Floer homology.

2.2 Composition and Triangle Counts

Definition 23. We say that $f \in \mathrm{Mor}^{\mathcal{A}}(Y_1, Y_2)$ is a *basic morphism* if there are left-module generators $\boldsymbol{u} \in \widehat{CFD}(Y_1)$ and $\boldsymbol{v} \in \widehat{CFD}(Y_2)$ and an algebra generator $\rho \in \mathcal{A}(-\mathcal{Z})$ such that $f(\boldsymbol{u}) = \rho\boldsymbol{v}$ and f vanishes on all other generators.

Lemma 2.2.1. *The set of basic morphisms forms an* $\mathbb{F}$*-basis for* $\mathrm{Mor}^{\mathcal{A}}(Y_1, Y_2)$.

Proof. Let $\boldsymbol{u}_1, \ldots, \boldsymbol{u}_m \in \widehat{CFD}(Y_1)$ and $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n \in \widehat{CFD}(Y_2)$ be the generators for a given choice of bordered Heegaard diagrams for the Y_i. For $j = 1, \ldots, m$, let $f_1^j, \ldots, f_{s_j}^j$ be the distinct basic morphisms for which $f_i^j(\boldsymbol{u}_j) = \rho_{ij}\boldsymbol{v}_{k(i,j)}$ is nonzero. Suppose that there is a linear dependence

$$\sum_{i,j} c_{ij} f_i^j = 0 \tag{2.7}$$

between them. For a given j, we then have a linear dependence

$$\sum_i c_{ij} f_i^j(\boldsymbol{u}_j) = \sum_i c_{ij}\rho_{ij}\boldsymbol{v}_{k(i,j)} = 0 \tag{2.8}$$

but the $\rho_{ij}\boldsymbol{v}_{k(i,j)}$ are all distinct, hence $\mathbb{F}$-linearly independent, since the f_i^j are basic and distinct so $c_{ij} = 0$ for all i and j. Now given $g \in \mathrm{Mor}^{\mathcal{A}}(Y_1, Y_2)$, write

$$g(\boldsymbol{u}_j) = \sum_i \sigma_{ij}\boldsymbol{v}_i. \tag{2.9}$$

For each i and j for which $\sigma_{ij}\boldsymbol{v}_i \neq 0$, one can then define a basic morphism $g_{i,j}$ by taking $g_{i,j}(\boldsymbol{u}_j) = \sigma_{ij}\boldsymbol{v}_i$ and $g_{i,j}(\boldsymbol{u}_k) = 0$ for $k \neq j$. We then have that

$$g = \sum_{i,j} g_{i,j} \tag{2.10}$$

by construction so the basic morphisms span $\mathrm{Mor}^{\mathcal{A}}(Y_1, Y_2)$. □

The identification

$$\mathrm{Mor}^{\mathcal{A}}(Y_1, Y_2) \cong \overline{\widehat{CFD}(Y_1)} \boxtimes \mathcal{A}(-\mathcal{Z}) \boxtimes \widehat{CFD}(Y_2) \tag{2.11}$$

can then be given in terms of basic morphisms as follows: suppose we have a basic morphism $f : \widehat{CFD}(Y_1) \to \widehat{CFD}(Y_2)$ defined by $f(\boldsymbol{u}) = \rho\boldsymbol{v}$, then f is sent under this isomorphism to the tensor product $\overline{\boldsymbol{u}} \boxtimes \rho \boxtimes \boldsymbol{v}$. If we have a second basic morphism $g : \widehat{CFD}(Y_2) \to \widehat{CFD}(Y_3)$ determined by $g(\boldsymbol{v}) = \sigma\boldsymbol{w}$, then the composition $g \circ f$ is given at the level of box tensor products by

$$(\overline{\boldsymbol{v}} \boxtimes \sigma \boxtimes \boldsymbol{w}) \circ (\overline{\boldsymbol{u}} \boxtimes \rho \boxtimes \boldsymbol{v}) = \overline{\boldsymbol{u}} \boxtimes \rho\sigma \boxtimes \boldsymbol{w}, \tag{2.12}$$

so we we may realize the composition map $f \otimes g \mapsto g \circ f$ explicitly in terms of the multiplication operation on $\mathcal{A}(\mathcal{Z})$ as:

$$\begin{aligned}(\overline{\boldsymbol{u}} \boxtimes \rho \boxtimes \boldsymbol{v}) \otimes (\overline{\boldsymbol{v}} \boxtimes \sigma \boxtimes \boldsymbol{w}) &\overset{ev}{\mapsto} \overline{\boldsymbol{u}} \boxtimes \rho \boxtimes \overline{\boldsymbol{v}}(\boldsymbol{v}) \boxtimes \sigma \boxtimes \boldsymbol{w} \\ = \overline{\boldsymbol{u}} \boxtimes \rho \boxtimes \iota_{\boldsymbol{v}} \boxtimes \sigma \boxtimes \boldsymbol{w} &\overset{\cong}{\mapsto} \overline{\boldsymbol{u}} \boxtimes \rho \boxtimes \sigma \boxtimes \boldsymbol{w} \overset{m}{\mapsto} \overline{\boldsymbol{u}} \boxtimes \rho\sigma \boxtimes \boldsymbol{w},\end{aligned} \tag{2.13}$$

where $ev : \widehat{CFD}(Y_2) \otimes_{\mathbb{F}} \overline{\widehat{CFD}(Y_2)} \to \mathcal{A}$ is the evaluation map $\boldsymbol{x} \otimes h \mapsto h(\boldsymbol{x})$ and the map preceding $m^{\mathcal{A}}$ is given by the isomorphism $\mathcal{A} \boxtimes \mathcal{I} \boxtimes \mathcal{A} \cong \mathcal{A} \boxtimes \mathcal{A}$. Note that this penultimate step is possible because $\boldsymbol{v}$ is a generator and the restriction of the evaluation map to the $\mathbb{F}$-vector subspace of $\widehat{CFD}(Y_2) \otimes_{\mathbb{F}} \overline{\widehat{CFD}(Y_2)}$ spanned by elements of the form $\boldsymbol{v} \otimes \overline{\boldsymbol{v}}$, where $\boldsymbol{v}$ is a generator as above, takes values in the subring $\mathcal{I}$ of idempotents of $\mathcal{A}(\mathcal{Z})$.

Small perturbations

In this subsection, we show that a small perturbation of the β-circles of a bordered Heegaard diagram $\mathcal{H} = (\overline{\Sigma}, \boldsymbol{\alpha}, \boldsymbol{\beta}, z)$ induces an isomorphism of type-D modules. Let $(\overline{\Sigma}, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}, z)$ be a provincially admissible bordered Heegaard triple with one boundary component such that $\boldsymbol{\beta}$ and $\boldsymbol{\gamma}$ consist entirely of circles. Then $(\overline{\Sigma}, \boldsymbol{\beta}, \boldsymbol{\gamma}, z)$ is an admissible balanced sutured Heegaard diagram for the sutured 3-manifold $Y_{\beta\gamma} \smallsetminus B^3$ with a single boundary suture. The corresponding sutured Floer complex $SFC(\overline{\Sigma}, \boldsymbol{\beta}, \boldsymbol{\gamma}, z)$ is isomorphic to the ordinary Heegaard Floer complex $\widehat{CF}(Y_{\beta\gamma})$ (cf. [Juh06, Proposition 9.1]). We may then define a type-D morphism

$$\widehat{f}_{\alpha\beta\gamma} : \widehat{CFD}(Y_{\alpha\beta}) \otimes \widehat{CF}(Y_{\beta\gamma}) \to \mathcal{A} \otimes \widehat{CFD}(Y_{\alpha\gamma})$$

by

$$\widehat{f}_{\alpha\beta\gamma}(\boldsymbol{x} \otimes \boldsymbol{y}) = \sum_{\boldsymbol{w} \in \mathfrak{S}(\boldsymbol{\alpha}, \boldsymbol{\gamma})} \sum_{B \in \pi_2(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{w})} a^B_{\boldsymbol{x},\boldsymbol{y},\boldsymbol{w}} \otimes \boldsymbol{w}, \tag{2.14}$$

where

$$a^B_{\boldsymbol{x},\boldsymbol{y},\boldsymbol{w}} = \sum_{\vec{\rho} \,|\, \mathrm{ind}(B,\rho)=0} \#\mathcal{M}^B(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{w}; \vec{\rho}) a(-\vec{\rho}). \tag{2.15}$$

Here, $\pi_2(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{w})$ is the space of homology classes of Whitney triangles connecting $\boldsymbol{x}$, $\boldsymbol{y}$, and $\boldsymbol{w}$, $\mathcal{M}^B(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}; \vec{\rho})$ is the moduli space of pseudoholomorphic representatives of B with asymptotic condition $\vec{\rho}$ at east infinity, and $a(-\vec{\rho})$ is defined as before. The fact that $\widehat{f}_{\alpha\beta\gamma}$ is a morphism of type-D structures follows from a straightforward variation on the usual proof that $\partial^2 = 0$ for $\widehat{CFD}(Y)$. Alternatively, it is a special case of [LOT16, Proposition 4.29].

For $\boldsymbol{\beta}^1$ a small Hamiltonian perturbation of $\boldsymbol{\beta}^0$, we will show that the map $\widehat{f}_{\alpha\beta^0\beta^1}$ induces an isomorphism $\widehat{CFD}(Y_{\alpha\beta^0}) \to \widehat{CFD}(Y_{\alpha\beta^1})$. We recall the following standard lemma [OS04b, Lemma 9.10].

Lemma 2.2.2. *Let $F : A \to B$ be a map of $\mathbb{R}$-filtered groups admitting a decomposition $F = F_0 + \ell$ where F_0 is a filtration-preserving isomorphism and $\ell(\boldsymbol{x}) < F_0(\boldsymbol{x})$ for all generators $\boldsymbol{x}$. Then, if the filtration on B is bounded below, F is an isomorphism.*

We recall here the definition of the energy filtration on $\widehat{CFD}(\overline{\Sigma}, \boldsymbol{\alpha}, \boldsymbol{\beta}, z)$ from [LOT18, Chapter 6], assuming that $(\overline{\Sigma}, \boldsymbol{\alpha}, \boldsymbol{\beta}, z)$ is admissible. Choose an area form on $\overline{\Sigma}$. Given a *Spin*^{c}-structure $\mathfrak{s}$ on Y, define $\mathcal{F} : \mathfrak{S}(\overline{\Sigma}, \boldsymbol{\alpha}, \boldsymbol{\beta}, \mathfrak{s}) \to \mathbb{R}$ as follows: choose any generator $\boldsymbol{x}_0 \in \mathfrak{S}(\overline{\Sigma}, \boldsymbol{\alpha}, \boldsymbol{\beta}, \mathfrak{s})$ and set $\mathcal{F}(\boldsymbol{x}_0) = 0$. For any other generator $\boldsymbol{x} \in \mathfrak{S}(\overline{\Sigma}, \boldsymbol{\alpha}, \boldsymbol{\beta}, \mathfrak{s})$, choose $A_{\boldsymbol{x}_0,\boldsymbol{x}} \in \pi_2(\boldsymbol{x}_0, \boldsymbol{x})$ and let

$$\mathcal{F}(\boldsymbol{x}) = -Area(A_{\boldsymbol{x}_0,\boldsymbol{x}}). \tag{2.16}$$

This definition is independent of the choice of $A_{\boldsymbol{x}_0,\boldsymbol{x}}$ since $(\overline{\Sigma}, \boldsymbol{\alpha}, \boldsymbol{\beta}, z)$ is admissible. For an algebra element $a \in \mathcal{A}$ such that $a\boldsymbol{x} \neq 0$, define $\mathcal{F}(a\boldsymbol{x}) = \mathcal{F}(\boldsymbol{x})$. Then $\mathcal{F}$ induces a filtration on $\widehat{CFD}(\overline{\Sigma}, \boldsymbol{\alpha}, \boldsymbol{\beta}, z)$.

Let $\mathcal{H}_{\alpha\beta} = (\overline{\Sigma}_g, \boldsymbol{\alpha}, \boldsymbol{\beta}^0, z)$ be an admissible genus g bordered Heegaard diagram. Provided $\boldsymbol{\beta}^1$ is a sufficiently small perturbation of $\boldsymbol{\beta}^0$, we may identify $\boldsymbol{x} \in \mathfrak{S}(\overline{\Sigma}, \boldsymbol{\alpha}, \boldsymbol{\beta}^0, \mathfrak{s})$ with its "nearest neighbor" $\boldsymbol{x}^1 \in \mathfrak{S}(\overline{\Sigma}, \boldsymbol{\alpha}, \boldsymbol{\beta}^1, \mathfrak{s})$. This identification extends to a vector space isomorphism $\widehat{CFD}(\overline{\Sigma}, \boldsymbol{\alpha}, \boldsymbol{\beta}^0, z) \to \widehat{CFD}(\overline{\Sigma}, \boldsymbol{\alpha}, \boldsymbol{\beta}^1, z)$ — which then extends automatically to an isomorphism $\Psi_{0\to 1}$ of type-D structures.

Note that if $\boldsymbol{\beta}^1$ is a small perturbation of $\boldsymbol{\beta}^0$ as above, then the homology of the complex $\widehat{CF}(\mathcal{H}_{\beta^0\beta^1})$ associated to the diagram $\mathcal{H}_{\beta^0\beta^1} = (\overline{\Sigma}_g, \boldsymbol{\beta}^0, \boldsymbol{\beta}^1, z)$ is given by $\widehat{HF}(\#^g S^2 \times S^1)$ since $\mathcal{H}_{\beta^0\beta^1}$ is an admissible balanced sutured Heegaard diagram for $\#^g S^2 \times S^1 \setminus B^3$.

Lemma 2.2.3. *Let $\Theta^{\text{top}}_{\beta^0\beta^1}$ denote the canonical top-dimensional homology class in $\widehat{HF}(\#^g S^2 \times S^1)$. Then the map $\widehat{F}^{\text{top}}_{\alpha\beta^0\beta^1} : \widehat{CFD}(\mathcal{H}_{\alpha\beta^0}) \to \mathcal{A} \otimes \widehat{CFD}(\mathcal{H}_{\alpha\beta^1})$ given by*

$$\boldsymbol{x} \mapsto \widehat{f}_{\alpha\beta^0\beta^1}(\boldsymbol{x} \otimes \Theta^{\text{top}}_{\beta^0\beta^1}) \tag{2.17}$$

is an isomorphism of type-D structures. Moreover, this map is homotopic to the nearest point map.

Proof. Let $T_{\boldsymbol{x}} \in \pi_2(\boldsymbol{x}, \Theta^{\text{top}}_{\beta^0\beta^1}, \boldsymbol{x}^1)$ be the canonical smallest triangle, which has an unique holomorphic representative by the Riemann mapping theorem. Provided our perturbation is small enough, we may assume that the area of $T_{\boldsymbol{x}}$ is smaller than the areas of all classes in $\pi_2(\boldsymbol{x}, \boldsymbol{y})$ for any generators $\boldsymbol{x}$ and $\boldsymbol{y}$ in either $\mathfrak{S}(\overline{\Sigma}, \boldsymbol{\alpha}, \boldsymbol{\beta}^0, \mathfrak{s})$ or $\mathfrak{S}(\overline{\Sigma}, \boldsymbol{\alpha}, \boldsymbol{\beta}^1, \mathfrak{s})$. Moreover, we may choose the area form so that $T_{\boldsymbol{x}}$ is the unique triangle of minimal area connecting $\boldsymbol{x}$, $\boldsymbol{y}$, and $\Theta^{\text{top}}_{\beta^0\beta^1}$ among all $\boldsymbol{y} \in \mathfrak{S}(\overline{\Sigma}, \boldsymbol{\alpha}, \boldsymbol{\beta}^1)$. Let $\mathcal{F}^1_0$ be the filtration on $\widehat{CFD}(\overline{\Sigma}, \boldsymbol{\alpha}, \boldsymbol{\beta}^1, z)$ defined as above. Define a new filtration $\mathcal{F}^1$ on $\widehat{CFD}(\overline{\Sigma}, \boldsymbol{\alpha}, \boldsymbol{\beta}^1, z)$ by taking $\mathcal{F}^1(\boldsymbol{x}^1) = \mathcal{F}^1_0(\boldsymbol{x}^1) - Area(T_{\boldsymbol{x}_0})$.

As in [LOT18, Proposition 6.41], the map $\widehat{F}^{\text{top}}_{\alpha\beta^0\beta^1}$ is filtered with respect to $\mathcal{F}$ and $\mathcal{F}^1$ and the filtration-preserving part of $\widehat{F}^{\text{top}}_{\alpha\beta^0\beta^1}$ is given by $\Psi_{0\to1}$. Note that we may promote $\widehat{F}^{\text{top}}_{\alpha\beta^0\beta^1}$ and $\Psi_{0\to1}$ to maps $\mathcal{A}\otimes\widehat{CFD}(\overline{\Sigma},\boldsymbol{\alpha},\boldsymbol{\beta}^0,z)\to\mathcal{A}\otimes\widehat{CFD}(\overline{\Sigma},\boldsymbol{\alpha},\boldsymbol{\beta}^1,z)$ of differential left $\mathcal{A}$-modules by taking $\widehat{F}^{\text{top}}_{\alpha\beta^0\beta^1}(a\otimes\boldsymbol{x}) = a\widehat{F}^{\text{top}}_{\alpha\beta^0\beta^1}(\boldsymbol{x})$ and similarly for $\Psi_{0\to1}$. Since $\Psi_{0\to1}$ is a vector space isomorphism, it follows from Lemma 2.2.2 that $\widehat{F}^{\text{top}}_{\alpha\beta^0\beta^1}$ is an isomorphism of differential left $\mathcal{A}$-modules and hence of type-D structures. One can easily adapt the argument given in [Gut22, Lemma 5.4] to show that $\widehat{F}^{\text{top}}_{\alpha\beta^0\beta^1}$ is homotopic to the nearest point map (cf. also [Lip06, Proposition 11.4]). □

We now recall a few definitions and results about holomorphic polygons with Reeb chord asymptotics. Denote by D_n an n-gon, i.e. a disk with n labeled punctures on its boundary. Label the boundary arcs clockwise as $e_0,\dots,e_{n-1}$ and let $p_{i,i+1}$ be the puncture between e_i and e_{i+1}. Define $\mathrm{Conf}(D_n)$ to be the moduli space of positively-oriented complex structures on D_n up to labeling-preserving biholomorphisms. Recall that this space has a Deligne–Mumford compactification $\overline{\mathrm{Conf}}(D_n)$ which is diffeomorphic to the associahedron and whose boundary $\partial\overline{\mathrm{Conf}}(D_n)$ consists of trees of equivalence classes of complex structures on polygons with each edge representing a gluing of two polygons along a vertex.

Definition 24 ([LOT16, Definition 3.5])**.** For a fixed symplectic form ω_Σ on a Riemann surface Σ, an *admissible collection of almost-complex structures* is a choice of $\mathbb{R}$-invariant almost complex structure J on $\Sigma\times[0,1]\times\mathbb{R}$ and a smooth family $\{J_j\}_{j\in\mathrm{Conf}(D_n)}$ of almost complex structures on $\Sigma\times D_n$ for each $n\geq 3$ such that the following conditions hold:

- For each $j\in\mathrm{Conf}(D_n)$, the projection $\pi_{\mathbb{D}}:\Sigma\times D_n\to D_n$ is (J_j,j)-holomorphic.

– For every $j \in \mathrm{Conf}(D_n)$, the fibers of $\pi_{\mathbb{D}}$ are J_j-holomorphic.

– Every J_j is adjusted to the split symplectic form $\omega_\Sigma \oplus \omega_j$ on $\Sigma \times D_n$.

– Each J_j agrees with J near the punctures of D_n in the sense that every puncture has a strip-like neighborhood U in D_n such that $(\Sigma \times U, J_j|_{\Sigma\times U})$ and $(\Sigma \times [0,1] \times (0,\infty), J)$ are biholomorphically equivalent.

– If (j_k) is a sequence in $\mathrm{Conf}(D_n)$ converging to some point $j_\infty \in \partial\overline{\mathrm{Conf}}(D_n)$ lying in the codimension-1 boundary stratum, i.e. a point $(j_{\infty,1}, j_{\infty,2}) \in \mathrm{Conf}(D_{m+1}) \times \mathrm{Conf}(D_{n-m+1})$ for some m, then the complex structures J_{j_k} converge to $J_{j_{\infty,1}} \sqcup J_{j_{\infty,2}}$ on $(\Sigma \times D_{m+1}) \sqcup (\Sigma \times D_{n-m+1})$. Convergence here is in the sense that, as $k \to \infty$, some arcs in D_{m+1} collapse and, over neighborhoods of these arcs, the complex structures J_{j_k} are obtained by inserting longer and longer necks the J_{j_k} converge in the C^∞-topology outside of these neighborhoods. The analogous compatibility condition is required for points lying in higher codimension boundary strata.

Definition 25 ([LOT16, Definition 4.5])**.** Let $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}^1, \dots, \boldsymbol{\beta}^n, z)$ be an admissible bordered Heegaard multidiagram in the sense of [LOT16, Definition 4.2], where $\boldsymbol{\alpha}$ is a complete set of bordered attaching curves compatible with $\mathcal{Z}$. Let S be a punctured Riemann surface and $\{J_j\}_{j\in\mathrm{Conf}(D_{n+1})}$ be an admissible collection of almost complex structures. Fix generators $\boldsymbol{x}^k \in \mathfrak{S}(\boldsymbol{\beta}^k, \boldsymbol{\beta}^{k+1})$ for $k = 1, \dots, n-1$ and $\boldsymbol{x}^0 \in \mathfrak{S}(\boldsymbol{\alpha}, \boldsymbol{\beta}^1)$, $\boldsymbol{x}^n \in \mathfrak{S}(\boldsymbol{\alpha}, \boldsymbol{\beta}^n)$, and let $q_i \in \partial D_{n+1}$ be points for $i = 1, \dots, k$. Consider maps of the form

$$u : (S, \partial S) \to (\Sigma \times D_{n+1}, (\boldsymbol{\alpha} \times e_0) \cup (\boldsymbol{\beta}^1 \times e_1) \cup \cdots \cup (\boldsymbol{\beta}^n \times e_n)) \tag{2.18}$$

such that the following hold:

– The projection map $\pi_\Sigma \circ u : S \to \Sigma$ has degree 0 at the region adjacent to the basepoint z.

– The punctures of S are mapped to the punctures $\{p_{i,i+1}\} \cup \{q_i\}$ of $D_{n+1} \setminus \{q_i\}$.

– The map u is asymptotic to $\boldsymbol{x}^i \times \{p_{i,i+1}\}$ at the preimage of $p_{i,i+1}$.

– u is asymptotic to $\boldsymbol{\rho}_i \times \{q_i\}$ at the punctures lying above q_i for some set $\boldsymbol{\rho}_i$ of Reeb chords in $\mathcal{Z}$.

– At each $q \in e_0 \smallsetminus \{q_i\}$, the g points $(\pi_\Sigma \circ u)((\pi_{\mathbb{D}} \circ u)^{-1}(q))$ lie in g distinct α-curves. Equivalently, $\boldsymbol{x} \otimes a(\boldsymbol{\rho}_1) \otimes \cdots \otimes a(\boldsymbol{\rho}_m)$ is nonzero, where tensor products are taken over the ring of idempotents in $\mathcal{A}(\mathcal{Z})$.

The set of maps of this type decomposes according to homology classes, the set of which we denote by $\pi_2(\boldsymbol{x}^n, \boldsymbol{x}^{n-1}, \ldots, \boldsymbol{x}^0; \boldsymbol{\rho}_1, \ldots, \boldsymbol{\rho}_m)$. For a fixed homology class $B \in \pi_2(\boldsymbol{x}^n, \boldsymbol{x}^{n-1}, \ldots, \boldsymbol{x}^0; \boldsymbol{\rho}_1, \ldots, \boldsymbol{\rho}_m)$, let

$$\mathcal{M}^B(\boldsymbol{x}^n, \boldsymbol{x}^{n-1}, \ldots, \boldsymbol{x}^0; \boldsymbol{\rho}_1, \ldots, \boldsymbol{\rho}_m; S) \tag{2.19}$$

denote the moduli space of pairs of the form (j, u) with $j \in \mathrm{Conf}(D_{n+1})$ and u a J_j-holomorphic representative of B.

Lemma 2.2.4 ([LOT16, Lemma 4.7])**.** *The expected dimension of the moduli space* $\mathcal{M}^B(\boldsymbol{x}^n, \boldsymbol{x}^{n-1}, \ldots, \boldsymbol{x}^0; \boldsymbol{\rho}_1, \ldots, \boldsymbol{\rho}_m; S)$ *is given by* $\mathrm{ind}(B, S; \boldsymbol{\rho}_1, \ldots, \boldsymbol{\rho}_m) + n - 2$, *where*

$$\mathrm{ind}(B, S; \boldsymbol{\rho}_1, \ldots, \boldsymbol{\rho}_m) = \left(\frac{3-n}{2}\right) g - \chi(S) + 2e(B) + m, \tag{2.20}$$

where g is the genus of Σ and $e(B)$ is the Euler measure of B.

Remark. The same statement holds if the multidiagram has more than one boundary component, each of which meets exactly one set of bordered attaching curves.

The Euler measure $e(B)$ can be characterized as follows: if D is a surface with boundary and corners equipped with a metric h such that ∂D is geodesic and has right-angled corners, then $e(D)$ is $\frac{1}{2\pi}$ times the integral over D of the curvature of h. From this definition, one can see that $e(D)$ is linear with respect to disjoint union and gluing along boundary segments so, if B is a formal sum $B = \sum_i n_i D_i$ of elementary domains D_i, then $e(B) = \sum_i n_i e(D_i)$. It follows from the Gauß–Bonnet theorem that if D is a surface as above with k corners with angle $\frac{\pi}{2}$ and ℓ with angle $\frac{3\pi}{2}$, then

$$e(D) = \chi(D) - \frac{k}{4} + \frac{\ell}{4}. \tag{2.21}$$

In particular, for a k-gon D with convex corners, we have $e(D) = 1 - \frac{k}{4}$. Now suppose that h is instead an arbitrary metric on D and that ∂D decomposes as $\partial D = c_1 \cup \cdots \cup c_k$. Parametrize each boundary segment c_i by $[0, 1]$. For each $i = 1, \ldots, k$, let θ_i be the angle by which the tangent vector to ∂D turns at the i^{th} corner $c_i(0)$, i.e. π minus the interior angle of D at $c_i(0)$, and define $t_i = \frac{\theta_i}{2\pi} - \frac{1}{4}$. A second application of the Gauß–Bonnet theorem allows us to rewrite $e(D)$ as

$$e(D) = \frac{1}{2\pi}\left(\int_D K dA + \sum_{i=1}^{k} \int_{c_i} \kappa_h ds\right) + \sum_{i=1}^{k} t_i, \tag{2.22}$$

where K and κ_h are the curvature and geodesic curvature of h, respectively. Therefore, if h is flat and D has geodesic boundary, we may then compute $e(D)$ by summing the contributions t_i from each corner. In particular, corners with interior angles of 60-, 90-, and 120-degrees contribute $+\frac{1}{12}$, 0, and $-\frac{1}{12}$, respectively, to the Euler measure of a flat polygon with geodesic boundary. We will use this fact momentarily.

In the case of triangles we have $n = 2$ so the dimension of the moduli space $\mathcal{M}^B(\boldsymbol{x}^2, \boldsymbol{x}^1, \boldsymbol{x}^0; \boldsymbol{\rho}_1, \ldots, \boldsymbol{\rho}_m; S)$ is given exactly by $\operatorname{ind}(B, S; \boldsymbol{\rho}_1, \ldots, \boldsymbol{\rho}_m)$, which we may write more succinctly as

$$\operatorname{ind}(B, S; \boldsymbol{\rho}_1, \ldots, \boldsymbol{\rho}_m) = \frac{g}{2} - \chi(S) + 2e(B) + m. \tag{2.23}$$

Lemma 2.2.5. *There are no positive domains for index zero holomorphic triangles in* AT *meeting* $\partial\mathsf{AT}$ *and having corners cyclically ordered according to* $(\gamma \cap \delta, \delta \cap \varepsilon, \gamma \cap \varepsilon)$.

Proof. We choose a metric on AT which is flat everywhere except on the component of $\mathsf{AT} \smallsetminus (\boldsymbol{\gamma} \cup \boldsymbol{\delta} \cup \boldsymbol{\varepsilon})$ containing $\boldsymbol{z}$. Moreover we choose this metric so that every γ-, δ, and ε-curve is geodesic and every intersection of two such curves occurs at 60 and 120 degree angles, the boundary components of AT are geodesic, and, for every $\eta \in \{\gamma, \delta, \varepsilon\}$, each η-curve meets $\partial\mathsf{AT}$ at the same angle: 120 degrees for the γ-curves, 90 degrees for the δ-curves, and 30 degrees for the ε-curves. To see that we can choose such a metric, note that the square $\square_k$ inherits a metric from its inclusion into the plane which descends to a metric on AT which is flat except on the region containing $\boldsymbol{z}$. Since the boundary of $\square_k$ is geodesic, it follows that $\partial\mathsf{AT}$ is geodesic. To see that every γ-, δ-, and ε-curve is geodesic and have the specified intersection angles, recall that we chose a particular modification of AT so that these curves arise from pairs g_i, d_i, and e_i of straight lines making an angle of 150 degrees, 90 degrees, and 30 degrees with the positive horizontal direction, respectively. Since the perturbations necessary to obtain the curves in AT can be achieved by planar translations of the lines in $\mathbb{R}^2$ corresponding to the pairs g_i and e_i, it follows that the γ-, δ-, and ε-curves are obtained as quotients of pairs of straight line segments with the same angle and hence are geodesic. The choice of angles of these segments guarantees that each of the intersections in AT occurs in one of the specified angles.

Suppose that B is a positive domain for an index zero holomorphic triangle in AT which has the above cyclic ordering on its corners and which does not meet the component of $\mathsf{AT} \smallsetminus (\boldsymbol{\gamma} \cup \boldsymbol{\delta} \cup \boldsymbol{\varepsilon})$ containing $\boldsymbol{z}$. As in the proof of [Aur10, Proposition 3.5], the Euler measure of B can be computed by summing the contributions from its corners because ∂B is geodesic: $+\frac{1}{12}$ for every corner with a 60-degree angle, 0 for every corner with a 90-degree angle, and $-\frac{1}{12}$ for every corner with a 120-degree angle. If p is an interior intersection point of two of the collections of curves in AT and B hits p at an interior point, then the local multiplicities of B in the four elementary domains meeting p are all equal so the local contribution of p to the Euler measure is zero. If B hits p at a point on the boundary which is not a corner, then B hits two of the four regions meeting at p. One of these regions meets p at a 60-degree angle and the other meets it at a 120-degree angle so the local contributions to the Euler measure cancel. If p is a genuine corner of B, then the cyclic ordering of the corners forces one of two scenarios: either B locally hits a region with a 60-degree angle at p or B locally hits two regions with a 60-degree angle at p and one with a 120-degree angle at p. In either of these two cases, the local contribution of such a corner is $+\frac{1}{12}$.

Now, if $p \in \boldsymbol{\eta} \cap \partial\mathsf{AT}$ for some $\eta \in \{\gamma, \delta, \varepsilon\}$, then there are two cases that we need to account for. Suppose, for the moment, that B meets exactly one

Reeb chord ρ in the η-boundary of AT. If p is contained in the interior of ρ, then the local multiplicities of B in the two regions meeting p are equal so the local contribution to the Euler measure is zero. Otherwise, p is an end of ρ, in which case there is a boundary intersection point q with $\partial\rho = \{p, q\}$ and the local contributions of these two corners to the Euler measure cancel since B meets p and q at complementary angles. In general, B could meet multiple boundary Reeb chords in which case the sum of the local contributions of the ends of all of the Reeb chords is zero since we can decompose this as a sum of single Reeb chord terms.

Summing over the $3g$ interior corners and all of the boundary Reeb chords of B, we see that $e(B) = \frac{g}{4}$ so, consequently, we have

$$\operatorname{ind}(B, S; \boldsymbol{\rho}_1, \ldots, \boldsymbol{\rho}_m) = g - \chi(S) + m. \tag{2.24}$$

For rigid triangles, this then tells us that $\chi(S) = g + m$ but S has at most g connected components so $\chi(S) \leq g$. Therefore, if B is a class represented by a rigid holomorphic triangle, then we must have $m = 0$, i.e. B does not meet the boundary of AT. □

Let $\mathcal{H}_i = (\overline{\Sigma}_i, \boldsymbol{\eta}_i, \boldsymbol{\beta}_i, z)$ be admissible bordered Heegaard diagrams for Y_i, $i = 1, 2, 3$, where $\eta_i = \gamma, \delta, \varepsilon$ according to the ordering $\gamma < \delta < \varepsilon$. Let $\mathcal{H}_i^+ = (\overline{\Sigma}_i, \boldsymbol{\eta}_i, \boldsymbol{\beta}_i^0, \boldsymbol{\beta}_i^1, z)$ be the result of creating a single parallel copy of each $\boldsymbol{\beta}$-circle and performing a finger move to create two intersection points between the resulting parallel pairs. Finally, let $\mathsf{AT}_{1,2,3} = \mathsf{AT}(\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3)$ be the result of gluing $\mathcal{H}_1^+$, $\mathcal{H}_2^+$, and $\mathcal{H}_3^+$ along the $\boldsymbol{\gamma}$-, $\boldsymbol{\delta}$-, and $\boldsymbol{\varepsilon}$-boundaries of $\mathsf{AT}(\mathcal{Z})$.

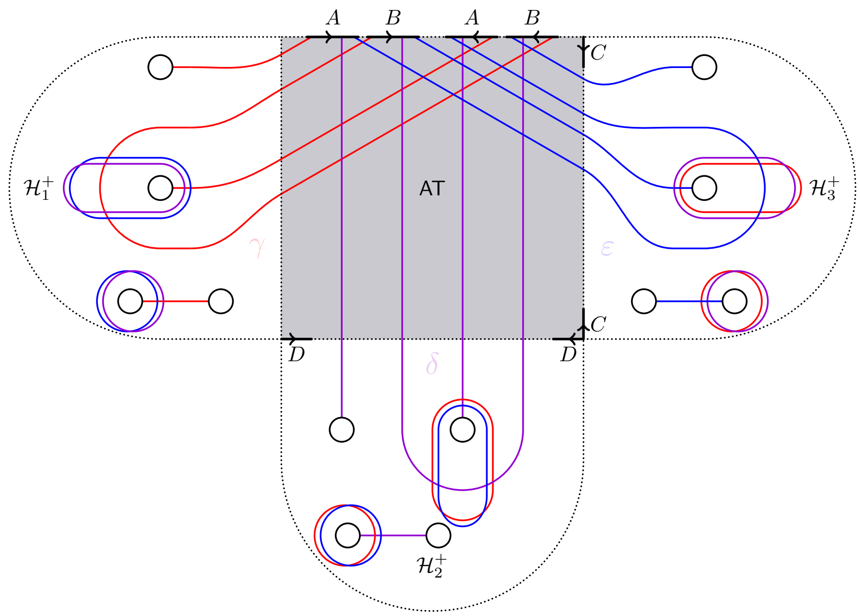


FIGURE 16. An example of an $\mathsf{AT}_{1,2,3}$ obtained by gluing triples to $\mathsf{AT}(\mathcal{Z})$.

Proposition 2.2.6. *If $\mathcal{H}_2$ is admissible, the dg-bimodule homomorphism*

$$F_{\delta,\delta} : \mathcal{A}_{\gamma,\boldsymbol{\delta}} \boxtimes \widehat{CFD}(\boldsymbol{\delta}, \boldsymbol{\beta}_2^0) \otimes \overline{\widehat{CFD}(\boldsymbol{\delta}, \boldsymbol{\beta}_2^1)} \boxtimes \mathcal{A}_{\boldsymbol{\delta},\boldsymbol{\varepsilon}} \to \mathcal{A}_{\gamma,\boldsymbol{\varepsilon}} \tag{2.25}$$

defined by counting triangles in $\mathsf{AT} \cup_\delta \mathcal{H}_2^+$ with one corner at the bottom-graded generator of $\widehat{CF}(\boldsymbol{\beta}_2^0, \boldsymbol{\beta}_2^1)$ is given up to homotopy by the map

$$\rho \boxtimes \boldsymbol{u}^0 \otimes \overline{\boldsymbol{v}}^1 \boxtimes \sigma \mapsto \rho \overline{\boldsymbol{v}}^1(\boldsymbol{u}^1)\sigma, \tag{2.26}$$

where we regard $\overline{\boldsymbol{v}}^1$ as a map from $\widehat{CFD}(\boldsymbol{\delta}, \boldsymbol{\beta}_2^1)$ to the ring of idempotents $\mathcal{I}$ in $\mathcal{A}$.

Proof. By definition, we have

$$F_{\delta,\delta}(\rho \boxtimes \boldsymbol{u}^0 \otimes \overline{\boldsymbol{v}}^1 \boxtimes \sigma) = \sum_{\tau \in \mathfrak{S}(\mathsf{AZ}_{\gamma,\varepsilon})} \sum_{\operatorname{ind}(C)=0} \#\mathcal{M}^C(\rho \boxtimes \boldsymbol{u}^0, \overline{\boldsymbol{v}}^1 \boxtimes \sigma, \tau \otimes \Theta^{\text{bot}}_{\beta_2^0 \beta_2^1})\tau, \tag{2.27}$$

where C ranges over $\pi_2(\rho \boxtimes \boldsymbol{u}^0, \overline{\boldsymbol{v}}^1 \boxtimes \sigma, \tau \otimes \Theta^{\mathrm{bot}}_{\beta_2^0\beta_2^1})$ and $\mathcal{M}^C(\rho \boxtimes \boldsymbol{u}^0, \overline{\boldsymbol{v}}^1 \boxtimes \sigma, \tau \otimes \Theta^{\mathrm{bot}}_{\beta_2^0\beta_2^1})$ is the moduli space of pseudoholomorphic representatives of the class C. By the pairing theorem for triangles [LOT16, Proposition 5.35], this map is homotopic to the one given by counting rigid triangles paired with sequences of bigons. Since there are no positive domains of rigid holomorphic triangles in AT which meet the boundary by Lemma 2.2.5, and because $\mathcal{H}_2^+$ is obtained by a small Hamiltonian translation, this tells us that $F_{\delta,\delta}$ is homotopic to the map

$$\rho \boxtimes \boldsymbol{u}^0 \otimes \overline{\boldsymbol{v}}^1 \boxtimes \sigma \mapsto \sum_{\tau \in \mathfrak{S}(\mathsf{AZ}_{\gamma,\varepsilon})} \sum_{\mathrm{ind}(C)=0} \#\mathcal{M}^C_{\times}(\rho, \sigma, \tau, \boldsymbol{u}^0, \overline{\boldsymbol{v}}^1, \Theta^{\mathrm{bot}}_{\beta_2^0\beta_2^1})\tau, \tag{2.28}$$

where the moduli space $\mathcal{M}^C_{\times}(\rho, \sigma, \tau, \boldsymbol{u}^0, \overline{\boldsymbol{v}}^1, \Theta^{\mathrm{bot}}_{\beta_2^0\beta_2^1})$ is defined by

$$\mathcal{M}^C_{\times}(\rho, \sigma, \tau, \boldsymbol{u}^0, \overline{\boldsymbol{v}}^1, \Theta^{\mathrm{bot}}_{\beta_2^0\beta_2^1}) = \bigsqcup_{A+B=C} \mathcal{M}^A(\rho, \sigma, \tau) \times \mathcal{M}^B(\boldsymbol{u}^0, \overline{\boldsymbol{v}}^1, \Theta^{\mathrm{bot}}_{\beta_2^0\beta_2^1}), \tag{2.29}$$

where A and B are provincial domains in AT and $\mathcal{H}_2^+$, respectively. Here, $\mathcal{M}^A(\rho, \sigma, \tau)$ is the moduli space of rigid pseudoholomorphic triangles of class A from $\rho \otimes \sigma$ to τ and $\mathcal{M}^B(\boldsymbol{u}^0, \boldsymbol{v}^1, \Theta^{\mathrm{bot}}_{\beta_2^0\beta_2^1})$ is the moduli space of rigid provincial triangles from $\boldsymbol{u}^0 \otimes \overline{\boldsymbol{v}}^1$ to $\Theta^{\mathrm{bot}}_{\beta_2^0\beta_2^1}$ representing the class B. Note that this latter moduli space is empty unless $\boldsymbol{u}^0$ and $\boldsymbol{v}^1$ have the same left-idempotent ι^{01}, which is then necessarily also the right-idempotent for ρ and the left-idempotent for σ in order for $\rho \boxtimes \boldsymbol{u}^0 \otimes \overline{\boldsymbol{v}}^1 \boxtimes \sigma$ to be nonzero. Together with additivity of the embedded index for disjoint unions and the fact that the index of a class with a pseudoholomorphic representative is non-negative, this then implies that

$$
\begin{aligned}
&F_{\delta,\delta}(\rho \boxtimes \boldsymbol{u}^0 \otimes \overline{\boldsymbol{v}}^1 \boxtimes \sigma) \\
&\simeq \sum_{\tau \in \mathfrak{S}(\mathsf{AZ}_{\gamma,\varepsilon})} \sum_{\operatorname{ind}(A)=\operatorname{ind}(B)=0} \#\mathcal{M}^A(\rho,\sigma,\tau)\#\mathcal{M}^B(\boldsymbol{u}^0,\overline{\boldsymbol{v}}^1,\Theta^{\mathrm{bot}}_{\beta_2^0\beta_2^1})\tau.
\end{aligned}
\tag{2.30}
$$

However, this gives us

$$
\begin{aligned}
&F_{\delta,\delta}(\rho \boxtimes \boldsymbol{u}^0 \otimes \overline{\boldsymbol{v}}^1 \boxtimes \sigma) \\
&\simeq \left(\sum_{\operatorname{ind}(B)=0} \#\mathcal{M}^B(\boldsymbol{u}^0,\overline{\boldsymbol{v}}^1,\Theta^{\mathrm{bot}}_{\beta_2^0\beta_2^1})\right) \sum_{\tau \in \mathfrak{S}(\mathsf{AZ}_{\gamma,\varepsilon})} \sum_{\operatorname{ind}(A)=0} \#\mathcal{M}^A(\rho,\sigma,\tau)\tau,
\end{aligned}
\tag{2.31}
$$

and the map

$$
\rho \otimes \sigma \mapsto \sum_{\tau \in \mathfrak{S}(\mathsf{AZ}_{\gamma,\varepsilon})} \sum_{\operatorname{ind}(A)=0} \#\mathcal{M}^A(\rho,\sigma,\tau)\tau \tag{2.32}
$$

is precisely the multiplication map $\mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$ by [Aur10, Proposition 4.8]. We then have

$$
F_{\delta,\delta}(\rho \boxtimes \boldsymbol{u}^0 \otimes \overline{\boldsymbol{v}}^1 \boxtimes \sigma) \simeq \rho \left(\sum_B \#\mathcal{M}^B(\boldsymbol{u}^0,\overline{\boldsymbol{v}}^1,\Theta^{\mathrm{bot}}_{\beta_2^0\beta_2^1})\iota^0\iota^1\right)\sigma, \tag{2.33}
$$

where ι^0 is the left-idempotent for $\boldsymbol{u}^0$ and ι^1 is the right-idempotent for $\overline{\boldsymbol{v}}^1$, which we may insert at no cost since the space $\mathcal{M}^B(\boldsymbol{u}^0,\overline{\boldsymbol{v}}^1,\Theta^{\mathrm{bot}}_{\beta_2^0\beta_2^1})$ of provincial triangles is empty unless $\iota^0 = \iota^1 = \iota^{01}$, in which case we have $\rho\sigma = \rho\iota^{01}\sigma$. We claim that the map $L : \widehat{CFD}(\boldsymbol{\delta},\boldsymbol{\beta}_2^0) \otimes \overline{\widehat{CFD}(\boldsymbol{\delta},\boldsymbol{\beta}^1)} \to \mathcal{A}$ given by

$$
\boldsymbol{u}^0 \otimes \overline{\boldsymbol{v}}^1 \mapsto \sum_{\operatorname{ind}(B)=0} \#\mathcal{M}^B(\boldsymbol{u}^0,\boldsymbol{v}^1,\Theta^{\mathrm{bot}}_{\beta_2^0\beta_2^1})\iota^0\iota^1 \tag{2.34}
$$

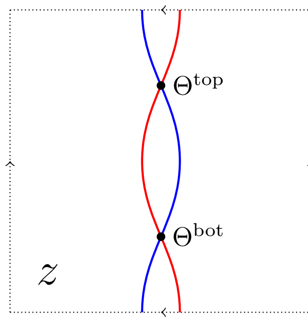


FIGURE 17. A standard genus 1 Heegaard diagram for $S^2 \times S^1$ with top- and bottom-graded generators labeled.

is homotopic to the perturbed evaluation map $ev \circ (\Psi_{0\to 1} \otimes \mathrm{id})$ given on generators by

$$\boldsymbol{u}^0 \otimes \overline{\boldsymbol{v}}^1 \mapsto \overline{\boldsymbol{v}}^1(\boldsymbol{u}^1). \tag{2.35}$$

However, L is dual to the type-D morphism $R : \widehat{CFD}(\boldsymbol{\delta}, \boldsymbol{\beta}_2^0) \to \mathcal{A} \otimes \widehat{CFD}(\boldsymbol{\delta}, \boldsymbol{\beta}^1)$ given by

$$\boldsymbol{u}^0 \mapsto \sum_{\boldsymbol{v}^1 \in \mathfrak{S}(\boldsymbol{\delta}, \boldsymbol{\beta}_2^1)} \sum_{\mathrm{ind}(B)=0} \#\mathcal{M}^B(\boldsymbol{u}^0, \Theta^{\rm top}_{\beta_2^0 \beta_2^1}, \boldsymbol{v}^1)\iota^0 \iota^1 \otimes \boldsymbol{v}^1 \tag{2.36}$$

which is filtered with respect to the the filtrations $\mathcal{F}$ and $\mathcal{F}^1$ defined in Lemma 2.2.3. As a filtered map, this has filtration preserving part given by $\Psi_{0\to 1}$ since $\Psi_{0\to 1}$ is a summand of R and R is a summand of $\widehat{F}^{\rm top}_{\delta \beta_2^0 \beta_2^1}$. This implies that R is an isomorphism and the same neck-stretching argument used in [Gut22, Lemma 5.4] to show that $\widehat{F}^{\rm top}_{\delta \beta_2^0 \beta_2^1}$ is homotopic to $\Psi_{0\to 1}$ can be used to show that R is homotopic to $\Psi_{0\to 1}$. Such a homotopy then induces a homotopy between the corresponding dual maps. Since the dual of $\Psi_{0\to 1}$ is $ev \circ (\Psi_{0\to 1} \otimes \mathrm{id})$, this proves the desired result. □

Theorem 2.2.7. *Let*

$$\widehat{G}_{\mathsf{AT}} : \mathrm{Mor}^{\mathcal{A}}(Y_1, Y_2) \otimes_{\mathbb{F}} \mathrm{Mor}^{\mathcal{A}}(Y_2, Y_3) \to \mathrm{Mor}^{\mathcal{A}}(Y_1, Y_3) \tag{2.37}$$

be the composite

$$\begin{array}{ccc}
\mathrm{Mor}^{\mathcal{A}}(Y_1, Y_2) \otimes_{\mathbb{F}} \mathrm{Mor}^{\mathcal{A}}(Y_2, Y_3) & \xrightarrow{\widehat{G}_{\mathsf{AT}}} & \mathrm{Mor}^{\mathcal{A}}(Y_1, Y_3) \\
\cong \downarrow & & \uparrow \cong \\
\widehat{CF}(\mathcal{H}_1 \cup \mathcal{H}_2) \otimes_{\mathbb{F}} \widehat{CF}(\mathcal{H}_2 \cup \mathcal{H}_3) & & \widehat{CF}(\mathcal{H}_1 \cup \mathcal{H}_3) \\
\text{1-handle} \downarrow & & \uparrow \text{3-handle} \\
(\widehat{CF}(\mathcal{H}_1 \cup \mathcal{H}_2) \otimes V^{\otimes g_3}) \otimes_{\mathbb{F}} (\widehat{CF}(\mathcal{H}_2 \cup \mathcal{H}_3) \otimes V^{\otimes g_1}) & \xrightarrow[\widehat{F}_{\mathsf{AT}_{1,2,3}}]{} & \widehat{CF}(\mathcal{H}_1 \cup \mathcal{H}_3) \otimes V^{\otimes g_2}
\end{array} \tag{2.38}$$

where we take the model $\overline{\widehat{CFD}(\mathcal{H}_i)} \boxtimes \mathcal{A} \boxtimes \widehat{CFD}(\mathcal{H}_j)$ *for* $\widehat{CF}(\mathcal{H}_i \cup \mathcal{H}_j)$*, the vertical isomorphisms are the ones described above,* V *is the two-dimensional model for* $\widehat{CF}(S^2 \times S^1)$ *given by the standard genus 1 Heegaard diagram for* $S^2 \times S^1$*,* $\widehat{F}_{\mathsf{AT}_{1,2,3}}$ *is the map determined by the Heegaard triple* $\mathsf{AT}_{1,2,3}$*, and*

$$\widehat{CF}(Y) \overset{\text{1-handle}}{\hookrightarrow} \widehat{CF}(Y) \otimes V^{\otimes m} \cong \widehat{CF}(Y \# (S^2 \times S^1)^{\# m})$$

and

$$\widehat{CF}(Y) \otimes V^{\otimes n} \cong \widehat{CF}(Y \# (S^2 \times S^1)^{\# n}) \overset{\text{3-handle}}{\twoheadrightarrow} \widehat{CF}(Y)$$

are the usual 1-handle and 3-handle maps defined on generators by

$$\boldsymbol{x} \mapsto \boldsymbol{x} \otimes \Theta^{\mathrm{top}}$$

and

$$\boldsymbol{y} \otimes \theta \mapsto \begin{cases} \boldsymbol{y} & \text{if } \theta = \Theta^{\text{bot}} \\ 0 & \text{else,} \end{cases} \tag{2.39}$$

respectively, where Θ^{bot} *is the bottom-graded generator. Then* $\widehat{G}_{\mathsf{AT}}$ *agrees up to homotopy with the composition map* $f \otimes g \mapsto g \circ f$*.*

Proof. We assume that each of the bordered Heegaard triples $\mathcal{H}_i^+ = (\overline{\Sigma}_i, \boldsymbol{\eta}, \boldsymbol{\beta}_i^0, \boldsymbol{\beta}_i^1)$ are obtained by suitable small Hamiltonian perturbations so that Lemma 2.2.3 applies. By construction and the pairing theorem for triangles [LOT16, Proposition 3.35], we have a decomposition $\widehat{G}_{\mathsf{AT}} \simeq \widehat{F}^{\text{top}}_{\gamma\beta_1^1\beta_1^0} \boxtimes F_{\delta,\delta} \boxtimes \widehat{F}^{\text{top}}_{\varepsilon\beta_3^0\beta_3^1}$ under the identifications $\text{Mor}^{\mathcal{A}}(Y_i, Y_j) \cong \overline{\widehat{CFD}(Y_i)} \boxtimes \mathcal{A} \boxtimes \widehat{CFD}(Y_j)$. Since the maps $\widehat{F}^{\text{top}}_{\gamma\beta_1^1\beta_1^0}$ and $\widehat{F}^{\text{top}}_{\varepsilon\beta_3^0\beta_3^1}$ are homotopic to the corresponding nearest point maps, Proposition 2.2.6 then tells us that $\widehat{G}_{\mathsf{AT}}$ is homotopic to the map given on basic morphisms by

$$(\overline{\boldsymbol{t}}^1 \boxtimes \rho \boxtimes \boldsymbol{u}^0) \otimes (\overline{\boldsymbol{v}}^1 \boxtimes \sigma \boxtimes \boldsymbol{w}^0) \mapsto \overline{\boldsymbol{t}}^0 \boxtimes \rho \overline{\boldsymbol{v}}^1(\boldsymbol{u}^1)\sigma \boxtimes \boldsymbol{w}^1, \tag{2.40}$$

which is precisely the composition map. □

Corollary 2.2.8. *Suppose that* $\mathcal{H}_1$ *and* $\mathcal{H}_1'$ *are bordered Heegaard diagrams for a bordered 3-manifold* Y_1 *differing by a single bordered Heegaard move, then the square*

$$\begin{array}{ccc} \text{Mor}^{\mathcal{A}}(\mathcal{H}_1, \mathcal{H}_2) \otimes \text{Mor}^{\mathcal{A}}(\mathcal{H}_2, \mathcal{H}_3) & \xrightarrow{f \otimes g \mapsto g \circ f} & \text{Mor}^{\mathcal{A}}(\mathcal{H}_1, \mathcal{H}_3) \\ \simeq\big\downarrow & & \big\downarrow\simeq \\ \text{Mor}^{\mathcal{A}}(\mathcal{H}_1', \mathcal{H}_2) \otimes \text{Mor}^{\mathcal{A}}(\mathcal{H}_2, \mathcal{H}_3) & \xrightarrow{f \otimes g \mapsto g \circ f} & \text{Mor}^{\mathcal{A}}(\mathcal{H}_1', \mathcal{H}_3) \end{array} \tag{2.41}$$

commutes up to homotopy, where the vertical maps are given by the homotopy equivalences $\mathrm{Mor}^{\mathcal{A}}(\mathcal{H}_1, \mathcal{H}_i) \to \mathrm{Mor}^{\mathcal{A}}(\mathcal{H}'_1, \mathcal{H}_i)$ *induced by the Heegaard move. The analogous statement also holds for* $\mathcal{H}_2$ *and* $\mathcal{H}_3$*.*

Proof. In the case of finger moves and handleslides, this follows from Theorem 2.2.7 by associativity of triangle counts. In the case of stabilizations, up to some number of finger moves and handleslides, one may assume that the stabilization is performed in a neighborhood of the basepoint, in which case the vertical maps are isomorphisms. □

2.3 4-manifolds with Corners from Bordered Heegaard Triples

Just as one may represent a 4-manifold with boundary by a closed Heegaard triple and bordered 3-manifolds may be represented using (arced) bordered Heegaard diagrams [LOT18], we may describe 4-manifolds with boundary and corners using a suitable amalgamation of the two notions.

Definition 26. A genus g *arced bordered Heegaard triple* with B boundary components is a quintuple $\mathcal{H} = (\overline{\Sigma}, \boldsymbol{\gamma}, \boldsymbol{\delta}, \boldsymbol{\varepsilon}, \boldsymbol{z})$, where:

- $\overline{\Sigma}$ is a compact connected surface of genus g with boundary components $\partial_1\overline{\Sigma}, \ldots, \partial_B\overline{\Sigma}$
- each $\boldsymbol{\eta} \in \{\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}\}$ is a pairwise disjoint collection

$$\boldsymbol{\eta} = \{\eta_1^c, \ldots, \eta_{g-T_\eta}\} \cup \bigcup_{i=1}^{B} \{\eta_1^i, \ldots, \eta_{2t_i^\eta}^i\},$$

where $T_\eta = \sum_{i=1}^{B} t_i^\eta$, consisting of embedded arcs η_j^i in $\overline{\Sigma}$ with boundary on $\partial_i\Sigma$ and circles η_k^c in the interior of $\overline{\Sigma}$. We further impose the condition that if

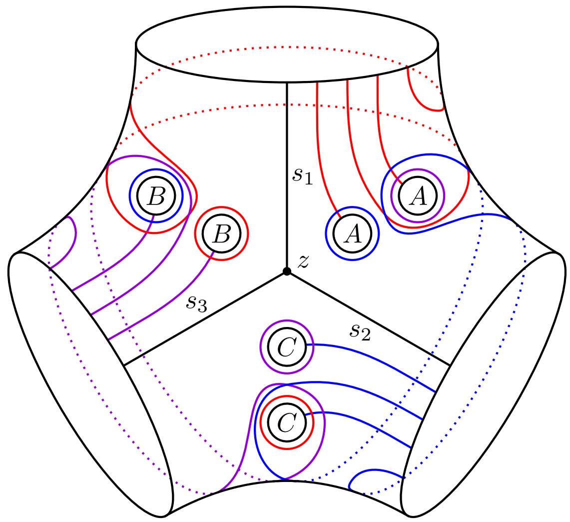


FIGURE 18. A genus 3 bordered Heegaard triple $\mathcal{H}$ with three boundary components.

$t_i^\eta \neq 0$, then $t_i^\theta = 0$ for $\boldsymbol{\theta} \neq \boldsymbol{\eta}$. In other words, this condition says that no two collections of curves meet the same boundary component nontrivially. For the sake of convenience, we denote the collection $\{\eta_1^c, \ldots, \eta_{g-T_\eta}^c\}$ by $\boldsymbol{\eta}^c$ and the collections $\{\eta_1^i, \ldots, \eta_{2t_i^\eta}^i\}$ by $\boldsymbol{\eta}^i$.

– $\boldsymbol{z} = (z; s_1, \ldots, s_b)$ consists of an interior point $z \in \overline{\Sigma}$ disjoint from $\boldsymbol{\gamma} \cup \boldsymbol{\delta} \cup \boldsymbol{\varepsilon}$ together with embedded arcs s_i in $\overline{\Sigma} \setminus (\boldsymbol{\gamma} \cup \boldsymbol{\delta} \cup \boldsymbol{\varepsilon})$ connecting z and $\partial_i \overline{\Sigma}$.

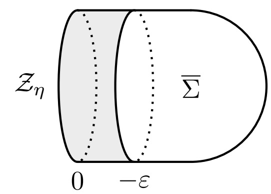
We also require that each of $\overline{\Sigma} \setminus \boldsymbol{\gamma}$, $\overline{\Sigma} \setminus \boldsymbol{\delta}$, and $\overline{\Sigma} \setminus \boldsymbol{\varepsilon}$ is connected and that the collections $\boldsymbol{\gamma}$, $\boldsymbol{\delta}$, and $\boldsymbol{\varepsilon}$ intersect pairwise transversely. Lastly, we require that each component of $\partial\overline{\Sigma}$ is met by some $\boldsymbol{\eta}$. If $\boldsymbol{\eta}^i$ is the collection of arcs meeting $\partial_i \Sigma$ nontrivially, we will denote the induced (as in Lemma 4.4 of [LOT18]) pointed matched circle by $\mathcal{Z}_i(\mathcal{H})$ or simply by $\mathcal{Z}_i$ when there is no risk of ambiguity.

Note that, for any two distinct collections $\boldsymbol{\eta}, \boldsymbol{\theta} \in \{\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}\}$, forgetting the third collection, filling in the now-empty boundary components with disks, and forgetting the arcs $s_{i_1}, \ldots, s_{i_f}$ which meet the filled boundary components, yields an arced bordered Heegaard diagram $\mathcal{H}^{\eta,\theta} = (\overline{\Sigma}_{\eta,\theta}, \boldsymbol{\eta}, \boldsymbol{\theta}, \boldsymbol{z}_{\eta,\theta})$. Such a diagram determines a (strongly bordered) 3-manifold $Y_{\eta\theta} = Y(\mathcal{H}^{\eta,\theta})$ with $B - f$ boundary components by attaching 2-handles to $\overline{\Sigma}_{\eta,\theta} \times [0,1]$, analogous to [LOT15, Constructions 5.3 and 5.6]. From an arced bordered Heegaard triple $\mathcal{H}$, we will define a 4-manifold $X(\mathcal{H})$ with connected boundary and corners.

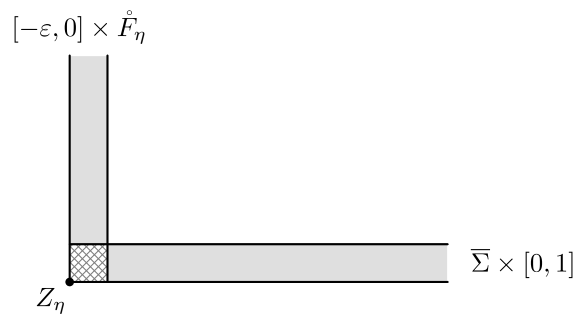
Remark. One could more generally allow bordered Heegaard triples $\mathcal{H}$ whose arcs connect multiple boundary components, in which case $\partial X(\mathcal{H})$ is a bordered sutured 3-manifold with corners following constructions analogous to those given by Zarev in [Zar11]. However, we will not explore this construction here; we content ourselves to only consider the case $B \leq 3$.

In addition to $Y_{\eta\theta}$, the arced bordered Heegaard diagram $\mathcal{H}^{\eta,\theta}$ specifies preferred disks $\Delta_j \subset \partial_j Y_{\eta\theta}$, which are obtained as the images in $Y_{\eta\theta}$ of the "faces" of the 2-handles attached in the last step of the above construction, points $z_j \in \partial\Delta_j$ coming from the endpoints of $\boldsymbol{z}_{\eta,\theta}$, and homeomorphisms of triples $\phi_i : (F(\mathcal{Z}_j), D_j, z_j) \to (\partial_j Y_{\eta\theta}, \Delta_j, z_j)$ for each $j \neq i_1, \ldots, i_f$, and an isotopy class $\nu_{\eta,\theta}$ of nowhere vanishing normal vector fields to $\boldsymbol{z}_{\eta,\theta}$ pointing into Δ_j at z_j. The data $(Y_{\eta\theta}, \boldsymbol{\phi}_{\eta,\theta}, \nu_{\eta,\theta})$, where $\boldsymbol{\phi}_{\eta,\theta} = \{\phi_j\}$ (note that this collection includes the data of the preferred disks and basepoints), is called the *strongly bordered* 3-manifold associated to $\mathcal{H}^{\eta,\theta}$. We will abbreviate this data as $Y_{\eta\theta}$.

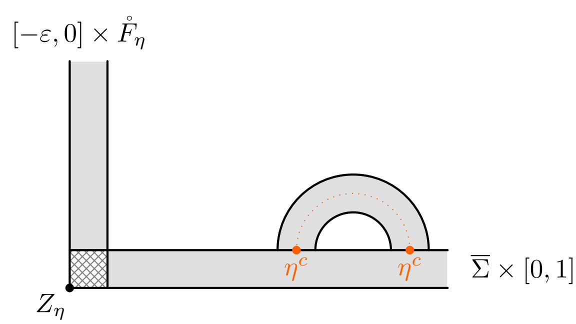
Construction 2.3.1. Let $\mathcal{H} = (\overline{\Sigma}, \boldsymbol{\gamma}, \boldsymbol{\delta}, \boldsymbol{\varepsilon}, \boldsymbol{z})$ be an an (arced) bordered Heegaard triple. For $\boldsymbol{\eta} \in \{\boldsymbol{\gamma}, \boldsymbol{\delta}, \boldsymbol{\varepsilon}\}$ meeting the boundary, construct a *cornered handlebody* $\overline{U}_\eta$ as follows. Let $\overline{U}_0 = \overline{\Sigma} \times [0,1]$ and let $\mathring{F}_\eta = F(\mathcal{Z}_\eta) \setminus \mathrm{int}(D^2_\eta)$, where D^2_η is the disk with $\partial D^2_\eta = Z_\eta$ used to construct $F(\mathcal{Z}_\eta)$ from the pointed matched circle $\mathcal{Z}_\eta = (Z_\eta, \boldsymbol{a}_\eta, M_\eta)$. Choose a closed collar neighborhood $[-\varepsilon, 0] \times Z_\eta$ of $Z_\eta \subset \overline{\Sigma}$ such that $\{0\} \times Z_\eta$ is identified with Z_η as in the following schematic figure.



Next, choose a closed tubular neighborhood $Z_\eta \times [0,1]$ of Z_η in $\mathring{F}_\eta$ and glue $\overline{U}_0$ to $[-\varepsilon, 0] \times \mathring{F}_\eta$ by identifying the subsets $([-\varepsilon, 0] \times Z_\eta) \times [0,1] \subset \overline{\Sigma} \times [0,1]$ and $[-\varepsilon, 0] \times (Z_\eta \times [0,1]) \subset [-\varepsilon, 0] \times F_\eta$ as in



and, similarly, attach a copy of $[-\varepsilon, 0] \times D^2$ at each boundary component not met by $\boldsymbol{\eta}$ to obtain a new cornered 3-manifold $\overline{U}_1$ with two cornered boundary components, both of which are of the form $\Sigma_\eta := \mathring{F}_\eta \cup_\eta \overline{\Sigma} \cup_\partial D^2 \cup_\partial \overset{B-1}{\cdots} \cup_\partial D^2$, where $B = \#\pi_0(\partial\overline{\Sigma})$ and each surface in this union is glued to $\overline{\Sigma}$ at a 90 degree angle. For $\boldsymbol{\eta}$ not meeting any boundary component, instead attach a copy of $[-\varepsilon, 0] \times D^2$ in this manner at each boundary component — in this case, the resulting cornered 3-manifold has boundary components of the form $\Sigma_\varnothing := \overline{\Sigma} \cup_\partial D^2 \cup_\partial \overset{B}{\cdots} \cup_\partial D^2$. Now attach 3-dimensional 2-handles to the η-circles $\eta_i^c \times \{0\} \subset \overline{\Sigma} \times [0,1]$ as in



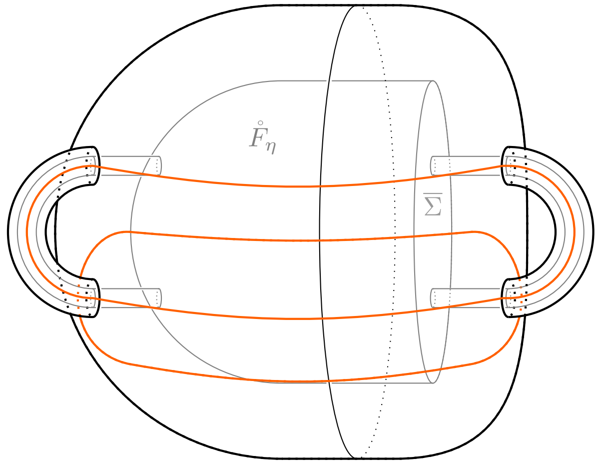


FIGURE 19. A genus 1 example of a $\overline{U}_2$ in the case that $\boldsymbol{\eta}$ does meet the boundary.

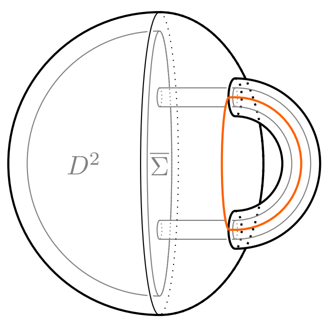
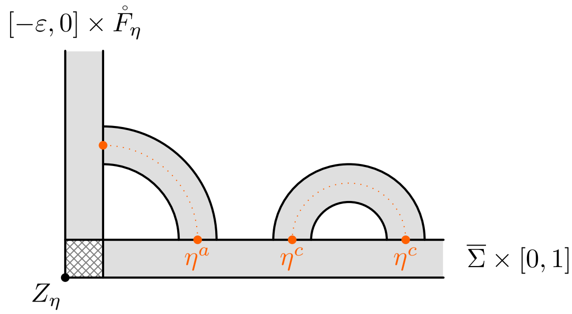
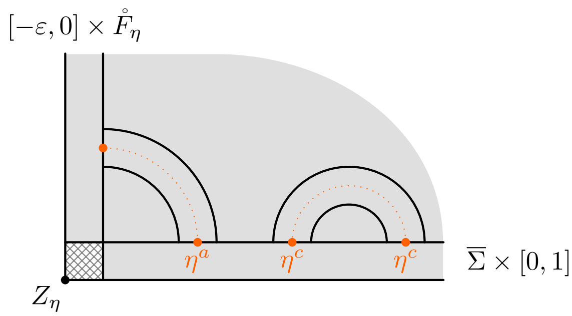


FIGURE 20. A genus 1 example of a $\overline{U}_2$ in the case that $\boldsymbol{\eta}$ does not meet the boundary.

to obtain a new 3-manifold $\overline{U}_2$ with two boundary components: a copy of Σ_η or $\Sigma_\varnothing$ meeting $\overline{\Sigma} \times \{1\}$ and a genus $2k_\eta$ surface S_η, where $4k_\eta$ is the number of points in the boundary pointed matched circle corresponding to $\boldsymbol{\eta}$ (which is zero if $\boldsymbol{\eta}$ does not meet the boundary), which meets $\overline{\Sigma} \times \{0\}$. Next, if $\boldsymbol{\eta}$ meets the boundary, join each η-arc $\eta_i^a \times \{0\} \subset S_\eta$ to the core of the corresponding handle in $\{-\varepsilon\} \times \mathring{F}_\eta$ to obtain a collection of closed curves and attach a 3-dimensional 2-handle along each as in the following figure. If $\boldsymbol{\eta}$ does not meet the boundary, instead go on immediately to the next step.



This has the effect of replacing the boundary component S_η with an S^2 boundary component. We then define $\overline{U}_\eta$ to be the result of filling this boundary component with a 3-ball as in



— the resulting space is a 3-manifold with boundary and corners, whose boundary stratum is $\partial_1\overline{U}_\eta = \Sigma_\eta$ or $\partial_1\overline{U}_\eta = \Sigma_\varnothing$, depending on whether or not $\boldsymbol{\eta}$ meets the boundary, and whose corner stratum is of the form $\partial_2\overline{U}_\eta = S^1 \sqcup \overset{B}{\cdots} \sqcup S^1$. We then define a cornered 4-manifold $X(\mathcal{H})$ by

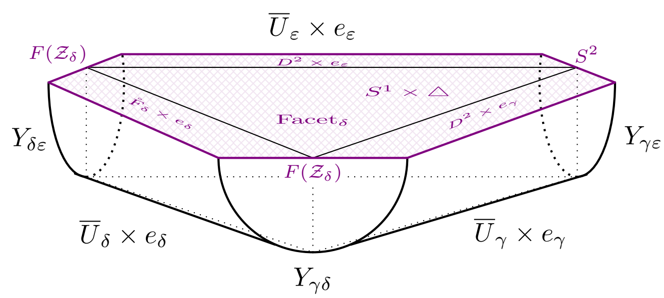
$$X(\mathcal{H}) = (\overline{\Sigma}\times\triangle)\cup_{\overline{\Sigma}\times e_\gamma}(\overline{U}_\gamma\times e_\gamma)\cup_{\overline{\Sigma}\times e_\delta}(\overline{U}_\delta\times e_\delta)\cup_{\overline{\Sigma}\times e_\varepsilon}(\overline{U}_\varepsilon\times e_\varepsilon), \tag{2.42}$$

where $\triangle$ is a triangle with edges labeled clockwise as e_γ, e_δ, and e_ε, smoothing corners between the $\overline{U}_\eta$'s at the vertices $\overline{\Sigma}\times(e_\eta\cap e_\theta)$. Note that the boundary stratum $\partial_1 X(\mathcal{H})$ is connected and consists of the following two pieces. First, it contains each of the bordered 3-manifolds $Y_{\eta\theta} = Y(\mathcal{H}_{\eta\theta})$, where the diagrams $\mathcal{H}_{\eta\theta} = (\overline{\Sigma},\boldsymbol{\eta},\boldsymbol{\theta},z)$ are the bordered Heegaard diagrams obtained from $\mathcal{H}$ by deleting one of the collections of curves and filling the corresponding boundary component with a disk. Second, if $\boldsymbol{\theta}_1$ and $\boldsymbol{\theta}_2$ are the collections of curves not meeting the η-boundary, it contains a copy of

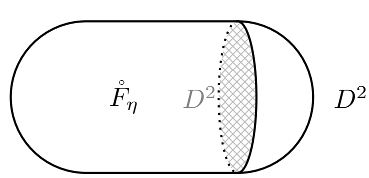
$$\mathrm{Facet}_\eta := S^1 \times \triangle \cup_{S^1 \times e_\eta} (\mathring{F}_\eta \times e_\eta) \cup_{S^1 \times e_{\theta_1}} (D^2 \times e_{\theta_1}) \cup_{S^1 \times e_{\theta_2}} (D^2 \times e_{\theta_2}), \quad (2.43)$$

and there is one such “facet” for each $\boldsymbol{\eta}$ meeting $\partial\overline{\Sigma}$. These two distinguished parts of the boundary stratum meet in two copies of $F(\mathcal{Z}_\eta)$ and one copy of S^2. The union of these surfaces over all $\boldsymbol{\eta}$ meeting $\partial\overline{\Sigma}$ forms the corner stratum $\partial_2 X(\mathcal{H})$.

In the single boundary component case, one may think of $X(\mathcal{H})$ schematically as in the following figure, which represents the δ-bordered case.

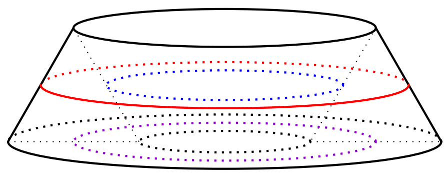


However, this representation of $X(\mathcal{H})$ may be somewhat misleading: topologically, the space Facet_η is a closed 3-dimensional regular neighborhood of the singular surface $\mathring{F}_\eta \cup_\partial D^2 \cup_\partial D^2$



— i.e. Facet_η is a 3-dimensional pair of pants cobordism $-F(\mathcal{Z}_\eta) \sqcup F(\mathcal{Z}_\eta) \to S^2$. To see this, note that Facet_η is the result of gluing $\mathring{F}_\eta \times [0,1]$ and two copies of

$D^2 \times [0,1]$ to $S^1 \times \triangle$ by identifying each of $\partial \mathring{F}_\eta \times [0,1]$ and the two copies of $\partial D^2 \times [0,1]$ with one of $S^1 \times e_\gamma$, $S^1 \times e_\delta$, and $S^1 \times e_\varepsilon$ so that $\partial \mathring{F}_\eta \times \{\frac{1}{2}\}$ and the two copies of $\partial D^2 \times \{\frac{1}{2}\}$ are identified with the circles $S^1 \times \{\text{midpoint}\}$ depicted in the following figure and smoothing corners.



Remark. More generally, an arced bordered Heegaard n-tuple

$$\mathcal{H} = (\overline{\Sigma}_g, \boldsymbol{\eta}_0, \ldots, \boldsymbol{\eta}_{n-1}, \boldsymbol{z}) \tag{2.44}$$

with B boundary components determines a cornered 4-manifold $X(\mathcal{H})$ whose boundary stratum consists of the bordered 3-manifolds $Y_{\eta_i \eta_{i+1}}$, with indices taken modulo n, together with facets Facet_{η_i} for each i for which $\boldsymbol{\eta}_i$ intersects $\partial \overline{\Sigma}_g$ nontrivially. The constructions of $X(\mathcal{H})$ and the facets Facet_{η_i} are identical to the $n = 3$ case except that we replace the triangle $\triangle$ with a planar n-gon.

<u>Gluing</u>

Let $\mathcal{H} = (\overline{\Sigma}_g, \boldsymbol{\gamma}, \boldsymbol{\delta}, \boldsymbol{\varepsilon}, \boldsymbol{z})$ be an arced bordered Heegaard triple with three boundary components and let $\mathcal{H}_1 = (\overline{\Sigma}_{g_1}, \boldsymbol{\gamma}_1, \boldsymbol{\delta}_1, z_1)$, $\mathcal{H}_2 = (\overline{\Sigma}_{g_2}, \boldsymbol{\delta}_2, \boldsymbol{\varepsilon}_2, z_2)$, and $\mathcal{H}_3 = (\overline{\Sigma}_{g_3}, \boldsymbol{\varepsilon}_3, \boldsymbol{\gamma}_3, z_3)$ be γ-, δ-, and ε-bordered Heegaard diagrams, respectively. Let $\mathcal{H}_{1,2,3} = \mathcal{H} \cup_\gamma \mathcal{H}_1^+ \cup_\delta \mathcal{H}_2^+ \cup_\varepsilon \mathcal{H}_3^+$ be the ordinary Heegaard triple that results from doubling the collections of curves in the $\mathcal{H}_i$ not meeting the boundary, labeling the new circles according to whichever label does not appear in $\mathcal{H}_i$, and gluing them to the corresponding boundary components of $\mathcal{H}$, as we did in the construction of $\mathsf{AT}_{1,2,3}$.

Proposition 2.3.1. *If $\mathcal{H}_1$ and $\mathcal{H}_2$ are bordered Heegaard triples sharing a common boundary matching and $\mathcal{H}_2$ has one boundary component, then there is a diffeomorphism $X(\mathcal{H}_1 \cup_\partial \mathcal{H}_2) \cong X(\mathcal{H}_1) \cup_{\mathrm{Facet}} X(\mathcal{H}_2)$, where* Facet *is the corresponding boundary facet. In particular, the 4-manifold $X(\mathcal{H}_{1,2,3})$ is diffeomorphic to*

$$X(\mathcal{H}) \cup_{\mathrm{Facet}_\gamma} X(\mathcal{H}_1^+) \cup_{\mathrm{Facet}_\delta} X(\mathcal{H}_2^+) \cup_{\mathrm{Facet}_\varepsilon} X(\mathcal{H}_3^+). \tag{2.45}$$

Proof. Suppose that $\mathcal{H}_2$ is an η-bordered Heegaard diagram. The effect of gluing $\mathcal{H}_2$ to the η-boundary of $\mathcal{H}_1$ is as follows. First, the underlying surface $\overline{\Sigma}_g$ is replaced by $\overline{\Sigma}_g \cup_\eta \overline{\Sigma}_{g_2}$ which has the effect of gluing $\overline{\Sigma}_g \times \triangle$ to $\overline{\Sigma}_{g_2} \times \triangle$ in the obvious manner. Second, gluing the η-arcs which meet the boundary along their common endpoints corresponds to gluing the 3-dimensional 2-handles along the corresponding cores of the 1-handles in $\mathring{F}_\eta$ determined by the arcs. This has the effect of gluing the respective η-handlebodies along their $\mathring{F}_\eta$-boundaries. Lastly, for $\theta \neq \eta$, the respective θ-handlebodies are glued along their disk boundaries. It is straightforward to see that these glued handlebodies are precisely the handlebodies obtained from the above construction using the glued diagram so this proves the result. □

Corollary 2.3.2. *The 4-manifold $X(\mathsf{AT}_{1,2,3})$ is diffeomorphic to the composition $W_2^{13,g_2} \circ W \circ (W_{-2}^{12,g_3} \sqcup W_{-2}^{23,g_1})$ of the pair of pants cobordism $W : Y_{12} \sqcup Y_{23} \to Y_{13}$ with the cobordisms $W_2^{ij,g_k} : Y_{ij} \to Y_{ij} \#(S^2 \times S^1)^{g_k}$ obtained by surgery on 0-framed g_k-component unlinks in Y_{ij} and their reverses $W_{-2}^{ij,g_k} : Y_{ij}\#(S^2 \times S^1)^{g_k} \to Y_{ij}$. Thus, if $W_1^{ij,g} : Y_{ij} \to Y_{ij}\#(S^2 \times S^1)^{\#g}$ and $W_3^{ij,g} : Y_{ij}\#(S^2 \times S^1)^{\#g} \to Y_{ij}$ are the*

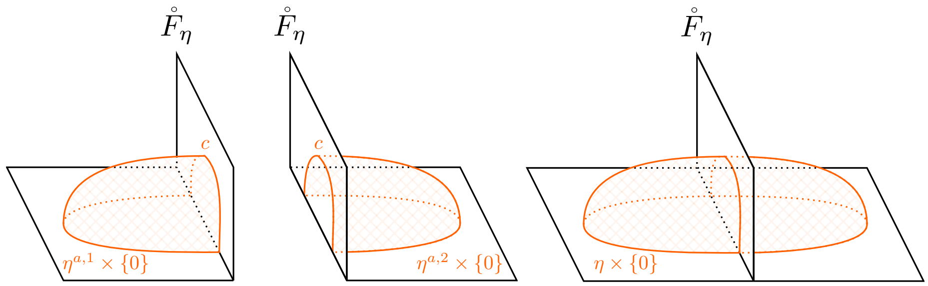


FIGURE 21. The effect of gluing bordered Heegaard triples on the 2-handles attached to matched pairs of curves of the form $\eta^a \quad \cup_\partial \quad c$, where c is the core of a 1-handle in $\mathring{F}_\eta$.

usual 1-handle and 3-handle cobordisms, then $W_3^{13,g_2} \circ X(\mathsf{AT}_{1,2,3}) \circ (W_1^{12,g_3} \sqcup W_1^{23,g_1})$ *is diffeomorphic to* W*.*

Proof. Suppose that $\overline{\mathcal{H}} = (\overline{\Sigma}_g, \boldsymbol{\alpha}, \boldsymbol{\beta}, z)$ is an α-bordered Heegaard triple and let $Y = Y(\overline{\mathcal{H}})$ be the corresponding bordered 3-manifold. We claim that the cornered 4-manifold $X = X(\overline{\mathcal{H}}^+)$ determined by the triple $\mathcal{H}^+ = (\overline{\Sigma}, \boldsymbol{\alpha}, \boldsymbol{\beta}^0, \boldsymbol{\beta}^1, z)$ obtained by doubling $\boldsymbol{\beta}$ is diffeomorphic to the cobordism of pairs

$$(-Y \sqcup Y, -\partial Y \sqcup \partial Y) \to ((S^2 \times S^1)^{\# g} \setminus B^3, S^2) \tag{2.46}$$

given by the complement of a regular neighborhood of the cornered handlebody $\overline{U}_\beta \times \{0\}$ in $Y \times [-1, 1]$. To see this, recall from [OS06, Proposition 4.3] that if $\mathcal{H}' = (\Sigma_0, \boldsymbol{\alpha}', \boldsymbol{\beta}', z)$ is any Heegaard diagram for a closed 3-manifold Y' and $(\Sigma_0, \boldsymbol{\alpha}', \boldsymbol{\beta}', \boldsymbol{\gamma}, z)$ is such that $\boldsymbol{\gamma}$ is obtained by a small Hamiltonian translation of $\boldsymbol{\beta}'$, then the 4-manifold $X_{\alpha'\beta'\gamma}$ determined by this diagram is diffeomorphic to $Y' \times [-1, 1]$ with a regular neighborhood of the handlebody $U_{\beta'} \times \{0\}$ deleted, i.e. the cobordism obtained by attaching 2-handles to a 0-framed unlink in a Euclidean ball in Y'. In particular, this is the case if $\mathcal{H}' = \mathcal{H}_0 \cup_\partial \mathcal{H}$ for some other α-bordered Heegaard diagram $\mathcal{H}_0$. The claim then follows from the previous proposition.

The first statement now follows from Proposition 2.3.1 together with the observation that the surface underlying the triple AT is naturally identified with $F(\mathcal{Z})$ with three disks removed and the fact that deleting any pair of curves from AT determines a bordered Heegaard diagram for $F(\mathcal{Z}) \times I$ after filling the now-empty boundary component with a disk. The second statement then follows from the fact that the 2- and 3-handles in $W_3^{13,g_2} \circ W_2^{13,g_2}$ and the 1- and 2-handles in $(W_{-2}^{12,g_3} \sqcup W_{-2}^{23,g_1}) \circ (W_1^{12,g_3} \sqcup W_1^{23,g_1})$ cancel. □

Another way of thinking about these results is as follows. Given a closed 3-manifold Y, we have two distinct ways of decomposing Y into 3-manifolds with boundary: we can either decompose Y as $Y = U_\alpha \cup_\Sigma U_\beta$, where U_α and U_β are handlebodies glued along a Heegaard surface Σ, or we can decompose it as $Y = -Y_1 \cup_{F(\mathcal{Z})} Y_2$, where Y_1 and Y_2 are bordered 3-manifolds which both have boundary parameterized by the same surface $F(\mathcal{Z})$. Here, we have chosen this second splitting to be one obtained by cutting a closed Heegaard diagram $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, z)$ for the decomposition $Y = U_\alpha \cup_\Sigma U_\beta$ along some circle which intersects one of the pairs of curves, giving us two bordered Heegaard diagrams with the same pointed matched circle $\mathcal{Z}$. In this case, the copy of the surface $F(\mathcal{Z})$ sitting inside of Y meets Σ transversely in a single separating copy of S^1. Therefore, each Y_i decomposes as a union of two cornered handlebodies $Y_i = \overline{U}_\alpha^i \cup_{\Sigma \cap Y_i} \overline{U}_\beta^i$ and each handlebody U_η decomposes similarly as $U_\eta = \overline{U}_\eta^1 \cup_{F(\mathcal{Z}) \cap U_\eta} \overline{U}_\eta^2$. This allows us to decompose Y into four "quadrants" which are compatible with the (restrictions of) the gluings in both decompositions of Y (cf. Figure 22). These quadrants are precisely the cornered handlebodies from Construction 2.3.1. If we had instead

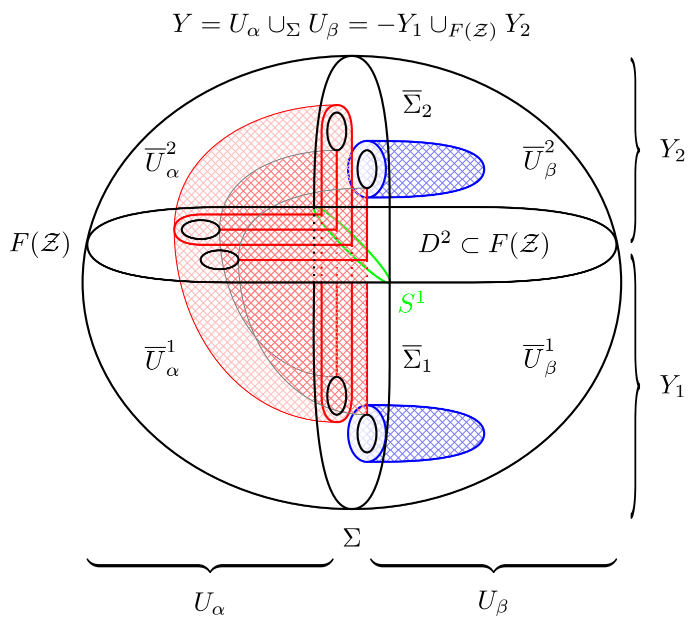


FIGURE 22. Splitting a closed 3-manifold into two handlebodies along a Heegaard surface Σ and into two bordered 3-manifolds along a surface $F(\mathcal{Z})$ transverse to the original. In each half-surface, the two small black circles are identified and, hence, such a pair represents a handle.

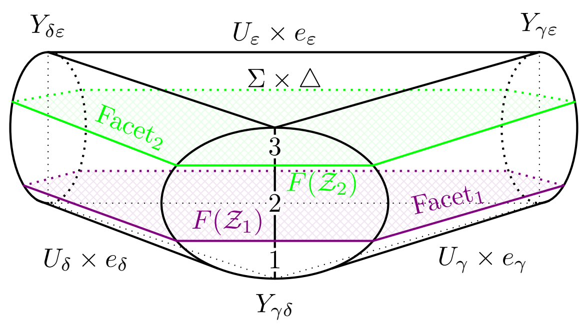


FIGURE 23. Slicing a 4-manifold with boundary obtained from a closed Heegaard triple $\mathcal{H} = (\Sigma, \boldsymbol{\gamma}, \boldsymbol{\delta}, \boldsymbol{\varepsilon}, \boldsymbol{z})$ along two facets. In this schematic example, the Heegaard surface Σ decomposes as $\Sigma = \overline{\Sigma}_1 \cup \overline{\Sigma}_2 \cup \overline{\Sigma}_3$ so each of the 3-manifolds $Y_{\eta\theta}$ decomposes into bordered 3-manifolds as $Y_{\eta\theta} = Y^1_{\eta\theta} \cup_{F_1} Y^2_{\eta\theta} \cup_{F_2} Y^3_{\eta\theta}$ and each handlebody U_η decomposes into cornered handlebodies as $U_\eta = \overline{U}^1_\eta \cup_{F_1 \cap U_\eta} \overline{U}^2_\eta \cup_{F_2 \cap U_\eta} \overline{U}^3_\eta$.

started with a closed Heegaard triple $\mathcal{H} = (\Sigma, \boldsymbol{\gamma}, \boldsymbol{\delta}, \boldsymbol{\varepsilon}, \boldsymbol{z})$, separated Σ along a circle intersecting exactly one of the sets of curves to obtain a decomposition $\Sigma = \overline{\Sigma}_1 \cup_\partial \overline{\Sigma}_2$, and glued the cornered handlebodies meeting $\overline{\Sigma}_i$ to $\overline{\Sigma}_i \times \triangle$ to obtain $X(\mathcal{H}_i)$, then the complement of the bordered 3-manifolds $Y_{\eta\theta}$ in $\partial_1 X(\mathcal{H}_i)$ is precisely the interior of a facet so gluing $X(\mathcal{H}_1)$ and $X(\mathcal{H}_2)$ along their respective boundary facets yields the original 4-manifold $X(\mathcal{H})$.

2.4 The Main Theorem

In [Zem21a, Zem21b], Zemke extends the minus and hat versions of Heegaard Floer homology to give monoidal functors out of the monoidal category of (multi)-pointed 3-manifolds and cobordisms between them equipped with embedded ribbon graphs connecting the basepoints. Given a closed Heegaard triple $(\Sigma, \boldsymbol{\gamma}, \boldsymbol{\delta}, \boldsymbol{\varepsilon}, \boldsymbol{z})$, let $X_{\gamma\delta\varepsilon}$ be the smooth 4-manifold with boundary $\partial X_{\gamma\delta\varepsilon} = -Y_{\gamma\delta} \sqcup -Y_{\delta\varepsilon} \sqcup Y_{\gamma\varepsilon}$ defined by

$$X_{\gamma\delta\varepsilon} = (\Sigma \times \triangle) \cup_{\Sigma\times e_\gamma} (U_\gamma \times e_\gamma) \cup_{\Sigma\times e_\delta} (U_\delta \times e_\delta) \cup_{\Sigma\times e_\varepsilon} (U_\varepsilon \times e_\varepsilon), \tag{2.47}$$

i.e. the pair of pants cobordism, as in [OS04b, Section 8]. In [Zem21a, Section 9], Zemke endows $X_{\gamma\delta\varepsilon}$ with an embedded trivalent graph $\Gamma_{\gamma\delta\varepsilon}$ as follows: let $v_0 \in \triangle$ be an interior point and define $\Gamma_0 \subset \triangle$ by attaching a straight line segment extending radially from v_0 to each of the three vertices of the triangle. Then one defines $\Gamma_{\gamma\delta\varepsilon} := \boldsymbol{z} \times \Gamma_0$ and gives this graph a ribbon structure by cyclically ordering the edges by endowing the ends of $X_{\gamma\delta\varepsilon}$ with the cyclic order $(-Y_{\gamma\delta}, -Y_{\delta\varepsilon}, Y_{\gamma\varepsilon})$.

Theorem 2.4.1 ([Zem21a, Theorem 9.1]). *Suppose that* $(\Sigma, \boldsymbol{\gamma}, \boldsymbol{\delta}, \boldsymbol{\varepsilon}, \boldsymbol{z})$ *is a closed pointed Heegaard triple. Let*

$$(X_{\gamma\delta\varepsilon}, \Gamma_{\gamma\delta\varepsilon}) : (Y_{\gamma\delta} \sqcup Y_{\delta\varepsilon}, \boldsymbol{z} \sqcup \boldsymbol{z}) \to (Y_{\gamma\varepsilon}, \boldsymbol{z}) \tag{2.48}$$

be the ribbon graph cobordism described above. Then, if $\mathfrak{s} \in Spin^c(X_{\gamma\delta\varepsilon})$*, the graph cobordism map*

$$F^B_{X_{\gamma\delta\varepsilon}, \Gamma_{\gamma\delta\varepsilon}, \mathfrak{s}} : CF^-(\Sigma, \boldsymbol{\gamma}, \boldsymbol{\delta}; \mathfrak{s}|_{Y_{\gamma\delta}}) \otimes_{\mathbb{F}[U]} CF^-(\Sigma, \boldsymbol{\delta}, \boldsymbol{\varepsilon}; \mathfrak{s}|_{Y_{\delta\varepsilon}}) \to CF^-(\Sigma, \boldsymbol{\gamma}, \boldsymbol{\varepsilon}; \mathfrak{s}|_{Y_{\gamma\varepsilon}})$$

is chain homotopic to the holomorphic triangle map $F^-_{\alpha,\beta,\gamma,\mathfrak{s}}$.

Corollary 2.4.2. *The hat Heegaard Floer analogue of [Zem21a, Theorem 9.1] holds.*

Theorem 2.4.3 ([Zem21b, Theorem 1.2]). *If* $(W, \Gamma) : (Y_0, \boldsymbol{p}_0) \to (Y_1, \boldsymbol{p}_1)$ *is a graph cobordism, then the graph cobordism map* $\widehat{F}_{W,\Gamma} : \widehat{CF}(Y_0, \boldsymbol{p}_0) \to \widehat{CF}(Y_1, \boldsymbol{p}_1)$ *is functorial with respect to composition of cobordisms and if* Γ *is a path connecting* $\boldsymbol{p}_0$ *to* $\boldsymbol{p}_1$*, then* $\widehat{F}_{W,\Gamma}$ *is homotopic to the cobordism map defined by Ozsváth–Szabó in* [OS06].

Note that the pair of pants cobordism with its embedded ribbon graph decomposes as $(X_{\gamma\delta\varepsilon}, \Gamma_{\gamma\delta\varepsilon}) = (W_1 \cup_{Y_{12}\#Y_{23}} W_2, \Gamma_1 \cup \Gamma_2)$, where $(W_1, \Gamma_1) : Y_{12} \sqcup Y_{23} \to Y_{12}\#Y_{23}$ is the connected sum cobordism with an embedded trivalent graph Γ_1, and $(W_2, \Gamma_2) : Y_{12}\#Y_{23} \to Y_{13}$ is the 2-handle cancellation cobordism equipped with an embedded path Γ_2 between basepoints. By [Zem21a, Proposition 8.1], the graph cobordism map $\widehat{F}_{W_1,\Gamma_1} : \widehat{CF}(Y_{12}) \otimes \widehat{CF}(Y_{23}) \to \widehat{CF}(Y_{12}\#Y_{23})$ is homotopic to Ozsváth–Szabó's connected sum isomorphism. By the previous theorem, the map $\widehat{F}_{W_2,\Gamma_2}$ is homotopic to the map $\widehat{F}_{W_2} : \widehat{CF}(Y_{12}\#Y_{23}) \to \widehat{CF}(Y_{13})$ defined by Ozsváth–Szabó in [OS06]. With these facts in hand, we are now ready to prove Theorem 2.0.1.

Theorem 2.4.4 (Theorem 2.0.1)**.** *Let Y_1, Y_2, and Y_3 be bordered 3-manifolds, all of which have boundaries parameterized by the same surface F, and let $\mathcal{A} = \mathcal{A}(-F)$ be the algebra associated to $-F$. Let $Y_{ij} = -Y_i \cup_\partial Y_j$ and consider the pair of pants cobordism $W : Y_{12} \sqcup Y_{23} \to Y_{13}$. Then the composition map*

$$\mathrm{Mor}^{\mathcal{A}}(Y_1, Y_2) \otimes \mathrm{Mor}^{\mathcal{A}}(Y_2, Y_3) \to \mathrm{Mor}^{\mathcal{A}}(Y_1, Y_3) \tag{2.49}$$

given by $f \otimes g \mapsto g \circ f$ fits into a homotopy commutative square of the form

$$\begin{array}{ccc} \mathrm{Mor}^{\mathcal{A}}(Y_1, Y_2) \otimes \mathrm{Mor}^{\mathcal{A}}(Y_2, Y_3) & \xrightarrow{f\otimes g \mapsto g\circ f} & \mathrm{Mor}^{\mathcal{A}}(Y_1, Y_3) \\ \simeq\downarrow & & \downarrow\simeq \\ \widehat{CF}(Y_{12}) \otimes \widehat{CF}(Y_{23}) & \xrightarrow{\widehat{F}_W} & \widehat{CF}(Y_{13}) \end{array} \tag{2.50}$$

where $\widehat{F}_W$ is the map induced by W and the vertical maps come from the pairing theorem of [LOT11].

Proof. By Corollary 2.4.2 and Theorem 2.4.3, the maps $\widehat{G}_{\mathsf{AT}}$ and $\widehat{F}_W$ are homotopic. The result then follows from Theorem 2.2.7. ☐

This immediately implies the following assertion of Lipshitz–Ozsváth–Thurston in [LOT11, Section 1.5].

Corollary 2.4.5. *The Yoneda composition map*

$$\mathrm{Ext}(Y_1, Y_2) \otimes_{\mathbb{F}} \mathrm{Ext}(Y_2, Y_3) \to \mathrm{Ext}(Y_1, Y_3), \tag{2.51}$$

where $\mathrm{Ext}(Y_i, Y_j) := \mathrm{Ext}(\widehat{CFD}(Y_i), \widehat{CFD}(Y_j))$, *coincides with the map*

$$\widehat{HF}(-Y_1 \cup_\partial Y_2) \otimes_{\mathbb{F}} \widehat{HF}(-Y_2 \cup_\partial Y_3) \to \widehat{HF}(-Y_1 \cup_\partial Y_3) \tag{2.52}$$

induced by W.

2.5 Application: an Algorithm for Computing $\widehat{F}_X$

As a consequence of Theorem 2.0.1, we describe an algorithm for computing the morphism $\widehat{HF}(Y_0) \to \widehat{HF}(Y_1)$ associated to an arbitrary cobordism $X : Y_0 \to Y_1$ between closed 3-manifolds. As in previous sections, we will abbreviate the notation for morphism spaces by omitting the symbols $\widehat{CFD}$ and $\widehat{CFDA}$: if Y_0 and Y_1 are 3-manifolds with boundary parametrized by $F(\mathcal{Z})$, then

$$\mathrm{Mor}^{\mathcal{A}}(Y_0, Y_1) := \mathrm{Mor}^{\mathcal{A}(-\mathcal{Z})}(\widehat{CFD}(Y_0), \widehat{CFD}(Y_1)) \tag{2.53}$$

and if $\varphi : F(\mathcal{Z}) \to F(\mathcal{Z})$ is a diffeomorphism, then we define

$$\mathrm{Mor}^{\mathcal{A}}(Y_0, \varphi \boxtimes Y_1) := \mathrm{Mor}^{\mathcal{A}(-\mathcal{Z})}(\widehat{CFD}(Y_0), \widehat{CFDA}(\varphi) \boxtimes \widehat{CFD}(Y_1)), \tag{2.54}$$

where $\widehat{CFDA}(\varphi)$ is the type-DA bimodule of the mapping cylinder of φ. In [OS06], Ozsváth–Szabó define a map $\widehat{F}_X$ as follows: first decompose X as $X = X_3 \circ X_2 \circ X_1$, where $X_1 : Y_0 \to Y_0'$ is a cobordism consisting entirely of 1-handles, $X_2 : Y_0' \to Y_1'$ is a cobordism consisting of 2-handles, and $X_3 : Y_1' \to Y_1$ is a cobordism consisting of 3-handles. They then define maps $\widehat{F}_{X_i}$, $i = 1, 2, 3$, between the Floer complexes of the respective 3-manifolds associated to each type of handle, take $\widehat{F}_X = \widehat{F}_{X_3} \circ \widehat{F}_{X_2} \circ \widehat{F}_{X_1}$, and show that the resulting map on homology is well-defined and invariant under Kirby moves and, hence, is a 4-manifold invariant (see also [JTZ21] and [Zem19]). The maps $\widehat{F}_{X_1}$ and $\widehat{F}_{X_3}$ are the same 1- and 3-handle maps described in Theorem 2.2.7. We now describe the 2-handle map $\widehat{F}_{X_2}$. For notational simplicity, assume that X is built entirely from 2-handles so that $\widehat{F}_X = \widehat{F}_{X_2}$. Then, X is given by surgery on some framed link $\mathbb{L} \subset Y_0$. We recall the following definitions from [OS06].

Definition 27. A *bouquet* for $\mathbb{L}$ is an embedded 1-complex $B(\mathbb{L}) \subset Y_0$ given by the union of $\mathbb{L} = K_1 \cup \cdots \cup K_k$ with a collection of arcs connecting the link components K_i to a fixed basepoint in Y_0.

Fix a bouquet $B(\mathbb{L})$ for $\mathbb{L}$. Let H_0 be a regular neighborhood of $B(\mathbb{L})$, $F = \partial H_0$, and let $H_1 = Y_0 \setminus \mathrm{int}(H_0)$ be the complementary handlebody. Now define $H_0(\mathbb{L})$ to be the result of performing surgery on $\mathbb{L} \subset H_0$. Then $H_0(\mathbb{L}) \cup_\partial H_1 = Y_1$ and $H_0(\mathbb{L}) \cup_\partial H_0 \cong \#^{g(F)-k} S^2 \times S^1$.

Definition 28. A Heegaard triple *subordinate to the bouquet* $B(\mathbb{L})$ is a Heegaard triple $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma})$ such that

1. $(\Sigma, \alpha_1, \ldots, \alpha_g, \beta_{k+1}, \ldots, \beta_g)$ is a diagram for the complement H_1 of the bouquet,
2. $\gamma_{k+1}, \ldots, \gamma_g$ are small Hamiltonian translates of the $\beta_{k+1}, \ldots, \beta_g$,
3. after surgering out the curves $\beta_{k+1}, \ldots, \beta_g$, the induced curves β_i and γ_i, for $i = 1, \ldots, k$, lie in punctured tori $F_i \subset \partial H_1$ given by the boundaries of regular neighborhoods of the components K_i,

4. the curves β_i, $i = 1, \ldots, k$, represent meridians of the K_i and

$$\#(\beta_i \cap \gamma_j) = \begin{cases} 0, & i \neq j \\ 1, & i = j \end{cases}$$

with transverse intersection in the latter case,

5. the curves γ_i, $i = 1, \ldots, k$, represent the framings of the components K_i under the natural identification of $H_1(\partial \mathrm{nbd}(K_1 \cup \cdots \cup K_k))$ with $H_1(\partial H_1)$.

The 4-manifold W specified by the triple $\mathcal{H} = (\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma})$ then has boundary components $-Y_0$, $\#^{g(F)-k} S^2 \times S^1$, and Y_1 (cf. [OS06, Proposition 4.3]) — indeed W is the pair of pants cobordism $Y_0 \sqcup \#^{g(F)-k} S^2 \times S^1 \to Y_1$ — and the map $\widehat{F}_X : \widehat{CF}(Y_0) \to \widehat{CF}(Y_1)$ is defined by taking $\widehat{F}_X(\boldsymbol{x}) = \widehat{F}_W(\boldsymbol{x} \otimes \Theta^{\mathrm{top}})$, where the right-hand side is the holomorphic triangle counting map determined by $\mathcal{H}$, i.e. the pair of pants map for the handlebodies H_0, $H_0(\mathbb{L})$, and H_1. We may realize this construction in the morphism spaces formulation of Heegaard Floer homology as follows: suppose that $\theta^{\mathrm{top}} \in \mathrm{Mor}^{\mathcal{A}(-F)}(-H_0(\mathbb{L}), H_0)$ is a representative of the top-graded class in $\widehat{HF}(\#^{g(F)-k} S^2 \times S^1)$. Then, Theorem 2.0.1 tells us that there is a homotopy commutative square

$$\begin{array}{ccc} \mathrm{Mor}^{\mathcal{A}(-F)}(-H_0, H_1) & \xrightarrow{-\circ\theta^{\mathrm{top}}} & \mathrm{Mor}^{\mathcal{A}(-F)}(-H_0(\mathbb{L}), H_1) \\ \simeq \big\uparrow & & \big\downarrow \simeq \\ \widehat{CF}(Y_0) & \xrightarrow{\widehat{F}_X} & \widehat{CF}(Y_1) \end{array} \tag{2.55}$$

where the vertical arrows come from the pairing theorem of [LOT11]. An algorithm for computing $\widehat{CFD}(H)$ for a handlebody H was given by Lipshitz–Ozsváth–Thurston in [LOT14a] (see also [Zha16]).

Now suppose that $X_1 : Y_0 \to Y_0'$ consists of a single 1-handle addition and let $\mathcal{A}_1 = \mathcal{A}(-F)$. Then the map $\widehat{F}_{X_1} : \widehat{CF}(Y_0) \to \widehat{CF}(Y_0')$ can be computed by decomposing Y_0' as $Y_0\#(S^2 \times S^1)$, in which case $\widehat{F}_{X_1}(\boldsymbol{x}) = \boldsymbol{x} \otimes \Theta^{\mathrm{top}}$. We now reinterpret this construction in the morphism spaces setting. If we take a Heegaard splitting $Y_0 = H_0 \cup_\varphi H_0$, where H_0 is a 0-framed handlebody of genus g and $\varphi : \partial H_0 \to \partial H_0$ is a diffeomorphism, then we automatically get a Heegaard splitting $Y_0' = H_0' \cup_{\varphi'} H_0'$, where H_0' is the genus $g+1$ handlebody $H_0' = H_0 \natural (D^2 \times S^1)$ and $\varphi' = \varphi \# \mathrm{id}_{\mathbb{T}^2}$. This then gives us

$$\widehat{CF}(Y_0') = \mathrm{Mor}^{\mathcal{A}_2}(-H_0', \varphi' \boxtimes H_0'), \tag{2.56}$$

where $\mathcal{A}_2 = \mathcal{A}(F(-\mathcal{Z})\#\mathbb{T}^2)$, by the pairing theorem. If $\mathcal{H}_0$ is a bordered Heegaard diagram for H_0 and $\mathcal{H}_\varphi$ is an (arced) bordered Heegaard diagram for φ, we may obtain bordered Heegaard diagrams $\mathcal{H}_0'$ and $\mathcal{H}_{\varphi'}$ by appending a copy of the standard diagrams for $D^2 \times S^1$ (with the 0-framing) and $\mathbb{T}^2 \times [0,1]$ to $\mathcal{H}_0$ and $\mathcal{H}_\varphi$, respectively. This gives us isomorphisms

$$\widehat{CFD}(H_0') \cong \widehat{CFD}(H_0) \otimes_{\mathbb{F}} \widehat{CFD}(D^2 \times S^1) \tag{2.57}$$

and

$$\begin{aligned} &\widehat{CFDA}(\varphi') \boxtimes \widehat{CFD}(H_0') \\ &\cong (\widehat{CFDA}(\varphi) \boxtimes \widehat{CFD}(H_0)) \otimes_{\mathbb{F}} (\widehat{CFDA}(\mathbb{T}^2 \times [0,1]) \boxtimes \widehat{CFD}(D^2 \times S^1)). \end{aligned} \tag{2.58}$$

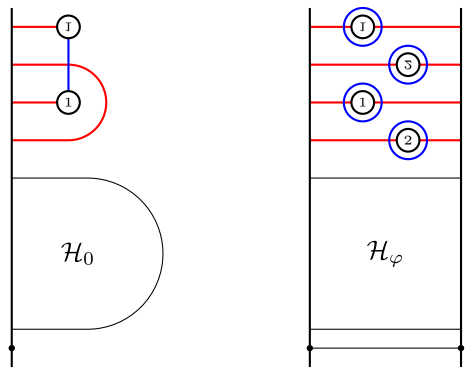
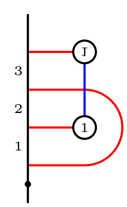


FIGURE 24. Bordered Heegaard diagrams $\mathcal{H}_0'$ (left) and $\mathcal{H}_{\varphi'}$ (right) obtained by appending standard diagrams to $\mathcal{H}_0$ and $\mathcal{H}_\varphi$.

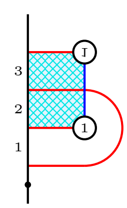
of $\mathcal{A}_2$-modules. Since $\mathbb{T}^2 \times [0,1] \cup D^2 \times S^1 \cong D^2 \times S^1$, by [HL19, Lemma 4.2], there is an unique homogeneous homotopy equivalence

$$h_1 : \widehat{CFD}(D^2 \times S^1) \to \widehat{CFDA}(\mathbb{T}^2 \times [0,1]) \boxtimes \widehat{CFD}(D^2 \times S^1). \tag{2.59}$$

Now, the standard diagram for $D^2 \times S^1$ with the 0-framing is



which has one generator, $\boldsymbol{s}$, and supports a single disk



with asymptotic condition $\rho_{23} \in \mathcal{A}(\mathbb{T}^2)$ so

$$\widehat{CFD}(D^2 \times S^1) = \boldsymbol{s} \circlearrowleft \rho_{23} \tag{2.60}$$

and, hence, $\widehat{CF}(S^2 \times S^1) \simeq \mathrm{End}^{\mathcal{A}(\mathbb{T}^2)}(\widehat{CFD}(D^2 \times S^1)) = \mathbb{F}\langle \theta_1, \theta_2 \rangle$, where $\theta_1(\boldsymbol{s}) = \boldsymbol{s}$ and $\theta_2(\boldsymbol{s}) = \rho_{23}\boldsymbol{s}$. One may easily check that $\partial\theta_1 = 2\theta_2 = 0$ and $\partial\theta_2 = 0$ so $\theta_1 = \theta^{\mathrm{top}}$ and $\theta_2 = \theta^{\mathrm{bot}}$. Under the above identifications, the 1-handle map

$$\widehat{F}_{X_1} : \mathrm{Mor}^{\mathcal{A}_1}(-H_0, \varphi \boxtimes H_0) \to \mathrm{Mor}^{\mathcal{A}_2}(-H_0', \varphi' \boxtimes H_0') \tag{2.61}$$

is given by $f \mapsto f^{\mathrm{top}}$, where $f^{\mathrm{top}} = (\mathrm{id} \otimes h_1) \circ (f \otimes \theta^{\mathrm{top}}) = f \otimes h_1$. The case of ℓ 1-handles is identical with the exception that one must instead append k copies of the standard diagram for $D^2 \times S^1$, in which case $\theta^{\mathrm{top}} = \theta_1^{\otimes \ell}$ and the codomain of $\widehat{F}_{X_1}$ is a space of morphisms of $\mathcal{A}(F(-\mathcal{Z})\#(\mathbb{T}^2)^{\#\ell})$-modules.

For the 2-handle map $\widehat{F}_{X_2} : \widehat{CF}(Y_0') \to \widehat{CF}(Y_1')$, we needed some potentially different Heegaard splitting $Y_0' = H \cup_\psi H$ (we again assume that H is 0-framed). However, by the Reidemeister–Singer theorem, after stabilizing sufficiently many times, we may arrange that $H_0' \cup_{\varphi'} H_0'$ and $H \cup_\psi H$ are isotopic Heegaard splittings so $H = H_0$ and $\psi = \xi^{-1} \circ \varphi' \circ \eta$, where $\eta, \xi : \partial H \to \partial H_0'$ are diffeomorphisms extending over $H = H_0$ (cf. [Pit08, Theorem 2.2]). Then we may compute $\widehat{CF}(Y_0')$

as

$$\begin{aligned}
&\mathrm{Mor}^{\mathcal{A}_2}(-H, \psi \boxtimes H) \\
&\cong \overline{\widehat{CFD}(-H)} \boxtimes \mathcal{A}_2 \boxtimes \widehat{CFDA}(\psi) \boxtimes \widehat{CFD}(H) \\
&\simeq \overline{\widehat{CFD}(-H)} \boxtimes \mathcal{A}_2 \boxtimes \widehat{CFDA}(\xi^{-1}) \boxtimes \widehat{CFDA}(\varphi') \boxtimes \widehat{CFDA}(\eta) \boxtimes \widehat{CFD}(H) \\
&\simeq \overline{\widehat{CFD}(-H)} \boxtimes \overline{\widehat{CFDA}(-\xi^{-1})} \boxtimes \mathcal{A}_2 \boxtimes \widehat{CFDA}(\varphi') \boxtimes \widehat{CFDA}(\eta) \boxtimes \widehat{CFD}(H) \qquad (2.62) \\
&\cong \overline{\widehat{CFDA}(-\xi^{-1}) \boxtimes \widehat{CFD}(-H)} \boxtimes \mathcal{A}_2 \boxtimes \widehat{CFDA}(\varphi') \boxtimes \widehat{CFDA}(\eta) \boxtimes \widehat{CFD}(H) \\
&\simeq \overline{\widehat{CFD}(-H_0')} \boxtimes \mathcal{A}_2 \boxtimes \widehat{CFDA}(\varphi') \boxtimes \widehat{CFD}(H_0') \\
&\cong \mathrm{Mor}^{\mathcal{A}_2}(-H_0', \varphi' \boxtimes H_0').
\end{aligned}$$

Here, the homotopy equivalence in the third line is given to us by [HL19, Lemma 4.5], which tells us that there is an unique homogeneous homotopy equivalence

$$\widehat{CFDA}(\psi) \simeq \widehat{CFDA}(\xi^{-1}) \boxtimes \widehat{CFDA}(\varphi') \boxtimes \widehat{CFDA}(\eta). \qquad (2.63)$$

By [HL19, Lemma 4.2], there are unique homogeneous homotopy equivalences $\widehat{CFD}(H) \to \widehat{CFD}(H_0')$ and $\widehat{CFD}(-H) \to \widehat{CFD}(-H_0')$ so this furnishes us with an algorithmically computable homotopy equivalence

$$h_2 : \mathrm{Mor}^{\mathcal{A}_2}(-H_0', \varphi' \boxtimes H_0') \to \mathrm{Mor}^{\mathcal{A}_2}(-H, \psi \boxtimes H) \qquad (2.64)$$

of morphism complexes. Moreover, this map agrees up to homotopy with the homotopy equivalence associated to the map associated to a sequence of Heegaard moves (cf. [HL19, proof of Theorem 5.1]). The map

$$\widehat{F}_{X_2} \circ \widehat{F}_{X_1} : \mathrm{Mor}^{\mathcal{A}_2}(-H_0, \varphi \boxtimes H_0) \to \mathrm{Mor}^{\mathcal{A}_2}(-H(\mathbb{L}), \psi \boxtimes H) \qquad (2.65)$$

is then given by $\widehat{F}_{X_2} \circ \widehat{F}_{X_1}(f) = h_2(f^{\text{top}}) \circ \theta^{\text{top}}$ by (2.55).

The case of 3-handles follows similarly to the case of 1-handles: if the cobordism $X_3 : Y_1' \to Y_1$ consists of a single 3-handle addition, then $\widehat{F}_{X_3} : \widehat{CF}(Y_1') \to \widehat{CF}(Y_1)$ can be computed by decomposing Y_1' as $Y_1 \#(S^2 \times S^1)$, in which case

$$\widehat{F}_X(\boldsymbol{y} \otimes \theta) = \begin{cases} \boldsymbol{y} & \text{if } \theta = \Theta^{\text{bot}} \\ 0 & \text{else.} \end{cases} \tag{2.66}$$

In the morphism spaces setting, we leverage the fact that we have Heegaard splittings $Y_1' = H(\mathbb{L}) \cup_\psi H = H_2' \cup_{\omega'} H_2'$, where $H_2' = H_2 \natural (D^2 \times S^1)$ and $\omega' = \omega \# \text{id}_{\mathbb{T}^2}$ for some Heegaard splitting $Y_1 = H_2 \cup_\omega H_2$. As before, we may stabilize sufficiently many times so that $H(\mathbb{L}) \cup_\psi H$ and $H_2' \cup_{\omega'} H_2'$ are isotopic Heegaard splittings and we obtain isomorphisms

$$\widehat{CFD}(H_2') \cong \widehat{CFD}(H_2) \otimes_{\mathbb{F}} \widehat{CFD}(D^2 \times S^1) \tag{2.67}$$

and

$$\begin{aligned} &\widehat{CFDA}(\omega') \boxtimes \widehat{CFD}(H_2') \\ &\cong (\widehat{CFDA}(\omega) \boxtimes \widehat{CFD}(H_2)) \otimes_{\mathbb{F}} (\widehat{CFDA}(\mathbb{T}^2 \times [0,1]) \boxtimes \widehat{CFD}(D^2 \times S^1)) \end{aligned} \tag{2.68}$$

of $\mathcal{A}(-\partial H_2 \# \mathbb{T}^2)$-modules. There is then an unique homogeneous homotopy equivalence

$$\begin{aligned} h_3 : &\text{Mor}^{\mathcal{A}(-\partial H_2 \# \mathbb{T}^2)}(-H(\mathbb{L}), \psi \boxtimes H) \\ &\to \text{Mor}^{\mathcal{A}(-\partial H_2 \# \mathbb{T}^2)}(-H_2 \otimes_{\mathbb{F}} (D^2 \times S^1), (\omega \boxtimes H_2) \otimes_{\mathbb{F}} (D^2 \times S^1)) \end{aligned} \tag{2.69}$$

induced by h_1^{-1}, which factors through $\mathrm{Mor}^{\mathcal{A}(-\partial H_2 \# \mathbb{T}^2)}(-H_2', \omega' \boxtimes H_2')$ so that the 3-handle map

$$\widehat{F}_{X_3} : \mathrm{Mor}^{\mathcal{A}_2}(-H(\mathbb{L}), \psi \boxtimes H) \to \mathrm{Mor}^{\mathcal{A}_3}(-H_2, \omega \boxtimes H_2), \tag{2.70}$$

where $\mathcal{A}_3 = \mathcal{A}(-\partial H_2)$, is then given by $((\mathrm{id} \otimes \overline{\theta}^{\mathrm{bot}}) \circ h_3)(f)$, where $\overline{\theta}^{\mathrm{bot}}$ is the $\mathcal{I}$-linear dual of θ^{bot}. In summary, if $X = X_3 \circ X_2 \circ X_1$, we may compute the map $\widehat{F}_X$ at the chain level via $\widehat{F}_X(f) = ((\mathrm{id} \otimes \overline{\theta}^{\mathrm{bot}}) \circ h_3)(h_2(f^{\mathrm{top}}) \circ \theta^{\mathrm{top}})$.

Since each of the 1-, 2-, and 3-handle maps and the homotopy equivalences of morphism complexes at each step are algorithmically computable, Theorem 2.0.1 and [LOT14a] furnish us with an algorithm for computing $\widehat{F}_X$, whose steps we outline below:

1. Fix a Heegaard splitting $Y_0 = H_0 \cup_\varphi H_0$ which has been stabilized sufficiently many times so that all of the pairs of Heegaard splittings in each step described above become isotopic, then pick a factorization of the gluing map φ into arcslides.

2. Compute a basis $\{f_1, \ldots, f_n\}$ for $H_*\mathrm{Mor}^{\mathcal{A}_1}(-H_0, \varphi \boxtimes H_0)$ consisting of explicit cycles in $\mathrm{Mor}^{\mathcal{A}_1}(-H_0, \varphi \boxtimes H_0)$.

3. For each f_i, compute the map $f_i^{\mathrm{top}} \in \mathrm{Mor}^{\mathcal{A}_2}(-H_0', \varphi' \boxtimes H_0')$.

4. Fix a (sufficiently stabilized) Heegaard splitting $Y_0' = H \cup_\psi H$ induced by a bouquet for a framed link $\mathbb{L} \subset Y_0'$ such that $Y_0'(\mathbb{L}) = Y_1'$ and compute $\widehat{CFD}(H)$ and a basis for $H_*\mathrm{Mor}^{\mathcal{A}_2}(H_0', H)$ in order to find the unique homogeneous homotopy equivalences which induce the homotopy equivalence h_2, and compute the latter.

5. Compute $\widehat{CFD}(-H(\mathbb{L}))$ and a basis for $H_*\mathrm{Mor}^{\mathcal{A}_2}(-H(\mathbb{L}), H)$ consisting of explicit cycles, identify $\theta^{\mathrm{top}} \in \mathrm{Mor}^{\mathcal{A}_2}(-H(\mathbb{L}), H)$ using this basis, and compute $h_2(f^{\mathrm{top}}) \circ \theta^{\mathrm{top}}$.

6. Compute $\widehat{CFDA}(\psi) \boxtimes \widehat{CFD}(H)$, a basis for $\mathrm{Mor}^{\mathcal{A}_2}(-H(\mathbb{L}), \psi \boxtimes H)$, and the homotopy equivalence h_3.

7. Compute $\widehat{F}_X(f_i) = ((\mathrm{id} \otimes \overline{\theta}^{\mathrm{bot}}) \circ h_3)(h_2(f_i^{\mathrm{top}}) \circ \theta^{\mathrm{top}})$ for $i = 1, \ldots, n$.

CHAPTER III

BIMODULES, BRANCHED COVERS, AND SPLITTINGS

3.1 Branched Arc Algebras

Branched double covers

Given a link $L \subset S^3$, one may construct a 3-manifold $\Sigma(L)$, called the *branched double cover of* L as follows: choose a Seifert surface F for L and let Y_L^0 be the complement of a tubular open neighborhood of $F \cap (S^3 \setminus \text{nbd}(L))$ in $S^3 \setminus \text{nbd}(L)$, where $\text{nbd}(L)$ is a tubular open neighborhood of L. The (cornered) 3-manifold Y_L^0 contains two copies of F, call them F_- and F_+. Let Y_L^1 be the manifold with boundary obtained by taking the quotient of $Y_L^0 \sqcup Y_L^0$ obtained by identifying $F_\pm$ in the first copy of Y_L^0 with $F_\mp$ in the second. Note that Y_L^1 has one toroidal boundary component for every component of L. The closed 3-manifold $\Sigma(L)$ is then obtained by Dehn filling each of these boundary components with respect to the Seifert framing induced by the copies of $F_\pm$ sitting inside of Y_L^1.

Example 10. The branched double cover of an unlink with k components is $\#^{k-1}(S^2 \times S^1)$. More generally, given two links L_0 and L_1, $\Sigma(L_0 \sqcup L_1) \cong \Sigma(L_0)\#\Sigma(L_1)\#(S^2 \times S^1)$.

Remark. A link cobordism $C : L_0 \to L_1$ induces a cobordism of 3-manifolds $\Sigma(C) : \Sigma(L_0) \to \Sigma(L_1)$, which we call the branched double cover of C.

Note that one may extend this definition to obtain branched double covers $\Sigma(T)$ of tangles T in the 3-ball, or in $S^2 \times [0, 1]$, which are 3-manifolds with boundary. For simplicity, we will restrict ourselves to the case of tangles with an even number of endpoints on the equator(s) of the boundary of their ambient

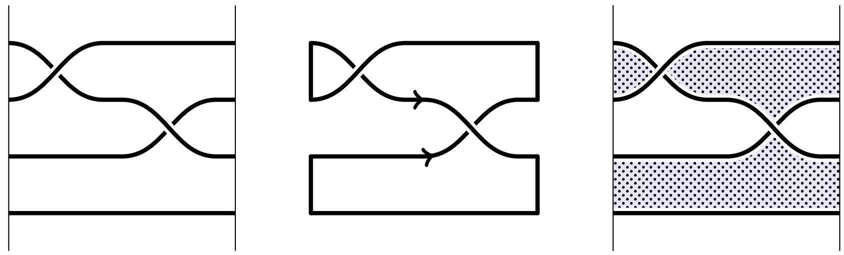


FIGURE 25. A diagram for a tangle $T \subset S^2 \times [0,1]$ (left), its plat closure $p(T)$ by equatorial arcs (middle), and the cornered Seifert surface obtained from applying Seifert's algorithm to $p(T)$ (right). Here, the vertical lines in the left- and right-hand figures represent the projections of the equators of $S^2 \times [0,1]$.

3-manifold. A *cornered Seifert surface* for such a tangle T is an orientable surface $F \subset Y$ with corners, where Y is either of B^3 or $S^2 \times [0,1]$, such that ∂F decomposes as the union of T and a collection of arcs in the equator(s) of Y. Such a surface always exists: T has an even number of endpoints on each boundary component of Y so the plat closure $p(T)$ of T embeds in Y, smoothly away from the endpoints of T. We may then apply Seifert's algorithm to any oriented diagram for $p(T)$ obtained by taking the plat closure of a diagram for T, using arcs in the projections of the equators for the closure, and regarding the resulting cornered surface as an embedded surface F in Y (see Figure 25 for an example). To construct $\Sigma(T)$, we take Y_T^0 to be the complement of a tubular open neighborhood of $F \cap (Y \smallsetminus \text{nbd}(T))$ and glue two copies of this space, as we did with Y_L^0 above, to obtain a cornered 3-manifold Y_T^1 whose codimension 1 stratum decomposes as $\partial_1 Y_T^1 = \overline{\Sigma} \cup_\partial \partial\text{nbd}(T)$, where $\overline{\Sigma}$ is a (possibly disconnected) surface with $\#\partial T$ boundary components. We then fill Y_T^1 with nbd(T) to obtain $\Sigma(T)$. If $T \subset B^3$ has $2n$ endpoints, then $\partial\Sigma(T)$ is an oriented surface of genus $n-1$. Similarly, if $T \subset S^2 \times [0,1]$ has $\#T \cap (S^2 \times \{0\}) = 2m$ and $\#T \cap (S^2 \cap \{1\}) = 2n$, then the boundary components of $\Sigma(T)$ have genus $m-1$ and $n-1$. One can see this by considering the branched double cover of the $2n$-stranded identity braid id_{2n} in $S^2 \times [0,1]$, which we may think of as a collar neighborhood of $\partial\Sigma(T)$. This 3-manifold is the product of an interval and the double cover Σ_g of S^2 branched along $2n$ points. Since the ramification index of each branch point is 2, the Riemann–Hurwitz formula tells us that $\chi(\Sigma_g) = 2\chi(S^2) - 2n = 2 - 2(n-1)$ so $g = n-1$ and $\Sigma(\mathrm{id}_{2n}) \cong \Sigma_{n-1} \times [0,1]$.

The algebras

In [OS05], Ozsváth–Szabó showed that, for any (based) link $L \subset S^3$, there is a spectral sequence $\widetilde{Kh}(mL;\mathbb{F}) \Rightarrow \widehat{HF}(\Sigma(L))$. They prove this result by constructing a filtration on $\widehat{CF}(\Sigma(L))$, associated to a diagram for L, such that the E^1-page of the induced spectral sequence is

$$\bigoplus_{\boldsymbol{v}\in\mathbf{2}^c} \widehat{HF}(\Sigma(L_{\boldsymbol{v}})), \tag{3.1}$$

where c is the number of crossings in the diagram, $\mathbf{2} = \{0,1\}$, and $L_{\boldsymbol{v}}$ is the complete resolution of the diagram determined by $\boldsymbol{v}$ and an ordering of the crossings. Since each $L_{\boldsymbol{v}}$ is a planar unlink, each summand is of the form $\widehat{HF}(\#^{k-1}(S^2 \times S^1))$, where k is the number of components of $L_{\boldsymbol{v}}$, which they show is isomorphic to $\widetilde{\mathcal{C}}_{Kh}(L_{\boldsymbol{v}})$. They then identify the d^1-differential, which is given by the maps on Heegaard Floer homology induced by the branched double covers of the saddle cobordisms making up the edges of the cube of resolutions, with the Khovanov differential. In the case that L is a planar unlink, the spectral sequence degenerates on the E^1-page, so one should expect there to be a Heegaard Floer

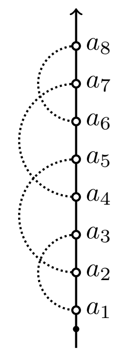


FIGURE 26. The genus 2 linear pointed matched circle.

analogue of the arc algebra H_n. Naïvely, this algebra might take the form

$$\bigoplus_{a,b\in\mathfrak{C}_n} \widehat{HF}(\Sigma(a^! b)) \tag{3.2}$$

with multiplication given by the maps induced by branched double covers of minimal saddle cobordisms. However there are some issues with this construction. First, the arc algebra H_n and its reduced version $\widetilde{H}_n$ have somewhat different properties as algebras — for example, $HH_*(H_1)$ is infinite-dimensional while $\widetilde{H}_1 \cong \mathbb{F}$ so $HH_*(H_1) \cong \mathbb{F}$ — though we will see later that this difference is only up to a tensor factor of the algebra V. Second, and more seriously, it is not immediately clear that this construction yields an algebra, or even a generalized algebra, in a sensible way. We will instead define a chain-level version of this structure and show that it is, in general, a nontrivial A_∞-deformation of H_n.

Definition 29. The *genus k linear pointed matched circle* $\mathcal{Z}_k$ is the pointed matched circle whose matching M matches the pairs $\{a_1, a_3\}$ and $\{a_{4k-2}, a_{4k}\}$ and, for each $n = 1, \ldots, 2k-2$, the pairs $\{a_{2n}, a_{2n+3}\}$ (see Figure 26). Note that $\mathcal{Z}_1$ is the usual pointed matched circle for the torus.

One may naturally view the branched double cover $\Sigma(T)$ of a tangle T in B^3 with $2n$ equatorial endpoints as having boundary parametrized by $\mathcal{Z}_{n-1}$ by using the algorithm given in [LOT16, Section 6.1] to construct an explicit bordered Heegaard diagram for $\Sigma(T)$. We recall this construction here for crossingless matchings, starting with a diagram $\mathcal{H}$ for the branched double cover of the *plat closure* on $2n$ points, i.e. the matching consisting of n caps stacked vertically. We illustrate the $n = 3$ case in Figure 27. First, draw a vertical line segment with a distinguished basepoint near its bottom end and, temporarily denoting the plat closure by a, identify ∂a with $[2n]$ by enumerating the endpoints from bottom to top. Step 1: to the right of this line draw $4n - 4$ horizontal line segments which each meet it at a single point, two corresponding to each of the endpoints 2 through $2n-2$ in ∂a and one each corresponding to 1 and $2n-1$, and enumerate these from bottom to top. Step 2: draw pairs of labeled circles representing handles at the other ends of the pairs of segments labeled $4k+2$ and $4k+5$ for $k = 1, 2, \ldots, n-2$ and one more pair for the segments labeled $4n-6$ and $4n-4$. Step 3: draw half-circular arcs to the right of the circles added in Step 2 which connect the endpoints of the segments labeled 1 and 3 and the pairs of segments labeled $4k$ and $4k+3$ for $k = 1, 2, \ldots, n-2$. Steps 2 and 3 completely specify the α-curves in $\mathcal{H}$. Step 4: draw a β-circle enclosing all of the circles contained in each region of the diagram bounded by an α-arc constructed in Step 3. The result is then a bordered Heegaard diagram for $\Sigma(a)$.

If $b \in \mathfrak{C}_n$ is any other crossingless matching, we may isotope the diagram for b so that it becomes the plat closure (on the right) by a of a product of cap-cup tangles (see Figure 28 for an example) which is minimal in the sense that there is no such presentation of b with fewer caps and cups. Note that, by minimality, no

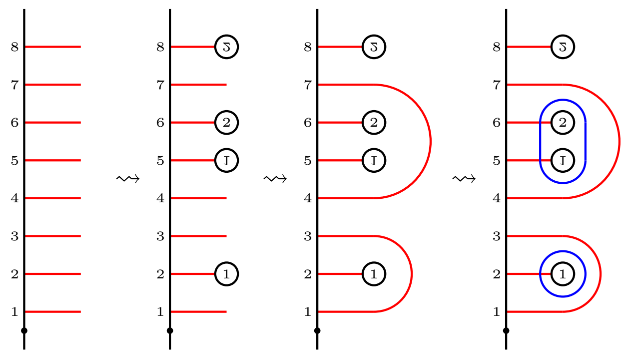


FIGURE 27. Construction of a bordered Heegaard diagram for the 6-ended plat closure. Here, steps 1 through 4 are illustrated from left to right.

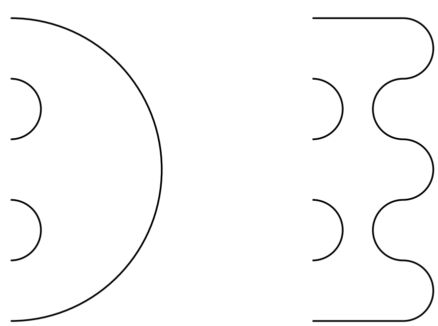
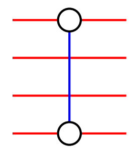
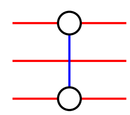


FIGURE 28. A crossingless matching on 6 points (left) and its minimal plat closure-form (right).

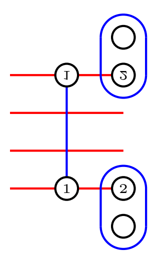
cap-cup pair will involve the bottom-most or top-most strands of this diagram for b. For each cap-cup pair, we insert a new handle and β-circle of the form



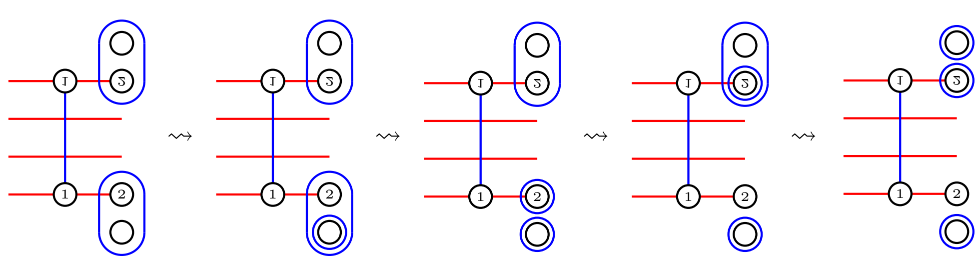
into the bordered Heegaard diagram for the plat closure, where the four α-curves are the arcs corresponding to the strands in which the cap-cup pair occurs, provided these strands are not the ones at heights $2n-2$ and $2n-1$. In the latter case, we instead insert



to modify the plat closure diagram. Inserting these handles and β-circles will always result in a diagram with some number of configurations of handles, α-curves, and β-circles of the form



where the two β-circles at right come from the original bordered Heegaard diagram for the plat closure. We may then perform a sequence of isotopies and handleslides



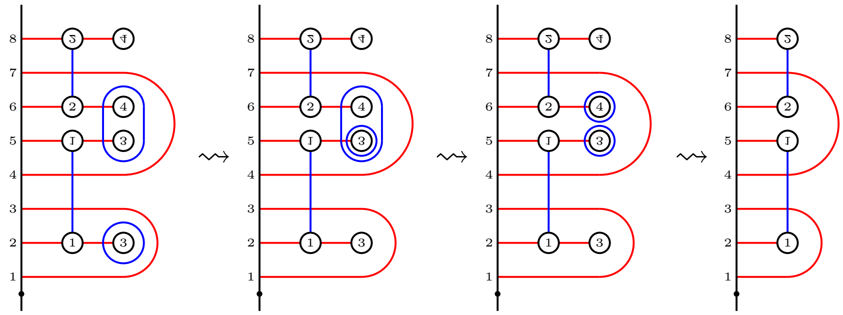


FIGURE 29. The bordered Heegaard diagram for the matching in Figure 28. Its destabilized form, at right, is obtained by performing an isotopy and a handleslide, shown in the two intermediate steps, followed by two destabilizations, which comprise the last step shown.

of the β-circles coming from the plat closure diagram, starting by isotoping the bottom-most circle — the one in the region adjacent to the boundary Reeb chord $[1, 3]$ — over the handle it encircles, until any such configuration in the diagram has been changed to be as at right. In the above schematic, the first step is an isotopy of a β-circle (which is not pictured in the first diagram) over a handle. After this sequence of Heegaard moves, each such resulting configuration contains a connected sum with a standard diagram for S^3 — here given by the handle corresponding to the two circles labeled 2, the β-circle enclosing the topmost of these circles, and the α-circle given by the two red line segments between the circles labeled 1 and the circles labeled 2. We then destabilize the diagram until all of these standard diagrams are removed to obtain the bordered Heegaard diagram $\mathcal{H}_b$ for $\Sigma(b)$ (see Figure 29 for an example).

Definition 30. Given $a \in \mathfrak{C}_n$, let $a_+ \in \mathfrak{C}_{n+1}$ be the crossingless matching obtained by adding a single extra arc below a. Regarding a_+ as a tangle in B^3, define $\widehat{CFD}(a_+) = \widehat{CFD}(\mathcal{H}_a)$, where $\mathcal{H}_a$ is the bordered Heegaard diagram for $\Sigma(a_+)$ constructed as above. The *branched arc algebra* $\mathfrak{h}_n$ on $2n$ points is then the differential algebra

$$\mathfrak{h}_n = \mathrm{End}^{\mathcal{A}_n}\left(\bigoplus_{a \in \mathfrak{C}_n} \widehat{CFD}(a_+)\right), \tag{3.3}$$

where $\mathcal{A}_n = \mathcal{A}(-\mathcal{Z}_n, 0)$, with algebra operation given by the composition map $\circ_{\mathrm{op}} : f \otimes g \mapsto g \circ f$ and the usual morphism space differential.

We will show that the algebras $H_*\mathfrak{h}_n$ and H_n agree. However, we first recall the following propositions from [OS05].

Proposition 3.1.1 ([OS05, Proposition 6.1])**.** *If $Y \cong \#^k(S^2 \times S^1)$, then $\widehat{HF}(Y)$ is a rank 1 free module over $\Lambda^* H_1(Y)$, generated by the class $\Theta^{\mathrm{top}} \in \widehat{HF}(Y)$. Moreover, if $K \subset Y$ is a curve representing an S^1 fiber in one of the $S^2 \times S^1$ summands, then the 3-manifold $Y' = Y_0(K)$ is diffeomorphic to $\#^{k-1}(S^2 \times S^1)$, with a natural identification $\pi : H_1(Y)/[K] \to H_1(Y')$. Under the 2-handle cobordism $W_1 : Y \to Y'$, the map $\widehat{F}_{W_1} : \widehat{HF}(Y) \to \widehat{HF}(Y')$ is determined by*

$$\widehat{F}_{W_1}(\xi \cdot \Theta^{\mathrm{top}}) = \pi(\xi) \cdot \Theta^{\mathrm{top}'}, \tag{3.4}$$

where $\Theta^{\mathrm{top}'} \in \widehat{HF}(Y')$ is the generator of $\widehat{HF}(Y')$ as a free $\Lambda^* H_1(Y')$-module and $\xi \in \Lambda^* H_1(Y)$. Dually, if $K \subset Y$ is a local unknot, then the manifold $Y''(K) = Y_0(K)$ is diffeomorphic to $\#^{k+1}(S^2 \times S^1)$, and there is a natural inclusion $i : H_1(Y) \to H_1(Y'')$. The map $\widehat{F}_{W_2} : \widehat{HF}(Y) \to \widehat{HF}(Y'')$ induced by the 2-handle cobordism $W_2 : Y \to Y''$ is then determined by

$$\widehat{F}_{W_2}(\xi \cdot \Theta^{\mathrm{top}}) = i(\xi) \wedge [K''] \cdot \Theta^{\mathrm{top}''}, \tag{3.5}$$

where $[K''] \in H_1(Y'')$ is a generator of $\ker(H_1(Y'') \to H_1(W_2))$.

In the case that Y is given as the branched double cover $\Sigma(\mathcal{D}) = \#^k(S^2 \times S^1)$ of a planar unlink $\mathcal{D} = S_0 \cup \cdots \cup S_k$, where S_0 is a distinguished component with a basepoint, this proposition furnishes us with the following variation of [OS05, Proposition 6.2].

Proposition 3.1.2 ([OS05, Proposition 6.2]). *If $\mathcal{D}$ is a planar unlink with one based component, then there is an isomorphism $\psi_{\mathcal{D}} : \widetilde{\mathcal{C}}_{Kh}(\mathcal{D}) \xrightarrow{\cong} \widehat{HF}(\Sigma(\mathcal{D}))$ which is natural under cobordisms in the sense that if $s : \mathcal{D} \to \mathcal{D}'$ is either a single merge or split cobordism, then the diagram*

$$\begin{array}{ccc} \widetilde{\mathcal{C}}_{Kh}(\mathcal{D}) & \xrightarrow{\widetilde{\mathcal{C}}_{Kh}(s)} & \widetilde{\mathcal{C}}_{Kh}(\mathcal{D}') \\ {\scriptstyle \psi_{\mathcal{D}}}\downarrow & & \downarrow{\scriptstyle \psi_{\mathcal{D}'}} \\ \widehat{HF}(\Sigma(\mathcal{D})) & \xrightarrow{\widehat{F}_{\Sigma(s)}} & \widehat{HF}(\Sigma(\mathcal{D}')) \end{array} \tag{3.6}$$

commutes.

We recall the proof of this statement in the case that s does not involve the marked component. We will not require the case that s involves the marked component in our proof that the algebras agree.

Proof. For $i > 0$, let γ_i be an arc in S^3 from S_0 to S_i which is disjoint from $\mathcal{D}$ away from its endpoints and let $\tilde{\gamma}_i$ be the preimage of γ_i in $\Sigma(\mathcal{D})$. Note that the preimages of any two choices of γ_i are homologous in $\Sigma(\mathcal{D})$. Then, by construction, $\{[\tilde{\gamma}_i]\}_{i=1}^k$ is a basis for $H_1(\Sigma(\mathcal{D}))$. Using [OS05, Proposition 6.1] and the identification, given in [OS05, Section 5], of $\widetilde{\mathcal{C}}_{Kh}(\mathcal{D})$ with the exterior algebra $\Lambda^* \widetilde{Z}(\mathcal{D})$, where $\widetilde{Z}(\mathcal{D})$ is the vector space formally spanned by the unmarked components $[S_1], \ldots, [S_k]$ of $\mathcal{D}$, the map $\psi_{\mathcal{D}}$ is then given by the isomorphism $\Lambda^* \widetilde{Z}(\mathcal{D}) \xrightarrow{\cong} \Lambda^* H_1(\Sigma(\mathcal{D}))$ determined by $[S_i] \mapsto [\tilde{\gamma}_i]$.

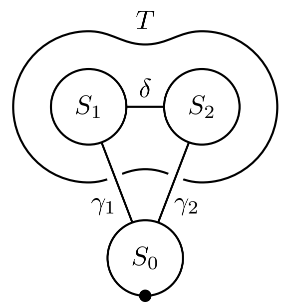


FIGURE 30. After merging S_1 and S_2, the curves $\tilde{\gamma}_1$ and $\tilde{\gamma}_2$ become homologous. Dually, if T is split into $S_1 \sqcup S_2$, the curve $\tilde{\delta} = \tilde{\gamma}_2 - \tilde{\gamma}_1$ becomes nullhomologous.

If s merges two circles S_1 and S_2 into a single circle T, then, in the cobordism $\Sigma(s)$, the curves $\tilde{\gamma}_1$ and $\tilde{\gamma}_2$ become homologous to the lift of the curve from S_0 to T in $\Sigma(\mathcal{D}')$. Commutativity of the above square then follows from [OS05, Proposition 6.1] and the definition of $\widetilde{\mathcal{C}}_{Kh}(s)$. Dually, if s splits a circle T into a disjoint union $S_1 \sqcup S_2$ of two circles, then the curve $\tilde{\delta} = \tilde{\gamma}_2 - \tilde{\gamma}_1$ is nullhomologous in $\Sigma(s)$ and commutativity of the square follows similarly. $\square$

Note that if $\mathcal{D}_0$ and $\mathcal{D}_0'$ are two planar unlinks, $\mathcal{D}$ and $\mathcal{D}'$ are the based unlink diagrams obtained by placing a based circle below each diagram, and $\mathcal{D}''$ is the diagram obtained from $\mathcal{D}_0 \sqcup \mathcal{D}_0'$ in the same manner, then there is automatically an isomorphism $\widetilde{Z}(\mathcal{D}) \oplus \widetilde{Z}(\mathcal{D}') \xrightarrow{\cong} \widetilde{Z}(\mathcal{D}'')$ because there is a canonical bijection between the set of unmarked components of $\mathcal{D} \sqcup \mathcal{D}'$, regarded as a single diagram with two marked components, and the unmarked components of $\mathcal{D}''$ which sends an unmarked component to itself (see Figure 31 for an example). This then induces an isomorphism $\Lambda^* \widetilde{Z}(\mathcal{D}) \otimes \Lambda^* \widetilde{Z}(\mathcal{D}') \xrightarrow{\cong} \Lambda^* \widetilde{Z}(\mathcal{D}'')$. We are now ready to prove that $H_* \mathfrak{h}_n$ and H_n are isomorphic.

Theorem 3.1.3. *Let $\overline{\circ}_{\mathrm{op}} : H_* \mathfrak{h}_n \otimes_{\mathbb{F}} H_* \mathfrak{h}_n \to H_* \mathfrak{h}_n$ denote the operation induced by $\circ_{\mathrm{op}}$ on homology. Then $(H_* \mathfrak{h}_n, \overline{\circ}_{\mathrm{op}}) \cong H_n$ as associative algebras.*

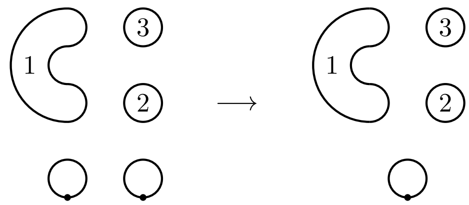


FIGURE 31. The canonical identification between the unmarked components of a diagram of the form $\mathcal{D} \sqcup \mathcal{D}'$ (left) and the corresponding diagram $\mathcal{D}''$ (right).

Proof. Note that we may regard H_n as the algebra

$$H_n = \bigoplus_{a,b \in \mathfrak{C}_n} \widetilde{\mathcal{C}}_{Kh}(a_+^! b_+), \tag{3.7}$$

where we place a basepoint on the bottom-most circle of $a_+^! b_+$ and regard $\widetilde{\mathcal{C}}_{Kh}(a_+^! b_+)$ as the quotient complex wherein the marked component is labeled 1. The multiplication m on H_n is then given by

$$m = \sum_{a,b,c \in \mathfrak{C}_n} \widetilde{\mathcal{C}}_{Kh}(C_{abc} \sqcup \mathrm{id}_{\bigcirc}), \tag{3.8}$$

where $C_{abc} : a^! b \sqcup b^! c \to a^! c$ is the minimal saddle cobordism.

Note that the pair-of-pants cobordism $W : \Sigma(a_+^! b_+) \sqcup \Sigma(b_+^! c_+) \to \Sigma(a_+^! c_+)$ decomposes as $W = \Sigma(C_{abc} \sqcup \mathrm{id}_{\bigcirc}) \circ W_\#$, where

$$W_\# : \Sigma(a_+^! b_+) \sqcup \Sigma(b_+^! c_+) \to \Sigma((a^! b \sqcup b^! c) \sqcup \bigcirc) \tag{3.9}$$

is the connected sum cobordism given by taking the connected sum at the preimages of the basepoints on the bottom-most circles in $a_+^! b_+$ and $b_+^! c_+$. We may decompose $C_{abc} \sqcup \mathrm{id}_{\bigcirc}$ as a movie $P_1 \xrightarrow{s_1} \cdots \xrightarrow{s_{k-1}} P_k$ of planar unlinks, where $P_1 = a^! b \sqcup b^! c \sqcup \bigcirc$, $P_k = a_+^! c_+$ and $P_i \xrightarrow{s_i} P_{i+1}$ is a single saddle cobordism, so that

$\Sigma(C_{abc} \sqcup \mathrm{id}_{\bigcirc}) = \Sigma(s_{k-1}) \circ \cdots \circ \Sigma(s_1)$. This then allows us to further decompose W as

$$W = \Sigma(s_{k-1}) \circ \cdots \circ \Sigma(s_1) \circ W_{\#} \tag{3.10}$$

Regarding P_i and P_{i+1} as successive resolutions $P_i = \mathcal{D}_i(0)$ and $P_{i+1} = \mathcal{D}_i(1)$ of a link diagram $\mathcal{D}_i$ with a single crossing, there is a commutative square

$$\begin{array}{ccc} \widetilde{\mathcal{C}}_{Kh}(P_i) & \xrightarrow{\widetilde{\mathcal{C}}_{Kh}(s_i)} & \widetilde{\mathcal{C}}_{Kh}(P_{i+1}) \\ \psi_i \downarrow & & \downarrow \psi_{i+1} \\ \widehat{HF}(\Sigma(P_i)) & \xrightarrow{\widehat{F}_{\Sigma(s_i)}} & \widehat{HF}(\Sigma(P_{i+1})), \end{array} \tag{3.11}$$

where $\psi_i = \psi_{\mathcal{D}_i(0)} : \widetilde{\mathcal{C}}_{Kh}(P_i) \to \widehat{HF}(\Sigma(P_i))$ is the isomorphism constructed in the proof of [OS05, Proposition 6.2] (see page 124). Note that, since the construction of each ψ_i depends only on the diagram P_i, we have $\psi_{i+1} = \psi_{\mathcal{D}_{i+1}(0)} = \psi_{\mathcal{D}_i(1)}$. For $a, b \in \mathfrak{C}_n$, let $\psi_{ab} = \psi_{a^!_+ b_+}$. We claim that the diagrams

$$\begin{array}{ccc} \widetilde{\mathcal{C}}_{Kh}(a^!_+ b_+) \otimes \widetilde{\mathcal{C}}_{Kh}(b^!_+ c_+) & \xrightarrow{f_{abc}} & \widetilde{\mathcal{C}}_{Kh}(P_1) \\ \psi_{ab} \otimes \psi_{bc} \downarrow & & \downarrow \psi_1 \\ \widehat{HF}(\Sigma(a^!_+ b_+)) \otimes \widehat{HF}(\Sigma(b^!_+ c_+)) & \xrightarrow{\widehat{F}_{W_{\#}}} & \widehat{HF}(\Sigma(P_1)) \end{array} \tag{3.12}$$

and

$$\begin{array}{ccc} \widetilde{\mathcal{C}}_{Kh}(P_1) & \xrightarrow{\widetilde{\mathcal{C}}_{Kh}(C_{abc} \sqcup \mathrm{id}_{\bigcirc})} & \widetilde{\mathcal{C}}_{Kh}(P_k) \\ \psi_1 \downarrow & & \downarrow \psi_k \\ \widehat{HF}(\Sigma(P_1)) & \xrightarrow{\widehat{F}_{\Sigma(C_{abc} \sqcup \mathrm{id}_{\bigcirc})}} & \widehat{HF}(\Sigma(P_k)) \end{array} \tag{3.13}$$

commute, where $f_{abc} : \widetilde{\mathcal{C}}_{Kh}(a^!_+ b_+) \otimes \widetilde{\mathcal{C}}_{Kh}(b^!_+ c_+) \to \widetilde{\mathcal{C}}_{Kh}(a^!_+ c_+)$ is the isomorphism given by

$$(a^! b \sqcup \bigcirc_1, \boldsymbol{v}) \otimes (b^! c \sqcup \bigcirc_1, \boldsymbol{w}) \mapsto ((a^! b \sqcup b^! c) \sqcup \bigcirc_1, \boldsymbol{v} \sqcup \boldsymbol{w}) \tag{3.14}$$

for any labelings $\boldsymbol{v}$ and $\boldsymbol{w}$ of $a^! b$ and $b^! c$. In other words, f_{abc} is the composite of the isomorphisms $\Lambda^* \widetilde{Z}(a^!_+ b_+) \otimes \Lambda^* \widetilde{Z}(b^!_+ c_+) \xrightarrow{\cong} \Lambda^*(\widetilde{Z}(a^!_+ b_+) \oplus \widetilde{Z}(b^!_+ c_+))$ and $\Lambda^*(\widetilde{Z}(a^!_+ b_+) \oplus \widetilde{Z}(b^!_+ c_+)) \xrightarrow{\cong} \Lambda^* \widetilde{Z}((a^! b \sqcup b^! c) \sqcup \bigcirc)$. Here, $\widehat{F}_{W_\#}$ is the map associated to $W_\#$, regarded as a graph cobordism $(\Sigma(a^!_+ b_+) \sqcup \Sigma(b^!_+ c_+), \{w_1, w_2\}) \to (\Sigma(P_1), w)$, as in [HMZ17, Proposition 5.2]. By [Zem21a, Proposition 8.1], this map computes the connected sum isomorphism of [OS04a, Proposition 6.1] given on generators at the chain level by the identification

$$\mathbb{T}_\gamma \cap \mathbb{T}_\delta = (\mathbb{T}_{\alpha_1} \cap \mathbb{T}_{\beta_1}) \times (\mathbb{T}_{\alpha_2} \cap \mathbb{T}_{\beta_2}), \tag{3.15}$$

where $(\Sigma, \boldsymbol{\gamma}, \boldsymbol{\delta}, z) = (\Sigma_1, \boldsymbol{\alpha}_1, \boldsymbol{\beta}_1, z_1) \#(\Sigma_2, \boldsymbol{\alpha}_2, \boldsymbol{\beta}_2, z_2)$ is the connected sum of Heegaard diagrams $(\Sigma_1, \boldsymbol{\alpha}_1, \boldsymbol{\beta}_1, z_1)$ and $(\Sigma_2, \boldsymbol{\alpha}_2, \boldsymbol{\beta}_2, z_2)$ for $\Sigma(a^!_+ b_+)$ and $\Sigma(b^!_+ c_+)$, respectively, with the connected sum taken at the basepoints z_1 and z_2, and z a basepoint in the connected sum region of Σ. More explicitly, $\widehat{F}_{W_\#}$ is given on basis elements by

$$\widehat{F}_{W_\#}(\xi \cdot \Theta^{\text{top}}_{ab} \otimes \xi' \cdot \Theta^{\text{top}}_{bc}) = \xi \otimes \xi' \cdot \Theta^{\text{top}}_{ac}, \tag{3.16}$$

where we identify $\xi \otimes \xi'$ with its image under the isomorphism

$$\Lambda^* H_1(\Sigma(a^!_+ b_+)) \otimes \Lambda^* H_1(\Sigma(b^!_+ c_+)) \to \Lambda^* H_1(\Sigma(a^!_+ c_+)) \tag{3.17}$$

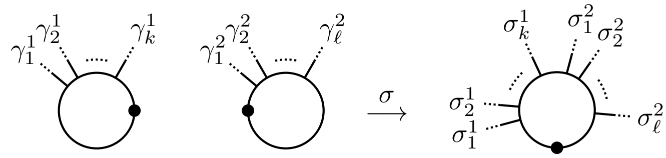


FIGURE 32. The bijection σ between the arcs for the diagrams $a^!_+b_+ \sqcup b^!_+c_+$ and P_1. Here, $\sigma_i^j = \sigma(\gamma_i^j)$.

induced by the identification of $H_1(\Sigma(a^!_+b_+)) \oplus H_1(\Sigma(b^!_+c_+))$ with $H_1(\Sigma(a^!_+c_+))$, which we outline as follows. Note that P_1 is obtained from the doubly-pointed diagram $a^!_+b_+ \sqcup b^!_+c_+$ by merging the two marked components into one. If $\gamma_1^1, \gamma_2^1, \ldots, \gamma_k^1$ are the arcs from the marked component of $a^!_+b_+$ to the remaining components and $\gamma_1^2, \gamma_2^2, \ldots, \gamma_\ell^2$ are the arcs for $b^!_+c$, then there is a natural choice of bijection σ between $\{\gamma_1^1, \gamma_2^1, \ldots, \gamma_k^1\} \sqcup \{\gamma_1^2, \gamma_2^2, \ldots, \gamma_\ell^2\}$ and the set of arcs for P_1 as illustrated in Figure 32. We then have an explicit isomorphism

$$H_1(\Sigma(a^!_+b_+)) \oplus H_1(\Sigma(b^!_+c_+)) \cong H_1(\Sigma(P_1)) \tag{3.18}$$

given by $[\tilde{\gamma}_i^j] \mapsto [\tilde{\sigma}_i^j]$, where $\tilde{\sigma}_i^j$ is the preimage of $\sigma(\gamma_i^j)$ in $\Sigma(P_1)$.

Now, since $\widehat{F}_{W_\#}$ agrees with the map of modules induced by the isomorphism

$$\Lambda^* H_1(\Sigma(a^!_+b_+)) \otimes \Lambda^* H_1(\Sigma(b^!_+c_+)) \cong \Lambda^* H_1(\Sigma(P_1)), \tag{3.19}$$

this tells us that $\widehat{F}_{W_\#} \circ (\psi_{ab} \otimes \psi_{bc}) = \psi_{ac} \circ f_{abc}$.

The fact that the second diagram commutes follows immediately from functoriality of reduced Khovanov and Heegaard Floer homology and the fact that the diagram

$$\begin{array}{ccccccc} \widetilde{\mathcal{C}}_{Kh}(P_1) & \xrightarrow{\widetilde{\mathcal{C}}_{Kh}(s_1)} & \widetilde{\mathcal{C}}_{Kh}(P_2) & \xrightarrow{\widetilde{\mathcal{C}}_{Kh}(s_2)} & \cdots & \xrightarrow{\widetilde{\mathcal{C}}_{Kh}(s_{k-1})} & \widetilde{\mathcal{C}}_{Kh}(P_k) \\ \downarrow \psi_1 & & \downarrow \psi_2 & & & & \downarrow \psi_k \\ \widehat{HF}(\Sigma(P_1)) & \xrightarrow{\widehat{F}_{\Sigma(s_1)}} & \widehat{HF}(\Sigma(P_2)) & \xrightarrow{\widehat{F}_{\Sigma(s_2)}} & \cdots & \xrightarrow{\widehat{F}_{\Sigma(s_{k-1})}} & \widehat{HF}(\Sigma(P_k)) \end{array} \tag{3.20}$$

commutes, which in turn follows from the fact that each individual square in this diagram commutes.

For the sake of brevity, define $\mathrm{Mor}^{\mathcal{A}_n}(a_+, b_+) = \mathrm{Mor}^{\mathcal{A}_n}(\widehat{CFD}(a_+), \widehat{CFD}(b_+))$. Since $\circ_{\mathrm{op}} : \mathrm{Mor}^{\mathcal{A}_n}(a_+, b_+) \otimes \mathrm{Mor}^{\mathcal{A}_n}(b_+, c_+) \to \mathrm{Mor}^{\mathcal{A}_n}(a_+, c_+)$ induces the cobordism map $\widehat{F}_W = \widehat{F}_{\Sigma(s_{k-1})} \circ \cdots \circ \widehat{F}_{\Sigma(s_1)} \circ \widehat{F}_{W_\#}$ on homology, it then follows that there is an isomorphism $(H_*\mathfrak{h}_n, \overline{\circ}_{\mathrm{op}}) \cong H_n$ of associative algebras since the square

$$\begin{array}{ccc} H_n \otimes H_n & \xrightarrow{m} & H_n \\ \downarrow \psi\otimes\psi & & \downarrow \psi \\ H_*\mathfrak{h}_n \otimes H_*\mathfrak{h}_n & \xrightarrow{\overline{\circ}_{\mathrm{op}}} & H_*\mathfrak{h}_n \end{array} \tag{3.21}$$

commutes, where $\psi = \sum_{a,b \in \mathfrak{C}_n} \psi_{ab} : H_n \to H_*\mathfrak{h}_n$ is the linear isomorphism assembled from the ψ_{ab}. □

Formality for A_∞-algebras

Homological perturbation theory allows one to transfer ∞-algebraic structures on chain complexes along certain types of morphisms. In particular, it allows one to construct a canonical A_∞-algebra structure on the homology of an A_∞-algebra.

Proposition 3.1.4 (Homological perturbation lemma for A_∞-algebras, [KS01]). *Let $\mathcal{A} = (A, \{m_i^A\})$ be an A_∞-algebra and let*

$$h \circlearrowleft \mathcal{A} \underset{p}{\overset{\iota}{\leftrightarrows}} H_*\mathcal{A} \tag{3.22}$$

be a retract of $\mathcal{A}$ onto its homology $H_*\mathcal{A}$, regarding (A, m_1) as a chain complex. That is to say chain maps $p : \mathcal{A} \to H_*\mathcal{A}$ and $\iota : H_*\mathcal{A} \to \mathcal{A}$, regarding $H_*\mathcal{A}$ as a complex with trivial differential, and a chain homotopy $h : \mathcal{A} \to \mathcal{A}$ such that

$$\iota p = \mathrm{id} + \partial h + h\partial \tag{3.23}$$

and

$$p\iota = \mathrm{id}. \tag{3.24}$$

Then $H_*\mathcal{A}$ admits an A_∞-algebra structure $\{m_i\}$ such that

1. $m_1 = 0$ and $m_2 = (m_2^A)_*$ and

2. there are A_∞ quasi-isomorphisms $p' : \mathcal{A} \to H_*\mathcal{A}$ and $\iota : H_*\mathcal{A} \to \mathcal{A}$ and an A_∞-homotopy $h' : \mathcal{A} \to \mathcal{A}$ which extend p, ι, and h.

   The structure maps $m_i : (H_*\mathcal{A})^{\otimes i} \to H_*\mathcal{A}[2-i]$ are given by

$$m_i = \sum_{T \in \mathscr{P}_i} m_i^T, \tag{3.25}$$

where $\mathscr{P}_i$ is the set of planar rooted trees with i leaves such that each internal vertex has degree at least 3, and m_i^T is given by labeling the leaves of T by ι,

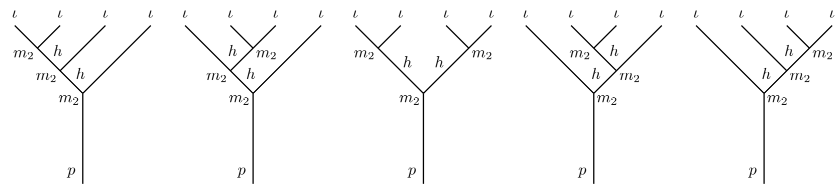


FIGURE 33. Trees contributing to the m_4 operation on the homology $H_*\mathcal{A}$ of a differential algebra $\mathcal{A}$.

interior edges by h, vertices by the A_∞-operations m_j^A, and the root by p and regarding this labeled tree as a composition of morphisms $(H_*\mathcal{A})^{\otimes i} \to H_*\mathcal{A}$. This A_∞-algebra structure on $H_*\mathcal{A}$ is independent of the choice of p, ι, and h up to A_∞-isomorphism.

Note, in particular, that if $\mathcal{A}$ is a genuine differential algebra, then the only trees T contributing to the A_∞-operations on $H_*\mathcal{A}$ are those whose internal vertices are all trivalent, i.e. the binary trees. For instance, there are two trees contributing to m_3 and the trees contributing to m_4 are those shown in Figure 33.

Proposition 3.1.5 ([KS01], Proposition 7)**.** *There is a canonical A_∞ quasi-isomorphism $q : H_*\mathcal{A} \to \mathcal{A}$.*

Sketch. The map $q_1 : H_*\mathcal{A} \to \mathcal{A}$ is defined to be the chain map ι while the higher q_i are defined by

$$q_i = \sum_{T \in \mathscr{P}_i} q_i^T, \tag{3.26}$$

where q_i^T is defined precisely as is m_i^T except that, instead of p, we label the root of each tree T by the homotopy h. One may then verify that $q = \{q_i\}$ is such a map. □

Definition 31. An A_∞-algebra $(\mathcal{A}, m)$ is called *formal* if there is an A_∞-algebra structure $\{\mu_i\}$ on $H_*\mathcal{A}$ with $\mu_i = 0$ for $i = 1$, $i > 2$, and $\mu_2 = \overline{m}_2$, together with an A_∞ quasi-isomorphism $i : (H_*\mathcal{A}, \mu) \to (\mathcal{A}, m)$ such that i_1 induces the identity on homology. In other words, if $\mathcal{A}$ is formal, then the higher operations on $\mathcal{A}$ are trivial up to a (canonical) quasi-isomorphism.

It is easy to show that $\mathfrak{h}_1$ is formal, but we will show that this is not the case for $\mathfrak{h}_n$ with $n > 1$. We first need a couple of technical propositions.

Proposition 3.1.6. *Let $\mathcal{A}_n$ be the weight-0 algebra for the genus n linear pointed matched circle. There are injective differential algebra homomorphisms $L_n : \mathcal{A}_n \hookrightarrow \mathcal{A}_{n+1}$.*

Proof. Consider the injective map $\iota_n : [4n] \hookrightarrow [4n+4]$ given by

$$\iota_n : i \mapsto \begin{cases} 4 & \text{if } i = 1 \\ i+4 & \text{else.} \end{cases} \tag{3.27}$$

Given any partial permutation $(S, T, \sigma) \in \mathcal{A}(n+1, -1)$ and $h \in [4n+4] \smallsetminus (S \cup T)$, define $(S, T, \sigma)_h \in \mathcal{A}(n+1, 0)$ by

$$(S, T, \sigma)_h = (S \cup \{h\}, T \cup \{h\}, \sigma_h) \tag{3.28}$$

where σ_h is the extension of σ to $S \cup \{h\}$ such that $\sigma_h(h) = h$. Suppose that $a \in \mathcal{A}_n$ is a basis element which decomposes into partial permutations as

$$a = \sum_{j=1}^{m} (S_j, T_j, \sigma_j) \tag{3.29}$$

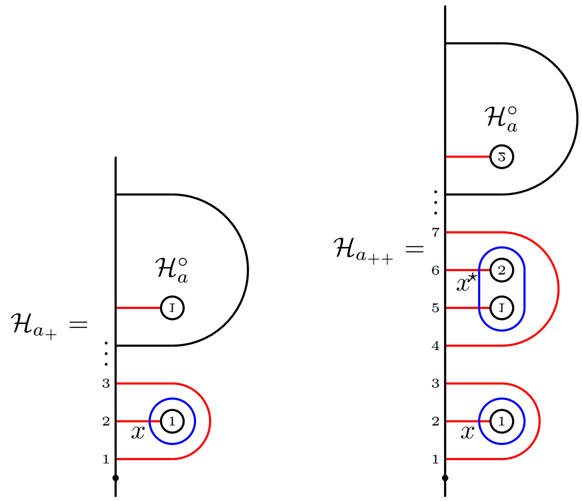


FIGURE 34. The diagrams $\mathcal{H}_{a_+}$ and $\mathcal{H}_{a_{++}}$.

and define

$$L_n(a) = \sum_{j=1}^{m} \sum_{h=1,3} (\iota_n(S), \iota_n(T), \iota_n \circ \sigma_j \circ \iota_n^{-1})_h,$$

where $\iota_n^{-1} : \iota_n([4n]) \to [4n]$ is the inverse of the bijection $\iota_n : [4n] \to \iota_n([4n])$, extending linearly to obtain a map $L_n : \mathcal{A}_n \to \mathcal{A}_{n+1}$. Since the height of each inserted strand is at most 3, it follows immediately that L_n is injective and that $L_n(\partial a) = \partial L_n(a)$ and $L_n(ab) = L_n(a)L_n(b)$ for all algebra elements $a, b \in \mathcal{A}_n$. $\square$

Proposition 3.1.7. *Given a crossingless matching $a \in \mathfrak{C}_n$, there is an $\mathbb{F}$-vector space isomorphism $\lambda_a : \widehat{CFD}(a_+) \to \widehat{CFD}(a_{++})$ such that $\boldsymbol{x}_i \xrightarrow{A_{ij}} \boldsymbol{x}_j$ is an arrow in the graph $\Gamma_{\widehat{CFD}(a_+)}$ if and only if $\lambda_a(\boldsymbol{x}_i) \xrightarrow{L_n(A_{ij})+\delta_{ij}\rho_{1,3}} \lambda_a(\boldsymbol{x}_j)$ is an arrow in the graph $\Gamma_{\widehat{CFD}(a_{++})}$.*

Proof. The diagrams $\mathcal{H}_{a_+}$ and $\mathcal{H}_{a_{++}}$ are of the form shown in Figure 34 and we claim that there is a bijection between the sets of generators for $\mathcal{H}_{a_+}$ and $\mathcal{H}_{a_{++}}$. To see this, note that if $\boldsymbol{x}$ is a generator for $\mathcal{H}_{a_{++}}$, then $x \in \boldsymbol{x}$ since x is the

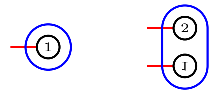


FIGURE 35. Regions adjacent to the basepoint in $\mathcal{H}_{a_+}$ (both) and $\mathcal{H}_{a_{++}}$ (right).

only intersection point on the bottom-most β-circle. Since there is exactly one intersection point in $\boldsymbol{x}$ lying on the next lowest β-circle and this point cannot lie on the same α-arc as x, we also have $x^\star \in \boldsymbol{x}$. Therefore, we have a decomposition $\boldsymbol{x} = \boldsymbol{x}^\circ \cup \{x, x^\star\}$, where $\boldsymbol{x}^\circ$ is a collection of intersection points in $\mathcal{H}_a^\circ$. The desired bijection is then given by $\boldsymbol{x}^\circ \cup \{x\} \mapsto \boldsymbol{x}^\circ \cup \{x, x^\star\}$ and λ_a is given by extending this bijection linearly. Note that the labels of the ends of the α-arcs in $\mathcal{H}_a^\circ \subset \mathcal{H}_{a_{++}}$ are obtained from the labels of the α-arcs in $\mathcal{H}_a^\circ \subset \mathcal{H}_{a_+}$ by applying ι_n and $\lambda_a(\boldsymbol{x})$ necessarily occupies the α-arc labeled 2 and 5 but not the arcs labeled 1 and 3 or 4 and 7 so $I_D(\lambda_a(\boldsymbol{x})) = L_n(I_D(\boldsymbol{x}))$ for all $\boldsymbol{x} \in \widehat{CFD}(a_+)$. By construction, the regions inside the β-circles shown in Figure 35 are adjacent to the basepoint in both $\mathcal{H}_{a_+}$ and $\mathcal{H}_{a_{++}}$ so the only domains contributing to the structure maps $\delta^1_{a_+}$ and $\delta^1_{a_{++}}$ are those supported in $\mathcal{H}_a^\circ$ and the annular domain $x \to x$ asymptotic to $\rho_{1,3}$ in both diagrams plus the annular domain $x^\star \to x^\star$ asymptotic to $\rho_{4,7}$ in $\mathcal{H}_{a_{++}}$ (see Figure 36). Now, there is a bijection between the sets of domains for index 1 holomorphic disks supported in $\mathcal{H}_a^\circ \subset \mathcal{H}_{a_+}$ and domains for index 1 holomorphic disks supported in $\mathcal{H}_a^\circ \subset \mathcal{H}_{a_{++}}$. This tells us that if $i \neq j$, then $\boldsymbol{x}_i \xrightarrow{A_{ij}} \boldsymbol{x}_j$ is an arrow in $\Gamma_{\widehat{CFD}(a_+)}$ if and only if $\lambda_a(\boldsymbol{x}_i) \xrightarrow{L_n(A_{ij})} \lambda_a(\boldsymbol{x}_j)$ is an arrow in $\Gamma_{\widehat{CFD}(a_{++})}$. This bijection, taken together with the existence of the annular domains $x \to x$ and $x^\star \to x^\star$, tells us that $\boldsymbol{x}_i \xrightarrow{A_i+\rho_{1,3}} \boldsymbol{x}_i$ is an arrow in $\Gamma_{\widehat{CFD}(a_+)}$ if and only if $\lambda_a(\boldsymbol{x}_i) \xrightarrow{L_n(A_i)+\rho_{1,3}+\rho_{4,7}} \lambda_a(\boldsymbol{x}_j)$ is an arrow in $\Gamma_{\widehat{CFD}(a_{++})}$. Since $\rho_{4,7} = L_n(\rho_{1,3})$, this proves the desired result. $\square$

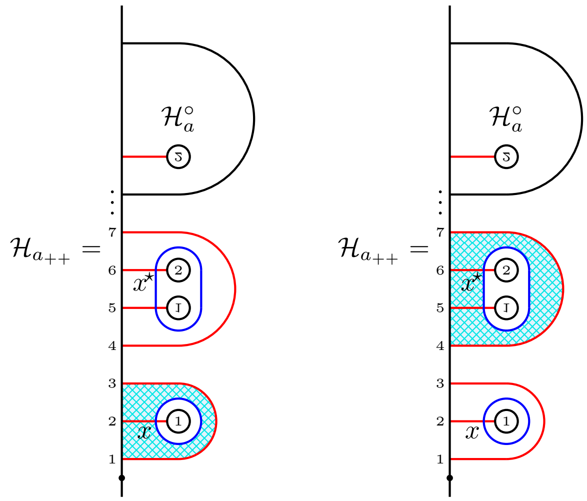


FIGURE 36. The domains asymptotic to $\rho_{1,3}$ (left) and $\rho_{4,7}$ (right) in $\mathcal{H}_{a_{++}}$.

Lemma 3.1.8. *Suppose that $\phi : C \to D$ is an injective chain map such that if $z \in \mathrm{im}(\phi) \cap \mathrm{im}(\partial_D)$, then $z = \partial_D y$ for some $y \in \mathrm{im}(\phi)$. Then the induced map $\phi_* : H_*(C) \to H_*(D)$ is injective.*

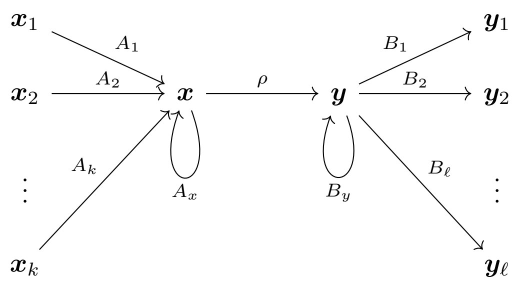
Proof. Suppose that $[x] \in \ker(\phi_*)$, then $0 = \phi_*([x]) = [\phi(x)]$ so $\phi(x) \in \mathrm{im}(\partial_D)$ and, hence, $\phi(x) = \partial_D w$ for some $w \in \mathrm{im}(\phi)$. Therefore, we have $\phi(x) = \partial_D(\phi(u)) = \phi(\partial_C u)$ for some $u \in C$. Since ϕ is injective, we then have that $x = \partial_C u$ so $[x] = 0$. Therefore, ϕ_* is injective. □

Corollary 3.1.9. *There is a homologically injective embedding $\Lambda_n : \mathfrak{h}_n \hookrightarrow \mathfrak{h}_{n+1}$ of differential algebras. Moreover, there is a direct sum decomposition*

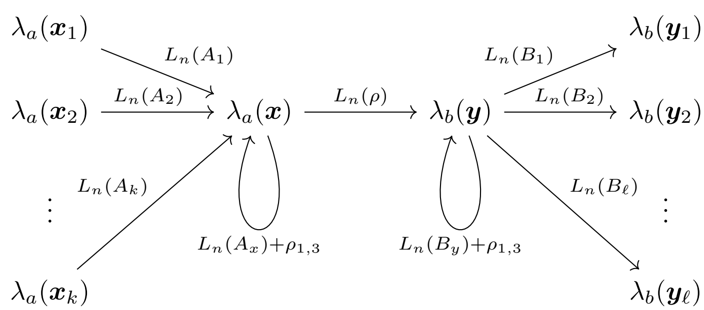
$$\mathrm{End}^{\mathcal{A}_{n+1}}\left(\bigoplus_{a \in \mathfrak{C}_n} \widehat{CFD}(a_{++})\right) = \mathrm{im}(\Lambda_n) \oplus \mathrm{im}(\rho_{1,3}\Lambda_n) \tag{3.30}$$

of vector spaces with respect to which the restriction of the differential on $\mathfrak{h}_{n+1}$ is block-diagonal. As a consequence, the map $(\Lambda_n)_* : H_*\mathfrak{h}_n \to H_*\mathfrak{h}_{n+1}$ is injective.

Proof. We claim that the injective linear map $\Lambda_n : \mathfrak{h}_n \hookrightarrow \mathfrak{h}_{n+1}$ given on a basic morphism $f : \boldsymbol{x} \mapsto \rho\boldsymbol{y}$ by the morphism $\Lambda_n f : \lambda_a(\boldsymbol{x}) \mapsto L_n(\rho)\lambda_a(\boldsymbol{y})$ is a differential algebra homomorphism. Note that if $g : \boldsymbol{y} \mapsto \sigma\boldsymbol{z}$ is another basic morphism with $\Lambda_n g : \lambda_b(\boldsymbol{y}) \mapsto L_n(\sigma)\lambda_c(\boldsymbol{z})$, then, by construction, we have $\Lambda_n(f \circ_{\text{op}} g) : \lambda_a(\boldsymbol{x}) \mapsto L_n(\rho\sigma)\lambda_c(\boldsymbol{z})$ and $L_n(\rho\sigma)\lambda_c(\boldsymbol{z}) = L_n(\rho)L_n(\sigma)\lambda_c(\boldsymbol{z})$ since L_n is an algebra homomorphism so $\Lambda_n(f \circ_{\text{op}} g) = \Lambda_n f \circ_{\text{op}} \Lambda_n g$. Therefore, Λ_n is an algebra homomorphism. Now consider the part



of the graph $\Gamma_{\text{Cone}(f)}$ contributing to ∂f and the corresponding part



of the graph $\Gamma_{\text{Cone}(\Lambda_n f)}$. We compute

$$\begin{aligned}\partial(\Lambda_n f) =& [\lambda_a(\boldsymbol{x}) \mapsto (L_n(A_x) + \rho_{1,3})L_n(\rho)\lambda_b(\boldsymbol{y})] \\ &+ \sum_{i=1}^{k}[\lambda_a(\boldsymbol{x}_i) \mapsto L_n(A_i)L_n(\rho)\lambda_b(\boldsymbol{y})] \\ &+ [\lambda_a(\boldsymbol{x}) \mapsto L_n(\rho)(L_n(A_y) + \rho_{1,3})\lambda_b(\boldsymbol{y})] \\ &+ \sum_{j=1}^{\ell}[\lambda_a(\boldsymbol{x}) \mapsto L_n(\rho)L_n(B_j)\lambda_b(\boldsymbol{y}_j)] \\ =& [\lambda_a(\boldsymbol{x}) \mapsto L_n(A_x\rho)\lambda_b(\boldsymbol{y})] + \sum_{i=1}^{k}[\lambda_a(\boldsymbol{x}_i) \mapsto L_n(A_i\rho)\lambda_b(\boldsymbol{y})] \\ &+ [\lambda_a(\boldsymbol{x}) \mapsto L_n(\rho A_y)\lambda_b(\boldsymbol{y})] + \sum_{j=1}^{\ell}[\lambda_a(\boldsymbol{x}) \mapsto L_n(\rho B_j)\lambda_b(\boldsymbol{y}_j)] \\ &+ [\lambda_a(\boldsymbol{x}) \mapsto \rho_{1,3}L_n(\rho)\lambda_b(\boldsymbol{y})] + [\lambda_a(\boldsymbol{x}) \mapsto L_n(\rho)\rho_{1,3}\lambda_b(\boldsymbol{y})],\end{aligned} \tag{3.31}$$

where the second equality follows from the fact that L_n is an algebra homomorphism. This then gives us

$$\begin{aligned}\partial(\Lambda_n f) =& \Lambda_n[\boldsymbol{x} \mapsto A_x\rho\boldsymbol{y}] + \sum_{i=1}^{k}\Lambda_n[\boldsymbol{x}_i \mapsto A_i\rho\boldsymbol{y}] \\ &+ \Lambda_n[\boldsymbol{x} \mapsto \rho A_y\boldsymbol{y}] + \sum_{j=1}^{\ell}\Lambda_n[\boldsymbol{x} \mapsto \rho B_j\boldsymbol{y}_j] \\ &+ [\lambda_a(\boldsymbol{x}) \mapsto \rho_{1,3}L_n(\rho)\lambda_b(\boldsymbol{y})] + [\lambda_a(\boldsymbol{x}) \mapsto L_n(\rho)\rho_{1,3}\lambda_b(\boldsymbol{y})] \\ =& \Lambda_n(\partial f) + [\lambda_a(\boldsymbol{x}) \mapsto [\rho_{1,3}, L_n(\rho)]\lambda_b(\boldsymbol{y})],\end{aligned} \tag{3.32}$$

where $[\rho_{1,3}, L_n(\rho)] = \rho_{1,3}L_n(\rho) + L_n(\rho)\rho_{1,3}$ is the commutator of $\rho_{1,3}$ and $L_n(\rho)$. However, by construction of L_n, we have that $[\rho_{1,3}, L_n(\rho)] = 0$ for all $\rho \in \mathcal{A}_n$ so $\partial(\Lambda_n f) = \Lambda_n(\partial f)$ and Λ_n is an injective differential algebra homomorphism. Now note that, again by construction of L_n, no element of $\text{im}(\Lambda_n)$ is of the form

$[\lambda_a(\boldsymbol{x}) \mapsto \rho\lambda_b(\boldsymbol{y})]$, where $\rho \in \rho_{1,3}\mathcal{A}_{n+1}$ so $\mathrm{im}(\Lambda_n) \cap \mathrm{im}(\rho_{1,3}\Lambda_n) = \{0\}$. Moreover, since $I_D(\lambda_a(\boldsymbol{x})) = L_n(I_D(\boldsymbol{x}))$ for any $\boldsymbol{x} \in \widehat{CFD}(a_+)$ and any generator of $\widehat{CFD}(a_{++})$ is of the form $\lambda_a(\boldsymbol{x})$, the only algebra elements acting nontrivially on the module $\bigoplus_{a\in\mathfrak{C}_n} \widehat{CFD}(a_{++})$ are those in $L_n(\mathcal{A}_n) \oplus \rho_{1,3}L_n(\mathcal{A}_n) \subset \mathcal{A}_{n+1}$. Therefore, if $f = [\lambda_a(\boldsymbol{x}) \mapsto \rho\lambda_b(\boldsymbol{y})] \in \mathrm{End}^{\mathcal{A}_{n+1}}\left(\bigoplus_{a\in\mathfrak{C}_n} \widehat{CFD}(a_{++})\right)$ is a basic morphism, then either $\rho \in L_n(\mathcal{A}_n)$ or $\rho \in \rho_{1,3}L_n(\mathcal{A}_n)$. Since the basic morphisms form a basis for $\mathrm{End}^{\mathcal{A}_{n+1}}\left(\bigoplus_{a\in\mathfrak{C}_n} \widehat{CFD}(a_{++})\right)$, this shows that

$$\mathrm{End}^{\mathcal{A}_{n+1}}\left(\bigoplus_{a\in\mathfrak{C}_n} \widehat{CFD}(a_{++})\right) = \mathrm{im}(\Lambda_n) \oplus \mathrm{im}(\rho_{1,3}\Lambda_n). \tag{3.33}$$

Since we have shown that Λ_n is a chain map, to show that the restriction of the differential is block diagonal with respect to this decomposition, it remains to show that $\mathrm{im}(\rho_{1,3}\Lambda_n)$ is closed under the differential. However, the computation showing that Λ_n is a chain map can be readily adapted, mutatis mutandis, to show that $\partial(\rho_{1,3}\Lambda_n f) = \rho_{1,3}\Lambda_n(\partial f)$. Lastly, note that the morphism spaces $\mathrm{Mor}^{\mathcal{A}_{n+1}}(c_+, d_+)$ are closed under the differential for all $c, d \in \mathfrak{C}_{n+1}$ so $g \in \mathfrak{h}_{n+1}$ is an element of $\mathrm{im}(\Lambda_n) \cap \mathrm{im}(\partial)$ if and only if $g = \partial f$ for some $f \in \mathrm{im}(\Lambda_n)$. Therefore, by Lemma 3.1.8, the map $(\Lambda_n)_*$ is injective. □

Theorem 3.1.10. *The differential algebras $\mathfrak{h}_n$ are not formal for $n > 1$.*

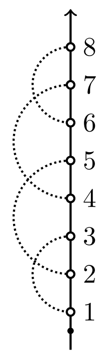
Proof. We will show in Section 3.2, by a lengthy but straightforward computation, that $\mathfrak{h}_2$ is non-formal with nontrivial m_3 operation. Since $\mathfrak{h}_2$ embeds homologically injectively in $\mathfrak{h}_n$ for all $n > 1$, this proves that $\mathfrak{h}_n$ is non-formal with nontrivial m_3 for all $n > 1$. □

Before we proceed, we will need a complete description of the algebra $\mathcal{A}_2$.

The algebra $\mathcal{A}_2$

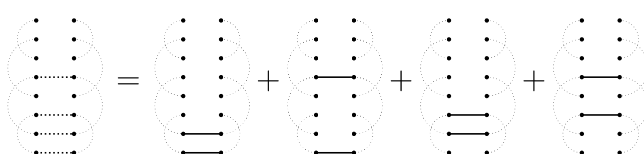
Consider the genus 2 linear pointed matched circle

$$\mathcal{Z}_2 = \quad (3.34)$$

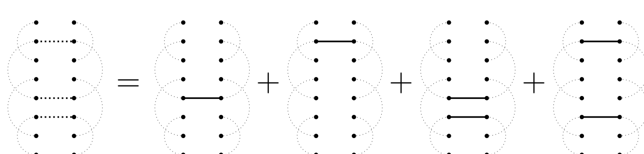


The matching $M : [8] \to [4]$ determining $\mathcal{Z}$ is given by $M(1) = M(3) = 1$, $M(2) = M(5) = 2$, $M(4) = M(7) = 3$, and $M(6) = M(8) = 4$. The algebra $\mathcal{A}_2$ contains six orthogonal idempotents $\iota_0 = I(\{1,2\})$, $\iota_1 = I(\{1,3\})$, $\iota_2 = I(\{1,4\})$, $\iota_3 = I(\{2,3\})$, $\iota_4 = I(\{2,4\})$, and $\iota_5 = I(\{3,4\})$, which are depicted below.

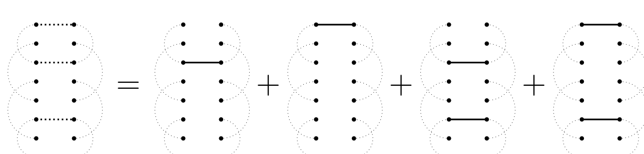
$\iota_0 =$

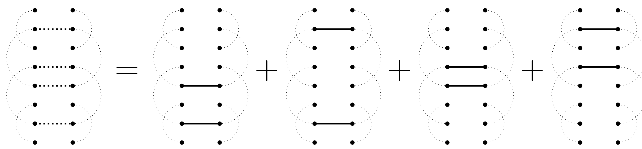


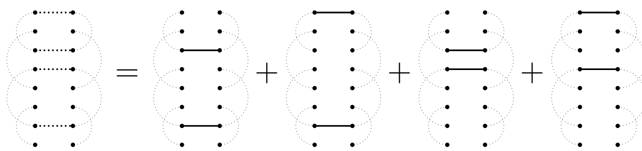
$\iota_1 =$

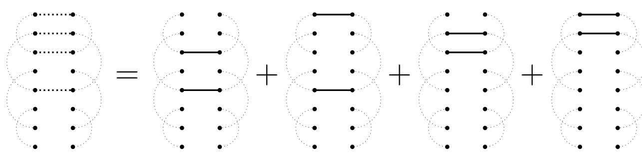


$\iota_2 =$



$\iota_3 =$ 

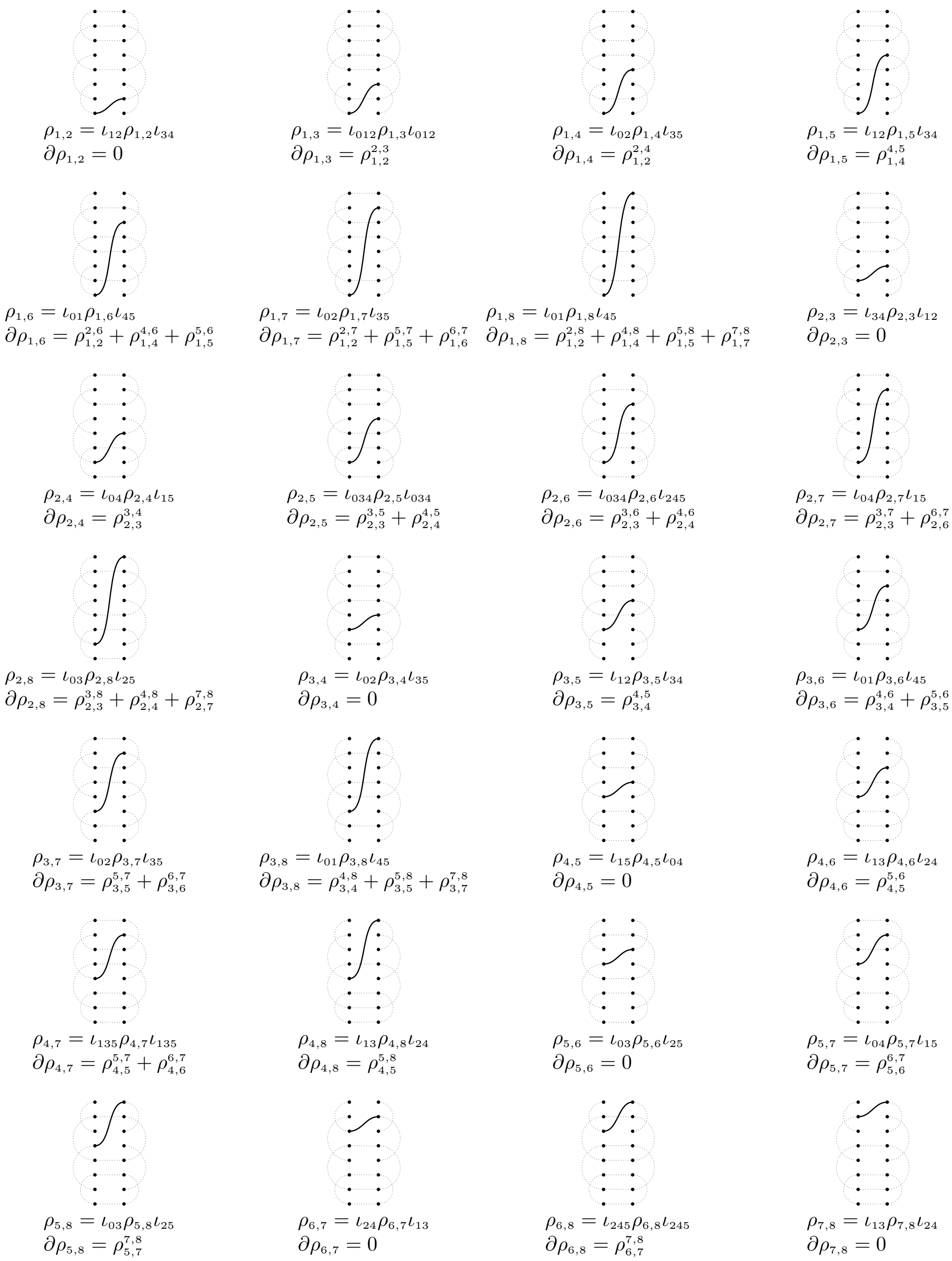
$\iota_4 =$ 

$\iota_5 =$ 

For a string $0 \leq a_1 < a_2 < \cdots < a_k \leq 5$, define an idempotent $\iota_{a_1 a_2 \cdots a_k}$ by

$$\iota_{a_1 a_2 \cdots a_k} = \sum_{i=1}^{k} \iota_{a_i} \tag{3.35}$$

and, for $0 \leq i < j \leq 7$, let $\rho_{i,j}$ be the strands algebra element determined by the Reeb chord in $\mathcal{Z}$ from i to j. In this notation, $\mathcal{A}_2$ has 28 single Reeb chord generators. $\mathcal{A}_2$ also has 179 double Reeb chord generators $\rho_{i,j}^{k,\ell} = \iota_a \rho_{i,j}^{k,\ell} \iota_b$, for $i < k$, corresponding to the sets of Reeb chords $\{[i,j],[k,\ell]\}$. However, many of these are redundant as they are products of single chord generators. For the sake of completeness, we list all of these generators, their idempotents, and their differentials below in Figures 37, 38, 39, 40, and 41.



$\rho_{1,2} = \iota_{12}\rho_{1,2}\iota_{34}$, $\partial\rho_{1,2} = 0$

$\rho_{1,3} = \iota_{012}\rho_{1,3}\iota_{012}$, $\partial\rho_{1,3} = \rho^{2,3}_{1,2}$

$\rho_{1,4} = \iota_{02}\rho_{1,4}\iota_{35}$, $\partial\rho_{1,4} = \rho^{2,4}_{1,2}$

$\rho_{1,5} = \iota_{12}\rho_{1,5}\iota_{34}$, $\partial\rho_{1,5} = \rho^{4,5}_{1,4}$

$\rho_{1,6} = \iota_{01}\rho_{1,6}\iota_{45}$, $\partial\rho_{1,6} = \rho^{2,6}_{1,2} + \rho^{4,6}_{1,4} + \rho^{5,6}_{1,5}$

$\rho_{1,7} = \iota_{02}\rho_{1,7}\iota_{35}$, $\partial\rho_{1,7} = \rho^{2,7}_{1,2} + \rho^{5,7}_{1,5} + \rho^{6,7}_{1,6}$

$\rho_{1,8} = \iota_{01}\rho_{1,8}\iota_{45}$, $\partial\rho_{1,8} = \rho^{2,8}_{1,2} + \rho^{4,8}_{1,4} + \rho^{5,8}_{1,5} + \rho^{7,8}_{1,7}$

$\rho_{2,3} = \iota_{34}\rho_{2,3}\iota_{12}$, $\partial\rho_{2,3} = 0$

$\rho_{2,4} = \iota_{04}\rho_{2,4}\iota_{15}$, $\partial\rho_{2,4} = \rho^{3,4}_{2,3}$

$\rho_{2,5} = \iota_{034}\rho_{2,5}\iota_{034}$, $\partial\rho_{2,5} = \rho^{3,5}_{2,3} + \rho^{4,5}_{2,4}$

$\rho_{2,6} = \iota_{034}\rho_{2,6}\iota_{245}$, $\partial\rho_{2,6} = \rho^{3,6}_{2,3} + \rho^{4,6}_{2,4}$

$\rho_{2,7} = \iota_{04}\rho_{2,7}\iota_{15}$, $\partial\rho_{2,7} = \rho^{3,7}_{2,3} + \rho^{6,7}_{2,6}$

$\rho_{2,8} = \iota_{03}\rho_{2,8}\iota_{25}$, $\partial\rho_{2,8} = \rho^{3,8}_{2,3} + \rho^{4,8}_{2,4} + \rho^{7,8}_{2,7}$

$\rho_{3,4} = \iota_{02}\rho_{3,4}\iota_{35}$, $\partial\rho_{3,4} = 0$

$\rho_{3,5} = \iota_{12}\rho_{3,5}\iota_{34}$, $\partial\rho_{3,5} = \rho^{4,5}_{3,4}$

$\rho_{3,6} = \iota_{01}\rho_{3,6}\iota_{45}$, $\partial\rho_{3,6} = \rho^{4,6}_{3,4} + \rho^{5,6}_{3,5}$

$\rho_{3,7} = \iota_{02}\rho_{3,7}\iota_{35}$, $\partial\rho_{3,7} = \rho^{5,7}_{3,5} + \rho^{6,7}_{3,6}$

$\rho_{3,8} = \iota_{01}\rho_{3,8}\iota_{45}$, $\partial\rho_{3,8} = \rho^{4,8}_{3,4} + \rho^{5,8}_{3,5} + \rho^{7,8}_{3,7}$

$\rho_{4,5} = \iota_{15}\rho_{4,5}\iota_{04}$, $\partial\rho_{4,5} = 0$

$\rho_{4,6} = \iota_{13}\rho_{4,6}\iota_{24}$, $\partial\rho_{4,6} = \rho^{5,6}_{4,5}$

$\rho_{4,7} = \iota_{135}\rho_{4,7}\iota_{135}$, $\partial\rho_{4,7} = \rho^{5,7}_{4,5} + \rho^{6,7}_{4,6}$

$\rho_{4,8} = \iota_{13}\rho_{4,8}\iota_{24}$, $\partial\rho_{4,8} = \rho^{5,8}_{4,5}$

$\rho_{5,6} = \iota_{03}\rho_{5,6}\iota_{25}$, $\partial\rho_{5,6} = 0$

$\rho_{5,7} = \iota_{04}\rho_{5,7}\iota_{15}$, $\partial\rho_{5,7} = \rho^{6,7}_{5,6}$

$\rho_{5,8} = \iota_{03}\rho_{5,8}\iota_{25}$, $\partial\rho_{5,8} = \rho^{7,8}_{5,7}$

$\rho_{6,7} = \iota_{24}\rho_{6,7}\iota_{13}$, $\partial\rho_{6,7} = 0$

$\rho_{6,8} = \iota_{245}\rho_{6,8}\iota_{245}$, $\partial\rho_{6,8} = \rho^{7,8}_{6,7}$

$\rho_{7,8} = \iota_{13}\rho_{7,8}\iota_{24}$, $\partial\rho_{7,8} = 0$

FIGURE 37. Single Reeb chord generators of $\mathcal{A}_2$. Dotted horizontal strands indicate that we sum over all ways of inserting a single horizontal strand at each corresponding height.

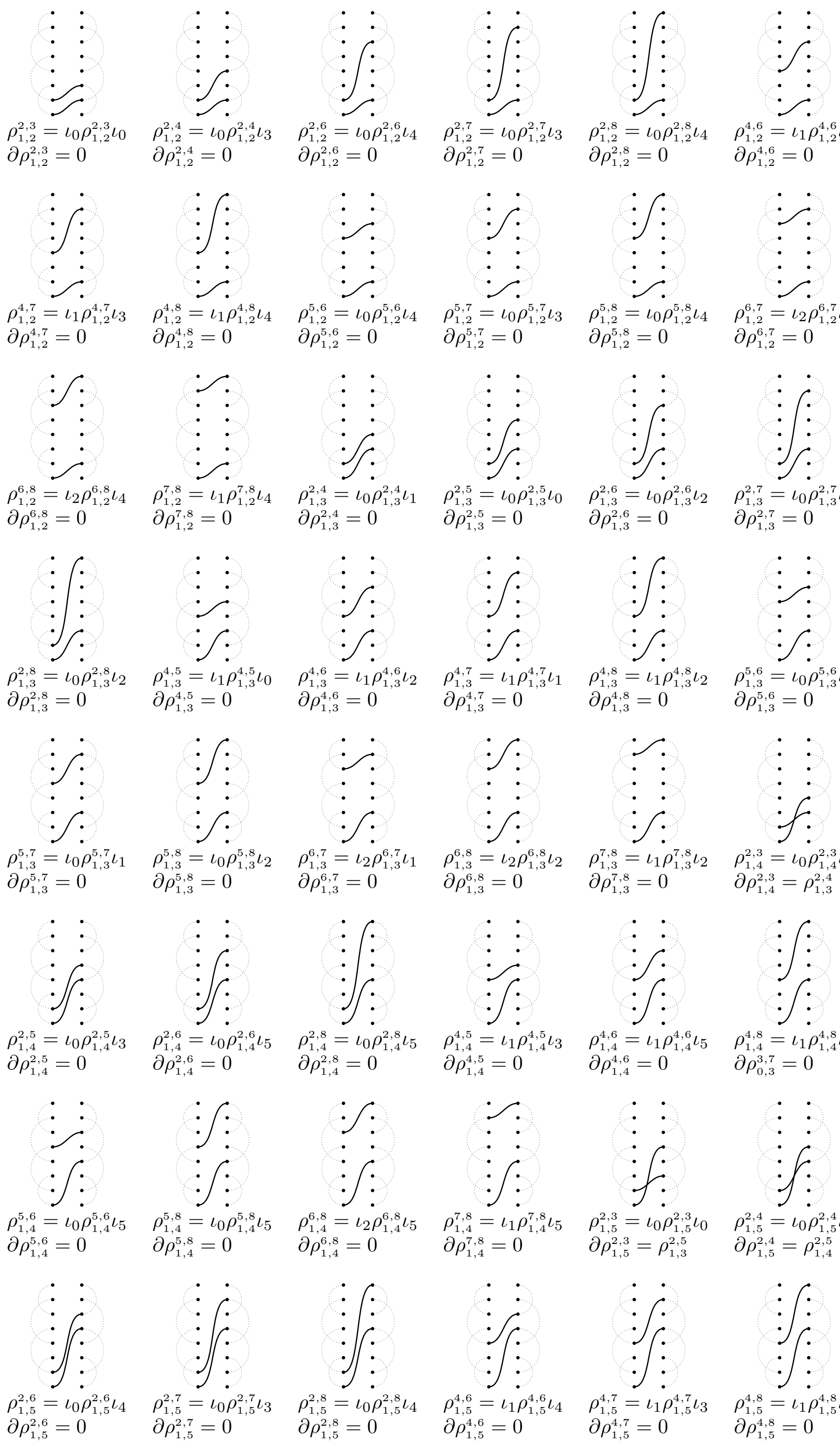


FIGURE 38. Double Reeb chord generators of $\mathcal{A}_2$ (Part I).

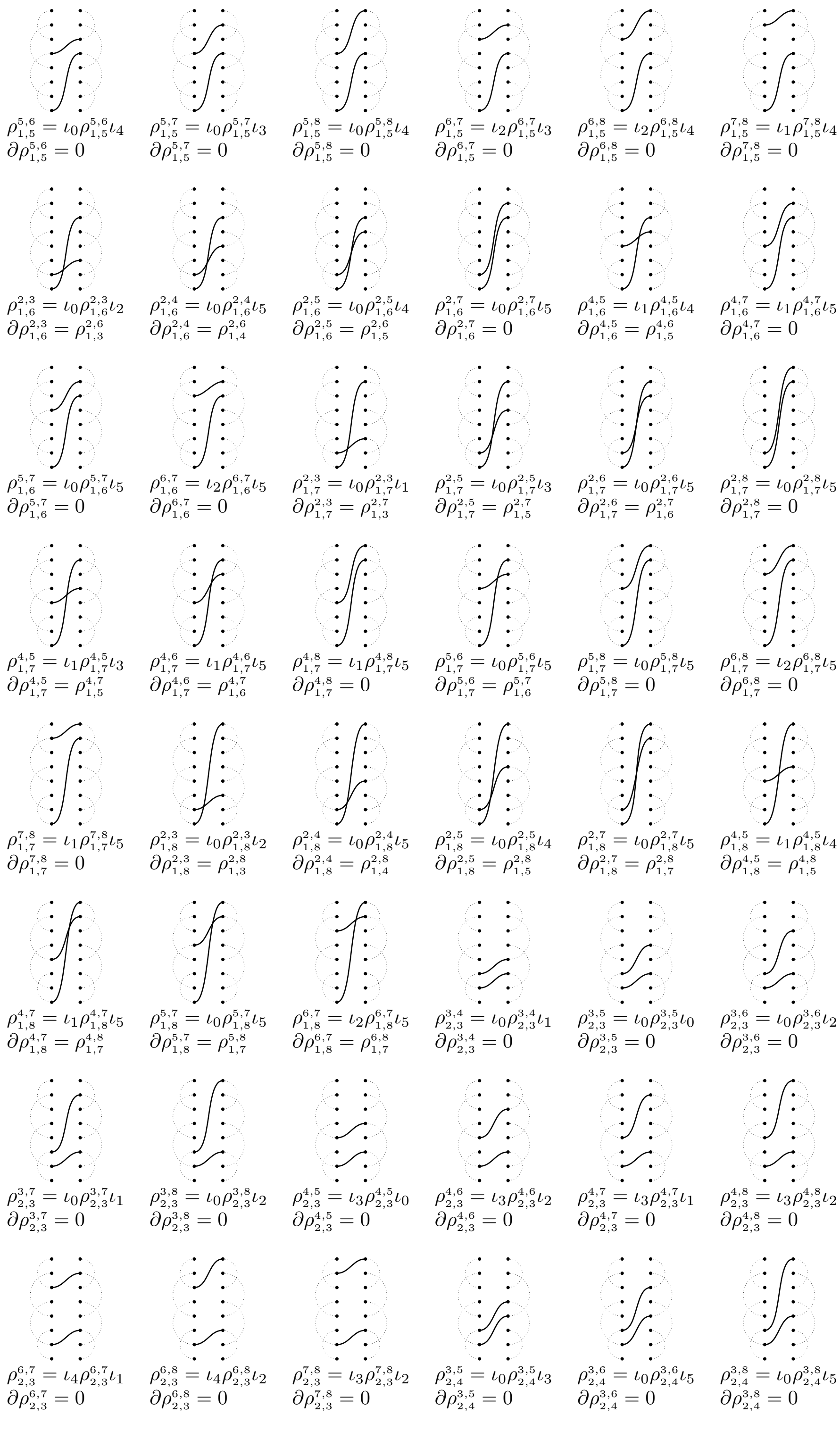


FIGURE 39. Double Reeb chord generators of $\mathcal{A}_2$ (Part II).

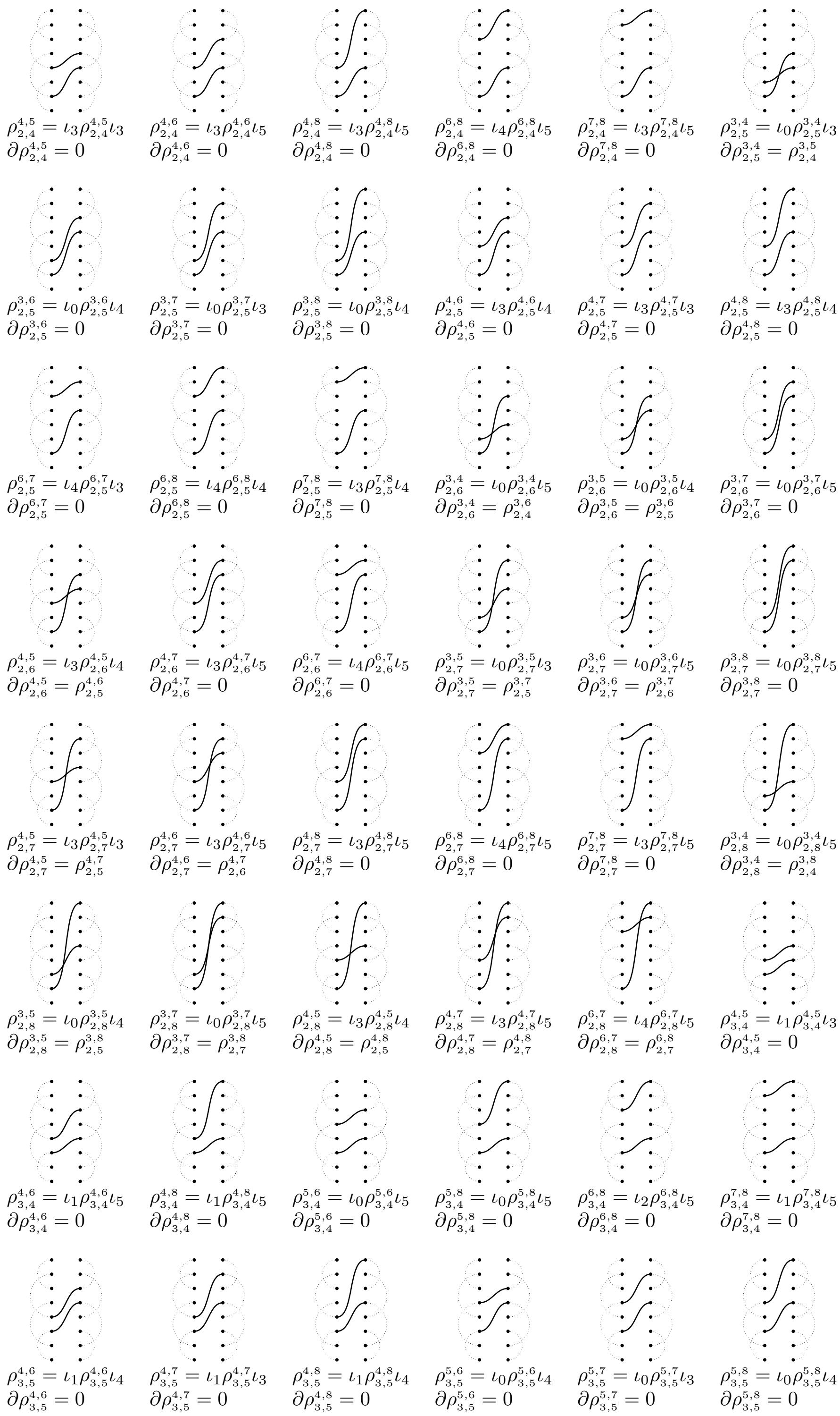


FIGURE 40. Double Reeb chord generators of $\mathcal{A}_2$ (Part III).

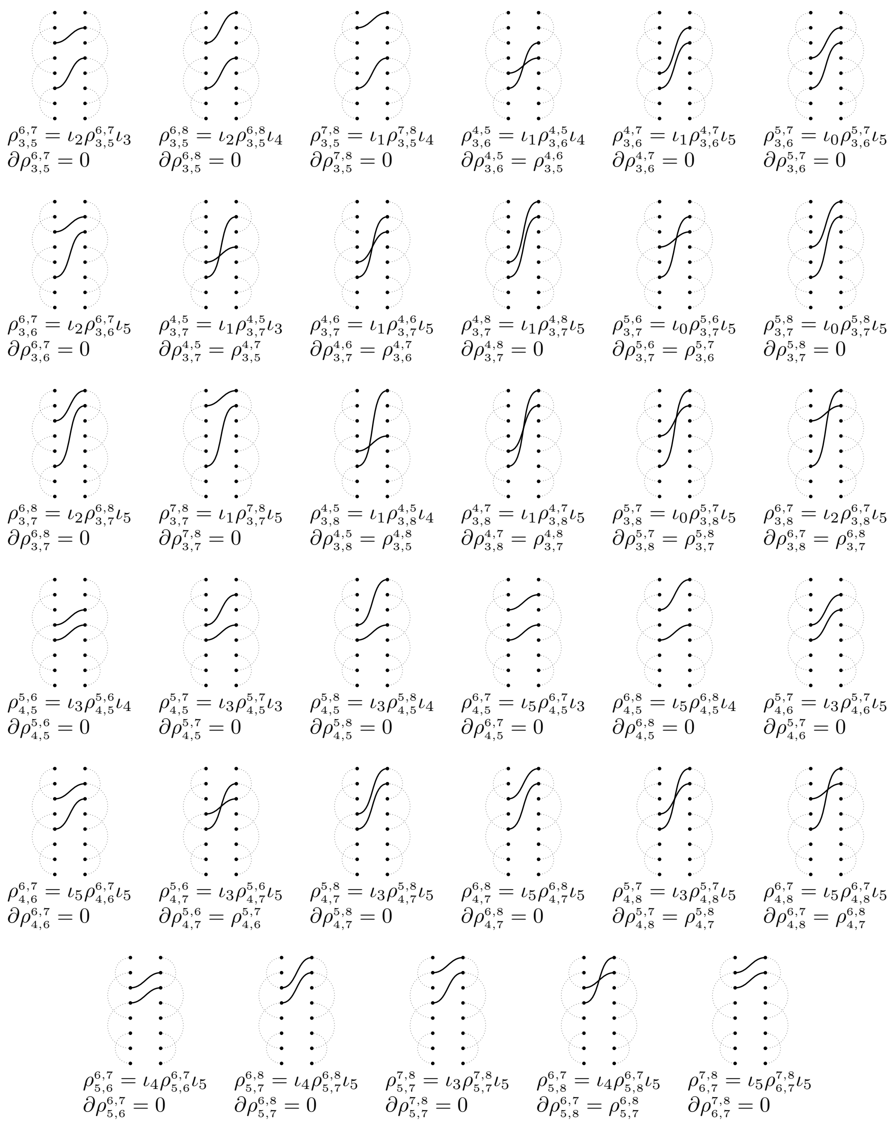


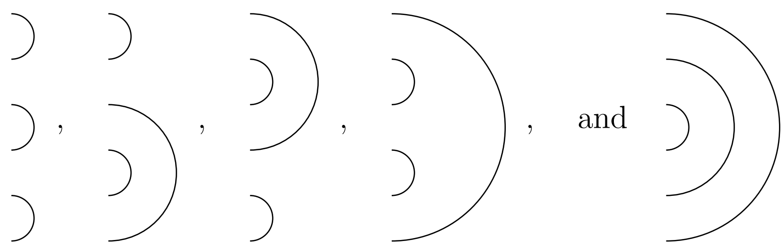
FIGURE 41. Double Reeb chord generators of $\mathcal{A}_2$ (Part IV).

3.2 The branched arc algebra $\mathfrak{h}_2$

The branched arc algebra $\mathfrak{h}_2$ is the endomorphism algebra

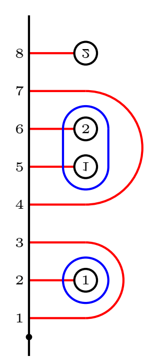
$$\mathfrak{h}_2 = \mathrm{End}^{\mathcal{A}_2}\left(\bigoplus_{a\in\mathfrak{C}_2} \widehat{CFD}(a_+)\right). \tag{3.36}$$

The set $\mathfrak{C}_3$ of crossingless matchings on six points consists of the five planar diagrams



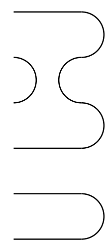
(3.37)

which we denote by a_1, a_2, a_3, a_4, and a_5, respectively. Of these, only a_1 and a_3 are of the form a_+ for some $a \in \mathfrak{C}_2$ so we restrict our attention to these. As a_1 is the one-ended plat closure of the six stranded identity braid, the first part of the algorithm given on page 120 furnishes us with the bordered Heegaard diagram $\mathcal{H}_1$ for $\Sigma(a_1)$ shown below.

$\mathcal{H}_1 =$ 

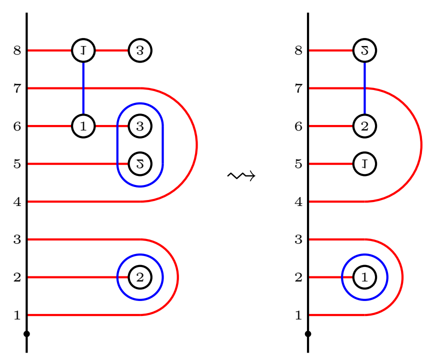
(3.38)

Isotope a_3 to obtain its minimal plat closure-form as follows.



(3.39)

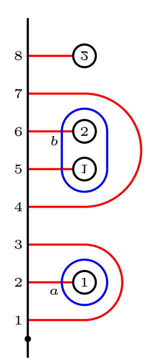
Inserting a new handle and β-curve into $\mathcal{H}_1$ for the cap-cup pair in this diagram, then simplifying using the destabilization procedure detailed on page 121, gives us the following Heegaard diagram

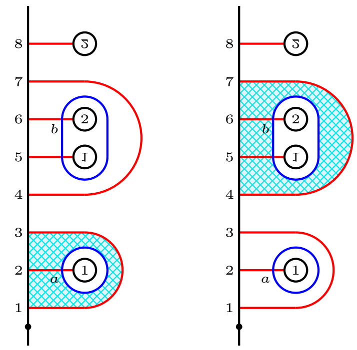
 $=: \mathcal{H}_3$ (3.40)

for $\Sigma(a_i)$. We now compute $\widehat{CFD}(a_i)$ for $i = 1, 3$.

<u>$\widehat{CFD}(a_1)$:</u>

It is not hard to see that

$\mathcal{H}_1 =$  (3.41) has a single generator $\boldsymbol{t} = \{a, b\}$ with $\iota_1 \boldsymbol{t} = \boldsymbol{t}$ and supports the following index 1 domains from $\boldsymbol{t}$ to itself:



$$\boldsymbol{t} \xrightarrow{\rho_{1,3}} \boldsymbol{t} \qquad \boldsymbol{t} \xrightarrow{\rho_{4,7}} \boldsymbol{t} \tag{3.42}$$

giving us

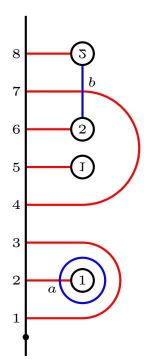
$$\widehat{CFD}(a_1) = \boldsymbol{t} \circlearrowright \rho_{1,3}+\rho_{4,7} \tag{3.43}$$

which is to say that $\widehat{CFD}(\mathcal{H}_1) = \mathbb{F}\langle \boldsymbol{t} \rangle$ with $\delta^1(\boldsymbol{t}) = (\rho_{1,3} + \rho_{4,7}) \otimes \boldsymbol{t}$. This coincides with the computation in §5.2 of [LOT14b].

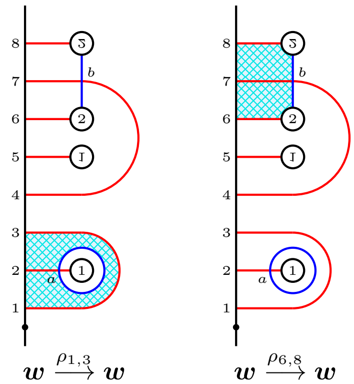
$\underline{\widehat{CFD}(a_3)}$:

By inspection, the diagram

$$\mathcal{H}_3 = \tag{3.44}$$



has a single generator $\boldsymbol{w} = \{a, b\}$ with $\iota_2 \boldsymbol{w} = \boldsymbol{w}$ and supports the domains



(3.45)

giving us

$$\widehat{CFD}(a_3) = \boldsymbol{w} \circlearrowleft^{\rho_{1,3}+\rho_{6,8}} \tag{3.46}$$

i.e. $\widehat{CFD}(\mathcal{H}_3) = \mathbb{F}\langle \boldsymbol{v} \rangle$ with $\delta^1(\boldsymbol{w}) = (\rho_{1,3} + \rho_{6,8}) \otimes \boldsymbol{w}$. Strictly speaking, the structure coefficients for these type-D structures should be of the form $\iota_i \rho$ but if $\boldsymbol{\xi}$ is some generator with $\iota_i \boldsymbol{\xi} = \boldsymbol{\xi}$, then we have that $\rho \otimes \boldsymbol{\xi} = \rho \otimes (\iota_i \boldsymbol{\xi}) = (\rho \iota_i) \otimes \boldsymbol{\xi}$ so this distinction is essentially cosmetic.

<u>The morphism spaces $\mathrm{Mor}(i, j)$</u>

Given $i, j \in \{1, 3\}$, let

$$\mathrm{Mor}(i, j) = \mathrm{Mor}^{\mathcal{A}_2} \left(\widehat{CFD}(a_i), \widehat{CFD}(a_j) \right) \tag{3.47}$$

be the space of $\mathcal{A}_2$-module homomorphisms $f : \widehat{CFD}(\mathcal{H}_i) \to \widehat{CFD}(\mathcal{H}_j)$. Then

$$\mathfrak{h}_2 = \mathrm{Mor}(1,1) \oplus \mathrm{Mor}(1,3) \oplus \mathrm{Mor}(3,1) \oplus \mathrm{Mor}(3,3). \tag{3.48}$$

We compute each summand separately.

$$\mathrm{Mor}^{\mathcal{A}_2}(1,1)$$

Since

$$\widehat{CFD}(a_1) = \boldsymbol{t} \circlearrowleft_{\rho_{1,3}+\rho_{4,7}} \tag{3.49}$$

and $\boldsymbol{t} = \iota_1\boldsymbol{t}$, a basic $\mathcal{A}_2$-module homomorphism $f : \widehat{CFD}(a_1) \to \widehat{CFD}(a_1)$ is determined by $f(\boldsymbol{t}) = \rho\boldsymbol{t}$ where $\rho \in \mathcal{A}_2$ satisfies $\rho = \iota_1\rho\iota_1$. One may verify that the possible values of ρ are ι_1, $\iota_1\rho_{1,3}\iota_1$, $\iota_1\rho_{4,7}\iota_1$, and $\rho^{4,7}_{1,3}$. Therefore, we have

$$\mathrm{Mor}(1,1) = \mathbb{F}\langle f^1_{1,1}, f^2_{1,1}, f^3_{1,1}, f^4_{1,1}\rangle, \tag{3.50}$$

where

$$\begin{aligned} f^1_{1,1}(\boldsymbol{t}) &= \boldsymbol{t} \\ f^2_{1,1}(\boldsymbol{t}) &= \iota_1\rho_{1,3}\iota_1\boldsymbol{t} \\ f^3_{1,1}(\boldsymbol{t}) &= \iota_1\rho_{4,7}\iota_1\boldsymbol{t} \\ f^4_{1,1}(\boldsymbol{t}) &= \rho^{4,7}_{1,3}\boldsymbol{t} \end{aligned} \tag{3.51}$$

and $\dim_{\mathbb{F}} H_*\mathrm{Mor}(1,1) = \dim_{\mathbb{F}} \widehat{HF}(\#^2 S^2 \times S^1) = 4$ so $\partial f = 0$ for every generator $f \in \mathrm{Mor}(1,1)$ and $H_*\mathrm{Mor}(1,1) = \mathbb{F}\langle [f^1_{1,1}], [f^2_{1,1}], [f^3_{1,1}], [f^4_{1,1}]\rangle$.

Mor(1, 3)

Here, we have

$$\widehat{CFD}(a_3) = \boldsymbol{w} \circlearrowright^{\rho_{1,3}+\rho_{6,8}} \tag{3.52}$$

with $\boldsymbol{w} = \iota_2 \boldsymbol{w}$ so a basic $\mathcal{A}_2$-module homomorphism $f : \widehat{CFD}(a_1) \to \widehat{CFD}(a_3)$ is determined by $f(\boldsymbol{t}) = \rho \boldsymbol{w}$ where $\rho = \iota_1 \rho \iota_2$. We then have that

$$\mathrm{Mor}(1,3) = \mathbb{F}\langle f^1_{1,3}, f^2_{1,3}, f^3_{1,3}, f^4_{1,3}, f^5_{1,3}, f^6_{1,3} \rangle \tag{3.53}$$

where

$$\begin{aligned} f^1_{1,3}(\boldsymbol{t}) &= \iota_1 \rho_{4,6} \iota_2 \boldsymbol{w} & f^4_{1,3}(\boldsymbol{t}) &= \rho^{4,6}_{1,3} \boldsymbol{w} \\ f^2_{1,3}(\boldsymbol{t}) &= \iota_1 \rho_{4,8} \iota_2 \boldsymbol{w} & f^5_{1,3}(\boldsymbol{t}) &= \rho^{4,8}_{1,3} \boldsymbol{w} \\ f^3_{1,3}(\boldsymbol{t}) &= \iota_1 \rho_{7,8} \iota_2 \boldsymbol{w} & f^6_{1,3}(\boldsymbol{t}) &= \rho^{7,8}_{1,3} \boldsymbol{w} \end{aligned} \tag{3.54}$$

and one may verify that

$$\begin{aligned} \partial f^1_{1,3} &= f^2_{1,3} & \partial f^4_{1,3} &= f^5_{1,3} \\ \partial f^2_{1,3} &= 0 & \partial f^5_{1,3} &= 0 \\ \partial f^3_{1,3} &= f^2_{1,3} & \partial f^6_{1,3} &= f^5_{1,3} \end{aligned} \tag{3.55}$$

so, as a chain complex, $\mathrm{Mor}(1,3)$ is given graphically by

$$\begin{array}{ccccc} f^1_{1,3} & \longrightarrow & f^2_{1,3} & \longleftarrow & f^3_{1,3} \\ & & & & \\ f^4_{1,3} & \longrightarrow & f^5_{1,3} & \longleftarrow & f^6_{1,3} \end{array}, \tag{3.56}$$

where an arrow $f^i_{1,3} \to f^j_{1,3}$ means that $f^j_{1,3}$ has coefficient 1 in $\partial f^i_{1,3}$. This complex has 2-dimensional homology with basis consisting of the classes $[f^1_{1,3} + f^3_{1,3}]$ and $[f^4_{1,3} + f^6_{1,3}]$.

$\mathrm{Mor}(3,1)$

Since

$$\widehat{CFD}(a_3) = \boldsymbol{w} \circlearrowleft_{\rho_{1,3}+\rho_{6,8}} \tag{3.57}$$

with $\iota_2 \boldsymbol{w} = \boldsymbol{w}$ and

$$\widehat{CFD}(a_1) = \boldsymbol{t} \circlearrowleft_{\rho_{1,3}+\rho_{4,7}} \tag{3.58}$$

with $\iota_1 \boldsymbol{t} = \boldsymbol{t}$, a basic morphism $f \in \mathrm{Mor}(3,1)$ is determined by $f(\boldsymbol{w}) = \rho \boldsymbol{t}$, where $\rho = \iota_2 \rho \iota_1$ so

$$\mathrm{Mor}(3,1) = \mathbb{F}\langle f^1_{3,1}, f^2_{3,1} \rangle \tag{3.59}$$

where

$$f^1_{3,1}(\boldsymbol{w}) = \iota_2\rho_{6,7}\iota_1\boldsymbol{t} \qquad f^2_{3,1}(\boldsymbol{w}) = \rho^{6,7}_{1,3}\boldsymbol{t} \tag{3.60}$$

and $\dim_{\mathbb{F}} H_*\mathrm{Mor}(3,1) = \dim_{\mathbb{F}} \widehat{HF}(S^2 \times S^1) = 2$ so it follows that $\partial f^1_{3,1} = \partial f^2_{3,1} = 0$ and $H_*\mathrm{Mor}(3,1) = \mathbb{F}\langle [f^1_{3,1}], [f^2_{3,1}]\rangle$.

$$\mathrm{Mor}(3,3)$$

Lastly, since

$$\widehat{CFD}(\mathcal{H}_3) = \boldsymbol{w} \circlearrowright^{\rho_{1,3}+\rho_{6,8}} \tag{3.61}$$

with $\iota_2\boldsymbol{w} = \boldsymbol{w}$, a basic morphism $f \in \mathrm{Mor}(3,3)$ is given by $f(\boldsymbol{w}) = \rho\boldsymbol{w}$, where $\rho = \iota_2\rho\iota_2$. Therefore,

$$\mathrm{Mor}(3,3) = \mathbb{F}\langle f^1_{3,3}, f^2_{3,3}, f^3_{3,3}, f^4_{3,3}\rangle, \tag{3.62}$$

where

$$\begin{aligned} f^1_{3,3}(\boldsymbol{w}) &= \boldsymbol{w} & f^3_{3,3}(\boldsymbol{w}) &= \iota_2\rho_{6,8}\iota_2\boldsymbol{w} \\ f^2_{3,3}(\boldsymbol{w}) &= \iota_2\rho_{1,3}\iota_2\boldsymbol{w} & f^4_{3,3}(\boldsymbol{w}) &= \rho^{6,8}_{1,3}\boldsymbol{w} \end{aligned} \tag{3.63}$$

and $\dim_{\mathbb{F}} H_*\mathrm{Mor}(3,3) = \dim_{\mathbb{F}} \widehat{HF}(\#^2 S^2 \times S^1) = 4$ so the differential on $\mathrm{Mor}(3,3)$ vanishes and $H_*\mathrm{Mor}(3,3) = \mathbb{F}\langle [f^1_{3,3}], [f^2_{3,3}], [f^3_{3,3}], [f^4_{3,3}]\rangle$.

$\mathfrak{h}_2$ and its homology

We now describe $\mathfrak{h}_2$ and its homology algebra $H_*\mathfrak{h}_2$ explicitly. One may verify using the above computations that $\mathfrak{h}_2$ has multiplication table with respect to the basis of basic morphisms as in Figure 42. The algebra $H_*\mathfrak{h}_2$ has a basis given by the homology classes $[f^1_{1,1}]$, $[f^2_{1,1}]$, $[f^3_{1,1}]$, $[f^4_{1,1}]$, $[f^1_{1,3}+f^3_{1,3}]$, $[f^4_{1,3}+f^6_{1,3}]$, $[f^1_{3,1}]$, $[f^2_{3,1}]$, $[f^1_{3,3}]$, $[f^2_{3,3}]$, $[f^3_{3,3}]$, and $[f^4_{3,3}]$. We define maps $p : \mathfrak{h}_2 \to H_*\mathfrak{h}_2$ and $\iota : H_*\mathfrak{h}_2 \to \mathfrak{h}_2$ by

$$\begin{aligned}
f^1_{1,1} &\mapsto [f^1_{1,1}] & f^5_{1,3} &\mapsto 0\\
f^2_{1,1} &\mapsto [f^2_{1,1}] & f^6_{1,3} &\mapsto 0\\
f^3_{1,1} &\mapsto [f^3_{1,1}] & f^1_{3,1} &\mapsto [f^1_{3,1}]\\
f^4_{1,1} &\mapsto [f^4_{1,1}] & f^2_{3,1} &\mapsto [f^2_{3,1}]\\
f^1_{1,3} &\mapsto [f^1_{1,3}+f^3_{1,3}] & f^1_{3,3} &\mapsto [f^1_{3,3}]\\
f^2_{1,3} &\mapsto 0 & f^2_{3,3} &\mapsto [f^2_{3,3}]\\
f^3_{1,3} &\mapsto 0 & f^3_{3,3} &\mapsto [f^3_{3,3}]\\
f^4_{1,3} &\mapsto [f^4_{1,3}+f^6_{1,3}] & f^4_{3,3} &\mapsto [f^4_{3,3}]
\end{aligned} \tag{3.64}$$

and

$$\begin{aligned}
[f^1_{1,1}] &\mapsto f^1_{1,1} & [f^1_{3,1}] &\mapsto f^1_{3,1}\\
[f^2_{1,1}] &\mapsto f^2_{1,1} & [f^2_{3,1}] &\mapsto f^2_{3,1}\\
[f^3_{1,1}] &\mapsto f^3_{1,1} & [f^1_{3,3}] &\mapsto f^1_{3,3}\\
[f^4_{1,1}] &\mapsto f^4_{1,1} & [f^2_{3,3}] &\mapsto f^2_{3,3}\\
[f^1_{1,3}+f^3_{1,3}] &\mapsto f^1_{1,3} & [f^3_{3,3}] &\mapsto f^3_{3,3}\\
[f^4_{1,3}+f^6_{1,3}] &\mapsto f^4_{1,3} & [f^4_{3,3}] &\mapsto f^4_{3,3},
\end{aligned} \tag{3.65}$$

respectively, so that $\iota p = \text{id}$ on $\langle f^2_{1,3}, f^3_{1,3}, f^5_{1,3}, f^6_{1,3}\rangle^\perp$ by construction. Now define $h : \mathfrak{h}_2 \to \mathfrak{h}_2$ by $h(f^2_{1,3}) = f^3_{1,3}$ and $h(f^5_{1,3}) = f^6_{1,3}$ and by zero on all other generators. Then $(\partial h + h\partial) = \text{id}$ on $\langle f^1_{1,3}, f^2_{1,3}, f^5_{1,3}, f^6_{1,3}\rangle$ and by zero on $\langle f^1_{1,3}, f^2_{1,3}, f^5_{1,3}, f^6_{1,3}\rangle^\perp$ so we have $\iota p = \text{id} + \partial h + h\partial$. Note that $p\iota = \text{id}$ by construction so p, ι, and h satisfy the hypotheses of the homological perturbation lemma. Using this retract, the homology algebra $H_*\mathfrak{h}_2$ has multiplication table as given in Figure 43 which we compare to the multiplication table for H_2 given in Figure 44 (note that we have used a nonstandard $\mathbb{F}$-basis for H_2). By inspection, the two tables coincide so identifying basis elements row-by-row provides us with an explicit algebra isomorphism $(H_*\mathfrak{h}_2, \overline{\circ}_{\text{op}}) \cong H_2$. Note that, under this isomorphism, the basis elements for $H_*\mathfrak{h}_2$ sitting in the summand $H_*\text{Mor}^{\mathcal{A}_2}(a_+, b_+)$ correspond to basis elements of H_2 sitting in the summand $\mathcal{C}_{Kh}(a^! b)$. One may verify directly that

$$m_3([f^1_{3,1}], [f^1_{1,3} + f^3_{1,3}], [f^3_{3,3}]) = [f^3_{3,3}]. \tag{3.66}$$

Note that $m_2([f^1_{1,3} + f^3_{1,3}], [f^3_{3,3}]) = m_2([f^1_{3,1}], [f^1_{1,3} + f^3_{1,3}]) = 0$ so the sequence of homology classes $[f^1_{3,1}], [f^1_{1,3} + f^3_{1,3}], [f^3_{3,3}] \in H_*\mathfrak{h}_2$ is Massey admissible in the sense of [LOT15, Definition 2.1.21]. One may then check that, for this sequence, the cycles $\xi_{i,j} = q_{j-i}(\alpha_{i+1}, \cdots, \alpha_j)$, where $\alpha_1 = [f^1_{3,1}]$, $\alpha_2 = [f^1_{1,3} + f^3_{1,3}]$, and $\alpha_3 = [f^3_{3,3}]$, are $\xi_{0,1} = f^1_{3,1}$, $\xi_{0,2} = 0$, $\xi_{1,3} = f^3_{1,3}$, and $\xi_{2,3} = f^3_{3,3}$ so the cycle $\sum_{0<k<3} \xi_{0,k}\xi_{k,3}$ representing $m_3([f^1_{3,1}], [f^3_{1,3}], [f^4_{1,1}])$ is $f^3_{3,3}$. Since this representing cycle is independent of the choices of the $\xi_{i,j}$ by [LOT15, Lemma 2.1.22], this shows that $\mathfrak{h}_2$ is not formal. This finishes the proof of Theorem 3.1.10.

row $\circ_{\text{op}}$ col	$f^1_{1,1}$	$f^2_{1,1}$	$f^3_{1,1}$	$f^4_{1,1}$	$f^1_{1,3}$	$f^2_{1,3}$	$f^3_{1,3}$	$f^4_{1,3}$	$f^5_{1,3}$	$f^6_{1,3}$	$f^1_{3,1}$	$f^2_{3,1}$	$f^1_{3,3}$	$f^2_{3,3}$	$f^3_{3,3}$	$f^4_{3,3}$
$f^1_{1,1}$	$f^1_{1,1}$	$f^2_{1,1}$	$f^3_{1,1}$	$f^4_{1,1}$							$f^1_{3,1}$	$f^2_{3,1}$				
$f^2_{1,1}$	$f^2_{1,1}$		$f^4_{1,1}$								$f^2_{3,1}$					
$f^3_{1,1}$	$f^3_{1,1}$	$f^4_{1,1}$														
$f^4_{1,1}$	$f^4_{1,1}$															
$f^1_{1,3}$	$f^1_{1,3}$	$f^4_{1,3}$														
$f^2_{1,3}$	$f^2_{1,3}$	$f^5_{1,3}$														
$f^3_{1,3}$	$f^3_{1,3}$	$f^6_{1,3}$	$f^2_{1,3}$	$f^5_{1,3}$							$f^3_{3,3}$	$f^4_{3,3}$				
$f^4_{1,3}$	$f^4_{1,3}$															
$f^5_{1,3}$	$f^5_{1,3}$															
$f^6_{1,3}$	$f^6_{1,3}$		$f^5_{1,3}$								$f^4_{3,3}$					
$f^1_{3,1}$					$f^3_{1,1}$			$f^4_{1,1}$					$f^1_{3,1}$	$f^2_{3,1}$		
$f^2_{3,1}$					$f^4_{1,1}$								$f^2_{3,1}$			
$f^1_{3,3}$					$f^1_{1,3}$	$f^2_{1,3}$	$f^3_{1,3}$	$f^4_{1,3}$	$f^5_{1,3}$	$f^6_{1,3}$			$f^1_{3,3}$	$f^2_{3,3}$	$f^3_{3,3}$	$f^4_{3,3}$
$f^2_{3,3}$					$f^4_{1,3}$	$f^5_{1,3}$	$f^6_{1,3}$						$f^2_{3,3}$		$f^4_{3,3}$	
$f^3_{3,3}$					$f^2_{1,3}$			$f^5_{1,3}$					$f^3_{3,3}$	$f^4_{3,3}$		
$f^4_{3,3}$					$f^5_{1,3}$								$f^4_{3,3}$			

FIGURE 42. A multiplication table for $\mathfrak{h}_2$ given by the basis of basic morphisms. Entries containing 0 are left blank for readability.

$[\text{row}]\overline{\circ}_{\text{op}}[\text{col}]$	$[f^1_{1,1}]$	$[f^2_{1,1}]$	$[f^3_{1,1}]$	$[f^4_{1,1}]$	$[f^1_{1,3}+f^3_{1,3}]$	$[f^4_{1,3}+f^6_{1,3}]$	$[f^1_{3,1}]$	$[f^2_{3,1}]$	$[f^1_{3,3}]$	$[f^2_{3,3}]$	$[f^3_{3,3}]$	$[f^4_{3,3}]$
$[f^1_{1,1}]$	$[f^1_{1,1}]$	$[f^2_{1,1}]$	$[f^3_{1,1}]$	$[f^4_{1,1}]$	$[f^1_{1,3}+f^3_{1,3}]$	$[f^4_{1,3}+f^6_{1,3}]$						
$[f^2_{1,1}]$	$[f^2_{1,1}]$		$[f^4_{1,1}]$		$[f^4_{1,3}+f^6_{1,3}]$							
$[f^3_{1,1}]$	$[f^3_{1,1}]$	$[f^4_{1,1}]$										
$[f^4_{1,1}]$	$[f^4_{1,1}]$											
$[f^1_{1,3}+f^3_{1,3}]$							$[f^3_{1,1}]$	$[f^4_{1,1}]$	$[f^1_{1,3}+f^3_{1,3}]$	$[f^4_{1,3}+f^6_{1,3}]$		
$[f^4_{1,3}+f^6_{1,3}]$							$[f^4_{1,1}]$		$[f^4_{1,3}+f^6_{1,3}]$			
$[f^1_{3,1}]$	$[f^1_{3,1}]$	$[f^2_{3,1}]$			$[f^3_{3,3}]$	$[f^4_{3,3}]$						
$[f^2_{3,1}]$	$[f^2_{3,1}]$				$[f^4_{3,3}]$							
$[f^1_{3,3}]$							$[f^1_{3,1}]$	$[f^2_{3,1}]$	$[f^1_{3,3}]$	$[f^2_{3,3}]$	$[f^3_{3,3}]$	$[f^4_{3,3}]$
$[f^2_{3,3}]$							$[f^2_{3,1}]$		$[f^2_{3,3}]$		$[f^4_{3,3}]$	
$[f^3_{3,3}]$									$[f^3_{3,3}]$	$[f^4_{3,3}]$		
$[f^4_{3,3}]$									$[f^4_{3,3}]$			

FIGURE 43. The multiplication table for $H_*\mathfrak{h}_2$ given by the retract defined above.

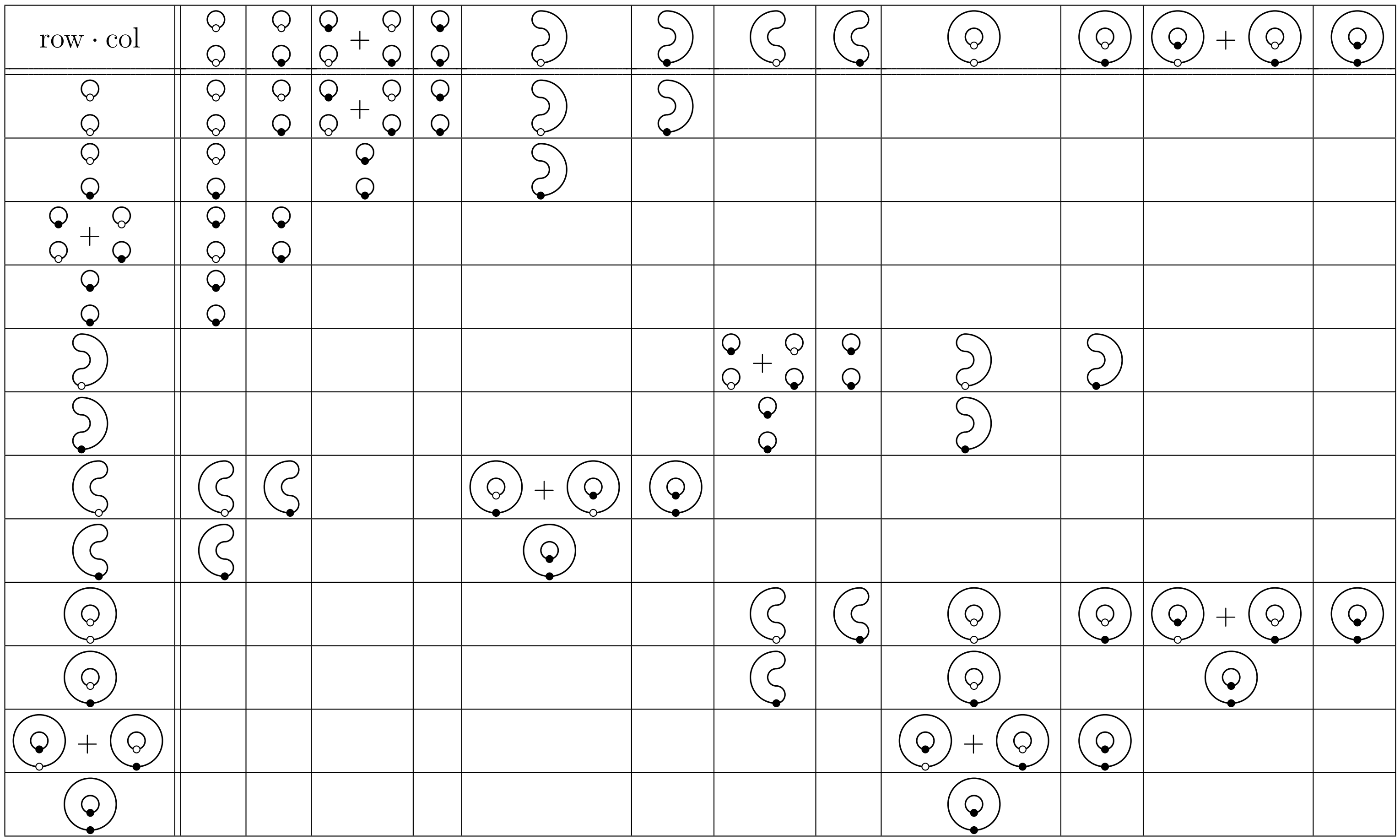


FIGURE 44. A multiplication table for H_2. Note that we are using a nonstandard basis for H_2.

3.3 Splitting results for Khovanov's arc algebras in characteristic 2

In this section, we prove that if R is a ring of characteristic 2, then Khovanov's arc algebra H_n over R on $2n$ points admits a tensor product decomposition $H_n \cong \widetilde{H}_n \otimes_R R[x]/(x^2)$ as algebras, where $\widetilde{H}_n$ is the *reduced arc algebra* over R on $2n$ points. We also prove a similar result for Khovanov's bimodules for tangles and show that no such splitting exists over $\mathbb{Z}$.

For now, fix an *arbitrary* base ring R. Recall that the *arc algebra* H_n over R is the unital associative graded R-algebra

$$H_n = q^{-n} \bigoplus_{a,b \in \mathfrak{C}_n} \mathcal{C}_{Kh}(a^! b), \tag{3.67}$$

where $a^!$ is the result of flipping a across the vertical axis, $a^! b$ is the result of gluing $a^!$ and b along their common endpoints, and $\mathcal{C}_{Kh} : \mathrm{Cob}^{1+1} \to R - \mathrm{Mod}$ is Khovanov's TQFT whose value on a single circle is given by

$$\mathcal{C}_{Kh}(\bigcirc) = V := R[x]/(x^2) \tag{3.68}$$

as a commutative Frobenius algebra with comultiplication defined on generators by $\Delta(1) = 1 \otimes x + x \otimes 1$ and $\Delta(x) = x \otimes x$. The elements 1 and x are endowed with an integer-valued *quantum grading* by taking $\mathrm{gr}_q(1) = 1$ and $\mathrm{gr}_q(x) = -1$ and the formal power q^{-n} in line (3.67) denotes a shift in this grading by $-n$. We take the convention that $H_0 = R$. The algebra structure on H_n is given by applying the functor $\mathcal{C}_{Kh}$ to the minimal saddle cobordisms $\Sigma_{a,b,c} : a^! b \sqcup b^! c \to a^! c$. More precisely, if $\boldsymbol{v}$ and $\boldsymbol{v}'$ are labelings of the components of $a^! b$ and $b^! c$, respectively, then the product $(a^!b, \boldsymbol{v})(b^!c, \boldsymbol{v}')$ is given by $Kh(\Sigma_{a,b,c})(\boldsymbol{v} \sqcup \boldsymbol{v}')$ and products of the form $(a^!b, \boldsymbol{v})(c^!d, \boldsymbol{v}')$ for $c \neq b$ vanish.

Definition 32. Given a crossingless matching $a \in \mathfrak{C}_n$, we distinguish the bottom-most of its $2n$ endpoints as a marked point. The *reduced arc algebra* over R on $2n$-points is then the associative graded R-algebra $\widetilde{H}_n$ defined by

$$\widetilde{H}_n = q^{-n} \bigoplus_{a,b \in \mathfrak{C}_n} \widetilde{\mathcal{C}}_{Kh}(a^!b). \tag{3.69}$$

Here, $\widetilde{\mathcal{C}}_{Kh}$ denotes the reduced Khovanov complex given by the choice of basepoint as the quotient complex in which the marked component of every generator is labeled with a 1 and the entire complex is endowed with a quantum grading shift of -1.

Lemma 3.3.1. *Let $\widetilde{m} : \widetilde{H}_n \otimes \widetilde{H}_n \to \widetilde{H}_n$ be the map induced by multiplication on H_n. Then $(\widetilde{H}_n, \widetilde{m})$ is a graded associative unital algebra.*

Proof. It is straightforward to see that the subgroup $I_x \subset H_n$ generated by elements in which the marked component is labeled by x is a homogeneous two-sided ideal. The statement then follows from the fact that $\widetilde{H}_n = H_n/I_x$. □

3.4 The Splitting Theorem

Given crossingless matchings $a, b \in \mathfrak{C}_n$, let $\kappa_0 \in \pi_0(a^!b)$ be the marked component of $a^!b$ and define $\pi_*(a^!b) = \pi_0(a^!b) \setminus \{\kappa_0\}$. We define a linear map $\lambda : \widetilde{H}_n \otimes V \to H_n$ as follows. Let

$$\widetilde{\mathcal{B}}_n = \bigcup_{a,b \in \mathfrak{C}_n} \left\{ (a^!b, \boldsymbol{v}) | \boldsymbol{v} \in \{1, x\}^{\pi_*(a^!b)} \right\} \tag{3.70}$$

be the "standard" basis for $\widetilde{H}_n$ consisting of two crossingless matchings $a, b \in \mathfrak{C}_n$ and a labelling $\boldsymbol{v} : \pi_*(a^!b) \to \{1, x\}$ of the unmarked components of $a^!b$ by either 1 or x. The marked component of a generator of $\widetilde{H}_n$ will always implicitly be labeled by 1 but, in light of the following, it will be convenient to think of the labeling restricted to unmarked components only. Given a basis element $(a^!b, \boldsymbol{v}) \in \widetilde{\mathcal{B}}_n$ and $s \in \{1, x\}$, let $(a^!b, \boldsymbol{v})_s \in H_n$ be the result of extending the labeling $\boldsymbol{v}$ to all of $\pi_0(a^!b)$ by taking $\boldsymbol{v}(\kappa_0) = s$. Now define

$$\mathfrak{X}(a^!b, \boldsymbol{v}) = \{\kappa \in \pi_*(a^!b) | \boldsymbol{v}(\kappa) = x\} \tag{3.71}$$

and, for a component $\kappa \in \mathfrak{X}(a^!b, \boldsymbol{v})$, define $(a^!b, \boldsymbol{v}_\kappa) \in H_n$ by taking $\boldsymbol{v}_\kappa(\kappa_0) = x$, $\boldsymbol{v}_\kappa(\kappa) = 1$, and $\boldsymbol{v}_\kappa(\kappa') = \boldsymbol{v}(\kappa')$ for all other components κ'. In other words, $(a^!b, \boldsymbol{v}_\kappa)$ is the result of labeling the marked component by x and relabeling κ with 1. We then define λ on basis elements $(a^!b, \boldsymbol{v}) \otimes s \in \widetilde{\mathcal{B}}_n \otimes \{1, x\}$ by

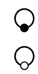
$$\lambda((a^!b, \boldsymbol{v}) \otimes s) = \begin{cases} (a^!b, \boldsymbol{v})_x & \text{if } s = x \\ (a^!b, \boldsymbol{v})_1 + \sum\limits_{\kappa \in \mathfrak{X}(a^!b, \boldsymbol{v})} (a^!b, \boldsymbol{v}_\kappa) & \text{otherwise.} \end{cases} \tag{3.72}$$

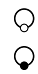
Example 11. Letting hollow and solid dots represent the labels of components via the convention $\circ = 1$ and $\bullet = x$, if

$$(a^!b, \boldsymbol{v}) = $$  (3.73)

i.e. $a = b$ is the first of the crossingless matchings in $\mathfrak{C}_2$ depicted in Figure 8 and $\boldsymbol{v} : \pi_*(a^! b) \to \{1, x\}$ is the map taking the unmarked component of $a^! b$ to x, then

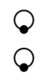
$$\lambda((a^! b, \boldsymbol{v}) \otimes 1) = \quad + \quad \tag{3.74}$$





and

$$\lambda((a^! b, \boldsymbol{v}) \otimes x) = \quad . \tag{3.75}$$



Lemma 3.4.1. *λ is a graded R-linear isomorphism.*

Proof. Note that the set

$$\mathcal{B}_n = \bigcup_{a,b \in \mathfrak{C}_n} \left\{ \lambda((a^! b, \boldsymbol{v}) \otimes s) | \boldsymbol{v} \in \{1, x\}^{\pi_*(a^! b)}, s \in \{1, x\} \right\} \tag{3.76}$$

forms an R-basis for H_n since there is a block lower-triangular matrix of the form

$$\begin{pmatrix} \mathrm{id} & 0 \\ B & \mathrm{id} \end{pmatrix}, \tag{3.77}$$

where B is a square matrix with entries in $\{0, 1\}$, taking the standard basis

$$\mathcal{B}_n^{\mathrm{std}} = \bigcup_{a,b \in \mathfrak{C}_n} \left\{ (a^! b, \boldsymbol{v}) | \boldsymbol{v} \in \{1, x\}^{\pi_0(a^! b)} \right\} \tag{3.78}$$

for H_n to $\mathcal{B}_n$. Here, we order $\mathcal{B}_n^{\mathrm{std}}$ so that those basis elements with $\boldsymbol{v}(\kappa_0) = 1$ appear first in the ordering. Now we have $\mathrm{rk}_R \widetilde{H}_n \otimes_R V = \mathrm{rk}_R H_n$ so λ is automatically an R-linear isomorphism since commutative rings have the invariant basis number property and both $\widetilde{H}_n \otimes_R V$ and H_n are free as R-modules. Note that $\mathrm{gr}_q((a^! b, \boldsymbol{v}) \otimes 1) = \mathrm{gr}_q((a^! b, \boldsymbol{v})_1) = \mathrm{gr}_q((a^! b, \boldsymbol{v}_\kappa))$ for any $\kappa \in \mathfrak{X}(a^! b, \boldsymbol{v})$ since each of these has the same number of tensor factors of 1 and x. For the same reason, we have $\mathrm{gr}_q((a^!b, \boldsymbol{v}) \otimes x) = \mathrm{gr}_q((a^!b, \boldsymbol{v})_x)$ so λ preserves quantum gradings and is, therefore, a graded isomorphism. $\square$

Theorem 3.4.2. *If R is a ring of characteristic 2, then λ is a graded R-algebra isomorphism.*

Proof. We have already shown that λ is a graded linear isomorphism so it suffices to show that it is multiplicative, i.e. that

$$\lambda((a^!b, \boldsymbol{v}) \otimes s_1)\lambda((b^!c, \boldsymbol{v}') \otimes s_2) = \lambda((a^!b, \boldsymbol{v})(b^!c, \boldsymbol{v}') \otimes s_1 s_2). \tag{3.79}$$

We do this by dividing into cases — note that we do not need to consider products of the form $(a^!b, \boldsymbol{v})(c^!d, \boldsymbol{v}')$ for $b \neq c$ since these are always zero in H_n and, therefore, also in $\widetilde{H}_n$.

Case 1: $s_1 = s_2 = x$. By far-commutation of saddles, we may always arrange for the marked components to merge first. Since $x^2 = 0$, we have

$$(a^!b, \boldsymbol{v})_x(b^!c, \boldsymbol{v}')_x = 0, \tag{3.80}$$

i.e. $0 = \lambda(((a^!b, \boldsymbol{v}) \otimes x)((b^!c, \boldsymbol{v}') \otimes x)) = \lambda((a^!b, \boldsymbol{v}) \otimes x)\lambda((b^!c, \boldsymbol{v}') \otimes x)$ for any basis elements $(a^!b, \boldsymbol{v}), (b^!c, \boldsymbol{v}') \in \widetilde{\mathcal{B}}_n$.

Case 2: $s_1 = 1$ and $s_2 = x$. Next, consider $\lambda((a^!b, \boldsymbol{v}) \otimes 1)\lambda((b^!c, \boldsymbol{v}') \otimes x)$: this is equal to $(a^!b, \boldsymbol{v})_1(b^!c, \boldsymbol{v}')_x$ since $(a^!b, \boldsymbol{v}_\kappa)(b^!c, \boldsymbol{v}')_x = 0$ for any $\kappa \in \mathfrak{X}(a^!b, \boldsymbol{v})$ as the marked components of both elements in this product are labeled x so their merger creates a label of $x^2 = 0$. Now suppose that the product $(a^!b, \boldsymbol{v})(b^!c, \boldsymbol{v}')$ in $\widetilde{H}_n$ is given as a linear combination of elements of the basis $\widetilde{\mathcal{B}}_n$ by

$$(a^!b, \boldsymbol{v})(b^!c, \boldsymbol{v}') = \sum_i (a^!c, \boldsymbol{v}_i''). \tag{3.81}$$

We claim that

$$(a^!b, \boldsymbol{v})_1(b^!c, \boldsymbol{v}')_x = \sum_i (a^!c, \boldsymbol{v}_i'')_x = \sum_i \lambda((a^!c; \boldsymbol{v}_i'') \otimes x). \tag{3.82}$$

Note that, under the saddle cobordism $a^!b \sqcup b^!c \to a^!c$, if the marked components merge and do not subsequently split, then this is true automatically. Otherwise, in $\widetilde{H}_n$, any splittings of the marked component produces some number of new components in the summands $(a^!c, \boldsymbol{v}_i'')$, each of which is labeled x. In H_n, after the first merger occuring in the saddle cobordism, the marked component of $(a^!b, \boldsymbol{v})_1(b^!c, \boldsymbol{v}')_x$ becomes labeled by x and any subsequent splittings produce the same new components as before, each of which is again labeled by x since we have $\Delta(x) = x \otimes x$. Therefore, we have

$$\lambda((a^!b, \boldsymbol{v}) \otimes 1)\lambda((b^!c, \boldsymbol{v}') \otimes x) = \lambda(((a^!b, \boldsymbol{v}) \otimes 1)((b^!c, \boldsymbol{v}') \otimes x)), \tag{3.83}$$

as desired.

Case 3: $s_1 = x$ and $s_2 = 1$. It follows from the previous case that

$$\lambda((a^!b, \boldsymbol{v}) \otimes x)\lambda((b^!c, \boldsymbol{v}') \otimes 1) = \lambda(((a^!b, \boldsymbol{v}) \otimes x)((b^!c, \boldsymbol{v}') \otimes 1)). \tag{3.84}$$

To see this, note that the algebra anti-automorphisms $\overline{(-)} : \widetilde{H}_n \otimes V \to \widetilde{H}_n \otimes V$ and $\overline{(-)} : H_n \to H_n$ given in both cases by $\overline{(a^!b, \boldsymbol{v})} = (b^!a, \boldsymbol{v})$ satisfy $\overline{\lambda((a^!b, \boldsymbol{v}) \otimes s)} =$

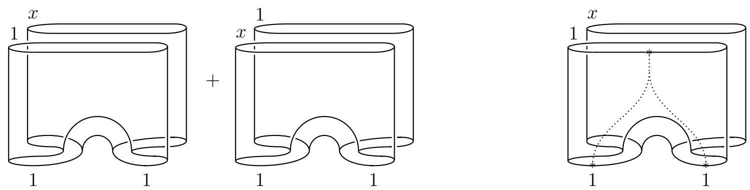


FIGURE 45. An example of $Kh(\Sigma)(\boldsymbol{v})$ (left) and $\widetilde{Kh}(\Sigma)(\boldsymbol{v})$ (right) in which the two differ.

$\lambda(\overline{(a^!b, \boldsymbol{v})} \otimes s)$ by construction. It is then straightforward to show that

$$\overline{\lambda((a^!b, \boldsymbol{v}) \otimes x)\lambda((b^!c, \boldsymbol{v}') \otimes 1)} = \overline{\lambda(((a^!b, \boldsymbol{v}) \otimes x)((b^!c, \boldsymbol{v}') \otimes 1))} \tag{3.85}$$

by a direct computation using Case 2.

Case 4: $s_1 = s_2 = 1$. Let $\Sigma : c \to c'$ be a connected, orientable, 2-dimensional cobordism, where c and c' are disjoint unions of planar circles. Recall that if $\boldsymbol{v}$ and $\boldsymbol{w}$ are labelings of c and c' by $\{1, x\}$ then $\boldsymbol{w}$ occurs as a summand in $Kh(\Sigma)(\boldsymbol{v})$ if and only if $g(\Sigma) = 0$ and $\#_x\boldsymbol{v} + \#_1\boldsymbol{w} = 1$. Here, for a labeling $\boldsymbol{u}$, the quantities $\#_1\boldsymbol{u}$ and $\#_x\boldsymbol{u}$ are the number of components labeled 1 and x by $\boldsymbol{u}$. The same holds true for $\widetilde{Kh}(\Sigma)(\boldsymbol{v})$ subject to the constraint that only those $\boldsymbol{v}$ and $\boldsymbol{w}$ which label the marked component 1 are permitted (cf. Figure 45). Now suppose we are given generators $(a^!b, \boldsymbol{v})$ and $(b^!c, \boldsymbol{v}')$ of $\widetilde{H}_n$ and consider the minimal saddle cobordism $\Sigma : a^!b \sqcup b^!c \to a^!c$. We claim that $\lambda(\widetilde{Kh}(\Sigma)(\boldsymbol{v} \sqcup \boldsymbol{v}') \otimes 1) = Kh(\Sigma)(\lambda \otimes \lambda((\boldsymbol{v} \sqcup \boldsymbol{v}') \otimes 1))$.

Subcase 1. We first consider the case that $\boldsymbol{v} \sqcup \boldsymbol{v}'$ labels all of the incoming circles of the component Σ_* of Σ which contains the marked incoming circles by 1. In the reduced product $\widetilde{Kh}(\Sigma)(\boldsymbol{v} \sqcup \boldsymbol{v}')$, each $\boldsymbol{w}$ occurring as a summand labels the marked outgoing circle by 1 and any other outgoing circles of Σ_* by x. The unreduced product $Kh(\Sigma)(\boldsymbol{v} \sqcup \boldsymbol{v}')$ consists of these terms plus terms in which the marked outgoing circle is labeled x and exactly one of the remaining outgoing circles of Σ_* is labeled 1. The summand of $\lambda(\widetilde{Kh}(\Sigma)(\boldsymbol{v} \sqcup \boldsymbol{v}') \otimes 1)$ consisting of $\widetilde{Kh}(\Sigma)(\boldsymbol{v} \sqcup \boldsymbol{v}')$ and those terms obtained only by summing over the x-labeled outgoing circles of Σ_* is precisely $Kh(\Sigma)(\boldsymbol{v} \sqcup \boldsymbol{v}')$. It thus suffices to show that the remaining terms either come in cancelling pairs or come from swapping the label on a marked incoming circle of Σ with that of an x-labeled circle. Consider a connected component Σ_1 of $\Sigma \setminus \Sigma_*$. If the incoming circles of Σ_1 are all labeled 1 and Σ_1 has ℓ outgoing circles, then any labeling $\boldsymbol{w}_1$ of these circles occuring as a sublabeling of a term in $\widetilde{Kh}(\Sigma)(\boldsymbol{v} \sqcup \boldsymbol{v}')$ labels $\ell - 1$ of them by x and one of them by 1. Moreover, if $\boldsymbol{w}$ is a summand of $\widetilde{Kh}(\Sigma)(\boldsymbol{v} \sqcup \boldsymbol{v}')$ and $\boldsymbol{w}_1$ occurs as a sublabeling of $\boldsymbol{w}$, then every possible labeling $\boldsymbol{w}'$ obtained from $\boldsymbol{w}$ by permutating $\boldsymbol{w}_1$ occurs exactly once as a summand of $\widetilde{Kh}(\Sigma)(\boldsymbol{v} \sqcup \boldsymbol{v}')$. Now, for any labeling $\boldsymbol{w}$ and sub-labeling $\boldsymbol{w}_1$ of the outgoing circles of Σ_1 and any choice of x-labeled component κ coming from $\boldsymbol{w}_1$, there exists a $\boldsymbol{w}'$ and $\boldsymbol{w}'_1$ such that $\boldsymbol{w}$ and $\boldsymbol{w}'$ agree away from $\boldsymbol{w}_1$ and $\boldsymbol{w}'_1$ and a choice of x-labeled component κ' coming from $\boldsymbol{w}'_1$ such that the labelings $\boldsymbol{w}_\kappa$ and $\boldsymbol{w}'_{\kappa'}$ agree. All such choices come in pairs so the summands of $\lambda(\widetilde{Kh}(\Sigma)(\boldsymbol{v} \sqcup \boldsymbol{v}') \otimes 1)$ coming from summing over the x-labeled outgoing circles of Σ_1 cancel.

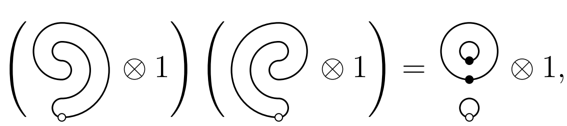
Note that if more than one incoming circle of Σ_1 is labeled x, then we have $\widetilde{Kh}(\Sigma)(\boldsymbol{v} \sqcup \boldsymbol{v}') = 0$. On the other hand, we also have $Kh(\Sigma)(\lambda \otimes \lambda((\boldsymbol{v} \sqcup \boldsymbol{v}') \otimes 1)) = 0$ since either more than two of the incoming circles is labeled x — in which case applying $Kh(\Sigma)$ to every term of $\lambda \otimes \lambda((\boldsymbol{v} \sqcup \boldsymbol{v}') \otimes 1)$ yields zero — or exactly two are, call them κ and κ'. In the latter case, the terms $Kh(\Sigma)((\boldsymbol{v} \sqcup \boldsymbol{v}')_\kappa)$ and $Kh(\Sigma)((\boldsymbol{v} \sqcup \boldsymbol{v}')_{\kappa'})$ agree and, hence, cancel modulo 2. If exactly one of the incoming circles κ_0 of Σ_1 is labeled by x, then every outgoing circle of Σ_1 is also labeled x.

If, as before, Σ_1 has ℓ outgoing circles $\kappa_1, \ldots, \kappa_\ell$, then the summand $\widetilde{Kh}(\Sigma)(\boldsymbol{v} \sqcup \boldsymbol{v}')_{\kappa_1} + \cdots + \widetilde{Kh}(\Sigma)(\boldsymbol{v} \sqcup \boldsymbol{v}')_{\kappa_\ell}$ of $\lambda(\widetilde{Kh}(\Sigma)(\boldsymbol{v} \sqcup \boldsymbol{v}') \otimes 1)$ coincides precisely with the summand $Kh(\Sigma)((\boldsymbol{v} \sqcup \boldsymbol{v}')_{\kappa_0})$ of $Kh(\Sigma)(\lambda \otimes \lambda((\boldsymbol{v} \sqcup \boldsymbol{v}') \otimes 1))$. Therefore, we have $\lambda(\widetilde{Kh}(\Sigma)(\boldsymbol{v} \sqcup \boldsymbol{v}') \otimes 1) = Kh(\Sigma)(\lambda \otimes \lambda((\boldsymbol{v} \sqcup \boldsymbol{v}') \otimes 1))$.

Subcase 2. If at least one of the incoming circles of the component Σ_* is labeled x, then $\widetilde{Kh}(\Sigma)(\boldsymbol{v} \sqcup \boldsymbol{v}') \otimes 1$ necessarily vanishes. If more than one of these incoming circles is labeled x, then, as before, every term of $\lambda((\boldsymbol{v} \sqcup \boldsymbol{v}') \otimes 1)$ necessarily also labels at least two of the incoming circles on this component by x so we also have that $Kh(\Sigma)(\lambda \otimes \lambda((\boldsymbol{v} \sqcup \boldsymbol{v}') \otimes 1)) = 0$. If exactly one incoming circle κ_0 of Σ_* is labeled x — assume for simplicity that this label comes from $\boldsymbol{v}$ — then the terms of $\lambda \otimes \lambda((\boldsymbol{v} \sqcup \boldsymbol{v}') \otimes 1)$ consist of $\boldsymbol{v} \sqcup \boldsymbol{v}'$, $\boldsymbol{v}_{\kappa_0} \sqcup \boldsymbol{v}'$, and terms of the form $\boldsymbol{v}_\kappa \sqcup \boldsymbol{v}'$, $\boldsymbol{v} \sqcup \boldsymbol{v}'_{\kappa'}$, and $\boldsymbol{v}_\kappa \sqcup \boldsymbol{v}'_{\kappa'}$ where κ and κ' are incoming circles of a component of $\Sigma \setminus \Sigma_*$ labeled x by $\boldsymbol{v}$ and $\boldsymbol{v}'$, respectively. In $Kh(\Sigma)(\lambda((\boldsymbol{v} \sqcup \boldsymbol{v}') \otimes 1))$, the first two of these terms contribute two identical and hence cancelling terms since we are working in characteristic 2 and the remaining terms contribute 0 since Σ merges at least two x-labeled circles in those cases.

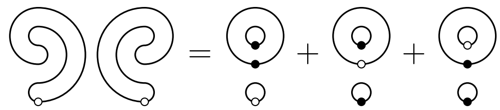
$\square$

Example 12. Using the same convention for hollow and filled dots as before, in $\widetilde{H}_3 \otimes V$, we have



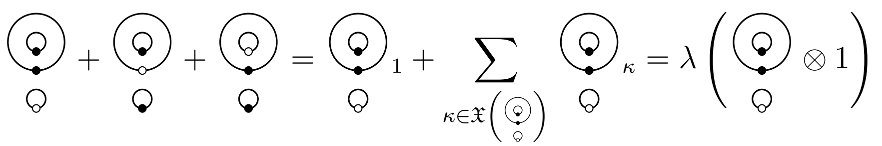
(3.86)

while in H_3, we have



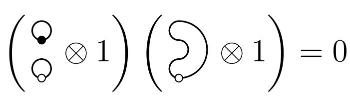
(3.87)

and



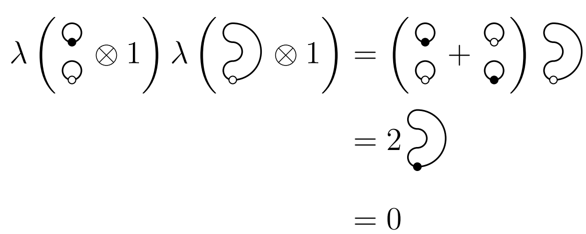
(3.88)

Example 13. In $\widetilde{H}_2 \otimes V$, we have



(3.89)

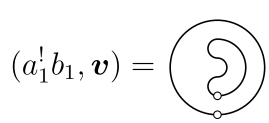
while



(3.90)

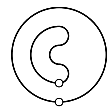
modulo 2, which shows that λ cannot possibly be a multiplicative map in characteristics other than 2.

Example 14. We consider two more examples to exhibit some of the phenomena that can occur when comparing $m \circ (\lambda \otimes \lambda)$ and $\lambda \circ \widetilde{m}$ in characteristic 2. Suppose that

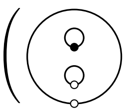
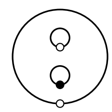


(3.91)

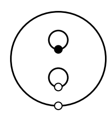
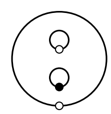
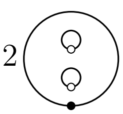
and

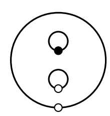
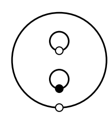
$$(b_1^! c_1, \boldsymbol{v}') =$$  , (3.92)

then

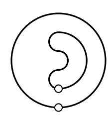
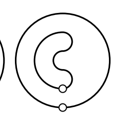
$$((a_1^! b_1, \boldsymbol{v}) \otimes 1)((b_1^! c_1, \boldsymbol{v}') \otimes 1) = \Big($$  $+$  $\Big) \otimes 1$ (3.93)

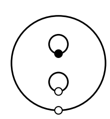
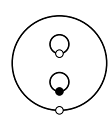
so

$$\lambda(((a_1^! b_1, \boldsymbol{v}) \otimes 1)((b_1^! c_1, \boldsymbol{v}') \otimes 1)) =$$  $+$  $+ 2$ 

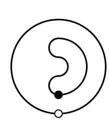
$=$  $+$  (3.94)

modulo 2. On the other hand, we have

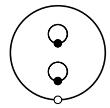
$$\lambda((a_1^! b_1, \boldsymbol{v}) \otimes 1)\lambda((b_1^! c_1, \boldsymbol{v}') \otimes 1) =$$  

$=$  $+$  . (3.95)

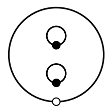
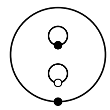
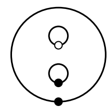
This is an instance of the first part of Case 4, Subcase 1, in the proof of the main theorem. Similarly, if

$$(a_2^! b_2, \boldsymbol{w}) =$$  (3.96)

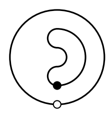
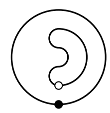
and $(b_2^! c_2, \boldsymbol{w}') = (b_1^! c_1, \boldsymbol{v}')$, then we have

$$((a_2^! b_2, \boldsymbol{w}) \otimes 1)((b_2^! c_2, \boldsymbol{w}') \otimes 1) =$$  $$\otimes 1 \quad (3.97)$$

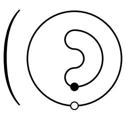
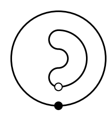
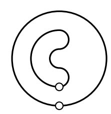
so

$$\lambda(((a_2^! b_2, \boldsymbol{w}) \otimes 1)((b_2^! c_2, \boldsymbol{w}') \otimes 1)) =$$  $+$  $+$  (3.98)

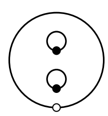
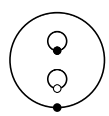
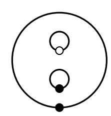
while

$$\lambda((a_2^! b_2, \boldsymbol{w}) \otimes 1) =$$  $+$  (3.99)

so

$$\lambda((a_2^! b_2, \boldsymbol{w}) \otimes 1)\lambda((b_2^! c_2, \boldsymbol{w}') \otimes 1) =$$ $\Big($  $+$  $\Big)$ 

(3.100)

$=$  $+$  $+$ .

This is an instance of the second part of Case 4, Subcase 1.

<u>Bimodules of planar tangles</u>

Now suppose that T is a planar (crossingless) $(2m, 2n)$-tangle diagram and let

$$\mathcal{C}_{Kh}(T) = q^{-n} \bigoplus_{a \in \mathfrak{C}_m, b \in \mathfrak{C}_n} \mathcal{C}_{Kh}(a^! T b) \tag{3.101}$$

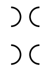
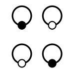
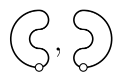
be the associated (H_m, H_n)-bimodule. Choose either the left bottom-most endpoint or the right bottom-most endpoint of T as a marked point for every $a^!Tb$ and denote the corresponding reduced bimodules by $\widetilde{\mathcal{C}}^L_{Kh}(T)$ and $\widetilde{\mathcal{C}}^R_{Kh}(T)$, respectively. We define a map $\lambda^L : \widetilde{\mathcal{C}}^L_{Kh}(T) \otimes V \to \mathcal{C}_{Kh}(T)$ as follows: given a labeling $\boldsymbol{v} : \pi_*(a^!Tb) \to \{1, x\}$, let $\mathfrak{X}(a^!Tb, \boldsymbol{v})$ denote the set of all components of $a^!Tb$ labeled x by $\boldsymbol{v}$. We then define

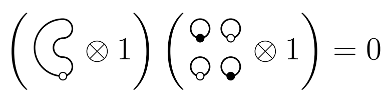
$$\lambda^L((a^!Tb, \boldsymbol{v}) \otimes s) = \begin{cases} (a^!Tb, \boldsymbol{v})_x & \text{if } s = x \\ (a^!Tb, \boldsymbol{v})_1 + \sum_{\kappa \in \mathfrak{X}(a^!Tb, \boldsymbol{v})} (a^!Tb, \boldsymbol{v}_\kappa) & \text{otherwise,} \end{cases} \tag{3.102}$$

where, as before, $(a^!Tb, \boldsymbol{v})_s$ and $(a^!Tb, \boldsymbol{v}_\kappa)$ are the elements of $\mathcal{C}_{Kh}(T)$ obtained by labeling the marked component by s and by swapping the label of κ and the marked component, respectively. We define λ^R similarly. Note that if the bottom left-most and bottom right-most endpoints of T are on the same connected component, then the two maps coincide.

Proposition 3.4.3. *If R is a ring of characteristic 2, then λ^L (resp. λ^R) is a graded linear isomorphism intertwining the left $\widetilde{H}_m \otimes V$- and H_m-module (resp. right $\widetilde{H}_n \otimes V$- and H_n-module) structures on $\widetilde{\mathcal{C}}^L_{Kh}(T) \otimes V$ (resp. $\widetilde{\mathcal{C}}^R_{Kh}(T) \otimes V$) and $\mathcal{C}_{Kh}(T)$. However, they are not bimodule isomorphisms in general.*

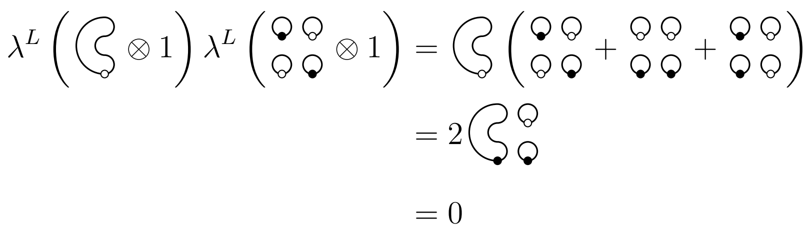
Proof. The proof for both is essentially identical to the proof of Theorem 3.4.2. The example that follows shows that λ^L and λ^R need not be bimodule isomorphisms when they are not equal. □

Example 15. Let $T =$  and consider  $\in \widetilde{\mathcal{C}}^L_{Kh}(T)$. Consider the left- and right-actions of the elements  $\in \widetilde{H}_2$: we have that



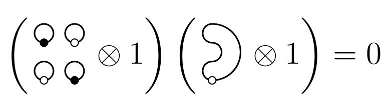
(3.103)

and



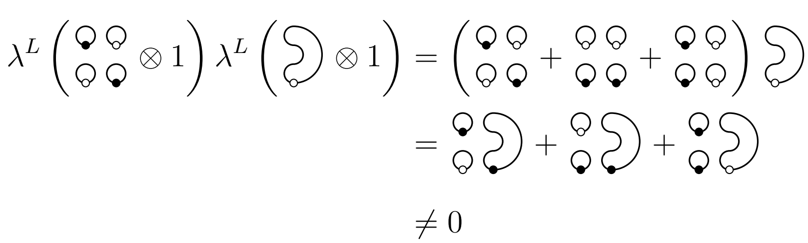
(3.104)

modulo 2, as expected, and, on the other hand, we have



(3.105)

while

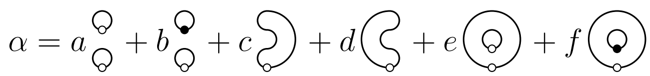


(3.106)

so λ^L is not a right-module homomorphism, even in characteristic 2.

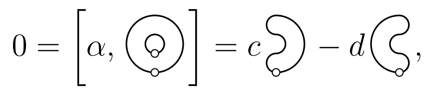
3.5 $\mathbb{Z}$-coefficients

We will now show that there is in general no such decomposition of arc algebras over $\mathbb{Z}$. To that end, let



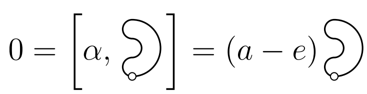
(3.107)

be an arbitrary central element in $\widetilde{H}_2$. Then we have



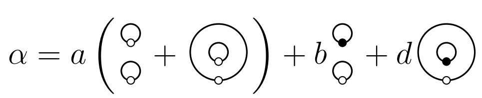
(3.108)

so $c = d = 0$. It then follows that

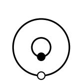


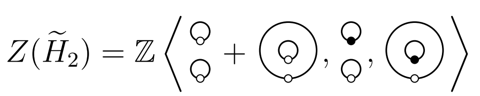
(3.109)

so $a = e$. Therefore, α is of the form



(3.110)

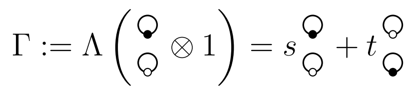
One can check that both  and  are themselves central so



(3.111)

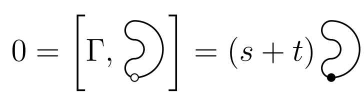
Now, since $\widetilde{H}_2$ and V are both free as abelian groups and V is commutative, we have $Z(\widetilde{H}_2 \otimes V) = Z(\widetilde{H}_2) \otimes V$.

In [Kho06], Khovanov showed that the only invertible central elements of degree 0 in H_n with $\mathbb{Z}$-coefficients are ± 1 and, as a consequence, that if M is an invertible complex of graded H_n-bimodules, then the only degree 0 automorphisms of M are $\pm\mathrm{id}$. The same argument holds, mutatis mutandis, in characteristic 2 to show that the only degree 0 automorphisms of $\widetilde{H}_n \otimes V$ and H_n are the respective identity maps. In particular, this tells us that if there were a graded algebra isomorphism $\Lambda : \widetilde{H}_2 \otimes V \to H_2$, then $\lambda = \Lambda$ modulo 2 so



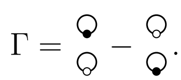
(3.112)

for some $s, t \in \{\pm 1\}$. Now Γ is central since Λ is an algebra isomorphism so



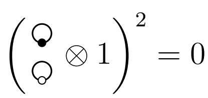
(3.113)

and hence $t = -s$. Up to composing Λ with $-\mathrm{id}$, we may assume $s = 1$ so



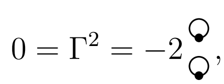
(3.114)

On the other hand, we have



(3.115)

so we would have to have



(3.116)

which is false. Therefore no such isomorphism can exist. Now note that $\widetilde{H}_2 \otimes V$ and H_2 include into $\widetilde{H}_n \otimes V$ and H_n, respectively, as subalgebras $\widetilde{J}_2 \otimes V$ and J_2 for any $n > 2$ by stacking $n-2$ round 1-labeled circles above every generator. If we had a

$\mathbb{Z}$-algebra isomorphism $\Lambda : \widetilde{H}_n \otimes V \to H_n$ and $e \in \widetilde{H}_n$ is a minimal idempotent, i.e. $e = (a^! a, \mathbf{1})$ for some $a \in \mathfrak{C}_n$, then we necessarily have that $\Lambda(e \otimes 1) = \pm e_1$ since $\lambda(e \otimes 1) = e_1$. This tells us that the restriction of Λ to $\widetilde{J}_2 \otimes V$ would give us an algebra isomorphism $\widetilde{J}_2 \otimes V \to J_2$ but we have shown this is impossible. Therefore, there is no graded $\mathbb{Z}$-algebra isomorphism $\widetilde{H}_n \otimes V \to H_n$ for any $n > 1$.

Further Directions

In [Wan21], Wang showed that there are bigraded R-module isomorphisms

$$KR_p(L; R) \cong \widetilde{KR}_p(L; R) \otimes_R R[x]/(x^p)$$

relating the unreduced and reduced Khovanov-Rozansky $\mathfrak{sl}_p$-link homologies (cf. [Kho04, KR08]) whenever R is a ring of characteristic p. Analogs of the arc algebras in the setting of $\mathfrak{sl}_p$ homology, the $\mathfrak{sl}_p$-web algebras, were introduced by Mackaay-Pan-Tubbenhauer, in the $p = 3$ case, and Mackaay in [MPT14, Mac14]. There is also an annular version of the arc algebra which was studied by Ehrig-Tubbenhauer in [ET21].

In [ORS13], Ozsváth, Rasmussen, and Szabó defined an "odd" version of Khovanov homology using an exterior version of the Frobenius algebra used in the original construction. This invariant also categorifies the Jones polynomial and agrees with ordinary Khovanov homology modulo 2. As in the characteristic 2 case, there is a splitting of odd Khovanov homology with $\mathbb{Z}$-coefficients (cf. [ORS13], Proposition 1.8). Moreover, other properties of Khovanov homology in characteristic 2 can be realized as the mod 2 reduction of a property of odd Khovanov homology. For instance, Wehrli proved in [Weh10] that Khovanov homology with $\mathbb{F}$-coefficients is mutation invariant and this was shown by Bloom for odd Khovanov homology in [Blo10]. The odd analogues of the arc algebras and bimodules for tangles were studied by Naisse-Vaz in [NV16] and Naisse-Putyra in [NP20], respectively. Unlike the ordinary arc algebras, however, odd arc algebras are only associative up to a sign depending on the elements being multiplied.

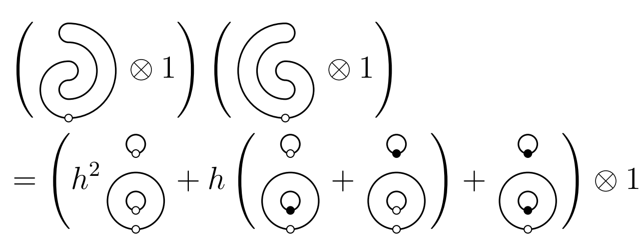
In [KR20], Khovanov and Robert studied an equivariant deformation V_α of the TQFT V, defined over the ring $R_\alpha = \mathbb{Z}[\alpha_0, \alpha_1] \cong H^*_{U(1)\times U(1)}(\mathrm{pt})$ as an R_α-algebra by

$$V_\alpha = R_\alpha[x]/((x-\alpha_0)(x-\alpha_1)) \cong H^*_{U(1)\times U(1)}(S^2) \tag{3.117}$$

with comultiplication given by

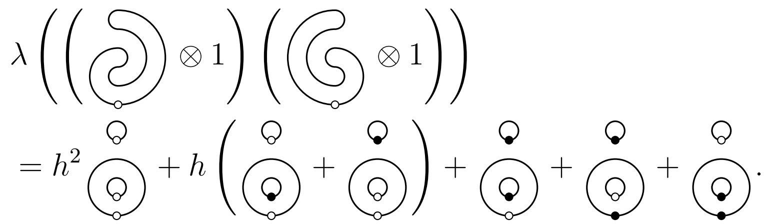
$$\begin{aligned} 1 &\mapsto 1\otimes x + x\otimes 1 - (\alpha_0+\alpha_1)1\otimes 1 \\ x &\mapsto x\otimes x - \alpha_0\alpha_1 1\otimes 1. \end{aligned} \tag{3.118}$$

This TQFT defines a link invariant in the same way as does V and, taking different values for the parameters α_0 and α_1 at the chain level, one can recover both Khovanov and Lee homology. One may define deformed arc algebras H_n^α and $\widetilde{H}_n^\alpha$ analogous to the unsual ones. However, even in characteristic 2, the naive R_α-linear extension of λ to a map $\widetilde{H}_n^\alpha \otimes V_\alpha \to H_n^\alpha$ is not multiplicative. For example, letting $h = \alpha_0 + \alpha_1$ and $t = \alpha_0\alpha_1$ for the sake of brevity, in $\widetilde{H}_3^\alpha \otimes V_\alpha$, we have



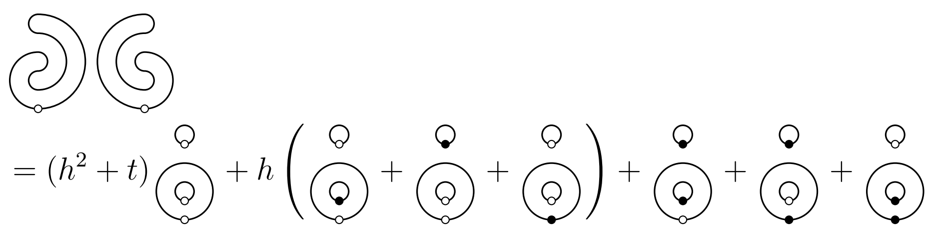
(3.119)

so



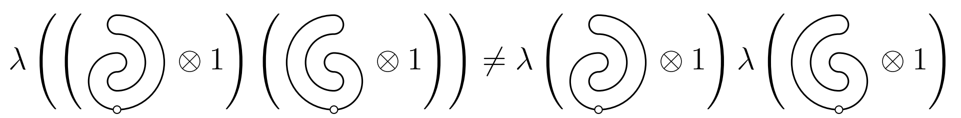
(3.120)

On the other hand, in H_3^α, we have



(3.121)

so



(3.122)

In light of the present result, it is natural to ask whether or not there are splittings analogous to ours in each of these settings: in characteristic p for the $\mathfrak{sl}_p$-web algebras, over $\mathbb{Z}$ for the odd arc algebras, and in characteristic 2 for the annular arc algebras, respectively. In the equivariant setting, this would take the form of an algebra isomorphism $\lambda^\alpha : \widetilde{H}_n^\alpha \otimes V_\alpha \to H_n^\alpha$ in characteristic 2 which recovers λ if we take $\alpha_0 = \alpha_1 = 0$.

REFERENCES CITED

[Aur10] Denis Auroux. Fukaya categories of symmetric products and Heegaard-Floer homology. *J. Gökova Geom. Topol.*, page 154, 2010.

[BHL19] John A. Baldwin, Matthew Hedden, and Andrew Lobb. On the functoriality of Khovanov-Floer theories. *Adv. Math.*, 345:1162–1205, 2019.

[Blo10] Jonathan M. Bloom. Odd Khovanov homology is mutation invariant. *Math. Res. Lett.*, 17(1):1–10, 2010.

[Blo11] Jonathan M. Bloom. A link surgery spectral sequence in monopole Floer homology. *Adv. Math.*, 226(4):3216–3281, 2011.

[BN05] Dror Bar-Natan. Khovanov's homology for tangles and cobordisms. *Geom. Topol.*, 9:1443–1499, 2005.

[BN07] Dror Bar-Natan. Fast Khovanov homology computations. *J. Knot Theory Ramifications*, 16(3):243–255, 2007.

[BPW19] Anna Beliakova, Krzysztof K. Putyra, and Stephan M. Wehrli. Quantum link homology via trace functor I. *Invent. Math.*, 215(2):383–492, 2019.

[BS15] Joshua Batson and Cotton Seed. A link-splitting spectral sequence in Khovanov homology. *Duke Math. J.*, 164(5):801–841, 2015.

[Dow18] Nathan Dowlin. A spectral sequence from Khovanov homology to knot Floer homology. *arXiv: Geometric Topology*, 2018.

[ET21] M. Ehrig and D. Tubbenhauer. Relative Cellular Algebras. *Transform. Groups*, 26(1):229–277, 2021.

[Gut22] Gary Guth. For exotic surfaces with boundary, one stabilization is not enough. *arXiv preprint*, arXiv:2207.11847, 2022.

[HL19] Kristen Hendricks and Robert Lipshitz. Involutive bordered Floer homology. *Transactions of the American Mathematical Society*, 372(1):389–424, apr 2019.

[HMZ17] Kristen Hendricks, Ciprian Manolescu, and Ian Zemke. A connected sum formula for involutive Heegaard Floer homology. *Selecta Mathematica*, 24(2):1183–1245, may 2017.

[HS21] Kyle Hayden and Isaac Sundberg. Khovanov homology and exotic surfaces in the 4-ball, 2021.

[Jon97] Vaughan F.R. Jones. A polynomial invariant for knots via von Neumann algebras. In *Fields Medallists' Lectures*, pages 448–458. World Scientific, 1997.

[JTZ21] András Juhász, Dylan Thurston, and Ian Zemke. Naturality and Mapping Class Groups in Heegaard Floer Homology. *Memoirs of the American Mathematical Society*, 273(1338), sep 2021.

[Juh06] András Juhász. Holomorphic discs and sutured manifolds. *Algebraic & Geometric Topology*, 6(3):1429–1457, oct 2006.

[Kau87] Louis H. Kauffman. State models and the Jones polynomial. *Topology*, 36:395–407, 1987.

[Kho00] Mikhail Khovanov. A categorification of the Jones polynomial. *Duke Math. J.*, 101(3):359–426, 2000.

[Kho02] Mikhail Khovanov. A functor-valued invariant of tangles. *Algebr. Geom. Topol.*, 2:665–741, 2002.

[Kho03] Mikhail Khovanov. Patterns in knot cohomology. I. *Experiment. Math.*, 12(3):365–374, 2003.

[Kho04] Mikhail Khovanov. sl(3) link homology. *Algebraic & Geometric Topology*, 4(2):1045–1081, nov 2004.

[Kho06] Mikhail Khovanov. An invariant of tangle cobordisms. *Trans. Amer. Math. Soc.*, 358(1):315–327, 2006.

[KM93] Peter B Kronheimer and Tomasz S Mrowka. Gauge theory for embedded surfaces, i. *Topology*, 32(4):773–826, 1993.

[KM11] P. B. Kronheimer and T. S. Mrowka. Khovanov homology is an unknot-detector. *Publ. Math. Inst. Hautes Études Sci.*, (113):97–208, 2011.

[KR08] Mikhail Khovanov and Lev Rozansky. Matrix factorizations and link homology. *Fundamenta Mathematicae*, 199(1):1–91, 2008.

[KR20] Mikhail Khovanov and Louis-Hadrien Robert. Link homology and frobenius extensions II, 2020.

[KS01] Maxim Kontsevich and Yan Soibelman. Homological Mirror Symmetry and Torus Fibrations. In K Fukaya, Y-G Oh, K Ono, and G Tian, editors, *Symplectic Geometry and Mirror Symmetry*. World Scientific, nov 2001.

[Lee02] Eun Soo Lee. An endomorphism of the khovanov invariant, 2002.

[Lev17] Adam Levine. Topics in Heegaard Floer homology I-IV. `http://www.newton.ac.uk/seminar/20170116100011001`, 2017.

[Lip06] Robert Lipshitz. A cylindrical reformulation of Heegaard Floer homology. *Geom. Topol.*, 10:955–1096, 2006. [Paging previously given as 955–1097].

[Lip20] Robert Lipshitz. A remark on quantum hochschild homology, 2020.

[LLS22] Tyler Lawson, Robert Lipshitz, and Sucharit Sarkar. Chen–Khovanov spectra for tangles. *Michigan Mathematical Journal*, 71(2), may 2022.

[LMW08] Robert Lipshitz, Ciprian Manolescu, and Jiajun Wang. Combinatorial cobordism maps in hat Heegaard Floer theory. *Duke Math. J.*, 145(2):207–247, 2008.

[LOT11] Robert Lipshitz, Peter Ozsvth, and Dylan Thurston. Heegaard Floer homology as morphism spaces. *Quantum Topology*, page 381449, 2011.

[LOT14a] Robert Lipshitz, Peter S Ozsváth, and Dylan P Thurston. Computing $\widehat{HF}$ by factoring mapping classes. *Geometry & Topology*, 18(5):2547–2681, dec 2014.

[LOT14b] Robert Lipshitz, Peter S. Ozsvth, and Dylan P. Thurston. Bordered floer homology and the spectral sequence of a branched double cover i. *Journal of Topology*, 7(4):1155–1199, 2014.

[LOT15] Robert Lipshitz, Peter S. Ozsváth, and Dylan P. Thurston. Bimodules in bordered Heegaard Floer homology. *Geom. Topol.*, 19(2):525–724, 2015.

[LOT16] Robert Lipshitz, Peter S. Ozsvth, and Dylan P. Thurston. Bordered floer homology and the spectral sequence of a branched double cover ii: the spectral sequences agree. *Journal of Topology*, 9(2):607–686, 2016.

[LOT18] Robert Lipshitz, Peter S. Ozsvath, and Dylan P. Thurston. Bordered Heegaard Floer homology. *Mem. Amer. Math. Soc.*, 254(1216):viii+279, 2018.

[Mac14] Marco Mackaay. The $\mathfrak{sl}_N$-web algebras and dual canonical bases. *J. Algebra*, 409:54–100, 2014.

[MMSW19] Ciprian Manolescu, Marco Marengon, Sucharit Sarkar, and Michael Willis. A generalization of Rasmussen's invariant, with applications to surfaces in some four-manifolds. *arXiv e-prints*, page arXiv:1910.08195, Oct 2019.

[MOT20] Ciprian Manolescu, Peter Ozsvath, and Dylan Thurston. Grid diagrams and heegaard floer invariants. 2020.

[MPT14] M. Mackaay, W. Pan, and D. Tubbenhauer. The $\mathfrak{sl}_3$-web algebra. *Math. Z.*, 277(1-2):401–479, 2014.

[MT93] William Menasco and Morwen Thistlethwaite. The classification of alternating links. *Annals of Mathematics*, 138(1):113–171, 1993.

[Mur88] Kunio Murasugi. Jones polynomials of periodic links. *Pacific J. Math.*, 131(2):319–329, 1988.

[NP20] Grégoire Naisse and Krzysztof Putyra. Odd Khovanov homology for tangles. *arXiv preprint arXiv:2003.14290*, 2020.

[NV16] Grégoire Naisse and Pedro Vaz. Odd Khovanov's arc algebra. *arXiv preprint arXiv:1604.05246*, 2016.

[ORS13] Peter S. Ozsváth, Jacob Rasmussen, and Zoltán Szabó. Odd Khovanov homology. *Algebr. Geom. Topol.*, 13(3):1465–1488, 2013.

[OS04a] Peter Ozsváth and Zoltán Szabó. Holomorphic disks and three-manifolds invariants: Properties and applications. *Annals of Mathematics*, 159(3):1159–1245, may 2004.

[OS04b] Peter Ozsváth and Zoltán Szabó. Holomorphic disks and topological invariants for closed three-manifolds. *Ann. of Math. (2)*, 159(3):1027–1158, 2004.

[OS05] Peter Ozsváth and Zoltán Szabó. On the Heegaard Floer homology of branched double-covers. *Adv. Math.*, 194(1):1–33, 2005.

[OS06] Peter Ozsváth and Zoltán Szabó. Holomorphic triangles and invariants for smooth four-manifolds. *Adv. Math.*, 202(2):326–400, 2006.

[Pic20] Lisa Piccirillo. The Conway knot is not slice. *Ann. of Math. (2)*, 191(2):581–591, 2020.

[Pit08] Wolfgang Pitsch. Integral homology 3-spheres and the Johnson filtration. *Transactions of the American Mathematical Society*, 360(06):2825–2848, jan 2008.

[Poi07] H. Poincaré. Sur l'uniformisation des fonctions analytiques. *Acta Math.*, 31:1–64, 1907.

[Ras05] Jacob Rasmussen. Knot polynomials and knot homologies. In *Geometry and topology of manifolds*, volume 47 of *Fields Inst. Commun.*, pages 261–280. Amer. Math. Soc., Providence, RI, 2005.

[Ras10] Jacob Rasmussen. Khovanov homology and the slice genus. *Inventiones mathematicae*, 182(2):419–447, sep 2010.

[Roz10] Lev Rozansky. A categorification of the stable $SU(2)$ Witten-Reshetikhin-Turaev invariant of links in $S^2 \times S^1$. *arXiv e-prints*, page arXiv:1011.1958, Nov 2010.

[Shu14] Alexander N. Shumakovitch. Torsion of Khovanov homology. *Fund. Math.*, 225(1):343–364, 2014.

[SW10] Sucharit Sarkar and Jiajun Wang. An algorithm for computing some Heegaard Floer homologies. *Annals of Mathematics*, 171(2):1213–1236, mar 2010.

[Tai98] Peter Guthrie Tait. On Knots I, II, II. *Scientific Papers*, 1:273–347, 1898.

[Thi87a] Morwen Thistlethwaite. Kauffman's polynomial and alternating links. *Topology*, 27(3):311–318, 1987.

[Thi87b] Morwen Thistlethwaite. A spanning tree expansion of the Jones polynomial. *Topology*, 26(3):297–309, 1987.

[Wan21] Joshua Wang. On sl(n) link homology with mod n coefficients, 2021.

[Weh10] Stephan M. Wehrli. Mutation invariance of Khovanov homology over $\mathbb{F}_2$. *Quantum Topol.*, 1(2):111–128, 2010.

[Wil21] Michael Willis. Khovanov homology for links in $\#^r(S^2 \times S^1)$. *Michigan Math. J.*, 70(4):675–748, 2021.

[Zar11] Rumen Zarev. *Bordered Sutured Floer Homology*. ProQuest LLC, Ann Arbor, MI, 2011. Thesis (Ph.D.)–Columbia University.

[Zem19] Ian Zemke. Graph cobordisms and Heegaard Floer homology. 2019.

[Zem21a] Ian Zemke. Duality and mapping tori in Heegaard Floer homology. *Journal of Topology*, 14(3):1027–1112, sep 2021.

[Zem21b] Ian Zemke. A graph TQFT for hat Heegaard Floer homology. *Quantum Topology*, 12(3):439–460, sep 2021.

[Zha16] Bohua Zhan. Combinatorial proofs in bordered Heegaard Floer homology. *Algebraic & Geometric Topology*, 16(5):2571–2636, nov 2016.