ON A SPECTRAL METHOD FOR CALCULATING THE ELECTRICAL RESISTIVITY OF A LOW TEMPERATURE METAL FROM THE LINEARIZED BOLTZMANN EQUATION

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While it is well known that transport equations may be derived diagrammatically, both this approach and that of Boltzmann inevitably encounter an integral equation that both is difficult to solve and, for the most part, has yielded only to uncontrolled approximations. Even though the popular approximations, which are typically either variational in nature or involve dropping memory effects, can be expected to capture the temperature scaling of the kinetic coefficients, it is desirable to exactly obtain the prefactor by way of a mathematically justifiable approximation. For the purpose of so precisely resolving the distribution function that governs the elementary excitations of a metal perturbed by an externally applied static electric field, a spectral method was developed that makes use of the temperature as a control parameter to facilitate an asymptotic inversion of the collision operator; the technique leverages a singularity that is inherent to the Boltzmann equation in the low temperature limit, i.e. when the dissipating Boson bath is all but frozen out.

The present dissertation is mainly concerned with the anomalous transport behavior that is commonly observed in quantum magnets; throughout a wide range of their phase diagram, materials such as the metallic ferromagnet

$ZrZn_2$ display a power law behavior of the electrical resistivity $\rho \propto T^s$ with $s < 2$. As is thoroughly established, this non-Fermi-liquid like exponent s does not arise solely due to the scattering of conduction electrons by phonons, magnons, or screened Coulomb fluctuations, for each of these soft excitations leads to $s > 2$ at temperatures $T \approx 10K$ (where $ZrZn_2$ exhibits $1.5 < s < 1.7$). After preliminarily investigating the electron-phonon system by way of rigorous reasoning, I will argue that the observed scaling of the residual resistivity $\rho \propto T^{3/2}$ in metallic ferromagnets can be attributed to interference between two scattering mechanisms: ferromagnons and static impurities.

This dissertation includes previously published co-authored material.

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CHAPTER I

INTRODUCTION

The present dissertation is mainly concerned with a spectral method for calculating the electrical conductivity of a metallic ferromagnet from the associated linearized Boltzmann equation. We strive to better understand the anomalous behavior of the resistivity $\rho \propto T^{3/2}$, where T is the temperature, that is observed in metallic ferromagnets such as $ZrZn_2$ throughout a wide range of their phase diagrams [1]. However, we initially focus on developing a rigorous approach for the general problem of electron-boson scattering at low temperature.

Our first achievement puts the problem of theoretically determining the linear response of a low temperature metal on a firm mathematical basis; we need not rely on unjustifiable approximations, for leverage to determine an asymptotically exact inversion of the collision operator is provided by the singularity that is inherent to Boltzmann's equation when the dissipative couplings only violate global electronic momentum conservation to a small degree.

As a preliminary application of our technique, we prove Bloch's law for the resistivity $\rho \propto T^5$ via a controlled expansion when $T \ll \omega_D \ll \epsilon_F$, where ω_D is the Debye temperature and ϵ_F is the Fermi temperature. For this purpose, we study the spectrum of Bloch's collision operator [2], which models the transition rates brought about by electron-phonon interactions; in the regime of asymptotically low temperature, where the lattice vibrations are nearly frozen out, it is shown that the resistivity is proportional to the lowest eigenvalue, which is both unique and isolated.

From here, we revisit the clean metallic ferromagnet [3]; this problem comes with an additional physical parameter T_0, which is the minimum unit of energy that can be transferred between electrons by a ferromagnon. Indeed, since both the electronic spin degeneracy is broken by the sample magnetization and, as polarized spin-waves of unit angular momentum, ferromagnons are incapable of mediating intraband transitions, there is a gap in the ferromagnon exchange spectrum. It follows that the transport rate is exponentially suppressed for $T \ll T_0$; we find $\rho \propto T e^{-T_0/T}$ when $T_0^2/\epsilon_F \ll T \ll T_0$. Additionally, the work of Moriya and Ueda [4] is verified; they found $\rho \propto T^2$ when $T_0 \ll T \ll T_1$, where $T_1 \ll \epsilon_F$ is analogous to the Debye temperature.

Lastly, weak damping due to quenched disorder is introduced in order to investigate the ballistic corrections to the transport rate of a realistic metallic ferromagnet. We establish that the leading beyond semi-classical contributions to the resistivity produce $\rho \propto T^{3/2}$ when $T \ll T_1$, which supplies an interpretation of experiment by accounting for interference between two scattering mechanisms: ferromagnons and static impurities.

This chapter includes previously published co-authored material.

1.1 Preliminary discussion

For the purpose of starting this theoretical investigation in an orderly manner, we shall advance a line of reasoning that begins by discussing Bloch's [2] work on the electron-phonon system (circa 1930) and then elaborates on the abstract. Next, we present our findings for the electrical conductivity of both the clean electron-phonon system [5] and the clean electron-magnon system [6]; afterwards, we give our prediction for the leading ballistic correction to the resistivity of a disordered metallic ferromagnet that results from the interference between impurity scattering and magnon exchange. Lastly, our work is briefly summarized in a section that reiterates the motivation behind our choice to put the problem of theoretically determining the linear response of a metal on a firm mathematical basis [7].

After demonstrating that a perfect crystal permits an infinite conductivity, as the electrons form wave packets which flow freely through the lattice [8], Bloch put forth a T^5 law for the electrical resistivity due to the scattering of electrons by sound waves at asymptotically low temperature; while this result is well established empirically, the task of rigorously establishing such a transport coefficient calls for the analysis of a singular integral equation. Indeed, the steady state distribution function that governs the elementary excitations of a metal perturbed by an applied static electric field must satisfy a kinetic equation[1] which both exhibits memory effects[2] and ceases to admit a solution in the limit of zero global electronic momentum dissipation. Moreover, even the associated linearized Boltzmann equation had not yielded to a controlled approximation until recently; only unjustifiable techniques had been employed towards overcoming the fact that an iterative approach is rendered invalid by the singularity which exists on account of the conductivity being infinite at zero temperature. To solve this problem, we obtain an asymptotically exact expression for the linear response of a low temperature material with a Fermi surface by leveraging the smallness of the degree to which the nearly frozen

[1]The kinetic equation admits as its solution the function that self-consistently balances, for each charge carrying state, the rate of incoming electrons with the rate of outgoing electrons, both of which are mutually dependent on one another insofar as the transition amplitude between two states is concerned with the associated occupation numbers.

[2]Electronic correlations develop in spacetime as a result of their incessant exchange of phonons.

out Bosonic processes violate global electronic momentum conservation in their action as a bath that strives, in opposition to the driving field, to thermalize the electrons.

The crux of our method is the existence of a unique lowest (and isolated) eigenvalue of the momentum relaxing collision operator that both vanishes with temperature and is proportional to the resistivity; we prove that Bloch's collision operator has this property. It follows that only a single eigenvalue problem need be overcome; also, the lowest eigenvalue can be determined by successive approximation. Here, the momentum relaxing collision operator is the integral operator specified by the interacting Hamiltonian that defines the relevant linearized Boltzmann equation.

From this position, we consider three other mechanisms that can be expected to contribute to the resistivity at asymptotically low temperature. Such irreversible processes can be realized by either collisions with imperfections of the lattice or interactions with soft modes of the system. The latter are massless excitations which have support at arbitrarily low energy; they naturally provide contributions to the relaxation rate that scale like T^s, with s an integer. In fact, a hallmark feature of the Fermi liquid is a T^2 scaling of the electrical resistivity due to scattering by charge density fluctuations [9]. And, the contribution due to scattering by spin-waves in a clean metallic ferromagnet is known to be $\rho \propto T^2$ when $T_0 \ll T \ll T_1$ [4], and $\rho \propto Te^{-T_0/T}$ when $T_0^2/\epsilon_F \ll T \ll T_0$; the parameter T_0 is the minimum unit of energy that can be transfered between electrons by a magnon [3]. That there exists a threshold temperature above which this spin-lattice coupling is activated can be deduced on the basis of physical reasoning, whereas the next-leading power law prefactor of the exponential term is difficult to ascertain; one must collect not only the charge carried by electrons on the Fermi surface, but also the effects of those states that are displaced by energies on the order of the exchange gap. Thus, it is not sufficient to include only the influence of the soft modes of a metallic ferromagnet on the conduction electrons when seeking an explanation for the non-Fermi-liquid like exponent $s = 3/2$ that is displayed in metallic ferromagnets such as $ZrZn_2$ throughout a wide range of their phase diagrams [1]; this behavior does not emerge solely as a result of the scattering of conduction electrons by either lattice vibrations or fluctuations of the internal electromagnetic field. Hence, we are compelled to introduce static impurities as a means for relaxing the nonequilibrium state.

When the crystal is irregular, the persistence of a nonzero residual value of the electrical resistivity at zero temperature can be modeled by subjecting electrons to an array of static scattering centers, which tend to restore spherical symmetry to the electron velocity distribution function, thereby suppressing the emergent current. Furthermore, the interference between impurities and dynamical modes is known to produce corrections to the semi-classical theory for the observable current that are nonanalytic in the temperature [10]; interestingly, these contributions may be either localizing or delocalizing. By using a diagrammatic method for generating the collision operator, we compute the so-called ballistic corrections[3] that arise from vertices containing both a magnon line and an impurity line; having familiarized ourselves with both the clean system and the zero temperature system, a perturbation series can be formed about each of these limiting cases. We find that the transport relaxation

[3]Since the observations we consider involve relatively clean samples, as indicated by their small residual resistivity, we do not suspect diffusive electron dynamics to be essential [11]; instead we focus on the ballistic regime, i.e. $\gamma \lesssim T \ll \epsilon_F$, with γ the elastic impurity rate [12, 13].

rate is given by the sum of the rates due to each vertex individually - a structure that is consistent with a generalized interpretation of Matthiessen's rule. In this way, the resistivity is shown to receive a correction that is proportional to $T^{3/2}$ due to multiple scattering events involving both impurities and spin-waves.

1.2 Results

To facilitate a complete presentation of our results, it is useful to introduce some notation; for simplicity, we shall omit details beyond those which are concerned with the algebraic structure encountered when computing the electrical resistivity.

In order to calculate the expectation value of the electric current that develops in response to an applied electric field, we seek a distribution function $\vec{\varphi}$ which both satisfies an equation of the form

$$\vec{k} = C_0 \circ \vec{\varphi} \tag{1.1}$$

and, when the inner product is normalized appropriately, determines the conductivity according to

$$\sigma = \frac{ne^2}{2m}\big(\vec{k};\vec{\varphi}\big), \tag{1.2}$$

where n is the electronic number density, e is the electron charge, and m is the electron mass. Here, C_0 is the collision operator; it acts on a Hilbert space $\vec{\mathfrak{H}}$ and is self-adjoint under the inner product $\big(\cdot;\cdot\big)$. Furthermore, the vector $\vec{k} \in \vec{\mathfrak{H}}$ is nearly a zero mode of C_0. Indeed, the Ward-Takahashi identity [14] associated with momentum balances implies that $C_0 \circ \vec{k} \neq 0$ if and only if global electronic momentum conservation is violated. Thus, Fredholm's orthogonality condition [15]

$$0 \neq \big(\vec{k};\vec{k}\,\big) = \big(C_0 \circ \vec{k};\vec{\varphi}\big), \tag{1.3}$$

shows that Equation 1.1 permits a solution that is singular in limit of vanishing temperature; such a control parameter does not exist in classical kinetic theory, for it is the finite extent of the Fermi sphere that renders nonzero the left-hand-side of Equation 1.3 at zero temperature. From here, we employed perturbation theory for linear operators to prove that there exists a unique lowest element λ_p in the spectrum of C_0 on $\vec{\mathfrak{H}}$. Additionally, we show that λ_p both is isolated and vanishes with temperature; in this limit, its corresponding eigenvector $\vec{\epsilon}_p$, the momentum mode, approaches $\vec{k}$, the bare momentum mode. Putting together the above, we have

$$\sigma \approx \frac{ne^2}{2m\lambda_p}\big[1 + \mathcal{O}(T)\big]. \tag{1.4}$$

As a result, the dominant behavior of the inverse charge transport lifetime $\tau_{tr}^{-1} = 2\lambda_p$ for the electron-phonon system is

$$\lambda_p = \nu_0 \pi \begin{cases} 480\zeta(5)\frac{T^5}{\omega_D^4}\left[1 - \frac{\omega_D^2}{16\epsilon_F^2}\right] & T \ll \omega_D \ll \epsilon_F \\ T & \omega_D \ll T \ll \epsilon_F \end{cases}, \tag{1.5}$$

where the dimensionless constant ν_0 is known exactly and ζ is the Riemann zeta function; Equation 1.5 proves Bloch's law [2]. By similar means, we determine both the thermopower[4]

$$S \approx \frac{\pi^2 T}{3|e|\epsilon_F}\big[1 + \mathcal{O}(T)\big], \tag{1.6}$$

which agrees with the findings of both Wilson [16] and Sondheimer [17], and the heat conductivity

$$\sigma_h \propto \frac{\omega_D^2}{T^2}, \tag{1.7}$$

[4]The thermopower S is the proportionality constant that measures the thermoelectric voltage which is induced in response to a macroscopic temperature gradient under conditions of zero mass flow.

where the dimensionful proportionality factor is temperature independent; for technical reasons,[5] we are limited to dimensional analysis in gathering the result that is Equation 1.7.

In the case of a clean metallic ferromagnet, where the resistivity develops due to scattering of electrons by spin-waves, we improve upon a previous investigation [3] by resolving the prefactor to the exponential

$$\lambda_p(T,\gamma=0) = 4\mathcal{V}_0 \begin{cases} \frac{TT_0}{T_1} e^{-T_{min}/T} & T_0^2/\epsilon_F \ll T \ll T_0 \\ \frac{\pi^2 T^2}{3T_1} & T_0 \ll T \ll T_1 \\ T & T_1 \ll T \ll \epsilon_F \end{cases}, \tag{1.8}$$

where T_{min} is the renormalized exchange gap and the dimensionless constant $\mathcal{V}_0$ is known exactly; it should be mentioned that the prefactor to the exponential is model dependent. While the activated behavior follows immediately when $T \ll T_{min}$, technical difficulties arise in the regime where $T \ll T_0^2/\epsilon_F$; since at such temperatures the magnon contribution to the resistivity is surely overwhelmed by some other mechanism for momentum dissipation, we feel that it is not worthwhile to endeavor this calculation, and instead stipulate that the prefactor will tend to a constant for temperatures sufficiently low.

Our final result is yet unpublished; to investigate the anomalous behavior of the metallic ferromagnet, we confront a linear integral equation of the form

$$\vec{k} = \left[C_0 + \gamma + L_0\right] \circ \vec{\varphi}, \tag{1.9}$$

where the collision operators C_0, γ, and L_0 correspond to a succession of scattering events involving a single magnon, a single impurity, and interaction processes involving both impurities and magnons, respectively. With both the

[5]See subsection 3.2.3.

zero temperature solution

$$\vec{\varphi}(T=0,\gamma) = \gamma^{-1} \circ \vec{k} \tag{1.10}$$

and the ideal solution

$$\vec{\varphi}(T,\gamma=0) = C_0^{-1} \circ \vec{k} \tag{1.11}$$

familiar, a perturbation series is constructed about each of these limiting cases in order to ascertain the effects of L_0; we find the content of Matthiessen's rule plus interference corrections, i.e.

$$\lambda_p(T,\gamma) = \lambda_p(T=0,\gamma) + \lambda_p(T,\gamma=0) + \Delta\lambda_p(T,\gamma), \tag{1.12}$$

with

$$\Delta\lambda_p(T,\gamma) \approx \mathcal{V}_0 \frac{\gamma}{\epsilon_F} \begin{cases} \frac{-\sqrt{T_1}T^{3/2}}{T_0} A^{\ll} & T_0^2/\epsilon_F \ll T \ll T_0 \\ \frac{T^{3/2}}{\sqrt{T_1}} A^{\gg} & T_0 \ll T \ll T_1 \end{cases}, \tag{1.13}$$

where

$$\lambda_p(T=0,\gamma) \approx \gamma \tag{1.14}$$

and

$$\begin{aligned} A^{\ll} &= \frac{1}{\pi}\int_0^\infty du \left\{ \frac{u^{3/2}}{\sinh^2 \frac{u}{2}} - \frac{2u^{-3/2}}{e^u - 1} \right\} \approx 3, \\ A^{\gg} &= \pi \int_0^\infty du \left\{ \frac{u^{3/2}}{4\sinh^2 \frac{u}{2}} + \frac{u/2}{e^u - 1} \right\} \approx 12. \end{aligned} \tag{1.15}$$

Evidently, an interpolation formula exists between the region of parameter space where scattering is magnon dominated (i.e. $\gamma \ll T^2/T_1$ when $T_0 \ll T \ll T_1$) and the regime where impurity scattering provides the leading contribution to the resistance (i.e. $T^2/T_1 \ll \gamma$ when $T_0 \ll T \ll T_1$). Note that both $T \ll \epsilon_F$ and $\gamma \ll \epsilon_F$ are assumed implicitly.

At last, we are now in a position to remark on the merit of exactly determining the transport coefficient; while Belitz and Kirkpatrick put forth qualitative arguments [11] which indicated that $\Delta\lambda_p \propto T^{3/2}$, their approach could not be expected to accurately capture the prefactor, so questions regarding convergence remained outstanding. Our calculation shows that the prefactor is orders of magnitude different than the previous prediction, in addition to exposing the crossover from localizing behavior when $T_0 \ll T \ll T_1$ to delocalizing behavior when $T_0^2/\epsilon_F \ll T \ll T_0$.

1.3 Summary

While it is well known [18] that transport equations may be derived diagrammatically, both this approach and that of Boltzmann inevitably encounter an integral equation that both is difficult to solve and, for the most part, has yielded only to uncontrolled approximations.[6] To our knowledge, not even Bloch's law had received a proof until recently. Even though the popular approximations, which are typically either variational in nature or involve dropping memory effects, can be expected to capture the temperature scaling of the kinetic coefficients, it is desirable to exactly obtain the prefactor by way of a mathematically justifiable approximation. For the purpose of so precisely resolving the distribution function that governs the elementary excitations of a metal perturbed by an externally applied static electric field, we developed a spectral method that makes use of the temperature T as a control parameter to facilitate an asymptotic inversion of the collision operator; our technique leverages a singularity that is inherent to the Boltzmann equation in the low temperature limit, i.e. when the dissipating Boson bath is all but frozen out.

Because the background fluctuations coupled to the charge carriers are limited to slowly modulating long wavelength processes with energies on

[6] Remarkably, Brooker and Sykes [19] have achieved a solution map for the equation of Abrikosov and Khalatnikov [20].

the order of T, the degree to which they violate global electronic momentum conservation is small; this feature of good conductors is reflected by the lowest element in the spectrum of the collision operator, which both vanishes with temperature and is non-degenerate[7]. Indeed, the existence of such a near zero mode is a necessary consequence of the Ward-Takahashi identity associated with momentum balance [14]; we prove its uniqueness by applying to the eigenproblem a series of manipulations that are standard within kinetic theory [21, 22].[8] Moreover, we show that this eigenvalue, which corresponds to the momentum mode, is isolated. After rigorously establishing that the inverse collision operator both is positive and admits a spectral representation in which the dominantly singular component projects onto the momentum mode, it follows that the smallest eigenvalue is proportional to the resistivity, since Kubo's formula calls for an inner product that in bra-ket terminology amounts to the matrix element formed by sandwiching the inverse collision operator between two bare momentum modes. Thus, the leading contributions to the current-current susceptibility are entirely contained in the nonconserving corrections to the momentum mode, which one can compute in a perturbation series about the known bare form; although the relevant linearized Boltzmann equation violates the condition for the existence of an iterative solution, such an eigenvalue problem permits the method of successive approximation.

[7]Note that the momentum relaxing collision operator (which, when the context prevents confusion, we simply refer to as the collision operator) is equipped with a backscattering factor; it is therefore distinguished from the fundamental collision operator, which possesses hydrodynamic modes related to mass, momentum, and energy.

[8]Physically, this argument succeeds on account of both detailed balance and the positivity of scattering amplitudes.

With a means of analysis now familiar, we seek to better understand the anomalous transport behavior that is commonly observed in quantum magnets [23]; throughout a wide range of their phase diagram, materials such as the metallic ferromagnet $ZrZn_2$ [1] display a power law behavior of the electrical resistivity $\rho \propto T^s$ with $s < 2$. As is thoroughly established, this non-Fermi-liquid like exponent s does not arise solely due to the scattering of conduction electrons by phonons, magnons, or screened Coulomb fluctuations, for each of these soft excitations leads to $s \geqslant 2$ at temperatures $T \approx 10\mathrm{K}$ (where $ZrZn_2$ exhibits $1.5 \lesssim s \lesssim 1.7$ [11]). Hence, the main goal of this work is to investigate the interplay between ferromagnons and quenched disorder, which is present even in relatively clean samples; however, before delving into this task, we preliminarily study the Fröhlich Hamiltonian.

Our first accomplishment is a proof of Bloch's law for the DC conductivity $\sigma \propto T^{-5}$. Then, we determine the two remaining transport coefficients that suffice to specify the thermoelectric behavior of an electron-phonon system: the thermopower $S \propto T$, and the heat conductivity $\sigma_h \propto T^{-2}$. It is assumed in these three computations that $T \ll \omega_D \ll \epsilon_F$, where ω_D is the Debye temperature and ϵ_F is the Fermi energy.

Once in a position to confidently handle the broken spin degeneracy that emerges in the presence of a magnetization field, we confirm the predictions of previous workers [3, 4] on the resistivity of an ideal ferromagnet: when $T_0 \ll T \ll T_1$, where T_0 is the minimum unit of energy that can be transferred by a magnon and $T_1 \ll \epsilon_F$ is analogous to the Debye temperature, $\rho \propto T^2$; when $T_0^2/\epsilon_F \ll T \ll T_0$, the transport relaxation rate $\rho \propto Te^{-T_0/T}$ is exponentially suppressed, for the inability of ferromagnons to mediate intraband transitions manifests as a gap in the exchange spectrum.

Despite the fact that significant attention is paid towards reaping widely-known truths from Bloch's collision integral, our primary achievement yields the dominant ballistic corrections to the electrical resistivity of a weakly disordered metallic ferromagnet; the response is derived, via an effective action that contains vertices for both magnon exchange and impurity scattering, by evaluating the Kubo formula in a Φ-derivable approximation [24]. In this way, one can algorithmically generate the beyond semi-classical contributions to the collision operator and then construct the ensuing corrections to the transport rate in a second von Neumann series about either the ideal solution or the zero temperature solution; we find a Matthiessen's rule-like interpolation formula between the regime of weaker magnons and the regime of weaker disorder.

As a result, we corroborate the arguments of Belitz and Kirkpatrick [11, 23], which indicated that the observed scaling of the resistivity $\rho \propto T^{3/2}$, when $T \ll T_1$, in metallic ferromagnets can be attributed to interference between two scattering mechanisms: ferromagnons and static impurities.

CHAPTER II

ELEMENTARY DEVELOPMENTS

To begin, we outline aspects of kinetic theory that are essential to the problem statement; before acquainting ourselves with the Boltzmann equation, the Drude theory of charge transport is briefly considered in order to most immediately become familiar with the notion of a transport lifetime. Following these advancements of the formalism, we define the three independent kinetic coefficients that characterize the thermoelectric behavior of a metal. Next, we outline a microscopic model for both the scattering of electrons by phonons and the scattering of electrons by magnons, each of which instantiates a momentum dissipating soft mode that is borne of the lattice.

2.1 Drude formula

By considering each electron individually, and treating interactions as instantaneous collisions that occur with a constant probability per unit time, one can construct a model [25] of charge transport in which carriers are unperturbed inbetween events. To this end, we investigate Ohm's law

$$j_e = \sigma E, \tag{2.1}$$

which prescribes the evolution of a current density j_e, as permitted by the conductivity σ, in linear response to an imposed electric field E. During the average time of free flight τ_{tr}, mobile electrons driven by the electric force develop an effective drift velocity $v_d = eE\tau_{tr}/m$ (since the electron velocity after each collision is taken to be equilibriated), from which there emerges a nonzero macroscopic current $j_e = nev_d$. Here, n is the number density of free electrons, $e = -|e|$ is the electric charge, and m is the elecron mass. Based on these simple physical assumptions, we arrive at the Drude formula for the electrical conductivity

$$\sigma = \frac{ne^2}{m}\tau_{tr}; \tag{2.2}$$

all details of the many-body physics are buried in the phenomenological parameter τ_{tr}, otherwise known as the transport lifetime.

2.2 Boltzmann equation

The single-particle picture of electrons allowed to travel freely except for momentary (and independent) collisions was eventually improved upon by Sommerfeld, who incorporated Fermi-Dirac statistics; from Pauli's principle arises the Fermi surface when $T \ll \epsilon_F$ (for metals the Fermi energy $\epsilon_F \sim 10^5$K), which delineates the degenerately packed region of phase space from those states with meager, temperature limited, occupation numbers.

In this approach, one assumes the existence of a distribution function F that obeys Boltzmann's integro-differential equation

$$\left[\frac{\partial}{\partial t} + \frac{dp_i}{dt}\frac{\partial}{\partial p_i} + \frac{dr_i}{dt}\frac{\partial}{\partial r_i}\right] F(\vec{r},\vec{p};t) = \mathcal{W}\left[F\right](\vec{r},\vec{p};t) \tag{2.3}$$

such that the average $\langle Q\rangle^t$ of any measureable (one-body) quantity Q is given by[1]

$$\langle Q\rangle^t = \int \frac{d\vec{r}d\vec{p}}{(2\pi)^3} Q(\vec{r},\vec{p};t)F(\vec{r},\vec{p};t); \tag{2.4}$$

here, $\mathcal{W}$ is a (to be determined) integral operator representing the collision processes that irreversibly draw the system towards equilibrium and the "streaming term"

$$\left[\frac{dp_i}{dt}\frac{\partial}{\partial p_i} + \frac{dr_i}{dt}\frac{\partial}{\partial r_i}\right] F(\vec{r},\vec{p};t) \tag{2.5}$$

[1]Throughout, we use natural units where $\hbar = c = k_B = 1$.

gives the Hamiltonian evolution of F along the phase space trajectory $(\vec{r}[t], \vec{p}[t])$. That we presume both the position $\vec{r}$ and the momentum $\vec{p}$ can be simultaneously specified (at time t) in spite of Heisenberg's uncertainty principle limits the applicability of Equation 2.3 to the quasi-classical regime.

Note that $\mathcal{W}$ must admit

$$\mathcal{W}\left[f\right] = 0, \tag{2.6}$$

where

$$f(\epsilon) = 1/(e^{\beta\epsilon} + 1) \tag{2.7}$$

is the Fermi-Dirac probability density; $f(\epsilon)$ gives the likelyhood for free electrons to occupy the state of energy ϵ when the system is in equilibrium with a thermal reservoir of inverse temperature $\beta = 1/T$. Indeed, detailed balanced is a feature of the equilibrium ensemble.

Now then, in the presence of a weak electric field that is both static and homogeneous, the low-lying excitations of a good, clean, single-band conductor (at temperatures small compared to $\epsilon_F = k_F^2/2m$) are nearly free electrons that propagate according to

$$\frac{d\vec{p}}{dt} = e\vec{E}, \tag{2.8}$$

and (after transient dynamics have subsided) the Boltzmann equation reads

$$e\vec{E} \cdot \vec{\nabla}_{\vec{p}} F(\vec{p}) = \mathcal{W}\left[F\right](\vec{p}); \tag{2.9}$$

if the right-hand side of Equation 2.9 does not conserve electronic momentum, then a steady-state will develop[2] where the current is

$$\vec{j}_e = \frac{e}{m} \int \frac{d^3p}{(2\pi)^3} \vec{p}\, F(\vec{p}). \tag{2.10}$$

[2]We are neglecting Joule heating, which is of order E^2 [8].

From here, we find it illuminating to study the relaxation time approximation

$$\mathcal{W}[F](\vec{p}) = \frac{f(\xi_p) - F(\vec{p})}{\tau_{tr}(\vec{p})}, \tag{2.11}$$

with $\xi_p = p^2/2m - \epsilon_F$, which predicts (for the linear response)

$$F(\vec{p}) = f(\xi_p) + \frac{e}{m} w(\xi_p)\tau_{tr}(p)\vec{p} \cdot \vec{E} \tag{2.12}$$

and

$$j_e^i = \frac{e^2}{m^2} \int \frac{d^3p}{(2\pi)^3} w(\xi_p)\tau_{tr}(p)p^i\vec{p} \cdot \vec{E}, \tag{2.13}$$

or,

$$\sigma \approx \frac{ne^2}{m}\tau_{tr}(k_F), \tag{2.14}$$

where

$$w(\xi_p) = -f'(\xi_p) = \frac{1}{4T} \cosh^{-2} \frac{\xi_p}{2T} \tag{2.15}$$

serves as a weight function. Notice that σ is proportional to $n = k_F^3/6\pi^2$; when a field is applied the entire Fermi sea is shifted in wavevector space. However, resistive thermalization is restricted to excitations near the Fermi surface [26], as only these states are energetic enough that a perturbation can connect them to vacant phase space.

One should be aware that experiment requires $\tau_{tr} \sim 10^{-14}$s for simple metals at room temperature. And therefore, the mean free path $l = v_F\tau_{tr}$, with v_F the Fermi velocity, is on the order of a hundred lattice units [27]; to assimilate this rather non-Newtonian aspect of metals, we must ground our reasoning on microscopic principles. Thus, it remains to realize the mechanism that gives rise to a finite lifetime of excited states; nevertheless, more information can be acquired insofar as certain ratios of physical quantities are independent of τ_{tr}.

2.2.1 Thermoelectric effects. In addition to charge, so too is heat carried by electrons; these two phenomena are not independent. Indeed, the metallic response to either an electric field or a temperature gradient results in both an electric current and a heat current; if

$$\begin{aligned} \vec{J}_1 &= \vec{j}_n, \quad \vec{X}_1 = \frac{-1}{T}\vec{\nabla}\tilde{\mu}, \\ \vec{J}_2 &= \frac{1}{T}\vec{j}_q, \quad \vec{X}_2 = \frac{-1}{T}\vec{\nabla}T, \end{aligned} \tag{2.16}$$

where $\vec{j}_n$ is the number current, $\tilde{\mu}$ is the electrochemical potential, and $\vec{j}_q$ is the heat current, then the linearized constitutive relations read

$$\vec{J}_i = \sum_j L_{ij}\vec{X}_j, \tag{2.17}$$

where, on account of the fact that the fluxes $\vec{J}_i$ and their conjugate (generalized) forces $\vec{X}_i$ are identified from the entropy balance equation[3]

$$\rho\frac{ds}{dt} + \vec{\nabla}\cdot\vec{j}_s = \sum_i \vec{J}_i \cdot \vec{X}_i, \tag{2.19}$$

with ρ the local particle number density, s the entropy per particle, and $\vec{j}_s = \vec{J}_2$ the entropy current, the matrix of transport coefficients L_{ij} obeys Onsager's reciprocal relation $L_{ij} = L_{ji}$ (as a consequence of the underlying microscopic reversibility of the equations of motion) [28].

Notice that this theory is postulated at the level of nonequilibrium thermodynamics. As a kludge, we appeal to the thermodynamic entropy in order to bypass the unitary evolution that is inherent to the von Neumann

[3]The system is assumed to maintain a state of mechanical equilibrium such that thermal equilibrium may be established locally; if only long-wavelength temperature variations exist, then each "physically infinitesimal" region is still governed by the Gibbs relation (for quasi-static irreversible processes) [28]

$$T ds = du + p dv - \mu dc_e, \tag{2.18}$$

where v is the specific volume, u is the (internal) energy per particle, and c_e is the electronic number fraction.

entropy of equilibrium statistical mechanics; beginning with a course grained description natively incorporates our inability to know the microscopic configuration of the macroscopic state when all that can be measured is a small number (relative to the many electronic degrees of freedom) of phenomenological variables. Furthermore, Equation 2.19 accounts only for the conduction electron contribution to the entropy source strength; the legitimacy of this approximation is well supported by the empirical fact that metals conduct heat better than (electrical) insulators do [29].

While the phase function $\vec{j}_n(\vec{p}) = \vec{p}/m$ takes an intuitive form, it remains to specify

$$\vec{j}_q = \vec{j}_u - \epsilon_F \vec{j}_n, \tag{2.20}$$

which involves the nontrivial energy current operator $\vec{j}_u$ [30]; we could find $\vec{j}_u$ from the energy balance equation, but it is a common tactic to neglect phonon drag and take $\vec{j}_u(\vec{p}) = \varepsilon_p \vec{j}_n(\vec{p})$, with $\varepsilon_p = p^2/2m$, to be simply the kinetic energy current of the electrons as seen by the lattice. In this way, Equation 2.20 reflects the fact that short range interactions, which tend to bring about a state of local equilibrium, prompt the flitting electrons to either absorb or release heat according to the local sign of $\varepsilon_p - \epsilon_F$.

Now then, consider a system in the presence of a weak temperature gradient and under conditions of no charge flow; in this case, there will arise a thermoelectric voltage

$$-\vec{\nabla}\tilde{\mu} = -eS\vec{\nabla}T, \quad -eS = \frac{L_{12}}{L_{11}} \tag{2.21}$$

of magnitude determined by the thermopower S. In fact, the (steady state) linearized Boltzmann equation (in the relaxation time approximation)

$$-w(\xi_p)\frac{\vec{p}}{m}\cdot\Big[e\vec{E}+T\vec{\nabla}\frac{\xi_p}{T}\Big]=\frac{f(\xi_p)-F(\vec{p})}{\tau_{tr}(\vec{p})} \tag{2.22}$$

admits the solution

$$F(\vec{p})=f(\xi_p)+w(\xi_p)\tau_{tr}(\xi_p)\frac{\vec{p}}{m}\cdot\Big[-\vec{\nabla}\tilde{\mu}-\frac{\xi_p}{T}\vec{\nabla}T\Big], \tag{2.23}$$

which yields

$$\begin{aligned}
L_{11} &= \frac{T}{3m^2}\int\frac{d^3p}{(2\pi)^3}w(\xi_p)\tau_{tr}(\xi_p)p^2,\\
L_{12} &= \frac{1}{3m^2}\int\frac{d^3p}{(2\pi)^3}w(\xi_p)\tau_{tr}(\xi_p)p^2\xi_p,\\
L_{22} &= \frac{1}{3m^2T}\int\frac{d^3p}{(2\pi)^3}w(\xi_p)\tau_{tr}(\xi_p)p^2\xi_p^2,
\end{aligned} \tag{2.24}$$

and therefore we have both

$$S\approx\frac{\pi^2T}{3|e|\epsilon_F}\Big[1+\mathcal{O}(T/\epsilon_F)\Big] \tag{2.25}$$

and the Wiedemann–Franz law for the ratio of the heat conductivity to the electrical conductivity

$$\frac{e^2\sigma_h}{\sigma}=\frac{T^2L_{22}}{L_{11}/T}=\frac{\pi^2T}{3}. \tag{2.26}$$

In order that Equation 2.25 comes with the correct numerical prefactor, it is crucial that

$$\tau_{tr}(\xi_p)N(\xi_p)\approx\tau_{tr}(k_F)N_F\big[1+\mathcal{O}(\xi_p^2/\epsilon_F^2)\big], \tag{2.27}$$

where $N(\epsilon)=N_F\sqrt{1+\epsilon/\epsilon_F}$, with $N_F=mk_F/2\pi^2$, is the electronic density of states at energy ϵ displaced from the Fermi surface; this necessity may be interpreted by analogy with the Lorentz gas (free electrons in a fixed array of hard spheres), where only the mean free-path is energy independent [31].

From this point, we desire a realized thermalized background of Bosonic excitations in order to both attain a definite expression for τ_{tr} and validate Equation 2.25.

2.3 Phonons, Ferromagnons, and Quenched Disorder

While electron Bloch waves travel freely in an ideal crystal at zero temperature, and therefore accelerate indefinitely under external forces, physical materials have inherent channels for the exchange of both energy and momentum. Indeed, equilibrium is established thanks to scattering by both impurities and lattices modes. Such interactions can be described by coupling an electron liquid to a Bosonic reservoir, which tends to relax the perturbation spawned electronic excitations by way of thermalizing processes; together with an external source of propulsion, this constitutes a mechanism capable of sustaining the steady state in which a constant flux of charge persists against the dissipating collisions that induce transitions across the Fermi surface.

Note that at asymptotically low temperature, the lattice is limited to its elementary excitations. Consequently, the conduction electrons effectively exchange energy only with soft modes, which characteristically exhibit an energy-momentum relation of the form $\omega \propto |\vec{k}|^n$, with n-integer; since $\omega \sim T$, the temperature scaling of the transport coefficients is intimately related to n, which can be captured via symmetry arguments. Because only slowly modulating long wavelength fluctuations endure the hydrodynamic limit, the free energy is definable by a small number of phenomenological parameters. To this end, we seek a wave equation which governs the free motion of those particle-like resonances that emerge from the cooperative behavior of many individual Bosonic degrees of freedom.

Consulting first the properties of an isotropic elastic medium [32], spatial translation invariance implies that the displacement field $\vec{d}$ is a Goldstone mode; these longtitudinal vibrations propagate according to

$$\partial_t^2 \vec{d} = c_s^2 \Delta \vec{d}, \tag{2.28}$$

where c_s is the speed of sound, at lowest order in the gradient expansion. On recognizing that the electrons sense charge density fluctuations brought about by both compression and expansion of the unit cell, it follows that the interaction energy [33]

$$H_{ep} = g \int d\vec{x}\, n(\vec{x}) \phi(\vec{x}), \tag{2.29}$$

where $\phi = \vec{\nabla} \cdot \vec{d}$ is the phonon field, g is a coupling constant, and $n = \psi^\dagger \psi$ is the local electron density.

To effect the magnetization field $\vec{M}$, we introduce a Zeeman term [3]

$$H_{em} = \Gamma_t \int d\vec{x}\, \vec{n}_s(\vec{x}) \cdot \vec{M}(\vec{x}), \tag{2.30}$$

where $\vec{n}_s = \psi^\dagger \vec{\sigma} \psi$, with $\vec{\sigma}$ the Pauli matrices, is the electronic spin density, and Γ_t is a coupling constant. The dynamics of $\vec{M}$ remain to be determined. On retaining only the massless fluctuations, it suffices to study the transverse susceptibility $\chi_\perp$, which is isotropic and therefore has two degrees of freedom; furthermore, the diagonal and off-diagonal components are of even and odd parity, respectfully, under time reversal. Thus,

$$\chi_\perp(k) = \frac{K}{2} \begin{pmatrix} \chi_+(k) + \chi_-(k) & -i\left[\chi_+(k) - \chi_-(k)\right] \\ i\left[\chi_+(k) - \chi_-(k)\right] & \chi_+(k) + \chi_-(k) \end{pmatrix}, \tag{2.31}$$

where the circularly polarized magnons χ_σ obey $\chi_\sigma(k) = \chi_{-\sigma}(-k)$ and K is a coupling constant. Indeed, $\chi_\perp(k)$ must be singular whenever either of the energy-momentum relations $i\Omega_\sigma = -\sigma D k^2$, with D the spin-wave stiffness, is satisfied; this yields[4]

$$\chi_\sigma(k) = \frac{\sigma}{i\Omega + \sigma D k^2}. \tag{2.32}$$

Despite the fact that the spectrum of this collective mode extends to arbitrarily low energy, the associated effective potential felt by electrons due to magnon exchange is gapped in light of the broken electron spin degeneracy, as ferromagnons are not capable of mediating intraband transitions.

Finally, we address the fact that real materials are fraught with structural defects; any amount of irregularity will result in a residual resistance at zero temperature. To model the collisions of electrons with distortions of the lattice, the sample is assumed to be clean enough such that the disorder may be treated as weak; in this case, the simplest way to realize an array of impurities is through a static random potential field n_i, viz.

$$H_{ei} = \int d\vec{x}\, n(\vec{x}) n_i(\vec{x}), \tag{2.33}$$

which both is spherically symmetric and takes the scattering sources to be independently distributed, i.e.

$$\left\{ n_i(\vec{x}) n_i(\vec{y}) \right\}_{\text{dis}} = \frac{1}{2\pi N_F \tau} \delta\left[\vec{x} - \vec{y}\right], \tag{2.34}$$

where τ is the elastic mean free time, and N_F is the electronic density of states on the Fermi surface.

Now then, with the Hamiltonian H specified, our primary goal is to compute the expected gauge current

$$\left\langle \vec{j}(\vec{x}, t) \right\rangle^t = \mathrm{Tr}\left\{ \vec{j}(\vec{x}) e^{-\beta H(t)} \right\}; \tag{2.35}$$

[4]Here, $k = (\vec{k}, i\Omega)$, with $i\Omega$ an imaginary frequency of the Matsubara technique.

to linear order in the electric field E, one expects an expression of the form

$$\begin{aligned} \left\langle j_i(\vec{x},t)\right\rangle^t &= \int d\vec{y}dt'\sigma_{ij}(\vec{x}-\vec{y},t-t')E^j(\vec{y},t') \\ &= \frac{ne^2}{m}\int d\vec{y}dt'\,\tau_{tr}(\vec{x}-\vec{y},t-t')E_i(\vec{y},t'), \end{aligned} \tag{2.36}$$

which involves a continuum of relaxation times, although the largest contributions come from those low-lying quasiparticles that are composed of an electron-hole pair whose constituents are near the Fermi surface. Evidently, the transport coefficient τ_{tr} may be interpretted as the materials's propensity to permit an organized drift of electrons; kinetic theory is generally concerned with the irreversible motions that both develop in response to macroscopic disturbances and draw the system back to its ground state.

CHAPTER III

ELECTRONS AND PHONONS

This chapter considers, by way of the linearized Boltzmann equation, the scattering of conduction electrons in the bulk of a metal by long-wavelength acoustic phonons with momentum-dependent resonance frequency $\omega(\vec{q}) = c_s|\vec{q}|$, where c_s is the speed of sound; the DC conductivity, the thermopower, and the heat conductivity are obtained.

This chapter includes previously published co-authored material.

3.1 Fröhlich's Hamiltonian

Our first task is to derive the collision operator from the microscopic Hamiltonian

$$H = H_0 + H_1, \tag{3.1}$$

where

$$H_0 = \sum_{\vec{k}} \left\{ \psi^\dagger(\vec{k}) \frac{k^2}{2m} \psi(\vec{k}) + \omega(k) \left[c^\dagger(\vec{k}) c(\vec{k}) + 1/2 \right] \right\} \tag{3.2}$$

is the bare Hamiltonian, with $\psi^\dagger, \psi$ and $c^\dagger, c$ the electron and phonon creation/annihilation operators, respectively, and [34]

$$H_1 = g \sum_{\vec{p},\vec{k}} \sqrt{\frac{\omega(k)}{2V}} \psi^\dagger(\vec{p} + \vec{k}/2) \psi(\vec{p} - \vec{k}/2) \left\{ c(\vec{k}) + c^\dagger(-\vec{k}) \right\} \tag{3.3}$$

is the interaction energy [35]; $g^2 \sim 1/N_F$ is a constant. Note that for a system quantized in a box of volume V, we have

$$\frac{1}{V} \sum_{\vec{k}} \to \int \frac{d^3k}{(2\pi)^3} \tag{3.4}$$

as V tends to infinity while intensive quantities are held fixed (the thermodynamic limit). In realspace,

$$H_1 = g \int d\vec{x}\, \psi^\dagger(\vec{x}) \psi(\vec{x}) \varphi(\vec{x}), \tag{3.5}$$

where

$$\varphi(\vec{x}) = \sum_{\vec{k}} \sqrt{\frac{\omega(k)}{2V}} \left\{ c(\vec{k}) e^{i\vec{k}\cdot\vec{x}} + c^{\dagger}(-\vec{k}) e^{-i\vec{k}\cdot\vec{x}} \right\} \tag{3.6}$$

is the phonon displacement operator. Evidently, H_1 is the energy that compels electrons to either accumulate near (or evade) deformations of the crystal, which result in a nonzero divergence of the local polarization field; indeed, the quasi-electrons participate in a screened electromagnetic interaction that eminates from the ions, which themselves react by undergoing oscillations (with presumably small displacements) about their equilibrium positions on the lattice.

If H_1 was adiabatically introduced in the far past, then the state of the system (at time t) $\left|\Psi(t)\right\rangle$ is continuously connected to the initial configuration $\left|\Psi(-\infty)\right\rangle$ according to

$$\left|\Psi(t)\right\rangle = \lim_{t_0 \to -\infty} \mathcal{U}_{H_1}(t, t_0) \left|\Psi(t_0)\right\rangle, \tag{3.7}$$

where

$$\mathcal{U}_{H_1}(t, t_0) = 1 - i \int_{t_0}^{t} dt' H_1^I(t') \mathcal{U}_{H_1}(t', t_0) \tag{3.8}$$

is the wave-operator; $H_1^I(t')$ is in the interaction representation. Therefore, insofar as the eigenstates of H_0 form a complete set in the full state space of H, H_1 may be viewed as a perturbation that connects the otherwise orthonormal noninteracting modes, which factor into a product of Bloch waves and the normal vibrations of the crystal.[1] Thus, if $\left|\Phi\right\rangle$ obeys $H_0\left|\Phi\right\rangle = E_\Phi\left|\Phi\right\rangle$, then the transition rate between two such eigenstates of H_0, $\left|\Phi\right\rangle$ and $\left|\Phi'\right\rangle$, is

[1]Note that the structure of the unit cell is irrelevant for the purpose of determining the long-wavelength bulk response; fluctuations on this scale are washed out.

approximately [36]

$$\frac{d}{dt}\left|\left\langle \Phi'\middle|\mathcal{U}_{H_1}(t,-\infty)\middle|\Phi\right\rangle\right|^2 \approx 2\pi\left|\left\langle \Phi'\middle|H_1\middle|\Phi\right\rangle\right|^2 \delta(E_{\Phi'}-E_{\Phi}); \tag{3.9}$$

that Fermi's golden rule is independent of time introduces irreversibility by neglecting initial correlations and suggests that this construction is valid only for time scales much longer than the typical collision time [37, 38].

Now then, since $\mathcal{W}$ is a rate, it is natural to express its action on F as

$$\begin{aligned}\mathcal{W}\big[F\big](\vec{q};t) = \sum_{\vec{p}} \Big\{ \big[1-F(\vec{q};t)\big]\mathcal{W}_{\text{in}}\big[F\big](\vec{q},\vec{p};t)F(\vec{p};t) \\ - F(\vec{q};t)\mathcal{W}_{\text{out}}\big[F\big](\vec{q},\vec{p};t)\big[1-F(\vec{p};t)\big]\Big\},\end{aligned} \tag{3.10}$$

where, in anticipation of our specifying the kernels $\mathcal{W}_{\text{in}}$ and $\mathcal{W}_{\text{out}}$ (which are associated with scattering-in and scattering-out processes, respectively) through the use of reasoning that is concerned only with the dynamics of a two-particle (electron-phonon) collision, the factors F and $1-F$ are explicitly incorporated such that the exclusion principle is manifest (statistically); we are led to consider the matrix elements

$$\begin{aligned}\left\langle N(\vec{k})+1;\vec{q}\middle|H_1\middle|N(\vec{k});\vec{p}\right\rangle &= g\sqrt{\frac{\omega(k)}{2V}}\delta(\vec{q}=\vec{p}-\vec{k})\sqrt{N(\vec{k})+1}, \\ \left\langle N(\vec{k})-1;\vec{q}\middle|H_1\middle|N(\vec{k});\vec{p}\right\rangle &= g\sqrt{\frac{\omega(k)}{2V}}\delta(\vec{q}=\vec{p}+\vec{k})\sqrt{N(\vec{k})},\end{aligned} \tag{3.11}$$

where $\left|N(\vec{k});\vec{p}\right\rangle$ is the state with both $N(\vec{k})$ phonons of wavevector $\vec{k}$ and an electron of momentum $\vec{p}$, from which one can form the amplitudes that correspond to the scattering of an electron from the state $\vec{p}$ to the state $\vec{q}$ as mediated by either the absorption or emission of a phonon of wavevector $\vec{k}$. In this approximation, we have for the number of electrons forced into $\vec{q}$ per unit time [25]

$$\sum_{\vec{p}} \big[1 - F(\vec{q})\big] \mathcal{W}_{\text{in}}\big[F\big](\vec{q}, \vec{p}; t) F(\vec{p}; t)$$

$$= \pi g^2 \int_{\vec{p}} \omega(p) \big\{ F(\vec{q} + \vec{p}) \big[N(\vec{p}) + 1\big] \big[1 - F(\vec{q})\big] \delta\big[\varepsilon(q) + \omega(p) - \varepsilon(|\vec{q} + \vec{p}|)\big] \tag{3.12}$$

$$+ F(\vec{q} - \vec{p}) N(\vec{p}) \big[1 - F(\vec{q})\big] \delta\big[\varepsilon(q) - \omega(p) - \varepsilon(|\vec{q} - \vec{p}|)\big] \big\},$$

and for those ejected from $\vec{q}$

$$\sum_{\vec{p}} F(\vec{q}; t) \mathcal{W}_{\text{out}}\big[F\big](\vec{q}, \vec{p}; t) \big[1 - F(\vec{p}; t)\big]$$

$$= \pi g^2 \int_{\vec{p}} \omega(p) \big\{ F(\vec{q}) \big[1 - F(\vec{q} - \vec{p})\big] \big[N(\vec{p}) + 1\big] \delta\big[\varepsilon(q) - \omega(p) - \varepsilon(|\vec{q} - \vec{p}|)\big] \tag{3.13}$$

$$+ F(\vec{q}) \big[1 - F(\vec{q} + \vec{p}) N(\vec{p}) \delta\big[\varepsilon(q) + \omega(p) - \varepsilon(|\vec{q} + \vec{p}|)\big] \big\}.$$

3.2 Linearized Boltzmann equation

Having specified a model collision operator, our present objective is to solve the linearized Boltzmann equation for the (externally driven) electron-phonon system; however, it is prudent to first examine the conservation laws [39]. To this end, we parameterize the distribution function as

$$F(\vec{q}) = f(\xi_q) + \frac{1}{2} w(\xi_q) \phi(\vec{q}), \tag{3.14}$$

make the unjustified replacement $N(\vec{k}) \to n(\omega[k])$, where the thermalized free phonon occupation numbers

$$n(\epsilon) = 1/(e^{\beta\epsilon} - 1) \tag{3.15}$$

are governed by Bose-Einstein statistics, and retain only the leading effects of the applied fields; these manipulations yield

$$\frac{\vec{q}}{m} \cdot \left[e\vec{E} + T\vec{\nabla} \frac{\xi_q}{T} \right] = \big[C_0 \circ \phi\big](\vec{q}), \tag{3.16}$$

where

$$
\begin{aligned}
&\left[C_0 \circ \phi\right](\vec{q}) \\
&= \pi g^2 \int \frac{d^3p}{(2\pi)^3} \left[\xi_p - \xi_q\right]^2 \mathrm{sgn}(\xi_p - \xi_q)\left[n(\xi_p - \xi_q) + f(\xi_p)\right] \\
&\qquad\qquad \times \delta\left[(\xi_q - \xi_p)^2 - \omega^2(|\vec{p} - \vec{q}|)\right]\left\{\phi(\vec{q}) - \phi(\vec{p})\right\}.
\end{aligned} \tag{3.17}
$$

At this point, it is useful to write

$$
\left[C_0 \circ \phi\right](\vec{q}) = \Gamma(\vec{q})\phi(\vec{q}) - \left[K_0 \circ \phi\right](\vec{q}), \tag{3.18}
$$

where the collision frequency

$$
\Gamma(\vec{q}) = \left[K_0 \circ 1\right](\vec{q}) \tag{3.19}
$$

is generated by

$$
K_0(\vec{q}, \vec{p}) = \left[n(\xi_p - \xi_q) + f(\xi_p)\right]\mathcal{V}''(\vec{q}, \vec{p}), \tag{3.20}
$$

with

$$
\mathcal{V}''(\vec{q}, \vec{p}) = \pi g^2 \left[\xi_p - \xi_q\right]^2 \mathrm{sgn}(\xi_p - \xi_q)\delta\left[(\xi_q - \xi_p)^2 - \omega^2(|\vec{p} - \vec{q}|)\right], \tag{3.21}
$$

proportional to the spectrum of the phonon susceptibility; the reciprocity relation

$$
w(\xi_q)K_0(\vec{q}, \vec{p}) = w(\xi_p)K_0(\vec{p}, \vec{q}) \tag{3.22}
$$

reflects the detailed balance of the underlying (equilibriated) microscopic processes.

Now then, while electron number conservation is equivalent to

$$
\left[C_0 \circ 1\right](\vec{q}) = 0, \tag{3.23}
$$

i.e. the constant function must be a collision invariant, the background Boson bath violates both the conservation of electronic momentum and electronic free energy. Indeed,

$$\begin{aligned}0 &\neq w(\xi_q)\frac{\vec{q}}{q^2}\cdot\left[C_0\circ\vec{k}\right](\vec{q})\\ &= \int\frac{d^3p}{(2\pi)^3}f(\xi_q)n(\xi_p-\xi_q)\left[1-f(\xi_p)\right]\\ &\qquad\qquad\times\frac{1}{2q^2}\left\{q^2-p^2+\left[\frac{\xi_p-\xi_q}{c_s}\right]^2\right\}\mathcal{V}''(\vec{q},\vec{p})\end{aligned}\tag{3.24}$$

shows that momentum is conserved iff the phonons are frozen out; likewise,

$$C_0\circ\xi\neq 0 \tag{3.25}$$

expresses the lack of energy conservation at nonzero temperature.

3.2.1 DC conductivity. We are finally in a position to determine the isothermal conductivity

$$\sigma=\frac{e^2}{6m^2}\int\frac{d^3q}{(2\pi)^3}w(\xi_q)\vec{q}\cdot\vec{\varphi}(\vec{q}), \tag{3.26}$$

where

$$\vec{q}=\left[C_0\circ\vec{\varphi}\right](\vec{q}), \tag{3.27}$$

with

$$\phi(\vec{q})=\frac{e}{m}\vec{E}\cdot\vec{\varphi}(\vec{q}),\quad \vec{\varphi}(\vec{q})=\hat{q}\varphi(\xi_q),\quad \hat{q}=\vec{q}/q. \tag{3.28}$$

Towards finding an asymptotically exact expression for φ, we equip the inner product

$$\left(\vec{\psi};\vec{\phi}\right)=\frac{1}{k_F^2N_F}\int\frac{d^3q}{(2\pi)^3}w(\xi_q)\left[\vec{\psi}\cdot\vec{\phi}\right](\vec{q}) \tag{3.29}$$

under which the collision operator C_0 is self-adjoint, i.e.

$$\left(\vec{\psi};C_0\circ\vec{\phi}\right)=\left(C_0\circ\vec{\psi};\vec{\phi}\right). \tag{3.30}$$

Now, Fredholm's orthogonality condition [15]

$$0\neq\left(\vec{k};\vec{k}\right)=\left(C_0\circ\vec{k};\vec{\varphi}\right), \tag{3.31}$$

proves that Equation 3.27 permits a solution that is singular in limit of vanishing temperature;[2] such a control parameter does not exist in classical kinetic theory, for it is the finite extent of the Fermi sphere that renders nonzero the left-hand-side of Equation 3.31 at zero temperature. Thus, we can write

$$\sigma = \frac{ne^2}{2m}\big(\vec{k};\vec{\varphi}\big) \approx \frac{ne^2}{2m}\big(\vec{k};\vec{\mathfrak{e}}_0\big)\frac{1}{\lambda_0\big(\vec{\mathfrak{e}}_0;\vec{\mathfrak{e}}_0\big)}\big(\vec{\mathfrak{e}}_0;\vec{k}\big)\big[1+\mathcal{O}(T)\big], \tag{3.32}$$

where the eigenproblem

$$C_0 \circ \vec{\mathfrak{e}}_i = \lambda_i \vec{\mathfrak{e}}_i, \tag{3.33}$$

which characterizes the spectrum of C_0 [on the Hilbert space $\vec{\mathfrak{H}}$ containing vector valued functions $\vec{v}(\vec{q}) = \hat{q}v(\xi_q)$ obeying $\big(\vec{v};\vec{v}\big) < \infty$], admits a unique lowest (and isolated) eigenvalue, which we denote by λ_0, with corresponding eigenvector $\vec{\mathfrak{e}}_0 = \hat{q}\mathfrak{e}_0$, that approaches zero as momentum conservation is restored; in the following, we prove this statement within the framework of perturbation theory for linear operators. To this end, consider the scalar equation

$$C \circ \mathfrak{e}_i = \lambda_i \mathfrak{e}_i, \tag{3.34}$$

where

$$\big[C \circ \phi\big](\vec{q}) = \Gamma(\vec{q})\phi(\vec{q}) - \big[K \circ \phi\big](\vec{q}) \tag{3.35}$$

[2] Note that while K_0 is endowed with a discrete spectral representation (on $\vec{\mathfrak{H}}$), C_0 is not. Nevertheless, both our use of Fredholm's alternative (by way of Equation 3.31) and the Laurent series (of the resolvent $R(z) = 1/[C_0 - z]$, where $z \in \mathbb{C}$) in Equation 3.32 remain valid because it is known both that C_0 is a positive operator (on $\vec{\mathfrak{H}}$ for $T > 0$) and that the spectrum of C_0 contains, in addition to a continuous portion that inhabits the interval $(\Gamma|_{\min}, \infty)$, a unique lowest (and isolated) eigenvalue that approaches zero as momentum conservation is restored. While we relegate proof of these statements to section A.1, it is worth mentioning that this structure is familiar from classical kinetic theory, e.g. the rarified gas of hard spheres [22, 40].

is the momentum relaxing collision operator, with

$$
\begin{aligned}
K(\vec{q},\vec{p}) &= \frac{\vec{q}\cdot\vec{p}}{qp}K_0(\vec{q},\vec{p}) \\
&= \frac{k_F^2}{qp}\left\{1-\left[\frac{2(\xi_q-\xi_p)^2}{\omega_D^2}-\frac{\xi_q+\xi_p}{2\epsilon_F}\right]\right\}K_0(\vec{q},\vec{p});
\end{aligned}
\tag{3.36}
$$

if the momentum nonconserving collision operator $L_1 + L_2$, as defined by the kernels

$$
\begin{aligned}
L_1(\vec{q},\vec{p}) &= \frac{k_F^2\xi_q}{q^2\epsilon_F}\Gamma(\xi_q)(2\pi)^3\delta(\vec{q}-\vec{p}) - \frac{k_F^2(\xi_q+\xi_p)}{2qp\epsilon_F}K_0(\vec{q},\vec{p}), \\
L_2(\vec{q},\vec{p}) &= \frac{2k_F^2(\xi_q-\xi_p)^2}{qp\omega_D^2}K_0(\vec{q},\vec{p}),
\end{aligned}
\tag{3.37}
$$

is weak[3] enough that C is asymptotically adjacent to

$$
L_0(\vec{q},\vec{p}) = \frac{k_F^2}{q^2}\Gamma(\xi_q)(2\pi)^3\delta(\vec{q}-\vec{p}) - \frac{k_F^2}{qp}K_0(\vec{q},\vec{p}),
\tag{3.38}
$$

the momentum conserving collision operator,[4] then the fact (see section A.1) that k is the unique (at nonzero temperature) nontrivial element [of the Hilbert space $\mathfrak{H}$ containing scalar valued functions $\nu = \nu(\xi_q)$ obeying $\big(\nu,\nu\big) < \infty$] in the nullspace of L_0, proves that λ_0 is of multiplicity one; here

$$
\big(\psi,\phi\big) = \frac{1}{k_F^2 N_F}\int\frac{d^3q}{(2\pi)^3}w(\xi_q)\psi(\xi_q)\phi(\xi_q)
\tag{3.39}
$$

is the induced inner product that allows for

$$
\big(\psi, L_0\circ\phi\big) = \big(L_0\circ\psi,\phi\big), \quad \big(\psi, C\circ\phi\big) = \big(C\circ\psi,\phi\big).
\tag{3.40}
$$

Note that the dissipation of a vector quantity depends not only on the time between scattering events, as there is an additional angular weight that gives proper importance to large angle scattering [26] while ensuring that glancing blows are less effective at transfering momentum.

---

[3]When $T \ll \omega_D$, we have $q,p \sim k_F\big[1+\mathcal{O}(T/\epsilon_F)\big]$ for the scaling of a typical electronic wavenumber.

[4]By construction, $L_0\circ k = 0$; indeed, particle number conservation implies momentum conservation in the absence of momentum dissipating interactions.

Finally, λ_0 can be systematically determined by matching coefficients of t, the book-keeping coefficient, where

$$C = L_0 + t\big[L_1 + L_2\big], \quad \lambda_0 = \mathcal{O}(t), \quad \mathfrak{e}_0 = k + \mathcal{O}(t); \tag{3.41}$$

in the physical case, $t = 1$. From the equation governing the first corrections

$$L_0 \circ \mathfrak{e}_0^{(1)} + \big[L_1 + L_2\big] \circ k = \lambda_0^{(1)} k \tag{3.42}$$

we conclude that

$$\lambda_0^{(1)}\big(k, k\big) = \big(k, \big[L_1 + L_2\big] \circ k\big) = \mathcal{O}(T^5/\omega_D^4), \quad T \ll \omega_D \tag{3.43}$$

is both necessary and sufficient for the existence of $\mathfrak{e}_0^{(1)}$. Next,

$$C_0 \circ \mathfrak{e}_0^{(2)} + K_2 \circ \mathfrak{e}_0^{(1)} = \lambda_0^{(2)} k + \lambda_0^{(1)} \mathfrak{e}_0^{(1)} \tag{3.44}$$

leads to[5]

$$\begin{aligned} \lambda_0^{(2)}\big(k, k\big) &= \big(k, \big[L_1 + L_2\big] \circ \mathfrak{e}_0^{(1)}\big) \\ &\approx \frac{-1}{2\epsilon_F}\big(k, \big[L_1 + L_2\big] \circ k\xi\big)\big[1 + \mathcal{O}(T^2/\omega_D)\big], \quad T \ll \omega_D \end{aligned} \tag{3.45}$$

since Equation 3.42 implies[6]

$$\mathfrak{e}_0^{(1)} = \frac{-1}{2\epsilon_F} k\xi + \mathcal{O}(T^2/\omega_D), \quad T \ll \omega_D; \tag{3.46}$$

evidently, we need not dive deeper. It follows that $L_1 + L_2$ perturbs the zero eigenvalue such that C is a positive operator.

[5]We have made use of our freedom to enforce $\big(1, \mathfrak{e}_0^{(i)}\big) = 0$ for integer $i > 0$.

[6]To see this, notice that $L_0 \circ k\xi = 2\epsilon_F L_1 \circ k$.

As a result, the dominant behavior of $\tau_{tr}^{-1} = 2\lambda_0$ (at asymptotically low temperature) is entirely captured by[7]

$$\lambda_0 \approx \left(1, K_2 \circ 1\right) \approx \nu_0 \pi \begin{cases} 480\zeta(5)\frac{T^5}{\omega_D^4}\left[1 - \frac{\omega_D^2}{16\epsilon_F^2}\right] & T \ll \omega_D \ll \epsilon_F \\ T & \omega_D \ll T \ll \epsilon_F \end{cases}, \tag{3.47}$$

where $\omega_D = 2c_s k_F$ is the Debye frequency, $\nu_0 = N_F g^2$ is a dimensionless constant of order one, and ζ is the Riemann zeta function; Equation 3.47 proves Bloch's law [2].

3.2.2 Thermopower. For an open circuit in which a temperature difference is maintained across the ends, the thermopower S, as defined by

$$-\vec{\nabla}\tilde{\mu} = -eS\vec{\nabla}T, \tag{3.48}$$

satisfies

$$\vec{h}(\vec{q}) = \left[C_0 \circ \vec{\varphi}\right](\vec{q}), \quad \vec{h}(\vec{q}) = \vec{q}\left[-eS - \frac{\xi_q}{T}\right] \tag{3.49}$$

under the condition of no mass flow

$$0 = \left(\vec{k}; \vec{\varphi}\right); \tag{3.50}$$

Equation 3.49 is obtained by letting

$$\phi(\vec{q}) = \frac{1}{m}\vec{\varphi}(\vec{q}) \cdot \vec{\nabla}T, \quad \vec{\varphi}(\vec{q}) = \hat{q}\varphi(\xi_q). \tag{3.51}$$

From here, it remains true that

$$\left(\vec{k}; \vec{\varphi}\right) \approx \left(\vec{k}; \vec{e}_0\right)\frac{1}{\lambda_0\left(\vec{e}_0; \vec{e}_0\right)}\left(\vec{e}_0; \vec{h}\right)\left[1 + \mathcal{O}(T)\right]. \tag{3.52}$$

---

[7]While we wish to emphasize that the prefactors in Equation 3.47 are exact consequences of the Fröhlich Hamiltonian in the first approximation, one should bear in mind that it is inconsistent to keep the term of relative order ω_D^2/ϵ_F^2 because, according to Migdal's theorem [41], we have (by taking electron-phonon coupling g as a constant) already neglected corrections to the interaction vertex of this order.

Thus, the requirement $0 = \left(\vec{\varepsilon}_0; \vec{h}\right)$ reads (after making the change of variables $\epsilon = \xi_q$)

$$0 = \int_{-\epsilon_F}^{\infty} d\epsilon\, w(\epsilon) \sqrt{1 + \frac{\epsilon}{\epsilon_F}} \left[1 + \frac{\epsilon}{\epsilon_F}\right] \left(1 - \frac{\epsilon}{2\epsilon_F}\right) \left\{eS + \frac{\epsilon}{T}\right\}, \tag{3.53}$$

which implies

$$S \approx \frac{\pi^2 T}{3|e|\epsilon_F}\left[1 + \mathcal{O}(T/\epsilon_F)\right], \tag{3.54}$$

as found by Wilson [16] and Sondheimer [17].

In passing, we stress the importance of carefully positing the Boltzmann equation; this result is incredibly sensitive to terms that are naively irrelevant at the outset.

3.2.3 Heat conductivity. Since energy is distributed among both the electronic and the vibronic degrees of freedom, one should not take the phonon occupation numbers to be given by the Bose distribution when computing the heat conductivity, for this omission of feedback violates energy conservation. Indeed, the effective collision operator employed so far in this section obeys[8]

$$C_0 \circ \omega \neq 0, \tag{3.55}$$

where $\omega(\vec{q}, \epsilon) = \epsilon$. Thus, before proceeding to evaluate Fourier's coefficient, we ought to introduce an operator $\mathcal{C}_0$ that admits both[9]

$$\mathcal{C}_0 \circ 1 = 0, \quad \mathcal{C}_0 \circ \omega = 0; \tag{3.57}$$

[8] In this situation, we find it convenient to use the many-body formalism wherein the Ward-Takahashi identities unequivocally imply that ω is in the nullspace of C_0 iff the total system energy is conserved; see also subsection B.8.2.

[9] If one insists on using C_0 rather than $\mathcal{C}_0$, then they would find for the energy-energy susceptibility

$$\Pi_{\varepsilon\varepsilon}(\vec{k} = 0, \Omega) \neq 0, \tag{3.56}$$

which cannot be true.

then, the heat diffusion coefficient is given by

$$D_\varepsilon = \frac{\left(\vec{k}\omega; \mathcal{C}_0^{-1} \circ \vec{k}\omega\right)}{m^2(\omega, \omega)}, \tag{3.58}$$

with $\vec{k}(\vec{q}, \epsilon) = \vec{q}$. While arriving at Equation 3.58 is not difficult, its evaluation is no simple task because one would need to realize[10] such a $\mathcal{C}_0$ and then ascertain the action of $\mathcal{C}_0^{-1}$ on $\vec{k}\omega$; we thus find ourselves in a predicament insofar as C_0 obeys Equation 3.55 despite the fact that the heat conductivity formula is derived under the assumption of energy conservation.[11] Towards alleviating our embarrassment, we'll adopt the accepted response to the above concern, which is to ignore it. Said less facetiously, one argues that it is a mistake to treat the electron-phonon system in isolation; in this way, the nonzero value of $C_0 \circ \omega$ is associated with energy lost to the environment.

Now then, we have for the heat conductivity

$$\sigma_h = \frac{\left(\vec{k}\omega; C_0^{-1} \circ \vec{k}\omega\right)}{m^2 T} \propto \frac{\omega_D^2}{T^2}, \tag{3.59}$$

where the dimensionful proportionality factor is temperature independent; since the inhomogeneity of the integral equation

$$C_0 \circ \vec{\varphi} = \vec{k}\omega \tag{3.60}$$

is not a bare hydrodynamic mode, we are limited to dimensional analysis in gathering the result that is Equation 3.59. To further approach the laboratory situation, we consider the thermal conductivity

$$\kappa = \frac{1}{m^2 T}\left[\left(\vec{k}\omega; C_0^{-1} \circ \vec{k}\omega\right) - \frac{\left(\vec{k}\omega; C_0^{-1} \circ \vec{k}\right)^2}{\left(\vec{k}; C_0^{-1} \circ \vec{k}\right)}\right], \tag{3.61}$$

[10] Note that Φ-derivability does not suffice to ensure that C_0 is conserving; the proofs supplied by Kadanoff, Baym, and Piman [24, 39, 42] apply only when the bare vertex is static.

[11] An alternate derivation of the Kubo formula for the heat conductivity that more directly involves the current $\vec{k}\omega$ is recited in subsubsection B.2.1.2.

which differs from the heat conductivity in that the condition of no mass flow is enforced; notice that the momentum mode does not supply a leading contribution toEquation 3.61.

CHAPTER IV

ELECTRONS, FERROMAGNONS, AND QUENCHED DISORDER

We now turn our attention towards the leading ballistic corrections to the DC conductivity as predicted by a Φ-derivable theory that couples otherwise free electrons to both a random potential field (which represents static impurities) and a bath of spin-waves (which models the magnetization fluctuations of a metallic ferromagnet). Although the magnon energy-momentum resonance $\omega = Dk^2$, with D the spin-wave stiffness, is soft, these Bosons are unable to mediate intraband transitions; thus, the exchange spectrum is gapped, for the electronic spin degeneracy is broken by a Zeeman term.

This chapter includes previously published co-authored material.

4.1 Kubo formula

Beginning with an action of the form

$$S\left[\bar{\psi}, \psi, u\right] = S_0\left[\bar{\psi}, \psi\right] + S_{\text{int}}\left[\bar{\psi}, \psi\right] + S_{\text{dis}}\left[\bar{\psi}, \psi, u\right], \tag{4.1}$$

where $\bar{\psi}, \psi$ are the (Grassmann valued [43]) electron field operators and u is a random (static) potential field, we proceed to study the Kubo function [34]

$$\Pi_R^{il}\left[x; x'\right] = -ie^2\theta\left[t - t'\right]\left\langle\left[j^i(x), j^l(x')\right]\right\rangle, \tag{4.2}$$

for the equilibrium fluctuations Π_R determine the linear response

$$e\left\{\left\langle\delta j^i(x)\right\rangle\right\}_{\text{dis}} = \text{Im}\left[\frac{-1}{\Omega}\Pi_R^{il}(k)e^{ikx}\right]E_0^l \tag{4.3}$$

to an applied field

$$\vec{E}_{\vec{k},\Omega}(\vec{x}, t) = \vec{E}_0 \cos\left[\vec{k}\cdot\vec{x} - \Omega t\right]. \tag{4.4}$$

Here, the disorder average

$$\left\{\Pi_R^{il}\left[p; p'\right]\right\}_{\text{dis}} = (2\pi)^4\delta\left[p - p'\right]\Pi_R^{il}(p) \tag{4.5}$$

restores spatial translation invariance, which is otherwise violated by any single impurity configuration.

Our goal is to compute the DC conductivity tensor

$$\sigma_{ij} = \lim_{\Omega \to 0} \frac{-1}{\Omega} \Pi''_{ij}(\Omega); \tag{4.6}$$

the spectrum Π'' is most conveniently obtained in the Matsubara (imaginary time) formalism [44], where

$$\Pi^{il}\left[\vec{k}, \tau \middle| \vec{k}', \tau'\right] = -e^2 \left\langle T_\tau \left[j^i(-\vec{k}, \tau) j^l(\vec{k}', \tau') \right] \right\rangle \tag{4.7}$$

is a timeordered (causal) correlation function that both admits a diagrammatic expansion and (on analytic continuation) [45]

$$\Pi''_{il}\left[\vec{k}, \vec{k}'; \Omega\right] = \mathrm{Im}\ \Pi_{il}\left[\vec{k}, \vec{k}'; i\Omega \to \Omega + i0\right]. \tag{4.8}$$

In order to calculate

$$\Pi^{il}(\vec{k}, i\Omega) = -e^2 \left\{ \left\langle j^i(-\vec{k}, -i\Omega) j^l(\vec{k}, i\Omega) \right\rangle \right\}_{\mathrm{dis}}, \tag{4.9}$$

we will employ the replica trick [46]. As a result of this procedure, we have

$$\begin{aligned} &\Pi^{ij}(2k) \\ &= \frac{e^2}{m^2} \lim_{N \to 0} \frac{1}{N} \sum_{\zeta=1}^{N} \frac{T}{V} \sum_{q,\sigma} q^i \mathfrak{G}^{\sigma}_{\zeta}(q_+) \mathfrak{G}^{\sigma}_{\zeta}(q_-) \\ &\qquad \times \left\{ q^j + \frac{T}{V} \sum_{p,\sigma'} \sum_{\zeta'=1}^{N} p^j \Lambda^{\sigma\sigma'}_{\zeta\zeta'}\left[q, p; 2k\right] \mathfrak{G}^{\sigma'}_{\zeta'}(p_+) \mathfrak{G}^{\sigma'}_{\zeta'}(p_-) \right\}, \end{aligned} \tag{4.10}$$

which is depicted graphically in Figure 1; Λ describes the bound, two-particle, interactions and is defined in Equation 4.37 through the four-point Green function [47]. We denote $q_\pm = q \pm k$, as is customary. The heavy, arrowed, lines on diagrams represent full Green functions $\mathfrak{G}$, which obey the Dyson equation

$$\mathfrak{G} = G_0^{-1} - \Sigma, \tag{4.11}$$

with G_0 the free electron [33]. To avoid clutter, we often suppress the frequency, wavevector, spin, and replica indices. Eventually, the symbol $\mathcal{G}$ will be used to denote the exact propagator in approximate theories (as obtained by retaining only a subset of the Λ processes).

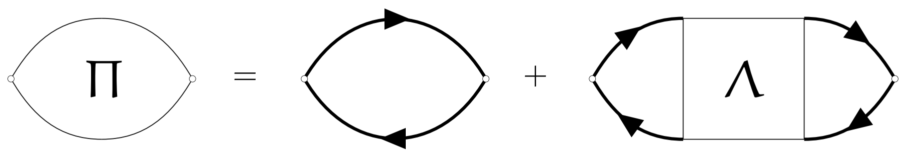


Figure 1. The polarization bubble of Equation 4.10; all vertex corrections are contained in Λ.

For the conductivity to be infinite in the limit of vanishing scattering amplitude, it is necessary to perform an all order analysis (rather than collecting a finite number of diagrams); in anticipation of our choosing to sum only a limited (albeit infinite) class of diagrams, we forewarn the reader of an obstacle that must be overcome: although the continuity equations are maintained if one performs a consistent expansion in powers of the coupling, this need not be true for approximate theories that are constructed by simultaneously selecting contributions at all orders while neglecting diagrams at finite orders (even if each vertex that the diagrams are composed of is individually conserving). Nevertheless, Kadanoff and Baym [24, 39] showed that the conservation laws are preserved if[1]

$$\tilde{\Lambda} = \frac{\delta \Sigma}{\delta \mathcal{G}}, \tag{4.12}$$

[1]Strictly speaking, the proofs supplied by Kadanoff, Baym, and Pičman [24, 39, 42] assume the bare vertex to be static; thus, while our vertex $\tilde{\Lambda}$ is derivable by summing all terms obtained on opening a single line from each self-energy diagram Σ, it is preferable to say that we work with a Φ-derivable theory rather than a conserving approximation.

where $\tilde{\Lambda}$ is the the irreducible[2] particle-hole vertex [see Figure 5], which implies that self-energy contributions and vertex corrections cannot be chosen independently. By considering [see Figure 2]

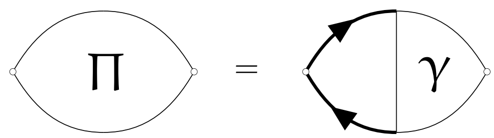


Figure 2. A diagrammatic definition of γ, the (current-)vertex part.

$$\gamma^{j}_{\sigma,\zeta}\left[q_{+},q_{-};2k\right]=q^{j}+\frac{T}{V}\sum_{p,\sigma'}\sum_{\zeta'=1}^{N}p^{j}\Lambda^{\sigma\sigma'}_{\zeta\zeta'}\left[q,p;2k\right]\mathfrak{G}^{\sigma'}_{\zeta'}(p_{+})\mathfrak{G}^{\sigma'}_{\zeta'}(p_{-}),\qquad(4.13)$$

and using the Bethe-Salpeter equation [48, 49],

$$\begin{aligned}&\Lambda\left[1,2;3,4\right]\\&=\tilde{\Lambda}\left[1,2;3,4\right]+\int_{i'}\tilde{\Lambda}\left[1,2';3',4\right]\mathfrak{G}\left[3';1'\right]\mathfrak{G}\left[4';2'\right]\Lambda\left[1',2;3,4'\right],\end{aligned}\qquad(4.14)$$

of Figure 3 to iterate, one achieves the analytical expression [see Figure 4]

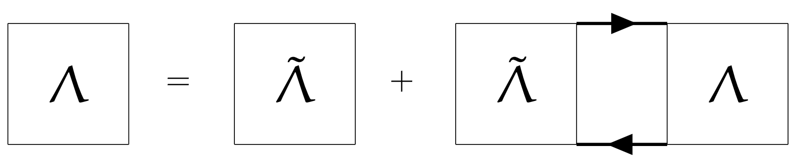


Figure 3. This (particle-hole channel) Bethe-Salpeter equation allows us to reorganize the diagrammatic expansion.

$$\begin{aligned}&\gamma^{j}_{\sigma,\zeta}\left[q_{+},q_{-};2k\right]\\&=q^{j}+\sum_{\zeta'=1}^{N}\int_{p,\sigma'}\tilde{\Lambda}^{\sigma\sigma'}_{\zeta\zeta'}\left[q,p;2k\right]\mathfrak{G}^{\sigma'}_{\zeta'}(p_{+})\mathfrak{G}^{\sigma'}_{\zeta'}(p_{-})\gamma^{j}_{\sigma',\zeta'}\left[p_{+},p_{-};2k\right].\end{aligned}\qquad(4.15)$$

Equation 4.15 represents the propagation of a quasi-particle pair that encounters an infinite chain of all possible collisions and requires solving an integral equation. Physically, a particle-like description of charge transport in many-electron systems must incorporate their essential intercorrelations [39], which

[2]When speaking of particle-hole irreducibility, we always refer to the $(1,4)\leftrightarrow(2,3)$ channel.

develop in this case due to incessant exchange of momentum through magnons and lattice defects.

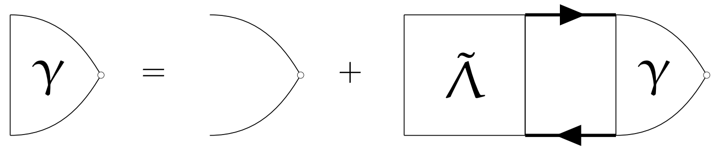


Figure 4. The ladder (integral) equation for γ has $\mathfrak{G}\tilde{\Lambda}\mathfrak{G}$ as its kernel.

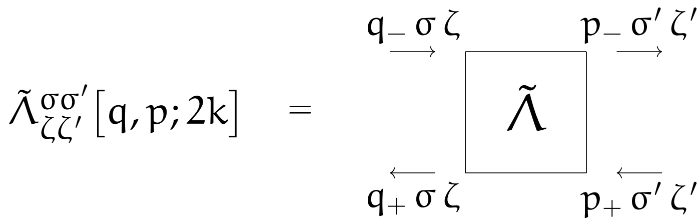


Figure 5. Our convention for labeling the irreducible particle-hole vertex; the energy-momentum conserving delta function is understood to be exhausted by an integral.

Witness that one need only specify $\tilde{\Lambda}$ in order to determine Π. Indeed, Toyoda [14] has shown that

$$\Sigma^{\sigma}_{\zeta}(q_+) - \Sigma^{\sigma}_{\zeta}(q_-) = \sum_{\zeta'=1}^{N} \int_{p,\sigma'} \left[\mathfrak{G}^{\sigma'}_{\zeta'}(p_+) - \mathfrak{G}^{\sigma'}_{\zeta'}(p_-)\right] \tilde{\Lambda}^{\sigma\sigma'}_{\zeta\zeta'}[p, q; 2k] \tag{4.16}$$

is a consequence of the continuity (operator) equation

$$\frac{d}{d\tau} n_{\sigma}(x) = -\vec{\nabla} \cdot \vec{j}_{\sigma}(x) + Q_{\sigma}(x), \qquad \sum_{\sigma} Q_{\sigma} = 0, \tag{4.17}$$

where $\vec{j}$ is the number current,

$$n_{\sigma}(x) = \bar{\psi}_{\sigma}(x)\psi_{\sigma}(x) \tag{4.18}$$

is the spin-σ number density, and Q_{σ} is an interband flux term whose realization is not needed. Furthermore, the Kramers-Kronig formula [50]

$$\Sigma'(\vec{p}, \epsilon) = \text{P.V.} \int \frac{du}{\pi} \frac{\Sigma''(\vec{p}, u)}{u - \epsilon}, \tag{4.19}$$

together with Equation 4.16, constitutes a definition of Σ that ensures the fundamental law governing mass transport is respected; particle number conservation has been baked in. Moreover, the momentum balance equation

$$m\frac{d}{d\tau}j^{i}_{\sigma}(x) = -\partial_j T^{\sigma}_{ij}(x) + \vec{Q}_{\sigma}(x), \tag{4.20}$$

where T_{ij} is the flux tensor and $\sum_{\sigma}\vec{Q}_{\sigma} \neq 0$ due to the dissipating bath, implies

$$\begin{aligned}&\vec{q}\big[\Sigma^{\sigma}_{\zeta}(q+i\Omega) - \Sigma^{\sigma}_{\zeta}(q-i\Omega)\big]\\ &= \vec{P}^{\sigma}_{\zeta}(q;i\Omega) + \sum_{\zeta'=1}^{N}\int_{p,\sigma'}\vec{p}\big[\mathfrak{G}^{\sigma'}_{\zeta'}(p+i\Omega) - \mathfrak{G}^{\sigma'}_{\zeta'}(p-i\Omega)\big]\tilde{\Lambda}^{\sigma\sigma'}_{\zeta\zeta'}\big[p,q;2i\Omega\big],\end{aligned} \tag{4.21}$$

where $\vec{P}$ vanishes iff momentum is conserved; for brevity, we omit the rather unwieldy expression that defines $\vec{P}$ as a functional of $\vec{Q}$, since $\vec{P}$ must be such that Equation 4.16 and Equation 4.21 are mutually consistent. In this way, Equation 4.16 and Equation 4.21, together with the corresponding relation that governs energy flow, supply knowledge pertaining to the hydrodynamic modes of the collision operator.

In the following, we develop a system of coupled integral equations (which should be interpreted as a generalized Boltzmann equation [51, 52]), governing the distribution functions γ, for which the Ward-Takahashi identities facilitate proof of existing solutions by way of orthogonality conditions.
Such tools of Fredholm theory [15] give one leverage to extract the transport coefficients; our technique improves upon Toyoda's [53] result (which is itself an extension of Éliashberg's [54] theory) for the relaxation rate in terms of the proper vertex part by including not only those electrons on the Fermi surface.

4.2 Effective action

Having taken advantage of fundamental symmetries, we motion to specify the isotropic metallic ferromagnet: the bare action

$$\tilde{S}_0 = \sum_{\zeta=1}^{N} \int dx \sum_{\sigma} \bar{\psi}_{\zeta}^{\sigma}(x) \left[\frac{-d}{d\tau} - \omega_{\sigma}(\vec{p}) \right] \psi_{\zeta}^{\sigma}(x), \quad \omega_{\sigma}(\vec{p}) = \frac{-\Delta}{2m} - \epsilon_F^{\sigma}, \tag{4.22}$$

propagates free electrons

$$G_0^{\sigma}(p) = \frac{1}{ip - \omega_{\sigma}(\vec{p})}, \quad \epsilon_F^{\sigma} = \epsilon_F + \sigma\lambda, \tag{4.23}$$

with band degeneracy broken by the Stoner gap λ, which is proportional to the average background magnetization, wherefrom spin-wave fluctuations originate that couple to the electron magnetic moments by an effective interaction [3]

$$\tilde{S}_{\text{int}} = \nu_0 \frac{1}{2} \sum_{\zeta=1}^{N} \int dx dy \sum_{\sigma,\sigma'} n_{\sigma'\sigma}^{\zeta}(x) \mathcal{V}_{\sigma\sigma'}(x-y) n_{\sigma\sigma'}^{\zeta}(y), \tag{4.24}$$

where $\mathcal{V}$ is a magnon exchange potential and

$$\nu_0 = \frac{2T_1}{N_F} \mathcal{V}_0, \tag{4.25}$$

with $\mathcal{V}_0$ a dimensionless number of order one that (in Stoner-Moriya mean-field theory) is given by [55]

$$\mathcal{V}_0 = \frac{\pi\lambda}{T_1}; \quad \text{(Stoner-Moriya)} \tag{4.26}$$

here, T_1 is analogous to the the Debye temperature. Lastly, the vertex [56]

$$\tilde{S}_{\text{dis}} = u_0 \frac{1}{2} \sum_{\zeta,\zeta'=1}^{N} \int d\vec{x} d\tau d\tau' \sum_{\sigma,\sigma'} n_{\zeta}^{\sigma}(\vec{x},\tau) n_{\zeta'}^{\sigma'}(\vec{x},\tau'), \tag{4.27}$$

with

$$u_0 = \frac{1}{2\pi N_F \tau} \tag{4.28}$$

the random potential strength, comes from marginalizing the electron-impurity coupling

$$\begin{aligned} S_{\rm dis} &= -\int dx \sum_{\sigma} \bar{\psi}_{\sigma}(x) u(\vec{x}) \psi_{\sigma}(x) \\ &= \frac{-1}{V} \sum_{\vec{k},\vec{p}} u(\vec{k}-\vec{p}) \sum_{i\omega,\sigma} \bar{\psi}_{\sigma}(\vec{k},i\omega)\psi_{\sigma}(\vec{p},i\omega) \end{aligned} \tag{4.29}$$

over the Gaussian ensemble of impurity configurations

$$\mathcal{P}[u] = \frac{1}{Z_u} \exp\Big\{ -\pi N_F \tau \int d\vec{x}\, u^2(\vec{x}) \Big\}, \tag{4.30}$$

which has both zero mean and variance

$$\big\{u(\vec{x})u(\vec{y})\big\}_{\rm dis} = \int_u u(\vec{x})u(\vec{y})\mathcal{P}[u] = \frac{1}{2\pi N_F \tau}\delta[\vec{x}-\vec{y}]; \tag{4.31}$$

$u(\vec{x})$ describes scattering by distortions of the lattice that are static, spherically symmetric, and uncorrelated in space. Here, Z_u is the normalization factor, τ is the elastic mean free time, and we denote by

$$\epsilon_F = \frac{k_F^2}{2m}, \tag{4.32}$$

the zero temperature Fermi level (when ν_0, u_0 interactions are neglected), so that

$$N_{\sigma}(\epsilon) = \frac{1}{\pi V} \sum_{\vec{k}} \big[-\mathfrak{G}''_{\sigma}(\vec{k},\epsilon) \big] \approx \frac{m k_F}{2\pi^2} \equiv N_F \tag{4.33}$$

is the density of states (on the Fermi surface) and

$$\int d\epsilon\, f(\epsilon) \sum_{\sigma} N_{\sigma}(\epsilon) \approx \frac{m k_F}{2\pi^2} \int_{-\epsilon_F}^{0} d\epsilon \sum_{\sigma} \sqrt{1+\epsilon/\epsilon_F} \approx \frac{k_F^3}{3\pi^2} \equiv n \tag{4.34}$$

is the electronic number density. Note that on Fourier transforming

$$u(\vec{x}) = \frac{1}{V} \sum_{\vec{k}} e^{i\vec{k}\cdot\vec{x}} u(\vec{k}), \tag{4.35}$$

we have

$$\big\{u(\vec{k})u(\vec{k'})\big\}_{\rm dis} = u_0 \delta(\vec{k}+\vec{k'}); \tag{4.36}$$

to properly restore translation invariance in the presence of quenched disorder, one must disorder average the observable of interest (rather than the partition function). Indeed, it is necessary to address the extreme separation of scales associated with the relaxation of electronic and impurity degrees of freedom; while the defects are always frozen in frustration, electrons zip about and (relatively) quickly equilibriate with the local Hamiltonian.

Our task is now to write and extract from the transport equation all leading beyond semi-classical corrections to the conductivity that arise from correlations between scattering mechanisms, i.e. those involving the $v_0 u_0$ product.

4.3 Transport equation

To obtain the relaxation kernel in perturbation theory, first gather the scattering amplitude by reading off Λ from the two-particle Green function

$$\begin{aligned} \mathfrak{G}^{\mathrm{II}}(1,2;3,4) &= \left\langle T_\tau\left\{\psi(1)\psi(2)\bar{\psi}(3)\bar{\psi}(4)\right\}\right\rangle_{\tilde{S}} \\ &= \mathfrak{G}(1;4)\mathfrak{G}(2;3) - \mathfrak{G}(1;3)\mathfrak{G}(2;4) \\ &\quad - \int_{i'} \mathfrak{G}\left[1;1'\right]\mathfrak{G}\left[2;2'\right]\Lambda\left[1',2';3',4'\right]\mathfrak{G}\left[3';3\right]\mathfrak{G}\left[4';4\right]. \end{aligned} \tag{4.37}$$

Using the notation

$$\begin{aligned} &\tilde{\Lambda}\left[x_1,x_2;x_3,x_4\right] \\ &= \int_{q,p,k} \tilde{\Lambda}\left[q,p;2k\right] e^{iq(x_1-x_4)+ip(x_2-x_3)+ik(x_1+x_4-x_2-x_3)}, \end{aligned} \tag{4.38}$$

one finds (to linear order in v_0, u_0)

$$\delta_{\zeta\zeta'}\left[v_0 \mathcal{V}_{\sigma\sigma'}(p-q) + u_0\beta\delta_{\omega_q\omega_p}\delta_{\sigma\sigma'}\right] \in \tilde{\Lambda}^{\sigma\sigma'}_{\zeta\zeta'}\left[q,p;2i\Omega\right]; \tag{4.39}$$

we've kept only exchange diagrams, for the ferromagnon doesn't mediate intraband transitions and the direct impurity term contributes at higher order in the replica number.

It follows that the single particle rate

$$\Gamma = -\Sigma'', \tag{4.40}$$

in the approximation of Equation 4.39, reads

$$\Gamma_\sigma(\vec{q}, \epsilon) = \Gamma_\sigma^{(v)}(\vec{q}, \epsilon) + \Gamma_\sigma^{(u)}(\epsilon), \tag{4.41}$$

where

$$\begin{aligned} &\Gamma_\sigma^{(v)}(\vec{q}, \epsilon) \\ &= v_0 \int \frac{du}{\pi} \{ n(u-\epsilon) + f(u) \} \frac{1}{V} \sum_{\vec{p}, \sigma'} \mathcal{V}''_{\sigma\sigma'}(\vec{p} - \vec{q}, u - \epsilon) \big[-\mathcal{G}''_{\sigma'}(\vec{p}, u) \big] \end{aligned} \tag{4.42}$$

and

$$\Gamma_\sigma^{(u)}(\epsilon) = u_0 \frac{1}{V} \sum_{\vec{p}} \big[-\mathcal{G}''_\sigma(\vec{p}, \epsilon) \big]. \tag{4.43}$$

Note that Equation 4.16 is an integral equation that we shall assume to admit an iterative solution. Corrections to the chemical potential are obtained through the use of

$$\begin{aligned} &\Sigma'_{\sigma,\zeta}(\vec{q}, \epsilon) - \Sigma'_{\sigma,\zeta}(\vec{q}, 0) \\ &= \lim_{iq \to \epsilon + i0} \lim_{i\Omega \to i0} \sum_{\zeta'=1}^{N} \int_{p, \sigma'} \big[\mathfrak{G}_{\sigma'}^{\zeta'}(p_+) - \mathfrak{G}_{\sigma'}^{\zeta'}(p_-) \big] \tilde{\Lambda}_{\sigma\sigma'}^{\zeta\zeta'}[p, q; 2i\Omega], \end{aligned} \tag{4.44}$$

with

$$\Sigma'(\vec{q}, 0) = \text{P.V.} \int \frac{d\epsilon}{\pi} \frac{\Sigma''(\vec{q}, \epsilon)}{\epsilon}. \tag{4.45}$$

Next, we transform[3]

$$\begin{aligned} &\Pi^{ij}(2i\Omega \to \Omega + i0) \\ &= \frac{e^2}{m^2} \lim_{i\Omega \to \Omega + i0} \lim_{N \to 0} \frac{1}{N} \sum_{\zeta=1}^{N} \int_{q, \sigma} q^i \mathfrak{G}_\zeta^\sigma(q_+) \gamma_{\sigma,\zeta}^j [q_+; q_-] \mathfrak{G}_\zeta^\sigma(q_-) \end{aligned} \tag{4.47}$$

---

[3]Since the DC conductivity requires only the zero wavenumber susceptibility, we know that

$$\gamma^j[q_+; q_-] = q^j \gamma[q_+; q_-], \quad q_\pm = q \pm i\Omega. \tag{4.46}$$

into an integral along the real line[4]

$$\sigma = \frac{e^2}{6m^2}\lim_{N\to 0}\sum_{\zeta=1}^{N}\int\frac{d\epsilon}{\pi}w(\epsilon)\frac{1}{V}\sum_{\vec{q},\sigma} \times \mathrm{Re}\Big\{\mathfrak{G}^{R}_{\sigma,\zeta}(\vec{q},\epsilon)\vec{q}\cdot\vec{\gamma}^{RA}_{\sigma,\zeta}(\vec{q},\epsilon)\mathfrak{G}^{A}_{\sigma,\zeta}(\vec{q},\epsilon) - \big[\mathfrak{G}^{R}_{\sigma,\zeta}(\vec{q},\epsilon)\big]^2\vec{q}\cdot\vec{\gamma}^{RR}_{\sigma,\zeta}(\vec{q},\epsilon)\Big\}; \tag{4.48}$$

the weight

$$w(\epsilon) = \frac{-\partial f}{\partial\epsilon}(\epsilon) = \frac{1}{4T}\cosh^{-2}\frac{\epsilon}{2T} \tag{4.49}$$

approaches Dirac's $\delta(\epsilon)$ in the (distribution) limit of vanishing temperature, and

$$\begin{aligned}\vec{\gamma}_{RA}(\vec{q},\epsilon) &= \lim_{i\Omega\to i0}\lim_{iq\to\epsilon+i0}\vec{\gamma}\big[q_+;q_-\big],\\ \vec{\gamma}_{RR}(\vec{q},\epsilon) &= \lim_{iq\to\epsilon+i0}\lim_{i\Omega\to i0}\vec{\gamma}\big[q_+;q_-\big],\\ \vec{\gamma}_{AA}(\vec{q},\epsilon) &= \lim_{iq\to\epsilon-i0}\lim_{i\Omega\to i0}\vec{\gamma}\big[q_+;q_-\big],\end{aligned} \tag{4.50}$$

with $\gamma_{RR} = \gamma^*_{AA}$, since γ is symmetric in its arguments. From Equation 4.16, one can derive Langer's identity [26, 57]

$$\vec{\gamma}_{RR}(\vec{q},\epsilon) = \vec{q} + m\frac{\partial}{\partial\vec{q}}\Sigma^R(\vec{q},\epsilon), \tag{4.51}$$

which demonstrates that the explicitly γ_{RR} dependent contribution to Equation 4.48 is regular in the limit of vanishing damping; these (nonhydrodynamic) contributions to the conductivity will be discarded. Additionally, the otherwise trio of coupled integral equations has been reduced to a single objective for γ_{RA}, with both γ_{RR} and γ_{AA} entering only to the effect of a (to be neglected) correction to the inhomogeneity. By retaining only those retarded-advanced terms corresponding to the long-lived excitations, we work in the hydrodynamic approximation

$$\sigma \sim \frac{e^2}{6m^2}\lim_{N\to 0}\sum_{\zeta=1}^{N}\int\frac{d\epsilon}{\pi}w(\epsilon)\frac{1}{V}\sum_{\vec{q},\sigma}\mathfrak{G}^{R}_{\sigma,\zeta}(\vec{q},\epsilon)\vec{q}\cdot\vec{\gamma}^{RA}_{\sigma,\zeta}(\vec{q},\epsilon)\mathfrak{G}^{A}_{\sigma,\zeta}(\vec{q},\epsilon), \tag{4.52}$$

[4]The analyic structure of γ is known from the work of Éliashberg [54].

with

$$\vec{\gamma}^{RA}_{\sigma,\zeta}(\vec{q},\epsilon)$$
$$= \vec{q} + u_0 \frac{1}{V}\sum_{\vec{p}} \left[\mathcal{G}\vec{\gamma}\mathcal{G}\right]^{RA}_{\sigma,\zeta}(\vec{p},\epsilon)$$
$$+ \nu_0 \int \frac{du}{\pi}\left\{n(u-\epsilon)+f(u)\right\}\frac{1}{V}\sum_{\vec{p},\sigma'} \mathcal{V}''_{\sigma\sigma'}(\vec{p}-\vec{q},u-\epsilon)\left[\mathcal{G}\vec{\gamma}\mathcal{G}\right]^{RA}_{\sigma',\zeta}(\vec{p},u). \tag{4.53}$$

At this point, it is convenient to use the identity

$$\mathfrak{G}^R\mathfrak{G}^A = \mathfrak{G}''/\Sigma'', \tag{4.54}$$

and seek

$$\vec{\varphi}_\sigma(\vec{q},\epsilon) = \vec{\gamma}^{RA}_\sigma(\vec{q},\epsilon)/\Gamma_\sigma(\vec{q},\epsilon), \tag{4.55}$$

as the solution to

$$\left[\mathfrak{C}^{(\nu)}_0 \circ \vec{\varphi}\right]_\sigma(\vec{q},\epsilon) + \left[\mathfrak{C}^{(u)}_0 \circ \vec{\varphi}\right]_\sigma(\vec{q},\epsilon) = \vec{q}, \tag{4.56}$$

where the two collision operators $\mathfrak{C}^{(\nu,u)}_0$ are defined by

$$\left[\mathfrak{C}^{(\nu,u)}_0 \circ \vec{\varphi}\right]_\sigma(\vec{q},\epsilon) = \Gamma^\sigma_{(\nu,u)}(\vec{q},\epsilon)\vec{\varphi}_\sigma(\vec{q},\epsilon) - \left[\mathfrak{K}^{(\nu,u)}_0 \circ \vec{\varphi}\right]_\sigma(\vec{q},\epsilon), \tag{4.57}$$

with both

$$\left[\mathfrak{K}^{(\nu)}_0 \circ \vec{\varphi}\right]_\sigma(\vec{q},\epsilon)$$
$$= \nu_0 \int \frac{du}{\pi}\left\{n(u-\epsilon)+f(u)\right\}\frac{1}{V}\sum_{\vec{p},\sigma'} \mathcal{V}''_{\sigma\sigma'}(\vec{p}-\vec{q},u-\epsilon)\left[-\mathcal{G}''\vec{\varphi}\right]_{\sigma'}(\vec{p},u) \tag{4.58}$$

and

$$\left[\mathfrak{K}^{(u)}_0 \circ \vec{\varphi}\right]_\sigma(\vec{q},\epsilon) = u_0 \frac{1}{V}\sum_{\vec{p}} \left[-\mathcal{G}''\vec{\varphi}\right]_\sigma(\vec{p},\epsilon) = 0, \tag{4.59}$$

which vanishes because the scattering centers are spherically symmetric.

Having demonstrated our ability to recover the structure as anticipated from within the simple disorder averaged technique [58], we move to incorporate those states which are connected to the vacuum only by ν_0 and u_0 in a combined presence. To this end, note that the potentials within

(a) Boson (b) Impurity

Figure 6. A diagrammatic representation of the (bare) effective interactions.

$$\begin{aligned} S^{\zeta}_{\text{int}} &= v_0 \frac{T}{2V} \sum_{k,\sigma,\sigma'} \mathcal{V}_{\sigma\sigma'}(k) \sum_{p_1,p_2} \\ &\qquad \times \bar{\psi}^{\sigma'}_{\zeta}(p_1)\psi^{\sigma}_{\zeta}(p_1-k)\bar{\psi}^{\sigma}_{\zeta}(p_1-k)\bar{\psi}^{\sigma}_{\zeta}(p_2)\psi^{\sigma'}_{\zeta}(p_2+k), \\ S^{\zeta}_{\text{dis}} &= u_0 \frac{1}{2V} \sum_{\{\vec{k}_i\}} \delta_{\vec{k}_1+\vec{k}_3,\vec{k}_2+\vec{k}_4} \sum_{\zeta'=1}^{N} \sum_{n,m,\sigma,\sigma'} \\ &\qquad \times \bar{\psi}^{\sigma}_{\zeta}(\vec{k}_1,i\omega_n)\psi^{\sigma}_{\zeta}(\vec{k}_2,i\omega_n)\bar{\psi}^{\sigma'}_{\zeta'}(\vec{k}_3,i\omega_m)\psi^{\sigma'}_{\zeta'}(\vec{k}_4,i\omega_m) \end{aligned} \tag{4.60}$$

can be associated with the diagrams in Figure 6. We already found

$$\begin{aligned} \tilde{\Lambda}^{(v)}_{\sigma\sigma'}[q,p;2i\Omega] &= v_0\mathcal{V}_{\sigma\sigma'}(p-q), \\ \tilde{\Lambda}^{(u)}_{\sigma\sigma'}[q,p;2i\Omega] &= u_0\beta\delta_{\omega_p\omega_q}\delta_{\sigma\sigma'}, \end{aligned} \tag{4.61}$$

and, with these as reference, the overall sign of additional diagrams [see Figure 7] is fixed according to the number of Fermion loops and the parity of the external leg permutation. Evidently, all of the terms to be considered are replica diagonal.

With labels added as in Figure 5, the five diagrams Figure 7a–Figure 7e are then in the exchange channel. Making a 90 degree rotation and then labeling gives the direct channel, while both 180 degree and 270 degree rotations (followed by labeling) supply the respective complex conjugate term that maintains the even, $\Pi(i\Omega) = \Pi(-i\Omega)$, symmetry. Of the particle-hole irreducible

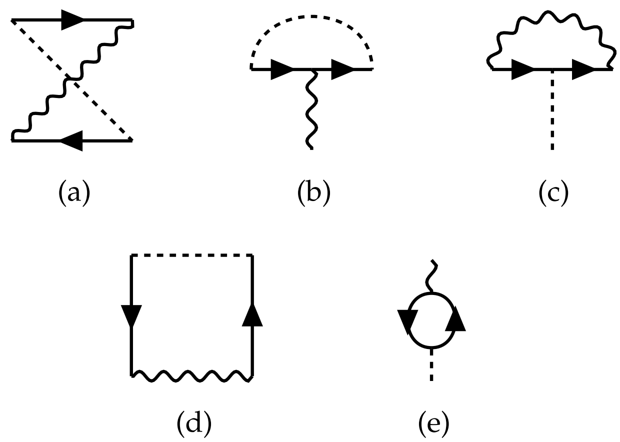


Figure 7. From these (unlabeled) diagrams may be obtained all vertices of order $v_0 u_0$.

contributions, only the following processes

$$
\begin{aligned}
&\tilde{\Lambda}^{(a)}_{\sigma\sigma'}[q, p; 2i\Omega] \\
&= 2u_0 v_0 \frac{1}{V} \sum_{\vec{k}} \mathcal{V}_{\sigma\sigma'}(p - q + \vec{k}) \\
&\qquad \times \left[\mathcal{G}_\sigma(q_- - \vec{k})\mathcal{G}_{\sigma'}(p_+ + \vec{k}) + \mathcal{G}_\sigma(q_+ - \vec{k})\mathcal{G}_{\sigma'}(p_- + \vec{k})\right], \\
&\tilde{\Lambda}^{(b)}_{\sigma\sigma'}[q, p; 2i\Omega] \\
&= u_0 v_0 \mathcal{V}_{\sigma\sigma'}(p - q) \frac{1}{V} \sum_{\vec{k}} \\
&\qquad \times \left[\mathcal{G}_\sigma(q_- - \vec{k})\mathcal{G}_{\sigma'}(p_- - \vec{k}) + \mathcal{G}_\sigma(q_+ - \vec{k})\mathcal{G}_{\sigma'}(p_+ - \vec{k})\right], \\
&\tilde{\Lambda}^{(c)}_{\sigma\sigma'}[q, p; 2i\Omega] \\
&= u_0 v_0 \beta \delta_{\omega_p \omega_q} \delta_{\sigma\sigma'} \frac{T}{V} \sum_{k,\rho} \mathcal{V}_{\rho\sigma}(k) \\
&\qquad \times \left[\mathcal{G}_\rho(q_- - k)\mathcal{G}_\rho(p_- - k) + \mathcal{G}_\rho(q_+ - k)\mathcal{G}_\rho(p_+ - k)\right]
\end{aligned}
\tag{4.62}
$$

surive the replica limit while satisfying the interband nature of ferromagnons; each vertex explicitly violates Matthiessen's rule and gives rise to a collision operator.

Now then, Equation 4.62 leads to

$$\begin{aligned}&\big[\mathfrak{K}_0^{(a+b)}\circ\phi\big]_\sigma(\vec{q},\epsilon)\\&=2u_0v_0\int\frac{du}{\pi}\big\{n(u-\epsilon)+f(u)\big\}\frac{1}{V}\sum_{\vec{p},\sigma'}\big[-\mathcal{G}''\phi\big]_{\sigma'}(\vec{p},u)\big]\\&\qquad\times\frac{1}{V}\sum_{\vec{k}}\big\{2\mathcal{V}''_{\sigma\sigma'}(\vec{p}-\vec{q}+\vec{k},u-\epsilon)\,\mathrm{Re}\big[\mathcal{G}^A_\sigma(\vec{q}-\vec{k},\epsilon)\mathcal{G}^R_{\sigma'}(\vec{p}+\vec{k},u)\big]\\&\qquad\qquad+\mathcal{V}''_{\sigma\sigma'}(\vec{p}-\vec{q},u-\epsilon)\,\mathrm{Re}\big[\mathcal{G}^A_\sigma(\vec{q}-\vec{k},\epsilon)\mathcal{G}^A_{\sigma'}(\vec{p}-\vec{k},u)\big]\big\}\end{aligned}\tag{4.63}$$

and

$$\begin{aligned}&\big[\mathfrak{K}_0^{(c)}\circ\phi\big]_\sigma(\vec{q},\epsilon)\\&=2u_0v_0\frac{1}{V}\sum_{\vec{p}}\big[-\mathcal{G}''\phi\big]_\sigma(\vec{p},\epsilon)\frac{1}{V}\sum_{\vec{k},\sigma'}\int\frac{du}{\pi}\\&\qquad\times\big\{\Phi_n(u-\epsilon)\mathcal{V}''_{\sigma\sigma'}(\vec{k},u-\epsilon)\,\mathrm{Re}\big[\mathcal{G}^A_{\sigma'}(\vec{q}+\vec{k},u)\mathcal{G}^A_{\sigma'}(\vec{p}+\vec{k},u)\big]\\&\qquad\qquad+\Phi_f(u)\mathcal{V}'_{\sigma\sigma'}(\vec{k},u-\epsilon)\,\mathrm{Im}\big[\mathcal{G}^A_\rho(\vec{q}+\vec{k},u)\mathcal{G}^A_{\sigma'}(\vec{p}+\vec{k},u)\big]\big\},\end{aligned}\tag{4.64}$$

where

$$\begin{aligned}\Phi_n(u)&=-\theta\big[u<0\big]n(-u)+\theta\big[u>0\big]n(u),\\\Phi_f(u)&=\theta\big[u<0\big]f(-u)-\theta\big[u>0\big]f(u).\end{aligned}\tag{4.65}$$

Thus, the transport equation is determined by collision integrals of the form

$$\big[\mathfrak{C}_0^{(i)}\circ\vec{\varphi}\big]_\sigma(\vec{q},\epsilon)=\Gamma^{(i)}_\sigma(\vec{q},\epsilon)\vec{\varphi}_\sigma(\vec{q},\epsilon)-\big[\mathfrak{K}^{(i)}\circ\vec{\varphi}\big]_\sigma(\vec{q},\epsilon),\tag{4.66}$$

where (compare with Equation 4.16)

$$\Gamma^{(i)}_\sigma(\vec{q},\epsilon)=\big[\mathfrak{K}_0^{(i)}\circ 1\big]_\sigma(\vec{q},\epsilon),\quad i\in\{u,v,a,b,c\},\tag{4.67}$$

and may be written

$$\vec{k}=\sum_i\mathfrak{C}_0^{(i)}\circ\vec{\varphi}\equiv\mathfrak{C}_0\circ\vec{\varphi};\tag{4.68}$$

we denote by $\vec{k}$ the vector of the linear space (upon which $\mathfrak{C}_0$ acts) with components $\vec{k}_\sigma(\vec{q},\epsilon)=\vec{q}$, and by 1, the constant function.

4.4 Asymptotic solutions

By truncating the second von Neumann series, this section achieves an asymptotically exact expression for the conductivity that is amenable to explicit evaluation.

4.4.1 Zero temperature. When the background Boson field is frozen out, the only available scattering mechanism is impurities [see Figure 8], so the

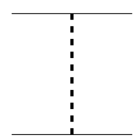


Figure 8. The bare impurity vertex.

relaxation kernel becomes (to leading order in the disorder)

$$\mathcal{C}_0^{(u)} \equiv \mathfrak{C}_0|_{\nu_0=0}, \tag{4.69}$$

which implies that

$$\vec{q} = \left[\mathcal{C}_0^{(u)} \circ \vec{\varphi}\right]_\sigma(\vec{q}, \epsilon) \tag{4.70}$$

trivially yields the Drude formula

$$\sigma(T=0,\gamma) \approx \frac{ne^2}{m}\tau_{\text{tr}}(T=0,\gamma), \tag{4.71}$$

where the transport lifetime

$$\begin{aligned}\tau_{\text{tr}}(T=0,\gamma) &= \int \frac{d\epsilon}{\pi} w(\epsilon) \frac{1}{2k_F^2 N_F V} \sum_{\vec{q},\sigma} q^2 \left[-\mathcal{G}''_\sigma(\vec{q},\epsilon)\right] \frac{1}{2\gamma_\sigma^{(u)}(\epsilon)} \\ &\approx \tau\left[1 + \mathcal{O}(T^2/\epsilon_F^2) + \mathcal{O}(\gamma/\epsilon_F)\right],\end{aligned} \tag{4.72}$$

with $\gamma = 1/2\tau$ the bare disorder rate, is determined by

$$\mathcal{G} \equiv \mathcal{G}|_{\nu_0=0}, \tag{4.73}$$

the Green function in the absence of magnons, and

$$\gamma^{\sigma}_{(u)}(\epsilon) \equiv \Gamma^{\sigma}_{(u)}(\epsilon)\big|_{\nu_0=0} = u_0 \frac{1}{V} \sum_{\vec{p}} \big[-\mathcal{G}''_{\sigma}(\vec{p}, \epsilon) \big]. \tag{4.74}$$

4.4.2 Ideal metal. When disorder effects are negligible, a model for the dominant momentum relaxing process is encapsulated in the collision

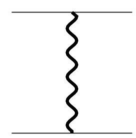


Figure 9. The bare magnon vertex.

operator corresponding to single magnon exchange [see Figure 9]

$$C^{(\nu)}_0 \equiv \mathfrak{C}_0|_{u_0=0}, \tag{4.75}$$

which yields for the transport equation

$$\vec{k} = C^{(\nu)}_0 \circ \vec{\varphi}, \quad \vec{\varphi} = \hat{k}\varphi, \quad \hat{k} = \vec{k}/k, \tag{4.76}$$

with both

$$C^{(\nu)}_0 = \gamma_{(\nu)} - K^{(\nu)}_0, \quad K^{(\nu)}_0 \equiv \mathfrak{K}^{(\nu)}_0\big|_{u_0=0}, \tag{4.77}$$

and

$$\begin{aligned} \gamma^{\sigma}_{(\nu)}(\vec{q}, \epsilon) &\equiv \Gamma^{\sigma}_{(\nu)}(\vec{q}, \epsilon)\big|_{u_0=0} \\ &= \nu_0 \int \frac{du}{\pi} \big\{ n(u-\epsilon) + f(u) \big\} \\ &\qquad \times \frac{1}{V} \sum_{\vec{p}, \sigma'} \mathcal{V}''_{\sigma\sigma'}(\vec{p} - \vec{q}, u - \epsilon) \big[-\mathfrak{G}''_{\sigma'}(\vec{p}, u) \big], \end{aligned} \tag{4.78}$$

dependent on both

$$\mathfrak{G} \equiv \mathcal{G}|_{u_0=0}, \tag{4.79}$$

the clean Green function, and

$$\mathcal{V}''_{\sigma\sigma'}(\vec{k}, u) = \sigma'(1 - \delta_{\sigma\sigma'})\delta\big[Dk^2 - \sigma' u\big]\theta\big[\sigma' u < T_1\big], \tag{4.80}$$

the spectrum of the (magnon exchange) potential; D is the spin-wave stiffness.

Now then, with the Coulomb interaction implicitly included, $\vec{k}$ is the only nontrivial vector of the form $\vec{v}_\sigma(\vec{q},\epsilon) = \vec{q}v_\sigma(\vec{q},\epsilon)$ that enters the nullspace of $C_0^{(\nu)}$ as $T \to 0$;[5] thus, Equation 4.76 is singular in the degree of momentum nonconservation. Indeed, the uniqueness of the momentum mode (and its coordinate representation for perturbatively weak dissipative scattering) may be ascertained by analyzing the eigenproblem

$$C^{(\nu)} \circ \mathfrak{e}_i = \lambda_i \mathfrak{e}_i, \tag{4.81}$$

where $\vec{\mathfrak{e}}_i = \hat{k}\mathfrak{e}_i$ is an eigenvector of $C_0^{(\nu)}$ corresponding to the eigenvalue λ_i iff $\mathfrak{e}_i$ obeys Equation 4.81, and

$$C^{(\nu)} = 1 - K^{(\nu)}, \tag{4.82}$$

with

$$\left[K^{(\nu)} \circ \vec{\varphi}\right]_\sigma(\vec{q},\epsilon) = \int \frac{du}{\pi}\frac{1}{V}\sum_{\vec{p},\sigma'} \frac{\vec{q}\cdot\vec{p}}{qp} K_0^{(\nu)}\left[\vec{q},\epsilon,\sigma\middle|\vec{p},u,\sigma'\right]\vec{\varphi}_{\sigma'}(\vec{p},u); \tag{4.83}$$

using the resonance of Equation 4.80, we rewrite

$$\begin{aligned} &\frac{\vec{q}\cdot\vec{p}}{qp}\mathcal{V}''_{\sigma\sigma'}(\vec{p}-\vec{q},u-\epsilon) \\ &= \frac{k_F^2}{qp}\left\{1 - \left[\frac{2\sigma'}{T_1}(u-\epsilon) - \frac{\xi_p+\xi_q}{2\epsilon_F}\right]\right\}\mathcal{V}''_{\sigma\sigma'}(\vec{p}-\vec{q},u-\epsilon), \end{aligned} \tag{4.84}$$

and define

$$L_0^{(\nu)}\left[\vec{q},\epsilon,\sigma\middle|\vec{p},u,\sigma'\right] = \frac{k_F^2}{q^2}\gamma^\sigma_{(\nu)}(\vec{q},\epsilon)\pi V\delta_{\sigma\sigma'}(q-p) - \frac{k_F^2}{qp}K_0^{(\nu)}\left[\vec{q},\epsilon,\sigma\middle|\vec{p},u,\sigma'\right], \tag{4.85}$$

so that

$$C^{(\nu)} = L_0^{(\nu)} + L_1^{(\nu)} + L_2^{(\nu)} \tag{4.86}$$

---

[5]Proof of this statement is omitted because it follows from manipulations that are in exact correspondence with those that we applied to Bloch's collision integral in section A.1.

would be momentum conserving if not for

$$L_1^{(\nu)}\big[\vec{q},\epsilon,\sigma\big|\vec{p},u,\sigma'\big] = \frac{k_F^2\xi_q}{q^2\epsilon_F}\gamma^{\sigma}_{(\nu)}(\vec{q},\epsilon)\pi V\delta_{\sigma\sigma'}(q-p) - \frac{k_F^2(\xi_p+\xi_q)}{2qp\epsilon_F}K_0^{(\nu)}\big[\vec{q},\epsilon,\sigma\big|\vec{p},u,\sigma'\big], \tag{4.87}$$

$$L_2^{(\nu)}\big[\vec{q},\epsilon,\sigma\big|\vec{p},u,\sigma'\big] = \frac{2k_F^2\sigma'(u-\epsilon)}{qpT_1}K_0^{(\nu)}\big[\vec{q},\epsilon,\sigma\big|\vec{p},u,\sigma'\big],$$

i.e. $L_0^{(\nu)}\circ k = 0$, but $L_1^{(\nu)}\circ k \neq 0$ and $L_2^{(\nu)}\circ k \neq 0$. Evidently, $L_1^{(\nu)}+L_2^{(\nu)}$ is suppressed[6] relative to $L_0^{(\nu)}$; it is therefore possible to consider $C^{(\nu)}$ as perturbatively near $L_0^{(\nu)}$. We are now in a position to determine the leading corrections to the eigenvalue λ_0 corresponding to the momentum mode $\mathfrak{e}_0$; this is achieved by writing

$$C^{(\nu)} = L_0^{(\nu)} + t\big[L_1^{(\nu)}+L_2^{(\nu)}\big], \tag{4.90}$$

making the ansatz

$$\lambda_0 = 0 + \mathcal{O}(t), \quad \mathfrak{e}_0 = k + \mathcal{O}(t), \tag{4.91}$$

and matching coefficients of t, which parameterizes the map $L_0^{(\nu)} \overset{t}{\mapsto} C^{(\nu)}$. In the physical case, $t=1$. Such a generalized asymptotic series is known to exist up to order t^2 if we use the metric [59]

$$\big(\psi,\phi\big) = \int\frac{d\epsilon}{\pi}w(\epsilon)\frac{1}{2k_F^2N_FV}\sum_{\vec{q},\alpha}\big[-\mathfrak{S}''\big]_\alpha(\vec{q},\epsilon)\psi_\alpha(\vec{q},\epsilon)\phi_\alpha(\vec{q},\epsilon), \tag{4.92}$$

such that (formally)

$$\big(\psi, L_\iota^{(\nu)}\circ\phi\big) = \big(L_\iota^{(\nu)}\circ\psi,\phi\big), \quad \iota\in\{0,1,2\}. \tag{4.93}$$

[6] Typical values of the electronic wavenumbers are

$$q \approx k_F^{\sigma}\big[1+\mathcal{O}(u_0,v_0)\big], \quad p \approx k_F^{\sigma'}\big[1+\mathcal{O}(u_0,v_0)\big], \tag{4.88}$$

where $\lambda/\epsilon_F \ll 1$ (for weak ferromagnets), while their frequencies are expected to be such that

$$\epsilon \approx \mathcal{O}(T_0,T), \quad u \approx \mathcal{O}(T_0,T). \tag{4.89}$$

In fact, both λ_0 and $\mathfrak{e}_0$ can be determined by the method of successive approximations [60], while the zero eigenvalue of $L_0^{(\nu)}$ violates the condition for the existence of iterative solutions [15] to the inhomogeneous

$$k = C^{(\nu)} \circ \varphi. \tag{4.94}$$

It follows that the equation governing the first corrections

$$L_0^{(\nu)} \circ \mathfrak{e}_0^{(1)} + \left[L_1^{(\nu)} + L_2^{(\nu)}\right] \circ k = \lambda_0^{(1)} k \tag{4.95}$$

admits a solution $\mathfrak{e}_0^{(1)}$ iff

$$\lambda_0^{(1)}\left(k, k\right) = \left(k, L_2^{(\nu)} \circ k\right), \tag{4.96}$$

where we've noticed that $L_0^{(\nu)} \circ k\xi = 2\epsilon_F L_1^{(\nu)} \circ k$ implies $\left(k, L_1^{(\nu)} \circ k\right) = 0$. From here, we write

$$\mathfrak{e}_0^{(1)} = \frac{-k\xi}{2\epsilon_F} + E_0^{(1)}, \tag{4.97}$$

which leads to

$$L_0^{(\nu)} \circ E_0^{(1)} + L_2^{(\nu)} \circ k = \lambda_0^{(1)} k; \tag{4.98}$$

consequently, knowledge of $E_0^{(1)}$ is not necessary.[7] At last, it is sufficient to extract from

$$L_0^{(\nu)} \circ \mathfrak{e}_0^{(2)} + \left[L_1^{(\nu)} + L_2^{(\nu)}\right] \circ \mathfrak{e}_0^{(1)} = \lambda_0^{(2)} k + \lambda_0^{(1)} \mathfrak{e}_0^{(1)} \tag{4.99}$$

only

$$\lambda_0^{(2)}\left(k, k\right) \approx \frac{-1}{2\epsilon_F}\left(k, L_1^{(\nu)} \circ k\xi\right). \tag{4.100}$$

In the preceeding, we have assumed that $C_0^{(\nu)}$ admits a spectral representation with a unique lowest eigenvalue λ_0, which corresponds with the eigenvector $\vec{\mathfrak{e}}_0 = \vec{k}\mathfrak{e}_0$, that remains isolated as it is perturbed away from its

[7] It is difficult to verify this assertion from our current position; for this reason, we relegate the argument to subsubsection A.2.2.1.

bare form $\lambda_0^{(0)} = 0$ in the absence of global electronic momentum dissipation (with $\vec{\mathfrak{e}}_0 = \vec{k}$); $C^{(\nu)}$ is also taken to have a positive definite spectrum. That it is judicious to formally grant these properties to $C_0^{(\nu)}$ (on $\vec{\mathfrak{H}}$) is made plausible (mathematically) by our proofs for Bloch's collision operator in section A.1, and (physically) by our expectation that of the five symmetries [22, 40] in equilibrium hydrodynamics, only particle number remains exactly conserved because the electron-lattice coupling dissolves both the momentum and the kinetic energy of the charge carriers.

Finally,

$$\sigma(T, \gamma = 0) = \frac{ne^2}{2m}\big(\vec{k}; \vec{\varphi}\big) \approx \frac{ne^2}{2m}\big(\vec{k}; \vec{\mathfrak{e}}_0\big)\frac{1}{\lambda_0\big(\vec{\mathfrak{e}}_0; \vec{\mathfrak{e}}_0\big)}\big(\vec{\mathfrak{e}}_0; \vec{k}\,\big)\big[1 + \mathcal{O}(T)\big], \tag{4.101}$$

where the inner product

$$\big(\vec{\psi}; \vec{\phi}\big) = \int \frac{d\epsilon}{\pi} w(\epsilon) \frac{1}{2k_F^2 N_F V} \sum_{\vec{q},\alpha} \big[-\mathfrak{S}''\big]_\alpha (\vec{q}, \epsilon)\big[\vec{\psi}_\alpha \cdot \vec{\phi}_\alpha\big](\vec{q}, \epsilon), \tag{4.102}$$

admits

$$\big(\vec{\psi}; C_0^{(\nu)} \circ \vec{\phi}\big) = \big(C_0^{(\nu)} \circ \vec{\psi}; \vec{\phi}\big); \tag{4.103}$$

as a result

$$\sigma(T, \gamma = 0) \approx \frac{ne^2}{2m\lambda_0}, \tag{4.104}$$

where

$$\lambda_0 = 4\mathcal{V}_0 \begin{cases} \frac{TT_0}{T_1} e^{-T_{min}/T} & T_0^2/\epsilon_F \ll T \ll T_0 \\ \frac{\pi^2 T^2}{3T_1} & T_0 \ll T \ll T_1 \\ T & T_1 \ll T \ll \epsilon_F \end{cases}, \tag{4.105}$$

with

$$T_{min} = T_0\Big[1 + \frac{T_0}{T_1} + \frac{T_0}{\lambda}\Big] \tag{4.106}$$

the renormalized exchange gap.

In the two hotter regimes of Equation 4.105, we reproduce the findings of Moriya and Ueda [4]. For the lower temperature range, our result improves upon a previous investigation [3] by resolving the next-leading temperature dependence; it should be mentioned that the prefactor to the exponential is model dependent.

While the activated behavior follows immediately when $T \ll T_{\rm min}$, technical difficulties arise in the regime where $T \ll T_0^2/\epsilon_F$; since such temperatures are astronomically low by experimental standards, we feel that it is not worthwhile to endeavor this calculation, and instead stipulate that the prefactor will tend to a constant for temperatures sufficiently low.

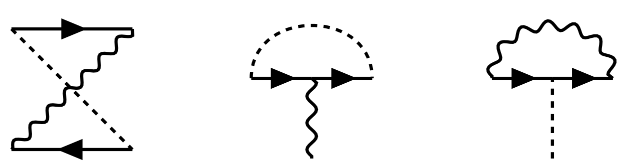
4.4.3 Beyond semi-classical. Now that we're familiar with both the clean ($u_0 = 0$) and the zero temperature ($\nu_0 = 0$) solutions, asymptotic expansions for the full problem

$$\vec{k} = \left\{\mathfrak{C}_0^{(\nu)} + \mathfrak{C}_0^{(u)} + \mathfrak{L}_0\right\} \circ \vec{\varphi} \tag{4.107}$$

are easily obtained. Here,

$$\mathfrak{L}_0 = \mathfrak{C}_0^{(a)} + \mathfrak{C}_0^{(b)} + \mathfrak{C}_0^{(c)} \tag{4.108}$$

corresponds to the vertices of Figure 10 and explicitly correlates the two



(a) (b) (c)

Figure 10. From these (unlabeled) diagrams may be obtained all contributions to the vertex of order $\nu_0 u_0$.

scattering mechanisms, while both $\mathfrak{C}_0^{(\nu)}$ and $\mathfrak{C}_0^{(u)}$ contain implicit magnon-impurity interference by way of self-energy corrections.

4.4.3.1 Weaker magnons. If we suppose that there exists a temperature scale (to be determined) beneath which the system both remains in the ballistic regime, i.e. $1/\tau \lesssim T \ll \epsilon_F$ [12], and features a resisitivity that is dominated by the bare impurity vertex, then we may assume that

$$\vec{\varphi} = \left[\mathcal{C}_0^{(u)}\right]^{-1} \circ \vec{k} + \mathcal{O}(\nu_0); \tag{4.109}$$

although we formally expand in ν_0, the true control parameter is T. After changing variables to

$$\vec{\varphi}_\sigma(\vec{q}, \epsilon) = \frac{\mathcal{G}''_\sigma(\vec{q}, \epsilon)}{\mathcal{G}''_\sigma(\vec{q}, \epsilon)} \vec{\chi}_\sigma^{(u)}(\vec{q}, \epsilon), \tag{4.110}$$

we identically rewrite Equation 4.107 as

$$\vec{k} = \left\{\mathcal{C}_0^{(u)} + \mathcal{L}_0^{(u)}\right\} \circ \vec{\chi}^{(u)} \equiv \mathcal{C}_0^{(u)} \circ \vec{\chi}^{(u)}, \tag{4.111}$$

where

$$\begin{aligned} &\left[\mathcal{L}_0^{(u)} \circ \vec{\chi}^{(u)}\right]_\sigma(\vec{q}, \epsilon) \\ &= \left[\left\{\mathfrak{C}_0^{(\nu)} + \mathfrak{L}_0\right\} \circ \frac{\mathcal{G}''}{\mathcal{G}''} \vec{\chi}^{(u)}\right]_\sigma(\vec{q}, \epsilon) \\ &\quad + \frac{1}{\mathcal{G}''_\sigma(\vec{q}, \epsilon)} \left[\Gamma_\sigma^{(u)}(\vec{q}, \epsilon) \mathcal{G}''_\sigma(\vec{q}, \epsilon) - \mathcal{G}''_\sigma(\vec{q}, \epsilon) \gamma_\sigma^{(u)}(\vec{q}, \epsilon)\right] \vec{\chi}_\sigma^{(u)}(\vec{q}, \epsilon). \end{aligned} \tag{4.112}$$

Thus, the desired series may be obtained from

$$\left[\mathcal{C}_0^{(u)}\right]^{-1} = \left[\mathcal{C}_0^{(u)}\right]^{-1} - \left[\mathcal{C}_0^{(u)}\right]^{-1} \circ \mathcal{L}_0^{(u)} \circ \left[\mathcal{C}_0^{(u)}\right]^{-1}, \tag{4.113}$$

which to leading order reads

$$\left[\mathcal{C}_0^{(u)}\right]^{-1} \approx \left[\mathcal{C}_0^{(u)}\right]^{-1} - \left[\mathcal{C}_0^{(u)}\right]^{-1} \circ \mathcal{L}_0^{(u)} \circ \left[\mathcal{C}_0^{(u)}\right]^{-1} + \mathcal{O}(\nu_0^2), \tag{4.114}$$

and therefore,

$$\begin{aligned}\sigma(T,\gamma) &- \sigma(T=0,\gamma) \\ &\approx \frac{-e^2}{6m^2}\int \frac{d\epsilon}{\pi}\, w(\epsilon)\frac{1}{V}\sum_{\vec{q},\sigma}\big[-\mathcal{G}''_\sigma(\vec{q},\epsilon)\big]\frac{1}{\gamma^\sigma_{(u)}(\epsilon)}\vec{q}\cdot\big[\mathcal{L}^{(u)}_0\circ\vec{k}\gamma^{-1}_{(u)}\big]_\sigma(\vec{q},\epsilon).\end{aligned} \tag{4.115}$$

4.4.3.2 Weaker disorder. If we have a relatively clean material at temperatures $T > T_{\min}$, then one might expect that

$$\vec{\varphi} = \big[C^{(\nu)}_0\big]^{-1}\circ\vec{k} + \mathcal{O}(u_0). \tag{4.116}$$

In this case, on redefining

$$\vec{\varphi}_\sigma(\vec{q},\epsilon) = \frac{\mathfrak{S}''_\sigma(\vec{q},\epsilon)}{\mathcal{G}''_\sigma(\vec{q},\epsilon)}\vec{\chi}^{(\nu)}_\sigma(\vec{q},\epsilon), \tag{4.117}$$

Equation 4.107 becomes

$$\vec{k} = \big\{C^{(\nu)}_0 + \mathcal{L}^{(\nu)}_0\big\}\circ\vec{\chi}^{(\nu)} \equiv \mathcal{C}^{(\nu)}_0\circ\vec{\chi}^{(\nu)}, \tag{4.118}$$

where

$$\begin{aligned}\big[\mathcal{L}^{(\nu)}_0 &\circ \vec{\chi}^{(\nu)}\big]_\sigma(\vec{q},\epsilon) \\ &= \Big[\big\{\mathfrak{C}^{(u)}_0 + \mathfrak{L}_0\big\}\circ\frac{\mathfrak{S}''}{\mathcal{G}''}\vec{\chi}^{(\nu)}\Big]_\sigma(\vec{q},\epsilon) \\ &\quad + \frac{1}{\mathcal{G}''_\sigma(\vec{q},\epsilon)}\big[\Gamma^{(\nu)}_\sigma(\vec{q},\epsilon)\mathfrak{S}''_\sigma(\vec{q},\epsilon) - \mathcal{G}''_\sigma(\vec{q},\epsilon)\gamma^{(\nu)}_\sigma(\vec{q},\epsilon)\big]\vec{\chi}^{(\nu)}_\sigma(\vec{q},\epsilon);\end{aligned} \tag{4.119}$$

it follows that

$$\big[\mathcal{C}^{(\nu)}_0\big]^{-1} = \big[C^{(\nu)}_0\big]^{-1} - \big[C^{(\nu)}_0\big]^{-1}\circ\mathcal{L}^{(\nu)}_0\circ\big[\mathcal{C}^{(\nu)}_0\big]^{-1}, \tag{4.120}$$

and therefore

$$\sigma(T,\gamma) - \sigma(T,\gamma=0) \approx \frac{-ne^2}{2m}\big(\vec{k};\vec{e}_0\big)\frac{1}{\lambda_0\big(\vec{e}_0;\vec{e}_0\big)}\Big(\vec{e}_0;\mathcal{L}^{(\nu)}_0\circ\big[\mathcal{C}^{(\nu)}_0\big]^{-1}\circ\vec{k}\Big), \tag{4.121}$$

which can be iterated and then resummed to

$$\sigma(T,\gamma) \approx \frac{ne^2}{2m}\big(\vec{k};\vec{e}_0\big)\frac{1}{\lambda_0\big(\vec{e}_0;\vec{e}_0\big)}\big(\vec{e}_0;\vec{k}\big)\Big[\lambda_0\big(\vec{e}_0;\vec{e}_0\big) + \big(\vec{e}_0;\mathcal{L}^{(\nu)}_0\circ\vec{e}_0\big)\Big]^{-1}. \tag{4.122}$$

4.5 DC conductivity

From both Equation 4.115 and Equation 4.122, we find the content of Matthiessen's rule plus interference corrections, i.e. if

$$\sigma(T,\gamma) = \frac{ne^2}{2m\Gamma_{tr}(T,\gamma)}, \tag{4.123}$$

then

$$\Gamma_{tr}(T,\gamma) = \Gamma_{tr}(T=0,\gamma) + \Gamma_{tr}(T,\gamma=0) + \Delta\Gamma_{tr}(T,\gamma), \tag{4.124}$$

where

$$\Delta\Gamma_{tr}(T,\gamma) \approx \mathcal{V}_0 \frac{\gamma}{\epsilon_F} \begin{cases} \frac{-\sqrt{T_1}T^{3/2}}{T_0} A^{\ll} & T_0^2/\epsilon_F \ll T \ll T_0 \\ \frac{T^{3/2}}{\sqrt{T_1}} A^{\gg} & T_0 \ll T \ll T_1 \end{cases}, \tag{4.125}$$

with

$$\begin{aligned} A^{\ll} &= \frac{1}{\pi}\int_0^\infty du \left\{ \frac{u^{3/2}}{\sinh^2 \frac{u}{2}} - \frac{2u^{-3/2}}{e^u - 1} \right\} \approx 3, \\ A^{\gg} &= \pi \int_0^\infty du \left\{ \frac{u^{3/2}}{4\sinh^2 \frac{u}{2}} + \frac{u/2}{e^u - 1} \right\} \approx 12, \end{aligned} \tag{4.126}$$

and

$$\Gamma_{tr}(T=0,\gamma) \approx \gamma, \tag{4.127}$$

and

$$\Gamma_{tr}(T,\gamma=0) \approx 4\mathcal{V}_0 \begin{cases} \frac{TT_0}{T_1} e^{-T_{min}/T} & T_0^2/\epsilon_F \ll T \ll T_0 \\ \frac{\pi^2 T^2}{3T_1} & T_0 \ll T \ll T_1 \\ T & T_1 \ll T \ll \epsilon_F \end{cases}; \tag{4.128}$$

only $\mathfrak{C}_0^{(a)}$ and $\mathfrak{C}_0^{(c)}$ contribute to $\Delta\Gamma_{tr}$, which must be considered much weaker than either $\Gamma_{tr}(T=0,\gamma)$ or $\Gamma_{tr}(T,\gamma=0)$. That is to say, an interpolation formula exists between the region of parameter space where scattering is magnon dominated (i.e. $\gamma \ll T^2/T_1$ when $T_0 \ll T \ll T_1$) and the regime where impurity scattering provides the leading contribution to the resistance (i.e. $T^2/T_1 \ll \gamma$ when $T_0 \ll T \ll T_1$). In stating these ranges, we have implicitly assumed that both $T \ll \epsilon_F$ and $\gamma \ll \epsilon_F$.

The non-Fermi-liquid temperature scaling that presents in Equation 4.125 has been predicted previously by Belitz and Kirkpatrick [11], albeit with a significantly different prefactor.

APPENDIX A

SUPPLEMENTARY MATERIAL

Here lie the technical details that underpin our analysis.

This chapter includes previously published co-authored material.

A.1 Spectral value decomposition

Throughout this work, the momentum mode $\vec{\mathfrak{e}}_0$ has played a critical role not only because its corresponding eigenvalue λ_0 is the lowest element in the spectrum of the collision operator C_0 (on $\vec{\mathfrak{H}}$) but also due to its appearance (in bare form) as the inhomogeneity in the transport equation

$$\vec{k} = C_0 \circ \vec{\varphi}, \tag{A.1}$$

which implies that $\vec{\varphi}$ is singular in the degree of momentum nonconservation; in order for

$$\left(\vec{k}; \vec{\varphi}\right) \approx \left(\vec{k}; \vec{\mathfrak{e}}_0\right) \frac{1}{\lambda_0\left(\vec{\mathfrak{e}}_0; \vec{\mathfrak{e}}_0\right)} \left(\vec{\mathfrak{e}}_0; \vec{k}\right) \left[1 + \mathcal{O}(T)\right] \tag{A.2}$$

it is crucial that λ_0 is both unique and isolated. Here, $\vec{\varphi} \in \vec{\mathfrak{H}}$ is an element of the Hilbert space consisting of functions $\vec{e}$ obeying

$$\left(\vec{e}_i; \vec{e}_j\right)_{\vec{\mathfrak{H}}} = \int \frac{d^3 q}{(2\pi)^3} w(\xi_q) \left[\vec{e}_i \cdot \vec{e}_j\right](\vec{q}) < \infty. \tag{A.3}$$

Furthermore, that C_0 is invertible (on $\vec{\mathfrak{H}}$) follows from the fact that its spectrum is positive definite.

In proceeding to formulate a proof of the above statements,[1] it is useful to introduce some notation.

Lemma 1. *Let both*

$$\left(\hat{e}_i, \hat{e}_j\right)_{L^2} = \int_{\mathbb{R}} d\epsilon\, \hat{e}_i(\epsilon) \hat{e}_j(\epsilon) < \infty \tag{A.4}$$

---

[1]For simplicity, we restrict our attention to the case of electron-phonon scattering.

denote the inner product in the Hilbert space L^2 *of square integrable functions* $\hat{e} : \mathbb{R} \to \mathbb{R}$ *and* $\mathfrak{H}$ *be the space of functions* $e : \mathbb{R}^3 \to \mathbb{R}$ *as defined by*

$$e(\vec{q}) = S[\hat{e}](\vec{q}) = \frac{1}{\sqrt{w(\xi_q)N(\xi_q)}}\theta\big[\xi_q > 0\big]\hat{e}(\xi_q) \quad \vec{q} \in \mathbb{R}^3, \tag{A.5}$$

where

$$\xi_q = \frac{|\vec{q}|^2}{2m} - \mu, \quad N(\xi_q) = \frac{mk(\xi_q)}{2\pi^2}, \quad k(\xi_q) = \sqrt{2m\mu\big[1 + \xi_q/\mu\big]}. \tag{A.6}$$

Consequently the map $S : L^2 \to \mathfrak{H}$ *as defined by* $e = S[\hat{e}]$ *is a Hilbert space isomorphism, i.e.* $\mathfrak{H}$ *is a Hilbert space under the induced inner product*

$$\big(e_i, e_j\big)_{\mathfrak{H}} = \int_{\mathbb{R}^3} \frac{d^3q}{(2\pi)^3} w(\xi_q) e_i(\vec{q}) e_j(\vec{q}). \tag{A.7}$$

Proof. By construction,

$$\big(e_i, e_j\big)_{\mathfrak{H}} = \big(\hat{e}_i, \hat{e}_j\big)_{L^2}. \tag{A.8}$$

□

Corollary 1. *When* $\vec{e} \in \vec{\mathfrak{H}}$ *iff*

$$\vec{e}(\vec{q}) = \frac{\vec{q}}{q} e(\vec{q}), \quad e \in \mathfrak{H}, \tag{A.9}$$

so too is $\vec{\mathfrak{H}}$ *isomorphic to* L^2.

Proof. Clearly,

$$\big(\vec{e}_i; \vec{e}_j\big)_{\vec{\mathfrak{H}}} = \big(e_i, e_j\big)_{\mathfrak{H}}. \tag{A.10}$$

□

From this perspective, we examine the operator $\hat{K}_0 : L^2 \to L^2$ that both satisfies

$$\big(\hat{e}_i, \hat{K}_0 \circ \hat{e}_j\big)_{L^2} = \big(e_i, K_0 \circ e_j\big)_{\mathfrak{H}} \tag{A.11}$$

and is generated by the kernel

$$
\begin{aligned}
\hat{K}_0(\epsilon,u) &= w(\epsilon)\frac{N(\epsilon)}{N_F}\frac{1}{\sqrt{w(\epsilon)N(\epsilon)}}K_0(\epsilon,u)\frac{1}{\sqrt{w(u)N(u)}} \\
&= \frac{1}{\sqrt{N(\epsilon)}}\frac{\bar{\mathcal{V}}''(\epsilon,u)}{2\sinh\frac{u-\epsilon}{2T}}\frac{1}{\sqrt{N(u)}},
\end{aligned}
\tag{A.12}
$$

where

$$
K_0(\epsilon,u) = \frac{N_F}{N(\epsilon)}\big[n(u-\epsilon)+f(u)\big]\bar{\mathcal{V}}''(\epsilon,u), \tag{A.13}
$$

with

$$
\bar{\mathcal{V}}''(\epsilon,u) = \frac{\pi\nu_0}{\omega_D^2}(u-\epsilon)^2\operatorname{sgn}(u-\epsilon)\theta(\epsilon,u), \tag{A.14}
$$

which involves the step function

$$
\begin{aligned}
\theta(\epsilon,u) = {}& \theta\big[\epsilon > -\epsilon_F\big]\theta\big[u > -\epsilon_F\big] \\
&\times\theta\Big[-2k(\epsilon)k(u) \leqslant k^2(\epsilon)+k^2(u)-\Big(\frac{u-\epsilon}{c}\Big)^2 \leqslant 2k(\epsilon)k(u)\Big].
\end{aligned}
\tag{A.15}
$$

Here,

$$
f(\epsilon) = 1/(e^{\epsilon/T}+1), \quad n(\epsilon) = 1/(e^{\epsilon/T}-1), \tag{A.16}
$$

are the Fermi-Dirac and Bose-Einstein distribution functions, respectively; the parameters $m, \mu, N_F, \nu_0, \omega_D, c, T$, with T the temperature, are positive real numbers.

In preparation for the next fact, notice that K_0 is self-adjoint in $\mathfrak{H}$.

Proposition 1. *The integral operator* $K_0 : \mathfrak{H} \to \mathfrak{H}$ *admits a discrete spectral representation.*

Proof. It suffices to show square integrability of the third iterate of $\hat{K}_0$

$$
\hat{K}_0^{(3)}(\epsilon,u) = \int_{\mathbb{R}} dx\, \hat{K}_0^{(2)}(\epsilon,x)\hat{K}_0(x,u), \tag{A.17}
$$

for then K_0 is Hilbert-Schmidt; here

$$
\hat{K}_0^{(2)}(\epsilon,u) = \int_{\mathbb{R}} dx\, \hat{K}_0(\epsilon,x)\hat{K}_0(x,u) \tag{A.18}
$$

is the second iterate of $\hat{K}_0$, which is bounded by [40]

$$\hat{K}_0^{(2)}(\epsilon, u) \leqslant \sqrt{\int_{\mathbb{R}} dx \left[\hat{K}_0(\epsilon, x)\right]^2} \sqrt{\int_{\mathbb{R}} dx \left[\hat{K}_0(x, u)\right]^2} \tag{A.19}$$

according to the Cauchy-Schwarz-Bunyakovskii inequality. Now then,

$$\begin{aligned} \int_{\mathbb{R}} dx \left[\hat{K}_0(\epsilon, x)\right]^2 &\leqslant \frac{C_1}{N(\epsilon)} \theta\left[\epsilon > -\mu\right] \int_{-\epsilon-\mu}^{\infty} \frac{du}{N(u+\epsilon)} \frac{u^4}{\sinh^2 \frac{u}{2T}} \\ &\leqslant \frac{C_2}{\sqrt{\mu+\epsilon}} \theta\left[\epsilon > -\mu\right] \int_{-\epsilon-\mu}^{\infty} \frac{du}{\sqrt{\mu+u+\epsilon}} e^{\frac{-|u|}{2T}}, \end{aligned} \tag{A.20}$$

with both C_1 and C_2 constants, implies

$$\hat{K}_0^{(2)}(\epsilon, u) \leqslant C_3 \sqrt{\frac{h(\epsilon) h(u)}{\sqrt{\mu+\epsilon}\sqrt{\mu+u}}} \theta\left[\epsilon > -\mu\right] \theta\left[u > -\mu\right], \tag{A.21}$$

where C_3 is a constant, $h(\epsilon) > 0$, and $h(\epsilon \to \infty) \sim 1/\sqrt{\epsilon}$. Thus,

$$\begin{aligned} \hat{K}_0^{(3)}(\epsilon, u) \leqslant C_4 &\sqrt{\frac{h(\epsilon)}{\sqrt{\mu+\epsilon}\sqrt{\mu+u}}} \theta\left[\epsilon > -\mu\right] \theta\left[u > -\mu\right] \\ &\times \int_{-u-\mu}^{\infty} \frac{dx}{N(x+u)} \sqrt{h(x)} \frac{x^2}{\sinh \frac{|x|}{2T}} \\ \leqslant C_5 &\sqrt{\frac{h(\epsilon)}{\sqrt{\mu+\epsilon}\sqrt{\mu+u}}} \theta\left[\epsilon > -\mu\right] \theta\left[u > -\mu\right] h(u/2), \end{aligned} \tag{A.22}$$

where both C_4 and C_5 are constants, can be used to show

$$\hat{K}_0^{(6)}(\epsilon, u) \leqslant C_6 \sqrt{\frac{h(\epsilon)}{\sqrt{\mu+\epsilon}\sqrt{\mu+u}}} \theta\left[\epsilon > -\mu\right] \theta\left[u > -\mu\right] h(u/2), \tag{A.23}$$

with C_6 constant, which yields

$$\hat{K}_0^{(6)}(\epsilon, \epsilon) \leqslant C_6 \sqrt{\frac{h(\epsilon)}{\mu+\epsilon}} \theta\left[\epsilon > -\mu\right] h(\epsilon/2), \tag{A.24}$$

and therefore

$$\int_{\mathbb{R}} d\epsilon \int_{\mathbb{R}} du \left[\hat{K}_0^{(3)}(\epsilon, u)\right]^2 = \int_{\mathbb{R}} d\epsilon \, \hat{K}_0^{(6)}(\epsilon, \epsilon) < \infty. \tag{A.25}$$

□

Before considering $\vec{\mathfrak{H}}$, we investigate the spectrum of the operator $C_0 : \mathfrak{H} \to \mathfrak{H}$ as defined by

$$\big[C_0 \circ e\big](\vec{q}) = \Gamma(\vec{q})e(\vec{q}) - \big[K_0 \circ e\big](\vec{q}), \quad \Gamma(\vec{q}) = \big[K_0 \circ 1\big](\vec{q}). \tag{A.26}$$

For this purpose, we first establish the following.

Lemma 2. *The function* $\Gamma : \mathbb{R}^3 \to \mathbb{R}$ *is positive definite.*

Proof. By definition,

$$\begin{aligned}
\Gamma(\vec{q}) &= \frac{1}{N(\xi_q)} \int_{\mathbb{R}} du \big[n(u-\xi_q) + f(u)\big] \tilde{\mathcal{V}}''(\xi_q, u) \\
&= \frac{\pi \nu_0}{\omega_D^2 N(\xi_q)} \int_{\mathbb{R}} du \big[n(u-\xi_q) + f(u)\big] (u-\xi_q)^2 \mathrm{sgn}(u-\xi_q) \theta(\xi_q, u) \\
&= \frac{\pi \nu_0}{\omega_D^2 N(\xi_q)} \int_0^\infty du\, u^2 \Big\{ \big[n(u) + f(u+\xi_q)\big] \theta(\xi_q, \xi_q + u) \\
&\qquad\qquad + \big[n(u) + f(u-\xi_q)\big] \theta(\xi_q, \xi_q - u) \Big\} \\
&> 0;
\end{aligned} \tag{A.27}$$

moreover

$$\Gamma(\vec{q}) \sim \xi_q^{5/2}, \quad |\vec{q}| \to \infty, \tag{A.28}$$

as follows by the asymptotic properties of the third order polylogarithm. □

Despite the fact that C_0 does not admit a purely discrete spectrum, Lemma 2 together with a textbook [59] theorem[2] of Weyl and von Neumann allows us to conclude that the zero eigenvalue of C_0 (on $\mathfrak{H}$) is isolated.

On supplementing the above logic with a standard manipulation [21, 22], we have our main result.

Theorem 1. *The linear operator* $C_0 : \mathfrak{H} \to \mathfrak{H}$ *is non-negative and its spectrum contains a zero eigenvalue that is both unique and isolated.*

[2]Which reads: "If a completely continuous symmetric transformation B is added to a symmetric transformation A, the set of limit points of the spectrum remains invariant."

Proof. Suppose that $\psi_\lambda \in \mathfrak{H}$ obeys $C_0 \circ \psi_\lambda = \lambda\psi_\lambda$. Then,

$$\begin{aligned}\lambda\big(\psi_\lambda, \psi_\lambda\big)_{\mathfrak{H}} &= \int_{\mathbb{R}} d\epsilon\, w(\epsilon)\frac{N(\epsilon)}{N_F}\psi_\lambda(\epsilon)\int_{\mathbb{R}} du\big\{\Gamma(\epsilon)\delta(\epsilon-u) - K_0(\epsilon,u)\big\}\psi_\lambda(u) \\ &= \frac{1}{2}\int_{\mathbb{R}} d\epsilon\int_{\mathbb{R}} du\big[\psi_\lambda(\epsilon)-\psi_\lambda(u)\big]^2 W_0(\epsilon,u),\end{aligned} \tag{A.29}$$

where

$$W_0(\epsilon,u) = w(\epsilon)\frac{N(\epsilon)}{N_F}K_0(\epsilon,u) = W_0(u,\epsilon) \tag{A.30}$$

is positive almost everywhere; $W_0(\epsilon,u) = 0$ iff $\epsilon = u$. Thus, $\lambda = 0$ requires that $\psi_\lambda(\epsilon) = \psi_\lambda(u)$, i.e. ψ_λ is the constant function; all other eigenvalues of C_0 (on $\mathfrak{H}$) must be positive. □

Finally, if we limit ourselves to the regime of weak momentum dissipation, then asymptotic perturbation theory may be employed towards determining both λ_0 and $\vec{\mathfrak{e}}_0$. To this end, we prefer to study the operator $C : \mathfrak{H} \to \mathfrak{H}$ as defined by

$$\big(\vec{e}_i; C_0 \circ \vec{e}_j\big)_{\vec{\mathfrak{H}}} = \big(e_i, C \circ e_j\big)_{\mathfrak{H}}, \tag{A.31}$$

which can be decomposed into a momentum conserving part L_0 and the remainder $L_1 + L_2$, i.e.

$$C = L_0 + L_1 + L_2, \tag{A.32}$$

with both $L_0 \circ k = 0$ and $\big[L_1 + L_2\big] \circ k \neq 0$.

Corollary 2. $C_0 : \vec{\mathfrak{H}} \to \vec{\mathfrak{H}}$ *is a positive operator which admits a unique lowest eigenvalue that is isolated (for* T *sufficiently low).*

Proof. Evidently, the spectra of C (on $\mathfrak{H}$) and C_0 (on $\vec{\mathfrak{H}}$) coincide; it remains to demonstrate that $0 < \lambda_0 < \Gamma_{min}$, which holds because the t-series

$$\mathfrak{e}_0 = k + \mathcal{O}(t), \quad \lambda_0 = 0 + \mathcal{O}(t), \tag{A.33}$$

with

$$\left[L_0 + t\left(L_1 + L_2\right)\right] \circ \mathfrak{e}_0 = \lambda_0 \mathfrak{e}_0, \tag{A.34}$$

yields $0 < \lambda_0 \sim T^5/\omega_D^4$ when $T \ll \omega_D$, while $\Gamma \sim T^3/\omega_D^2$ when $T \ll \omega_D$; thus there exists a low temperature regime where the zero eigenvalue of L_0 is stable under the momentum nonconserving perturbation $L_1 + L_2$ [59]. $\square$

A.2 Computations

We sequester to this section all the technical details associated with evaluating our integral expressions for the conductivity.

A.2.1 Phonons. It is enlightening to derive the linearized Boltzmann equation from the Kubo formula for the linear response. To this end, we apply the T-approximation [39] to a theory of free electrons supplemented by the density-density vertex of Figure A.1; the wiggly line represents the dynamical potential $\mathcal{V}$, which is proportional to the longtitudinal phonon susceptibility and appears as an effective exchange interaction after integrating out the bosonic fluctuations.

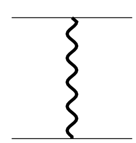


Figure A.1. A diagrammatic representation of the effective interaction.

In this particular truncation of the series for the scattering amplitude, Kubo's formula for the singular in temperature contribution to the bulk DC conductivity reads

$$\sigma = \frac{ne^2}{2m} \int \frac{d\epsilon}{\pi} w(\epsilon) \frac{1}{N_F V} \sum_{\vec{q}} \frac{q^2}{k_F^2} \left[-G''(\vec{q}, \epsilon) \right] \varphi(\vec{q}, \epsilon), \tag{A.35}$$

where φ obeys the ladder equation

$$\Gamma(\vec{q},\epsilon)\varphi(\vec{q},\epsilon) = 1 + \big[K \circ \varphi\big](\vec{q},\epsilon), \tag{A.36}$$

with

$$\begin{aligned} &\big[K \circ \varphi\big](\vec{q},\epsilon) \\ &= \int \frac{du}{\pi}\{n(u-\epsilon)+f(u)\}\frac{1}{V}\sum_{\vec{k}}\frac{\vec{q}\cdot\vec{k}}{q^2}\mathcal{V}''(\vec{k}-\vec{q},u-\epsilon)\big[-G''(\vec{k},u)\big]\varphi(\vec{k},u), \end{aligned} \tag{A.37}$$

and

$$\Gamma(\vec{q},\epsilon) = \int \frac{du}{\pi}\{n(u-\epsilon)+f(u)\}\frac{1}{V}\sum_{\vec{k}}\mathcal{V}''(\vec{k}-\vec{q},u-\epsilon)\big[-G''(\vec{k},u)\big], \tag{A.38}$$

which is generated by the kernel K_0, as defined by

$$\big[K \circ \varphi\big](\vec{q},\epsilon) = \int \frac{du}{\pi}\frac{1}{V}\sum_{\vec{k}}\frac{\vec{q}\cdot\vec{k}}{q^2}K_0\big[\vec{q},\epsilon\big|\vec{k},u\big]\varphi(\vec{k},u), \tag{A.39}$$

according to

$$\Gamma(\vec{q},\epsilon) = \big[K_0 \circ 1\big](\vec{q},\epsilon). \tag{A.40}$$

Now then, as the distribution functions n,f, w, $\mathcal{V}''$, and G'' ensure the scaling relationships

$$\begin{aligned} \epsilon \sim T, \quad & q^2 \sim k_F^2 + 2m\epsilon, \\ u \sim T, \quad & k^2 \sim K_F^2 + 2mu, \end{aligned} \tag{A.41}$$

it is helpful to rewrite

$$\begin{aligned} &\vec{q}\cdot\vec{k}\,\mathcal{V}''(\vec{k}-\vec{q},u-\epsilon) \\ &= q^2\Big\{1 - \frac{1}{2q^2}\Big[\Big(q^2-k^2\Big) + \Big(\frac{\epsilon-u}{c}\Big)^2\Big]\Big\}\mathcal{V}''(\vec{k}-\vec{q},u-\epsilon), \end{aligned} \tag{A.42}$$

by using the resonance [3]

$$\mathcal{V}''(\vec{k},u) = \pi g^2 u^2 \mathrm{sgn}(u)\delta\big[u^2 - c^2k^2\big]. \tag{A.43}$$

It follows that

$$\Gamma(\vec{q},\epsilon)\varphi(\vec{q},\epsilon) = 1 + \Big[\Big\{K_0 - \frac{k_F^2}{q^2}\Big(K_1 + K_2\Big)\Big\} \circ \varphi\Big](\vec{q},\epsilon), \tag{A.44}$$

where

$$K_1\big[\vec{q},\epsilon\big|\vec{k},u\big] = \frac{1}{2k_F^2}(q^2-k^2)K_0\big[\vec{q},\epsilon\big|\vec{k},u\big],$$
$$K_2\big[\vec{q},\epsilon\big|\vec{k},u\big] = \frac{2}{\omega_D^2}(\epsilon-u)^2K_0\big[\vec{q},\epsilon\big|\vec{k},u\big], \tag{A.45}$$

with $\omega_D = 2ck_F$ the Debye frequency. Next, we restrict our attention to the regime with $T \ll \omega_D$, whereupon $\Gamma \sim T^3/\omega_D^2$ implies that it is consistent (in the sense of counting powers of the control parameter T) to make the quasi-particle approximation

$$-G''(\vec{q},\epsilon) \to Z(\epsilon)\pi\delta\big[\mu(\epsilon)-\varepsilon_q\big], \tag{A.46}$$

where the renormalized chemical potential $\mu(\epsilon)$ is the solution to

$$\mu[\epsilon] = \epsilon_F + \epsilon - \Sigma'(\mu[\epsilon];\epsilon), \tag{A.47}$$

and the inverse quasi-particle weight $Z^{-1}(\epsilon)$ is given by

$$Z^{-1}(\epsilon) = 1 + \frac{\partial}{\partial\varepsilon_p}\bigg|_{\mu(\epsilon)}\Sigma'(\varepsilon_p;\epsilon); \tag{A.48}$$

here $\varepsilon_p = p^2/2m$ and $\Sigma'(\varepsilon_p;\epsilon) = \Sigma'(p=\sqrt{2m\varepsilon_p},\epsilon)$. Then, in the notation

$$\frac{1}{V}\sum_{\vec{q}}\big[-G''(\vec{q},\epsilon)\big]\psi(\vec{q},\epsilon) = Z(\epsilon)\pi N(\epsilon)\bar{\psi}(\epsilon), \tag{A.49}$$

we have

$$\sigma = \frac{ne^2}{2m}\int\frac{d\epsilon}{\pi}\,w(\epsilon)\frac{Z(\epsilon)\pi N(\epsilon)k^2(\epsilon)}{N_Fk_F^2}\bar{\varphi}(\epsilon), \tag{A.50}$$

and need to confront

$$\bar{\Gamma}(\epsilon)\bar{\varphi}(\epsilon) = 1 + \int\frac{du}{\pi}\Big\{\bar{K}_0(\epsilon,u) - \frac{k_F^2}{k^2(\epsilon)}\Big[\bar{K}_1(\epsilon,u)+\bar{K}_2(\epsilon,u)\Big]\Big\}\bar{\varphi}(u), \tag{A.51}$$

where the kernels

$$\bar{K}_1(\epsilon,u) = \frac{1}{2\epsilon_F}(\mu[\epsilon]-\mu[u])\bar{K}_0(\epsilon,u),$$
$$\bar{K}_2(\epsilon,u) = \frac{2}{\omega_D^2}(\epsilon-u)^2\bar{K}_0(\epsilon,u) \tag{A.52}$$

are specified by

$$\big[\bar{K}_0 \circ \bar{\varphi}\big](\epsilon) = \frac{1}{Z(\epsilon)\pi N(\epsilon)} \int \frac{du}{\pi} \big\{n(u-\epsilon) + f(u)\big\} \bar{\mathcal{V}}''(\epsilon, u)\bar{\varphi}(u), \tag{A.53}$$

with

$$\begin{aligned} \bar{\mathcal{V}}''(\epsilon, u) &= \frac{1}{V^2} \sum_{\vec{k},\vec{q}} G''(\vec{q}, \epsilon)\mathcal{V}''(\vec{k}-\vec{q}, u-\epsilon)G''(\vec{k}, u) \\ &= \pi g^2 \frac{\pi^2 N_F^2}{\omega_D^2} Z(\epsilon)Z(u)(u-\epsilon)^2 \mathrm{sgn}\big[u-\epsilon\big]\theta(\epsilon, u), \end{aligned} \tag{A.54}$$

which involves the Heaviside step function

$$\begin{aligned} \theta(\epsilon, u) = \theta\big[\mu(\epsilon) \geqslant 0\big]\theta\big[\mu(u) \geqslant 0\big] \\ \times \theta\Big[-2k(u)k(\epsilon) \leqslant k^2(u) + k^2(\epsilon) - \Big(\frac{\epsilon - u}{c}\Big)^2 \leqslant 2k(u)k(\epsilon)\Big]; \end{aligned} \tag{A.55}$$

recall that

$$N(\epsilon) = \frac{mk(\epsilon)}{2\pi^2}, \quad k^2(\epsilon) = 2m\mu(\epsilon). \tag{A.56}$$

Towards extracting the leading temperature dependence of the conductivity, we use

$$\begin{aligned} \frac{k_F^2}{k^2(\epsilon)} &= 1 + \frac{1}{k^2(\epsilon)}\big[k_F^2 - k^2(\epsilon)\big] \\ &= 1 + \frac{1}{2k^2(\epsilon)}\big\{\big[k^2(u) - k^2(\epsilon)\big] + \big[2k_F^2 - k^2(u) - k^2(\epsilon)\big]\big\} \end{aligned} \tag{A.57}$$

to write

$$\frac{k_F^2}{k^2(\epsilon)}\Big[\bar{K}_1(\epsilon, u) + \bar{K}_2(\epsilon, u)\Big] \approx \big\{1 - \frac{1}{2k_F^2}\big[k^2(\epsilon) - k^2(u)\big]\big\}\bar{K}_1(\epsilon, u) + \bar{K}_2(\epsilon, u); \tag{A.58}$$

the term

$$\frac{1}{2k_F^2}\big[2k_F^2 - k^2(\epsilon) - k^2(u)\big]\bar{K}_1(\epsilon, u) \tag{A.59}$$

is irrelevant because its strongest contribution to the resistivity is of order T^6. Thus, by neglecting kernels weaker (in the sense of their temperature scaling) than $\bar{K}_2$, we arrive at

$$\Big[\Big\{\frac{k_F^2}{k^2(\epsilon)}\Big(\bar{K}_1 + \bar{K}_2\Big)\Big\} \circ \bar{\varphi}\Big](\epsilon) \approx \Big[\Big\{\Big(\bar{K}_1 + \bar{K}_2 - \bar{K}_4\Big)\Big\} \circ \bar{\varphi}\Big](\epsilon), \tag{A.60}$$

where

$$\bar{K}_4(\epsilon, u) = \frac{1}{2k_F^2}\left[k^2(\epsilon) - k^2(u)\right]\bar{K}_1(\epsilon, u). \tag{A.61}$$

Now, the rate balancing equation reads

$$\left[\bar{C} \circ \bar{\varphi}\right](\epsilon) = 1, \tag{A.62}$$

where the collision operator $\bar{C} = \bar{C}_0 + t\bar{C}_1$, with both

$$\left[\bar{C}_0 \circ \bar{\varphi}\right](\epsilon) = \bar{\Gamma}(\epsilon)\bar{\varphi}(\epsilon) - \left[\bar{K}_0 \circ \bar{\varphi}\right](\epsilon) \tag{A.63}$$

and

$$\left[\bar{C}_1 \circ \bar{\varphi}\right](\epsilon) = \left[\left\{\bar{K}_1 + \bar{K}_2 - \bar{K}_4\right\} \circ \bar{\varphi}\right](\epsilon), \tag{A.64}$$

will be assumed to admit a spectral representation under the inner product

$$\left(\bar{\psi}, \bar{\phi}\right) = \int \frac{d\epsilon}{\pi} w(\epsilon) \frac{Z(\epsilon)\pi N(\epsilon)}{N_F} \bar{\psi}(\epsilon)\bar{\phi}(\epsilon), \tag{A.65}$$

such that the perturbed momentum mode $\bar{\mathfrak{e}}_0$ obeying

$$\bar{C} \circ \bar{\mathfrak{e}}_0 = \lambda_0 \bar{\mathfrak{e}}_0, \quad \bar{\mathfrak{e}}_0 = 1 + \mathcal{O}(t), \quad \lambda_0 = 0 + \mathcal{O}(t), \tag{A.66}$$

corresponds to an eigenvalue λ_0 that is both isolated and the unique smallest element in the spectrum of $\bar{C}_0$; note that Equation A.65 facilitates the expression

$$\sigma = \frac{e^2 N_F}{6m^2}\left(\bar{k}^2, \bar{\varphi}\right), \quad t = 1, \tag{A.67}$$

and allows $\bar{C}_0$ to be formally self-adjoint, i.e.

$$\left(\bar{\psi}, \bar{C}_0 \circ \bar{\phi}\right) = \left(\bar{C}_0 \circ \bar{\psi}, \bar{\phi}\right), \tag{A.68}$$

as may be checked by direct computation. From here, we recognize that the approximate momentum conservation expressed by[3]

$$\bar{C}_0 \circ 1 = 0 \tag{A.69}$$

[3] Indeed, when momentum dissipating interactions are absent, particle number conservation implies global momentum conservation.

demonstrates that Equation A.62 is singular in the perturbation $\bar{C}_1$, which justifies expanding the resolvent of $\bar{C}$ in a Laurent series to obtain

$$\begin{aligned}\sigma &\approx \frac{ne^2}{2m}\big(\bar{k}^2/k_F^2, \bar{\mathfrak{e}}_0\big)\frac{1}{\lambda_0\big(\bar{\mathfrak{e}}_0, \bar{\mathfrak{e}}_0\big)}\big(\bar{\mathfrak{e}}_0, 1\big)\big[1 + \mathcal{O}(T\omega_D)\big] \\ &\approx \frac{ne^2}{2m}\frac{Z(0)}{\lambda_0}\big[1 + \mathcal{O}(T/\omega_D)\big],\end{aligned} \tag{A.70}$$

where

$$\big(1, 1\big) \approx Z(0) + \mathcal{O}(T/\omega_D), \quad \big(\bar{\mathfrak{e}}_0, \bar{\mathfrak{e}}_0\big) \approx \big(1, 1\big) + \mathcal{O}(T/\omega_D). \tag{A.71}$$

It remains to determine λ_0; the equation governing the first corrections

$$\bar{C}_0 \circ \bar{\mathfrak{e}}_0^{(1)} + \bar{C}_1 \circ 1 = \lambda_0^{(1)} 1 \tag{A.72}$$

admits a solution $\bar{\mathfrak{e}}_0^{(1)}$ iff

$$\lambda_0^{(1)}\big(1, 1\big) = \big(1, \big[\bar{K}_2 - \bar{K}_4\big] \circ 1\big) = \mathcal{O}(T^5/\omega_D^4), \tag{A.73}$$

where we've used the fact that $\bar{K}_1$ is skew-adjoint. Notice that if

$$\bar{\mathfrak{e}}_0^{(1)}(\epsilon) = \frac{-1}{2k_F^2}\Big[\bar{k}^2(\epsilon) - \frac{(1, \bar{k}^2)}{(1, 1)}1\Big] + \bar{E}_0^{(1)}, \tag{A.74}$$

then

$$\bar{C}_0 \circ \bar{\mathfrak{e}}_0^{(1)} = -\bar{K}_1 \circ 1 + \bar{C}_0 \circ \bar{E}_0^{(1)} \tag{A.75}$$

implies

$$\bar{C}_0 \circ \bar{E}_0^{(1)} = \lambda_0^{(1)} 1 - \big[\bar{K}_2 - \bar{K}_4\big] \circ 1; \tag{A.76}$$

knowledge of $\bar{E}_0^{(1)} = \mathcal{O}(T^2/\omega_D^2)$ is therefore not necessary. Next,

$$\bar{C}_0 \circ \bar{\mathfrak{e}}_0^{(2)} + \bar{C}_1 \circ \bar{\mathfrak{e}}_0^{(1)} = \lambda_0^{(2)} 1 \tag{A.77}$$

requires

$$\lambda_0^{(2)}\big(1, 1\big) = \big(1, \bar{C}_1 \circ \bar{\mathfrak{e}}_0^{(1)}\big) \approx \big(1, \bar{K}_1 \circ \bar{\mathfrak{e}}_0^{(1)}\big) + \mathcal{O}\big(T^6/\omega_D^5\big). \tag{A.78}$$

Having established that

$$\sigma \approx \frac{ne^2}{2m}\frac{Z(0)}{\lambda_0^{(1)}+\lambda_0^{(2)}}\big[1+\mathcal{O}(T/\omega_D)\big], \tag{A.79}$$

we motion to evaluate λ_0; evidently, it is sufficient to take

$$\mu[\epsilon] \approx \epsilon_F + \epsilon - \Sigma'(\mu[0];0) - \epsilon\frac{d}{d\nu}\Big|_{\nu=0}\Sigma'(\mu[\nu];\nu), \tag{A.80}$$

whereupon,

$$\bar{K}_1(\epsilon,u) \approx \frac{\epsilon-u}{2\epsilon_F}\Big[1-\frac{d}{d\nu}\Big|_{\nu=0}\Sigma'(\mu[\nu];\nu)\Big]\bar{K}_0(\epsilon,u) \tag{A.81}$$

and

$$\begin{aligned}\bar{K}_4(\epsilon,u) &\approx \frac{\epsilon-u}{2\epsilon_F}\Big[1-\frac{d}{d\nu}\Big|_{\nu=0}\Sigma'(\mu[\nu];\nu)\Big]\bar{K}_1(\epsilon,u)\\ &= \frac{\omega_D^2}{8\epsilon_F^2}\Big[1-\frac{d}{d\nu}\Big|_{\nu=0}\Sigma'(\mu[\nu];\nu)\Big]^2\bar{K}_2(\epsilon,u).\end{aligned} \tag{A.82}$$

From the symmetry property

$$w(\epsilon)Z(\epsilon)N(\epsilon)\overline{K}_0(\epsilon,u) = w(u)Z(u)N(u)\overline{K}_0(u,\epsilon), \tag{A.83}$$

it is easy to see that

$$\big(1,\bar{K}_1\circ\bar{\mathfrak{e}}_0^{(1)}\big) \approx \frac{\omega_D^2}{16\epsilon_F^2}\Big[1-\frac{d}{d\nu}\Big|_{\nu=0}\Sigma'(\mu[\nu];\nu)\Big]^2\big(1,\bar{K}_2\circ 1\big), \tag{A.84}$$

which implies

$$\lambda_0\big(1,1\big) \approx \Big\{1-\frac{\omega_D^2}{16\epsilon_F^2}\Big[1-\frac{d}{d\nu}\Big|_{\nu=0}\Sigma'(\mu[\nu];\nu)\Big]^2\Big\}\big(1,\bar{K}_2\circ 1\big)+\mathcal{O}\big(T^6/\omega_D^5\big). \tag{A.85}$$

Finally, the Bloch–Grüneisen law [2] is found on performing the integral

$$\begin{aligned}&\big(1,\bar{K}_2\circ 1\big)\\ &= \frac{2}{N_F\omega_D^2}\int\frac{d\epsilon\, du}{\pi^2}w(\epsilon)\big\{n(u-\epsilon)+f(u)\big\}(\epsilon-u)^2\,\bar{\mathcal{V}}''(\epsilon,u)\\ &= \frac{2}{N_F\omega_D^2}\int\frac{d\epsilon}{\pi}w(\epsilon)\int_0^\infty\frac{du}{\pi}\big\{n(u)+f(u+\epsilon)\big\}u^2\\ &\qquad\qquad \times\big[\bar{\mathcal{V}}''(\epsilon,u+\epsilon)-\bar{\mathcal{V}}''(-\epsilon,-u-\epsilon)\big],\end{aligned} \tag{A.86}$$

which contains

$$\theta(\pm\epsilon, \pm u \pm \epsilon) = \theta\big[0 \leqslant u \leqslant T_1^{\pm}(\epsilon)\big], \tag{A.87}$$

where the generalized Debye frequencies

$$T_1^{\pm}(\epsilon) \approx \omega_D\Big[1 \pm \frac{\omega_D}{4\epsilon_F} \pm \frac{\epsilon}{2\epsilon_F}\Big] + \mathcal{O}(\epsilon^2) \tag{A.88}$$

allow for[4]

$$\bar{\mathcal{V}}''(\pm\epsilon, \pm u \pm \epsilon) \approx \pi g^2 \frac{\pi^2 N_F^2}{\omega_D^2} Z^2(0) u^2 \mathrm{sgn}(\pm u)\theta\big[0 \leqslant u \leqslant \omega_D\big] \tag{A.89}$$

in Equation A.86. As a result,

$$\begin{aligned}\big(1, \bar{K}_2 \circ 1\big) &\approx Z^2(0)\frac{\pi\nu_0}{\omega_D^4}\int d\epsilon\, w(\epsilon)\int_0^{\omega_D} du\big\{n(u) + f(u+\epsilon)\big\}u^4 \\ &= \pi\nu_0 Z^2(0)\frac{T^5}{\omega_D^4}\int_0^{\omega_D/T} du \frac{u^5}{4\sinh^2\frac{u}{2}} \\ &\approx \frac{T^5}{\omega_D^4}120\pi\zeta(5)\nu_0 Z^2(0)\big[1 + \mathcal{O}(T/\omega_D)\big]\end{aligned} \tag{A.90}$$

yields

$$\frac{1}{2\tau_{tr}(T)} = \frac{T^5}{\omega_D^4}\Big[1 - \frac{\tilde{\omega}_D^2}{16\epsilon_F^2}\Big]\nu_0 120\zeta(5) + \mathcal{O}(T^6/\omega_D^5), \quad T \ll \omega_D, \tag{A.91}$$

where

$$\tilde{\omega}_D = \omega_D\Big[1 - \frac{d}{d\nu}\Big|_{\nu=0}\Sigma'(\mu[\nu];\nu)\Big]; \tag{A.92}$$

as found by Prange and Kadanoff [61], the field strength renormalization factor $Z(0)$ has canceled out.

A.2.2 Magnons. Without further ado, we commence an evaluation of both the clean conductivity and its leading, Matthiessen's rule violating, corrections in the presence of quenched disorder.

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[4]The particle-hole bound difference, $T_1^+ - T_1^-$, leads only to exponentially suppressed corrections and the ϵ-dependence of $T_1^{\pm}(\epsilon)$ is of higher order in temperature.

A.2.2.1 Ideal metal. When impurity effects can be safely ignored, determining the conductivity

$$\sigma = \frac{e^2}{6m^2}\int \frac{d\epsilon}{\pi} w(\epsilon)\frac{1}{V}\sum_{\vec{q},\alpha} q^2\left[G''\varphi\right]_\alpha(\vec{q},\epsilon) \tag{A.93}$$

requires solving

$$\gamma_\alpha^{(\nu)}(\vec{q},\epsilon)\varphi_\alpha(\vec{q},\epsilon) = 1 + \left[K^{(\nu)}\circ\varphi\right]_\alpha(\vec{q},\epsilon), \tag{A.94}$$

where

$$K^{(\nu)}\left[\vec{q},\epsilon,\alpha\middle|\vec{p},u,\beta\right] = \frac{\vec{q}\cdot\vec{p}}{q^2}K_0^{(\nu)}\left[\vec{q},\epsilon,\alpha\middle|\vec{p},u,\beta\right], \tag{A.95}$$

with both

$$\begin{aligned}&\left[K_0\circ\varphi\right]_\alpha(\vec{q},\epsilon)\\ &= \int\frac{du}{\pi}\left\{n(u-\epsilon)+f(u)\right\}\frac{1}{V}\sum_{\vec{k},\beta}\mathcal{V}''_{\alpha\beta}(\vec{k}-\vec{q},u-\epsilon)\left[-G''_\beta(\vec{k},u)\right]\varphi_\beta(\vec{k},u)\end{aligned} \tag{A.96}$$

and

$$\gamma_{(\nu)} = K_0^{(\nu)}\circ 1. \tag{A.97}$$

As usual, we first use

$$\mathcal{V}''_{\alpha\beta}(\vec{k},u) = \mathcal{V}_0\frac{2T_1}{N_F}\beta\delta\left[Dk^2-\beta u\right](1-\delta_{\alpha\beta}) \tag{A.98}$$

to write

$$\begin{aligned}K^{\alpha\beta}\left[\vec{q},\epsilon\middle|\vec{k},u\right] &= \frac{\vec{k}\cdot\vec{q}}{q^2}K_0^{\alpha\beta}\left[\vec{q},\epsilon\middle|\vec{k},u\right]\\ &= \left\{1-\frac{1}{2q^2}\left[(q^2-k^2)+\frac{\beta}{D}(u-\epsilon)\right]\right\}K_0^{\alpha\beta}\left[\vec{q},\epsilon\middle|\vec{k},u\right],\end{aligned} \tag{A.99}$$

and then limit ourselves to contributions that are linear in $\mathcal{V}_0$ by taking

$$-G''_\sigma(\vec{q},\epsilon)\to\pi\delta\left[\epsilon-\omega_\sigma(\vec{q})\right] \tag{A.100}$$

and pinning all wavevectors to the ϵ-Fermi shell

$$\bar{\phi}_\alpha(\epsilon) = \frac{1}{\pi N_\alpha(\epsilon)V}\sum_{\vec{q}}\left[-G''_\alpha(\vec{q},\epsilon)\right]\phi_\alpha(\vec{q},\epsilon), \tag{A.101}$$

which produces

$$\left[\bar{K}_0 \circ \bar{\varphi}\right]_\alpha(\vec{q}, \epsilon) = \frac{1}{\pi N_\alpha(\epsilon)} \int \frac{du}{\pi} \left\{n(u-\epsilon) + f(u)\right\} \sum_\beta \bar{\mathcal{V}}''_{\alpha\beta}(\epsilon, u) \bar{\varphi}_\beta(u), \tag{A.102}$$

where

$$\begin{aligned} \bar{\mathcal{V}}''_{\alpha\beta}(\epsilon, u) &= \frac{1}{V^2} \sum_{\vec{k},\vec{q}} G''_\alpha(\vec{q}, \epsilon) \mathcal{V}''_{\alpha\beta}(\vec{k}-\vec{q}, u-\epsilon) G''_\beta(\vec{k}, u) \\ &= 2\pi^2 N_F \mathcal{V}_0 \beta (1-\delta_{\alpha\beta}) \theta_{\alpha\beta}(\epsilon, u) \end{aligned} \tag{A.103}$$

involves

$$\begin{aligned} &\theta_{\alpha\beta}(\epsilon, u) \\ &= \theta\left[\epsilon \geqslant -\epsilon_F^\alpha\right] \theta\left[u \geqslant -\epsilon_F^\beta\right] \\ &\quad \times \theta\left[-2\bar{k}_\alpha(\epsilon)\bar{k}_\beta(u) \leqslant \bar{k}_\alpha^2(\epsilon) + \bar{k}_\beta^2(u) - \frac{\beta}{D}(u-\epsilon) \leqslant 2\bar{k}_\alpha(\epsilon)\bar{k}_\beta(u)\right]; \end{aligned} \tag{A.104}$$

as a result of this procedure, Equation A.94 is reduced to

$$\begin{aligned} &\bar{\gamma}^\alpha_{(\nu)}(\epsilon)\bar{\varphi}_\alpha(\epsilon) \\ &= 1 + \int \frac{du}{\pi} \sum_\beta \left\{1 - \frac{1}{2\bar{k}_\alpha^2(\epsilon)} \left[2m(\epsilon + \alpha\lambda - u - \beta\lambda) + \frac{\beta}{D}(u-\epsilon)\right]\right\} \\ &\qquad \times \bar{K}_0^{\alpha\beta}(\epsilon, u) \bar{\varphi}_\beta(u), \end{aligned} \tag{A.105}$$

with

$$\bar{k}_\alpha^2(\epsilon) = k_F^2 + 2m(\epsilon + \alpha\lambda). \tag{A.106}$$

Note that

$$\beta(u-\epsilon)\mathcal{V}''_{\alpha\beta}(\vec{k}-\vec{q}, u-\epsilon) = |u-\epsilon|\, \mathcal{V}''_{\alpha\beta}(\vec{k}-\vec{q}, u-\epsilon). \tag{A.107}$$

Next, we use the identity

$$\frac{k_F^2}{k_\alpha^2(\epsilon)} = \frac{1}{(1+\epsilon/\epsilon_F)^2 - (\lambda/\epsilon_F)^2} \left[1 + \frac{\epsilon - \alpha\lambda}{\epsilon_F}\right] \tag{A.108}$$

to expand

$$
\begin{aligned}
&\frac{1}{2k_\alpha^2(\epsilon)}\left[4m\alpha\lambda + 2m(\epsilon-u) + \frac{\alpha}{D}(\epsilon-u)\right] \\
&= \frac{1}{(1+\epsilon/\epsilon_F)^2-(\lambda/\epsilon_F)^2}\left[1+\frac{\epsilon-\alpha\lambda}{\epsilon_F}\right]\left[\alpha\frac{\lambda}{\epsilon_F}+\frac{\epsilon-u}{2\epsilon_F}+2\alpha\frac{\epsilon-u}{T_1}\right] \\
&\approx \frac{1}{(1+\epsilon/\epsilon_F)^2-(\lambda/\epsilon_F)^2} \\
&\qquad \times\left\{\left[\alpha\frac{\lambda}{\epsilon_F}+\left(1-\frac{4\lambda}{T_1}\right)\frac{\epsilon-u}{2\epsilon_F}+\mathcal{O}\left(\frac{\lambda T_0}{\epsilon_F^2}\right)\right]\right. \\
&\qquad\qquad \left.+\left[\frac{-\lambda^2}{\epsilon_F^2}+2\alpha\frac{\epsilon-u}{T_1}+\mathcal{O}\left(\frac{T_0^2}{T_1\epsilon_F}\right)\right]\right\};
\end{aligned}
\tag{A.109}
$$

in the above, a self-adjoint kernel (see Equation A.119 and Equation A.126)

$$\hat{\hat{K}}_{2,c}^{\alpha\beta}(\epsilon,u) = \alpha\frac{\epsilon^2-u^2}{T_1\epsilon_F}\,\hat{\hat{K}}_0^{\alpha\beta}(\epsilon,u) \tag{A.110}$$

has been discarded (along with all those weaker than it) because its first eigenvalue correction

$$\left(1,\hat{\hat{K}}_{2,c}\circ 1\right) \approx \mathcal{V}_0\frac{T_0^2}{T_1}\left[0+\mathcal{O}\left(\frac{T_0}{\epsilon_F}\right)\right]e^{-T_{min}/T} \tag{A.111}$$

is outside the reach of Equation A.125; to work beyond this level of precision requires more effort than seems worthwhile, as the prefactor to the exponential is model dependent and the terms neglected so far are important only when $T \ll T_0^2/\epsilon_F$, where the (exponentially suppressed) magnon contribution to the resistivity is surely overwhelmed by some other mechanism for momentum dissipation. The minimum unit of energy transfer T_{min} is defined by Equation A.123.

Additionally, the skew-adjoint kernel

$$\hat{\hat{K}}_{1,c}^{\alpha\beta}(\epsilon,u) = \alpha\lambda\frac{\epsilon+u}{2\epsilon_F^2}\,\hat{\hat{K}}_0^{\alpha\beta}(\epsilon,u) \tag{A.112}$$

(along with all those weaker than it) has been neglected because its strongest contribution comes in as

$$\frac{-\lambda}{2\epsilon_F}\left(\sigma,\hat{\hat{K}}_{1,c}\circ 1\right) \approx \mathcal{V}_0\frac{T_0^2}{T_1}\left[0+\mathcal{O}\left(\frac{T_0}{\epsilon_F}\right)\right]e^{-T_{\min}/T}. \tag{A.113}$$

In the temperature range $T_0^2/\epsilon_F \ll T \ll T_0$, it will turn out that

$$\begin{aligned}\lambda_0(1,1) \approx \mathcal{V}_0\frac{4T_0^2}{T_1}\left[0+0\frac{T_0}{T_1}+0\frac{T_0}{\lambda}+\mathcal{O}\left(\frac{T_0}{\epsilon_F}\right)\right]e^{-T_{\min}/T}\\ +\mathcal{V}_0\frac{4T\,T_0}{T_1}\left[1+o(1)\right]e^{-T_{\min}/T},\end{aligned} \tag{A.114}$$

where the hydrodynamic eigenvalue λ_0 obeys Equation A.129. Onwards, with

$$\frac{1}{(1+\epsilon/\epsilon_F)^2-(\lambda/\epsilon_F)^2} \approx \Delta\left[1-2\Delta\frac{\epsilon}{\epsilon_F}+\mathcal{O}\left(\frac{T_0^2}{\epsilon_F^2}\right)\right], \tag{A.115}$$

where

$$\Delta = \frac{1}{1-(\lambda/\epsilon_F)^2} = \frac{1}{1-4T_0/T_1}, \tag{A.116}$$

we truncate at[5]

$$\begin{aligned}&\frac{1}{2k_\alpha^2(\epsilon)}\left[4m\alpha\lambda+2m(\epsilon-u)+\frac{\alpha}{D}(\epsilon-u)\right]\\ &\approx \Delta\left\{\left[\alpha\frac{\lambda}{\epsilon_F}+\left(1-\frac{4\lambda}{T_1}\right)\frac{\epsilon-u}{2\epsilon_F}+\mathcal{O}\left(\frac{\lambda T_0}{\epsilon_F^2}\right)\right]\right.\\ &\qquad\left.+\left[-\frac{\lambda^2}{\epsilon_F^2}+2\alpha\frac{\epsilon-u}{T_1}\left(1-\Delta\frac{\lambda T_1}{2\eta\epsilon_F^2}\right)+\mathcal{O}\left(\frac{T_0^2}{T_1\epsilon_F}\right)\right]\right\},\end{aligned} \tag{A.117}$$

which allows the decomposition

$$\bar{\varphi}_\alpha(\epsilon) = 1/\bar{\gamma}^\alpha_{(\nu)}(\epsilon)+\left[\left\{\hat{\hat{K}}_0-\hat{\hat{K}}_1-\hat{\hat{K}}_2\right\}\circ\bar{\varphi}\right]_\alpha(\epsilon), \tag{A.118}$$

where

$$\begin{aligned}\left[\hat{\hat{K}}_0\circ\bar{\varphi}\right]_\alpha(\epsilon) &= \int\frac{du}{\pi}\sum_\beta \hat{\hat{K}}_0^{\alpha\beta}(\epsilon,u)\bar{\varphi}_\beta(u)\\ &= \frac{1}{\pi N_\alpha(\epsilon)\bar{\gamma}^\alpha_{(\nu)}(\epsilon)}\int\frac{du}{\pi}\left\{n(u-\epsilon)+f(u)\right\}\sum_\beta \bar{\mathcal{V}}''_{\alpha\beta}(\epsilon,u)\bar{\varphi}_\beta(u),\end{aligned} \tag{A.119}$$

[5] In the paper [5], we mistakenly set the right-hand side of Equation A.115 to Δ, which yields $\eta = 2$; on keeping the ϵ/ϵ_F correction, $\eta = 1$.

and

$$\begin{aligned}
\hat{\hat{K}}^{\alpha\beta}_{1,a}(\epsilon,u) &= \alpha\Delta\frac{\lambda}{\epsilon_F}\hat{\hat{K}}^{\alpha\beta}_0(\epsilon,u),\\
\hat{\hat{K}}^{\alpha\beta}_{1,b}(\epsilon,u) &= \Delta\frac{\epsilon-u}{2\epsilon_F}\left[1-\frac{4\lambda}{T_1}\right]\hat{\hat{K}}^{\alpha\beta}_0(\epsilon,u),\\
\hat{\hat{K}}^{\alpha\beta}_{2,a}(\epsilon,u) &= \Delta\frac{-\lambda^2}{\epsilon_F^2}\hat{\hat{K}}^{\alpha\beta}_0(\epsilon,u),\\
\hat{\hat{K}}^{\alpha\beta}_{2,b}(\epsilon,u) &= 2\alpha\Delta\frac{\epsilon-u}{T_1}\left[1-\Delta\frac{\lambda T_1}{2\eta\epsilon_F^2}\right]\hat{\hat{K}}^{\alpha\beta}_0(\epsilon,u).
\end{aligned} \tag{A.120}$$

When evaluating the scalar product integrals, we'll encounter

$$(1-\delta_{\alpha\beta})\theta_{\alpha\beta}(\epsilon,\epsilon-\alpha u) = (1-\delta_{\alpha\beta})\theta\left[T^\alpha_0(\epsilon)\leqslant u\leqslant T^\alpha_1(\epsilon)\right], \tag{A.121}$$

where

$$T^\alpha_{0,1}(\epsilon) = T^\alpha_{0,1}(\epsilon=0)\left[1+\mathcal{O}\left(\frac{T_0}{\epsilon_F},\frac{T}{\epsilon_F}\right)\right], \quad T_0 = 4D(\lambda/\nu_F)^2; \tag{A.122}$$

the electron excitation frequencies, ϵ and u, scale with both T and T_0. To the same approximation,

$$T_{\min}\equiv T^\alpha_0(\epsilon=0)\approx T_0\left[1+\frac{T_0}{T_1}+\frac{T_0}{\lambda}+\mathcal{O}\left(\frac{T_0}{\epsilon_F},\frac{T}{\epsilon_F}\right)\right] \tag{A.123}$$

and

$$T_{\max}\equiv T^\alpha_1(\epsilon=0)\approx T_1 = 4Dk_F^2; \tag{A.124}$$

therefore,

$$\bar{\mathcal{V}}''_{\alpha\beta}(\epsilon,u)\approx 2\pi^2 N_F\mathcal{V}_0\beta(1-\delta_{\alpha\beta})\left\{\theta\left[T_{\min}\leqslant u\leqslant T_1\right]+\mathcal{O}\left(\frac{T_0}{\epsilon_F},\frac{T}{\epsilon_F}\right)\right\}. \tag{A.125}$$

Now then, having divided through by $\bar{\gamma}_{(\nu)}$ to form $\hat{\hat{K}}$ from $\bar{K}$, we equip

$$\left(\bar{\psi},\bar{\phi}\right) = \int\frac{d\epsilon}{\pi}\,w(\epsilon)\frac{1}{2N_F}\sum_\alpha \pi N_\alpha(\epsilon)\bar{\gamma}^\alpha_{(\nu)}(\epsilon)\bar{\psi}_\alpha(\epsilon)\bar{\phi}_\alpha(\epsilon), \tag{A.126}$$

which is motivated by the symmetry

$$w(\epsilon)N_\alpha(\epsilon)\bar{\gamma}^\alpha_{(\nu)}(\epsilon)\,\hat{\hat{K}}^{\alpha\beta}_0(\epsilon,u) = w(u)N_\beta(u)\bar{\gamma}^\beta_{(\nu)}(u)\,\hat{\hat{K}}^{\beta\alpha}_0(u,\epsilon), \tag{A.127}$$

so that[6]

$$
\begin{aligned}
\sigma &= \frac{e^2}{6m^2}\int d\epsilon\, w(\epsilon)\sum_\alpha \bar{k}_\alpha^2(\epsilon)N_\alpha(\epsilon)\bar{\varphi}_\alpha(\epsilon) \\
&= \frac{2N_F e^2}{6m^2}\left(\bar{k}^2\bar{\gamma}_{(\nu)}^{-1}, \bar{\varphi}\right) \\
&\approx \frac{2N_F e^2}{6m^2}\left(\bar{k}^2\bar{\gamma}_{(\nu)}^{-1}, \bar{e}_0\right)\frac{1}{\lambda_0\left(\bar{e}_0, \bar{e}_0\right)}\left(\bar{e}_0, \bar{\gamma}_{(\nu)}^{-1}\right) \\
&\approx \frac{ne^2}{2m}\frac{1}{\lambda_0\left(1,1\right)},
\end{aligned}
\tag{A.128}
$$

where

$$
\hat{\bar{C}}^{(\nu)}\circ\bar{e}_0 = \lambda_0\bar{e}_0, \quad \bar{e}_0 = 1 + \mathcal{O}(t), \quad \lambda_0 = 0 + \mathcal{O}(t), \tag{A.129}
$$

defines the momentum mode $\bar{e}_0$, with corresponding eigenvalue λ_0, in an asymptotic t-series when

$$
\hat{\bar{C}}^{(\nu)} = \hat{\bar{C}}_0^{(\nu)} + t\hat{\bar{C}}_1^{(\nu)}, \quad \hat{\bar{C}}_0^{(\nu)} = 1 - \hat{\bar{K}}_0, \quad \hat{\bar{C}}_1^{(\nu)} = \hat{\bar{K}}_1^{(\nu)} + \hat{\bar{K}}_2^{(\nu)}; \tag{A.130}
$$

the collision operator $\hat{\bar{C}}^{(\nu)}$ conserves momentum iff $t = 0$, i.e.

$$
\hat{\bar{C}}^{(\nu)}\circ 1 = \mathcal{O}(t); \tag{A.131}
$$

Note that the equality $\hat{\bar{C}}_0^{(\nu)}\circ 1 = 0$ as it appears in Equation A.131 should be interpretted as a consequence of approximate momentum conservation; in the absence of momentum dissipating interactions, particle number conservation implies global momentum conservation.

It remains to solve the eigenproblem of Equation A.129; at first order

$$
\hat{\bar{C}}_0^{(\nu)}\circ\bar{e}_0^{(1)} + \hat{\bar{C}}_1^{(\nu)}\circ 1 = \lambda_0^{(1)}1 \tag{A.132}
$$

[6]The spectrum of $\hat{\bar{C}}_0^{(\nu)}$ is contained in the interval $[-1,1]$; consequently, $\hat{\bar{C}}^{(\nu)}$ is a positive operator.

requires

$$\lambda_0^{(1)}\big(1,1\big) = \big(1, \hat{\hat{K}}_2^{(\nu)} \circ 1\big). \tag{A.133}$$

Towards finding $\bar{\mathfrak{e}}_0^{(1)}$, note that both

$$\hat{\hat{K}}_{1,a}^{(\nu)} \circ 1 = \Delta \frac{\lambda}{\epsilon_F} \sigma, \quad \hat{\hat{C}}_0^{(\nu)} \circ \sigma = 2\sigma \tag{A.134}$$

and

$$\hat{\hat{K}}_{1,b}^{(\nu)} \circ 1 = \Delta \left[1 - \frac{4\lambda}{T_1}\right] \frac{1}{2\epsilon_F} \hat{\hat{C}}_0^{(\nu)} \circ \omega; \tag{A.135}$$

here $\sigma_\pm(\epsilon) = \pm 1$ and $\omega_\alpha(\epsilon) = \epsilon$. So, if we let

$$\bar{\mathfrak{e}}_0^{(1)} = \Delta \frac{-\lambda}{2\epsilon_F} \sigma - \Delta \left(1 - \frac{4\lambda}{T_1}\right) \frac{1}{2\epsilon_F} \left[\omega - \frac{\big(1,\omega\big)}{\big(1,1\big)} 1\right] + \bar{E}_0^{(1)}, \tag{A.136}$$

then

$$\hat{\hat{C}}_0^{(\nu)} \circ \bar{E}_0^{(1)} = \left[\lambda_0^{(1)} + \Delta \frac{\lambda^2}{\epsilon_F^2}\right] 1 - \hat{\hat{K}}_{2,b}^{(\nu)} \circ 1; \tag{A.137}$$

while the right-hand side of Equation A.137 is not visibly small, it is orthogonal to both 1 and σ, which follows (respectfully) by the condition for the existence of $\mathfrak{e}_0^{(1)}$ [i.e. the definition of $\lambda_0^{(1)}$] and the combination of $\big(1,\sigma\big) = 0$ with the identity

$$\big(\sigma, \hat{\hat{K}}_{2,b}^{(\nu)} \circ 1\big) = \Delta \left[1 - \Delta \frac{\lambda T_1}{2\eta \epsilon_F^2}\right] \frac{2}{T_1} \big(1, \hat{\hat{C}}_0^{(\nu)} \circ \omega\big) = 0, \tag{A.138}$$

which can be seen from

$$\hat{\hat{K}}_{2,b}^{(\nu)} \circ 1 = \Delta \left[1 - \Delta \frac{\lambda T_1}{2\eta \epsilon_F^2}\right] \frac{2}{T_1} \sigma \hat{\hat{C}}_0^{(\nu)} \circ \omega. \tag{A.139}$$

To determine $\bar{E}_0^{(1)}$ would require solving a nontrivial integral equation; we'll argue (see Equation A.162 and the attached footnote) that $\bar{E}_0^{(1)}$ contributes only in a subdominant manner. Returning to the task of finding $\lambda_0^{(1)}$, the term

$$\big(1, \hat{\hat{K}}_{2,a}^{(\nu)} \circ 1\big) = \Delta \frac{-\lambda^2}{\epsilon_F^2} \big(1,1\big) \tag{A.140}$$

calls for

$$
\begin{aligned}
(1,1) &= \int \frac{d\epsilon}{\pi}\, w(\epsilon) \frac{1}{2N_F} \sum_{\alpha} \pi N_\alpha(\epsilon) \bar{\gamma}^{\alpha}_{(\nu)}(\epsilon) \\
&= \int \frac{d\epsilon du}{\pi^2}\, w(\epsilon) \frac{1}{2N_F} \sum_{\alpha,\beta} \pi N_\alpha(\epsilon) \bar{K}_0^{\alpha\beta}(\epsilon,u) \\
&= \int \frac{d\epsilon du}{\pi^2}\, w(\epsilon) \frac{1}{2N_F} \sum_{\alpha,\beta} \{n(u-\epsilon)+f(u)\} \bar{\mathcal{V}}''_{\alpha\beta}(\epsilon,u) \\
&\approx \mathcal{V}_0 \sum_{\alpha} \int_{T_{\min}}^{T_1} du \int d\epsilon\, w(\epsilon) \{n(u)+f(u-\alpha\epsilon)\} \\
&\approx \mathcal{V}_0 \sum_{\alpha} \int_{T_{\min}}^{\infty} du \frac{u}{4T\sinh^2\frac{u}{2T}} \\
&\approx 2\mathcal{V}_0 (T_{\min}+T) e^{-T_{\min}/T},
\end{aligned} \tag{A.141}
$$

which implies

$$
(1,\hat{\hat{K}}^{(\nu)}_{2,a}\circ 1) \approx -8\mathcal{V}_0\Delta \frac{T_0^2}{T_1}\left[\frac{T_{\min}}{T_0}+\frac{T}{T_0}\right]\left\{1+\mathcal{O}\left(\frac{T_0}{\epsilon_F},\frac{T}{\epsilon_F}\right)\right\} e^{-T_{\min}/T}. \tag{A.142}
$$

Next,

$$
\begin{aligned}
&(1,\hat{\hat{K}}^{(\nu)}_{2,b}\circ 1) \\
&\approx \mathcal{V}_0\Delta\left[1-\Delta\frac{\lambda T_1}{2\eta\epsilon_F^2}\right] \sum_{\alpha} \frac{2}{T_1} \int_{T_{\min}}^{\infty} \frac{u^2 du}{4T\sinh^2\frac{u}{2T}} \\
&\approx \mathcal{V}_0\Delta\left[1-\Delta\frac{\lambda T_1}{2\eta\epsilon_F^2}\right] \frac{4T_0^2}{T_1}\left[\frac{T_{\min}^2}{T_0^2}+\frac{2T_{\min}T}{T_0^2}+\mathcal{O}\left(\frac{T^2}{T_0^2}\right)\right] e^{-T_{\min}/T}.
\end{aligned} \tag{A.143}
$$

Note that $T_0/\lambda = \sqrt{T_0T_1}/(2\epsilon_F)$ and

$$
T_{\min}^2 = T_0^2\left[1+\frac{2T_0}{T_1}+\frac{2T_0}{\lambda}+\mathcal{O}\left(\frac{T_0}{\epsilon_F}\right)\right]. \tag{A.144}
$$

So,

$$
(1,\hat{\hat{K}}^{(\nu)}_{2,b}\circ 1) \approx 4\mathcal{V}_0\Delta \frac{T_0^2}{T_1}\left[1+\frac{2T_0}{T_1}+\frac{2T_0}{\lambda}-\Delta\frac{\lambda T_1}{2\eta\epsilon_F^2}+\frac{2T}{T_0}\right] e^{-T_{\min}/T}. \tag{A.145}
$$

In total,

$$
\begin{aligned}
\lambda_0^{(1)}(1,1) \approx{}& \mathcal{V}_0\Delta\frac{T_0^2}{T_1}\left\{-8\left(1+\frac{T_0}{T_1}+\frac{T_0}{\lambda}+\frac{T}{T_0}\right)\right\} e^{-T_{\min}/T} \\
&+ \mathcal{V}_0\Delta\frac{T_0^2}{T_1}\left\{4\left(1+\frac{2T_0}{T_1}+\frac{2T_0}{\lambda}-\frac{\lambda T_1}{2\eta\epsilon_F^2}+\frac{2T}{T_0}\right)\right\} e^{-T_{\min}/T}
\end{aligned} \tag{A.146}
$$

comes out negative. At order t^2,

$$\hat{\hat{C}}_0^{(\nu)} \circ \bar{\mathfrak{e}}_0^{(2)} + \hat{\hat{C}}_1^{(\nu)} \circ \bar{\mathfrak{e}}_0^{(1)} = \lambda_0^{(2)} 1 + \lambda_0^{(1)} \bar{\mathfrak{e}}_0^{(1)} \tag{A.147}$$

implies

$$\lambda_0^{(2)} \big(1, 1\big) = \big(1, \hat{\hat{C}}_1^{(\nu)} \circ \bar{\mathfrak{e}}_0^{(1)}\big), \tag{A.148}$$

which requires a number of computations. To begin,

$$\begin{aligned} \big(1, \hat{\hat{K}}_{1,b}^{(\nu)} \circ \bar{\mathfrak{e}}_0^{(1)}\big) &= -\big(\hat{\hat{K}}_{1,b}^{(\nu)} \circ 1, \bar{\mathfrak{e}}_0^{(1)}\big) \\ &= -\Delta \left[1 - \frac{4\lambda}{T_1}\right] \frac{1}{2\epsilon_F} \big(\hat{\hat{C}}_0^{(\nu)} \circ \omega, \bar{\mathfrak{e}}_0^{(1)}\big) \\ &= -\Delta \left[1 - \frac{4\lambda}{T_1}\right] \frac{1}{2\epsilon_F} \big(\omega, \lambda_0^{(1)} 1 - \hat{\hat{C}}_1^{(\nu)} \circ 1\big), \end{aligned} \tag{A.149}$$

is concerned with both

$$\begin{aligned} \big(\omega, 1\big) &\approx \mathcal{V}_0 \sum_\alpha \int_{T_{\min}}^{T_{\max}} du \int d\epsilon \, w\big(\epsilon\big) \epsilon \big\{n(u) + f(u - \alpha\epsilon)\big\} \\ &\approx \mathcal{V}_0 \sum_\alpha \int_{T_{\min}}^{\infty} \frac{u^2 du}{4T \sinh^2 \frac{u}{2T}} \frac{\alpha}{2} \\ &\approx \mathcal{V}_0 T_{\min}^2 \left\{0 + \mathcal{O}\left(\frac{T_0}{\epsilon_F}, \frac{T}{\epsilon_F}, \frac{T^2}{T_0^2}\right)\right\} e^{-T_{\min}/T} \end{aligned} \tag{A.150}$$

and

$$\big(\omega, \sigma\big) \approx \mathcal{V}_0 T_{\min}^2 \left\{1 + \frac{2T}{T_{\min}} + \mathcal{O}\left(\frac{T_0}{\epsilon_F}, \frac{T}{\epsilon_F}, \frac{T^2}{T_0^2}\right)\right\} e^{-T_{\min}/T}, \tag{A.151}$$

in addition to the two matrix elements

$$\begin{aligned} \big(\omega, \hat{\hat{K}}_{1,b}^{(\nu)} \circ 1\big) &= \Delta \left[1 - \frac{4\lambda}{T_1}\right] \frac{1}{2\epsilon_F} \big(\omega, \hat{\hat{C}}_0^{(\nu)} \circ \omega\big), \\ \big(\omega, \hat{\hat{K}}_{2,b}^{(\nu)} \circ 1\big) &= 2\Delta \left[1 - \Delta \frac{\lambda T_1}{2\eta \epsilon_F^2}\right] \frac{1}{T_1} \big(\sigma\omega, \hat{\hat{C}}_0^{(\nu)} \circ \omega\big), \end{aligned} \tag{A.152}$$

which can be evaluated easily, viz.

$$
\begin{aligned}
&\left(\omega, \hat{\hat{C}}_0^{(\nu)} \circ \omega\right) \\
&= \int \frac{d\epsilon du}{\pi^2} w(\epsilon) \sum_{\alpha,\beta} \epsilon(\epsilon - u)\pi N_\alpha(\epsilon) \bar{K}_0^{\alpha\beta}(\epsilon, u) \\
&= \frac{1}{2} \int \frac{d\epsilon du}{\pi^2} w(\epsilon) \sum_{\alpha,\beta} (\epsilon - u)^2 \pi N_\alpha(\epsilon) \bar{K}_0^{\alpha\beta}(\epsilon, u) \\
&\approx \mathcal{V}_0 \sum_\alpha \int_{T_{min}}^{T_{max}} du \, \frac{u^2}{2} \int d\epsilon \, w(\epsilon) \left\{ n(u) + f(u - \alpha\epsilon) \right\} \\
&\approx \mathcal{V}_0 \sum_\alpha \frac{1}{2} \int_{T_{min}}^{\infty} \frac{u^3 du}{4T \sinh^2 \frac{u}{2T}} \\
&\approx \mathcal{V}_0 T_{min}^3 \left\{ 1 + \mathcal{O}\left(\frac{T_0}{\epsilon_F}, \frac{T}{T_0}, \frac{T^2}{T_0^2} \right) \right\} e^{-T_{min}/T}
\end{aligned}
\tag{A.153}
$$

and

$$
\left(\sigma\omega, \hat{\hat{C}}_0^{(\nu)} \circ \omega\right) \approx \mathcal{V}_0 T_{min}^3 \left\{ 0 + \mathcal{O}\left(\frac{T_0}{\epsilon_F}, \frac{T}{T_0}, \frac{T^2}{T_0^2} \right) \right\} e^{-T_{min}/T}; \tag{A.154}
$$

hence,

$$
\left(\omega, \hat{\hat{K}}_{1,b}^{(\nu)} \circ 1\right) \approx \Delta \left[1 - \frac{4\lambda}{T_1} \right] \frac{T_{min}^3}{2\epsilon_F} e^{-T_{min}/T} \tag{A.155}
$$

and

$$
\left(\omega, \hat{\hat{K}}_{2,b}^{(\nu)} \circ 1\right) \approx 0. \tag{A.156}
$$

Thus,

$$\begin{aligned}
&\big(1, \hat{\hat{K}}_{1,b}^{(\nu)} \circ \bar{\mathfrak{e}}_0^{(1)}\big) \\
&\approx \Delta\left[1 - \frac{4\lambda}{T_1}\right]\frac{1}{2\epsilon_F}\big(\omega, \hat{\hat{K}}_1^{(\nu)} \circ 1\big) \\
&= \Delta\left[1 - \frac{4\lambda}{T_1}\right]\frac{1}{2\epsilon_F}\left\{\Delta\frac{\lambda}{\epsilon_F}(\omega, \sigma) + \Delta\left[1 - \frac{4\lambda}{T_1}\right]\frac{1}{2\epsilon_F}\big(\omega, \hat{\hat{C}}_0^{(\nu)} \circ \omega\big)\right\} \\
&\approx \mathcal{V}_0\Delta^2\left[1 - \frac{4\lambda}{T_1}\right]\left\{\frac{\lambda}{2\epsilon_F^2}T_{\min}^2 + \left[1 - \frac{4\lambda}{T_1}\right]\frac{T_{\min}^3}{4\epsilon_F^2}\right\}e^{-T_{\min}/T} \\
&\approx \mathcal{V}_0\Delta^2\left(1 - \frac{4\lambda}{T_1}\right)\left[\frac{\lambda}{2\epsilon_F^2}T_0^2\left\{1 + \frac{2T_0}{T_1} + \frac{2T_0}{\lambda}\right\}\right]e^{-T_{\min}/T} \\
&\approx \mathcal{V}_0\frac{T_0^2}{T_1}\Delta^2\left\{\frac{\lambda T_1}{2\epsilon_F^2} - \frac{2\lambda^2}{\epsilon_F^2}\right\}e^{-T_{\min}/T}.
\end{aligned} \tag{A.157}$$

Continuing on our quest to calculate

$$\lambda_0^{(2)}(1,1) = \big(1, \hat{\hat{K}}_1^{(\nu)} \circ \bar{\mathfrak{e}}_0^{(1)}\big) + \big(1, \hat{\hat{K}}_{2,b}^{(\nu)} \circ \bar{\mathfrak{e}}_0^{(1)}\big), \tag{A.158}$$

we still need

$$\begin{aligned}
&\big(1, \hat{\hat{K}}_{1,a}^{(\nu)} \circ \bar{\mathfrak{e}}_0^{(1)}\big) \\
&= \Delta\frac{-\lambda}{\epsilon_F}\big(\sigma, \bar{\mathfrak{e}}_0^{(1)}\big) \\
&= \Delta\frac{-\lambda}{\epsilon_F}\left\{\Delta\frac{-\lambda}{2\epsilon_F}(\sigma, \sigma) - \Delta\left[1 - \frac{4\lambda}{T_1}\right]\frac{1}{2\epsilon_F}(\sigma, \omega)\right\} \\
&= \mathcal{V}_0\Delta^2\frac{\lambda}{\epsilon_F}\left\{\frac{\lambda}{2\epsilon_F}2\big(T_{\min} + T\big) + \left[1 - \frac{4\lambda}{T_1}\right]\frac{T_{\min}^2}{2\epsilon_F}\right\}e^{-T_{\min}/T},
\end{aligned} \tag{A.159}$$

which implies both

$$\begin{aligned}
&\big(1, \hat{\hat{K}}_{1,a}^{(\nu)} \circ \bar{\mathfrak{e}}_0^{(1)}\big) \\
&\approx \mathcal{V}_0\Delta^2\frac{T_0^2}{T_1}\left\{4\left(1 + \frac{T_0}{T_1} + \frac{T_0}{\lambda} + \frac{T}{T_0}\right) + \left[\frac{\lambda T_1}{2\epsilon_F^2} - \frac{2\lambda^2}{\epsilon_F^2}\right]\right\}e^{-T_{\min}/T}
\end{aligned} \tag{A.160}$$

and

$$\begin{aligned}
\lambda_0^{(2)}(1,1) \approx{}& \mathcal{V}_0\Delta^2\frac{T_0^2}{T_1}\left\{4\left(1 + \frac{T_0}{T_1} + \frac{T_0}{\lambda} + \frac{T}{T_0}\right) + \frac{\lambda T_1}{2\epsilon_F^2} - 8\frac{T_0}{T_1}\right\}e^{-T_{\min}/T} \\
&+ \mathcal{V}_0\Delta^2\frac{T_0^2}{T_1}\left\{\frac{\lambda T_1}{2\epsilon_F^2} - 8\frac{T_0}{T_1}\right\}e^{-T_{\min}/T} + \big(\hat{\hat{K}}_{2,b} \circ 1, \bar{\mathfrak{e}}_0^{(1)}\big),
\end{aligned} \tag{A.161}$$

where

$$
\begin{aligned}
&\big(\hat{\bar{K}}^{(\nu)}_{2,b}\circ 1, \bar{\mathfrak{e}}^{(1)}_0\big)\\
&= \Delta\left[1-\Delta\frac{\lambda T_1}{2\eta\epsilon_F^2}\right]\frac{2}{T_1}\big(\hat{\bar{C}}^{(\nu)}_0\circ\omega, \sigma\bar{\mathfrak{e}}^{(1)}_0\big)\\
&= \Delta\left[1-\Delta\frac{\lambda T_1}{2\eta\epsilon_F^2}\right]\frac{2}{T_1}\big(\hat{\bar{C}}^{(\nu)}_0\circ\omega, \left\{\frac{-\Delta}{2\epsilon_F}\left(1-\frac{4\lambda}{T_1}\right)\sigma\left[\omega-\frac{(1,\omega)}{(1,1)}1\right]+\sigma\bar{E}_1\right\}\big)\\
&\approx \Delta\left[1-\frac{\lambda T_1}{2\eta\epsilon_F^2}\right]\frac{2}{T_1}\big(\hat{\bar{C}}^{(\nu)}_0\circ\omega, \sigma\bar{E}^{(1)}\big)
\end{aligned}
\tag{A.162}
$$

is suppressed by T_0/ϵ_F even in the worst case where $\bar{E}^{(1)}_0 \sim \frac{x}{T_1}+\sigma\frac{\lambda x}{\epsilon_F T_1}$ scales as strongly as possible.[7] Surprisingly, there is another zero prefactor; we must continue. At third order,

$$
\hat{\bar{C}}^{(\nu)}_0\circ\bar{\mathfrak{e}}^{(3)}_0+\hat{\bar{C}}^{(\nu)}_1\circ\bar{\mathfrak{e}}^{(2)}_0=\lambda^{(3)}_0 1+\lambda^{(2)}_0\bar{\mathfrak{e}}^{(1)}_0+\lambda^{(1)}_0\bar{\mathfrak{e}}^{(2)}_0
\tag{A.163}
$$

requires

$$
\lambda^{(3)}_0(1,1)=\big(1,\hat{\bar{C}}^{(\nu)}_1\circ\bar{\mathfrak{e}}^{(2)}_0\big)=\big(1,\big[\hat{\bar{K}}^{(\nu)}_1+\hat{\bar{K}}^{(\nu)}_{2,b}\big]\circ\bar{\mathfrak{e}}^{(2)}_0\big);
\tag{A.164}
$$

with

$$
\bar{\mathfrak{e}}^{(2)}_0=\frac{\big(\sigma,\bar{\mathfrak{e}}^{(2)}_0\big)}{(\sigma,\sigma)}\sigma+\bar{E}^{(2)},\quad \big(1,E^{(2)}\big)=\big(\sigma,\bar{E}^{(2)}\big)=0,
\tag{A.165}
$$

and discarding $\bar{E}^{(2)}_0$ terms for the same reason as $\bar{E}^{(1)}_0$ is not necessary, it follows that

$$
\begin{aligned}
\lambda^{(3)}_0(1,1) &\approx \frac{-\lambda}{\epsilon_F}\big(\sigma,\bar{\mathfrak{e}}^{(2)}_0\big)\\
&= \frac{-\lambda}{2\epsilon_F}\big(\sigma,\lambda^{(1)}_0\bar{\mathfrak{e}}^{(1)}_0-\hat{\bar{C}}^{(\nu)}_1\circ\bar{\mathfrak{e}}^{(1)}_0\big)\\
&\approx \frac{-\lambda}{2\epsilon_F}\left\{\Delta\frac{-\lambda}{2\epsilon_F}\lambda^{(1)}_0(\sigma,\sigma)-\Delta\frac{-\lambda}{2\epsilon_F}\big(\sigma,\hat{\bar{K}}_2\circ\sigma\big)\right\}\\
&= \Delta\frac{\lambda^2}{2\epsilon_F^2}\lambda^{(1)}_0(1,1).
\end{aligned}
\tag{A.166}
$$

---

[7] That the x^n and σx^n monomials span the space of functions inhabited by $E^{(1)}$ is plausible insofar as it can be proved to be true in the presence of a UV frequency cutoff [62] because (x^n, x^n) exists for all n. We then expect that $E^{(1)}$ vanishes as $T_1 \to \infty$ and that any leading σ structure must come with a factor of λ/ϵ_F.

The straggler

$$\begin{aligned}\lambda_0^{(4)}(1,1) &= \left(1, \hat{\mathfrak{C}}_1^{(\nu)} \circ \bar{\mathfrak{e}}_0^{(3)}\right) \\ &\approx \Delta \frac{-\lambda}{\epsilon_F}\left(\sigma, \bar{\mathfrak{e}}_0^{(3)}\right) \\ &\approx \Delta \frac{-\lambda}{2\epsilon_F}\lambda_0^{(2)}\left(\sigma, \bar{\mathfrak{e}}_0^{(1)}\right) \\ &= \Delta^2 \frac{\lambda^2}{4\epsilon_F^2}\lambda_0^{(2)}(1,1),\end{aligned} \tag{A.167}$$

where $\bar{E}_0^{(3)}$ is similarly irrelevant; one can check that all remaining eigenvalue corrections are negligible

On summing the relevant contributions to λ_0, we find

$$\lambda_0(1,1) \approx \mathcal{V}_0 \frac{4T_0^2}{T_1}\left[(\eta-1)\frac{T_0}{\lambda} + \mathcal{O}\left(\frac{T_0}{\epsilon_F}\right)\right] e^{-T_{\min}/T} + \mathcal{V}_0 \frac{4T\,T_0}{T_1} e^{-T_{\min}/T}, \tag{A.168}$$

which yields (on restoring $\eta = 1$)

$$\frac{1}{2\tau_{tr}(T)} = \mathcal{V}_0 \frac{4T\,T_0}{T_1} e^{-T_{\min}/T}, \quad T_0^2/\epsilon_F \ll T \ll T_0. \tag{A.169}$$

A.2.2.2 Beyond semi-classical. Of the contributions to Equation 4.115 and Equation 4.122, only $\left(\vec{k}; \mathfrak{C}_0^{(a)} \circ \vec{k}\right)$ and $\left(\vec{k}; \mathfrak{C}_0^{(c)} \circ \vec{k}\right)$ are important; we shall provide an explicit evaluation of the former. To this end, let

$$F_{\sigma\sigma'}(\epsilon, u) = \frac{4u_0 v_0}{N_F V} \sum_{\vec{k}} \mathcal{V}''_{\sigma\sigma'}(\vec{k}, u-\epsilon) f_{\sigma\sigma'}(\epsilon, u; k) \tag{A.170}$$

where

$$\begin{aligned}f_{\sigma\sigma'}(\epsilon, u; k) = \frac{1}{V^2}\sum_{\vec{q},\vec{p}} G''_\sigma(\vec{q}, \epsilon) G_{\sigma'}(\vec{p}, u)\left\{q^2 - \vec{q}\cdot\vec{p}\right\}\frac{1}{k_F^2} \\ \times \operatorname{Re}\left[G^A_\sigma(\vec{p}-\vec{k}, \epsilon) G^R_{\sigma'}(\vec{q}+\vec{k}, u)\right],\end{aligned} \tag{A.171}$$

and

$$u_0 = \frac{\gamma}{\pi N_F}, \quad v_0 = \frac{2T_1}{N_F}\mathcal{V}_0, \tag{A.172}$$

with γ the elastic impurity rate and $\mathcal{V}_0$ a dimensionless constant of order one.

Then,

$$\left(\vec{k};\mathfrak{C}_0^{(a)}\circ\vec{k}\right)=\frac{1}{2}\sum_{\sigma,\sigma'}\int\frac{d\epsilon du}{\pi^2}w(\epsilon)\left\{n(u-\epsilon)+f(u)\right\}F_{\sigma\sigma'}(\epsilon,u), \tag{A.173}$$

where

$$F_{\sigma\sigma'}(\epsilon,u)=\frac{u_0\nu_0}{\pi^2N_FD}\bar{k}f_{\sigma\sigma'}(\epsilon,u;\bar{k})\theta\left[\sigma'(u-\epsilon)>0\right]\sigma'\left(1-\delta_{\sigma\sigma'}\right), \tag{A.174}$$

with $\bar{k}=\sqrt{|u-\epsilon|/D}$. From here, we write

$$f_{\sigma\sigma'}(\epsilon,u;\bar{k})=f^{(')}_{\sigma\sigma'}(\epsilon,u;\bar{k})+f^{('')}_{\sigma\sigma'}(\epsilon,u;\bar{k}) \tag{A.175}$$

in terms of

$$\begin{aligned}f^{('')}_{\sigma\sigma'}(\epsilon,u;\bar{k})=\frac{\pi^4m^2N_F^2\bar{q}^2}{4k_F^4\bar{k}^2}&\left\{1+\frac{m^2}{\bar{q}^2\bar{k}^2}\zeta^{(A)}_{\sigma}\zeta^{(R)}_{\sigma'}\right\}\\ &\times\theta\left[\epsilon+\epsilon_F^{\sigma}>0\right]\theta\left[-\bar{k}\bar{p}\leqslant m\zeta^{(A)}_{\sigma}\leqslant\bar{k}\bar{p}\right]\\ &\times\theta\left[u+\epsilon_F^{\sigma'}>0\right]\theta\left[-\bar{k}\bar{q}\leqslant m\zeta^{(R)}_{\sigma'}\leqslant\bar{k}\bar{q}\right],\end{aligned} \tag{A.176}$$

and

$$\begin{aligned}&f^{(')}_{\sigma\sigma'}(\epsilon,u;\bar{k})\\ &=\frac{m^4N_F^2\pi^2}{4k_F^4\bar{k}^4}\left\{\left[\frac{\bar{q}^2\bar{k}^2}{m^2}+\zeta^{(R)}_{\sigma'}\zeta^{(A)}_{\sigma}\right]\log\left|\frac{m\zeta^{(R)}_{\sigma'}+\bar{q}\bar{k}}{m\zeta^{(R)}_{\sigma'}-\bar{q}\bar{k}}\right|\log\left|\frac{m\zeta^{(A)}_{\sigma}+\bar{p}\bar{k}}{m\zeta^{(A)}_{\sigma}-\bar{p}\bar{k}}\right|\right.\\ &\qquad-\frac{2\bar{p}\bar{k}}{m}\zeta^{(R)}_{\sigma'}\log\left|\frac{m\zeta^{(R)}_{\sigma'}+\bar{q}\bar{k}}{m\zeta^{(R)}_{\sigma'}-\bar{q}\bar{k}}\right|\\ &\qquad-\frac{2\bar{q}\bar{k}}{m}\zeta^{(A)}_{\sigma}\log\left|\frac{m\zeta^{(A)}_{\sigma}+\bar{p}\bar{k}}{m\zeta^{(A)}_{\sigma}-\bar{p}\bar{k}}\right|\\ &\qquad\left.+\frac{4\bar{q}\bar{p}\bar{k}^2}{m^2}\right\};\end{aligned} \tag{A.177}$$

here, $\bar{q}=k_\sigma(\epsilon)$, $\bar{p}=k_{\sigma'}(u)$, and

$$\begin{aligned}\zeta^{(A)}_{\sigma}&=2\epsilon_F\left[\frac{\epsilon-u}{2\epsilon_F}-\frac{2|u-\epsilon|}{T_1}+\sigma\frac{\lambda}{\epsilon_F}\right],\\ \zeta^{(R)}_{\sigma'}&=-2\epsilon_F\left[\frac{\epsilon-u}{2\epsilon_F}+\frac{2|u-\epsilon|}{T_1}-\sigma'\frac{\lambda}{\epsilon_F}\right].\end{aligned} \tag{A.178}$$

Having proceeded exactly until now, further progress entails inflicting an approximation upon the integrand of

$$\begin{aligned}&\left(\vec{k};\mathfrak{C}_0^{(a)}\circ\vec{k}\right)\\&=\frac{u_0\nu_0}{2\pi^2 N_F D}\int\frac{d\epsilon}{\pi}w(\epsilon)\int_0^\infty\frac{du}{\pi}\sum_\sigma\left\{n(u)+f(u-\sigma\epsilon)\right\}\underline{k}f_{\sigma\sigma'}(\epsilon,\epsilon-\sigma u;\underline{k}),\end{aligned}\tag{A.179}$$

where $\underline{k}=\sqrt{u/D}$ and $\sigma'=-\sigma$; for the leading temperature dependence, it is sufficient to take

$$\begin{aligned}f^{('')}_{\sigma\sigma'}(\epsilon,\epsilon-\sigma u;\underline{k})\approx\left[\frac{\pi^2 m N_F k_F^\sigma}{2k_F^2\underline{k}}\right]^2\left\{1+\left[\frac{m}{k_F^\sigma\underline{k}}\right]^2\underline{\zeta}_\sigma^{(A)}\underline{\zeta}_{\sigma'}^{(R)}\right\}\\\times\theta\left[T_{\min}<u<T_1\right]\left\{1+\mathcal{O}\left(\frac{\epsilon}{\epsilon_F},\frac{u}{\epsilon_F}\right)\right\}\end{aligned}\tag{A.180}$$

where

$$\begin{aligned}\underline{\zeta}_\sigma^{(A)}&=2\epsilon_F\left[\frac{-2u}{T_1}+\sigma\frac{\lambda}{\epsilon_F}\right],\\\underline{\zeta}_{\sigma'}^{(R)}&=-2\epsilon_F\left[\frac{2u}{T_1}-\sigma'\frac{\lambda}{\epsilon_F}\right],\end{aligned}\tag{A.181}$$

and

$$\begin{aligned}&f^{(')}_{\sigma\sigma'}(\epsilon,u;\underline{k})\\&\approx\frac{m^4N_F^2\pi^2}{4k_F^4\underline{k}^4}\left\{\left[\left(\nu_F^\sigma\underline{k}\right)^2+\underline{\zeta}_{\sigma'}^{(R)}\underline{\zeta}_\sigma^{(A)}\right]\log\left|\frac{\underline{\zeta}_{\sigma'}^{(R)}+\nu_F^\sigma\underline{k}}{\underline{\zeta}_{\sigma'}^{(R)}-\nu_F^\sigma\underline{k}}\right|\log\left|\frac{\underline{\zeta}_\sigma^{(A)}+\nu_F^{\sigma'}\underline{k}}{\underline{\zeta}_\sigma^{(A)}-\nu_F^{\sigma'}\underline{k}}\right|\right.\\&\qquad-2\nu_F^{\sigma'}\underline{k}\underline{\zeta}_{\sigma'}^{(R)}\log\left|\frac{\underline{\zeta}_{\sigma'}^{(R)}+\nu_F^\sigma\underline{k}}{\underline{\zeta}_{\sigma'}^{(R)}-\nu_F^\sigma\underline{k}}\right|\\&\qquad-2\nu_F^\sigma\underline{k}\underline{\zeta}_\sigma^{(A)}\log\left|\frac{\underline{\zeta}_\sigma^{(A)}+\nu_F^{\sigma'}\underline{k}}{\underline{\zeta}_\sigma^{(A)}-\nu_F^{\sigma'}\underline{k}}\right|\\&\qquad\left.+4\nu_F^\sigma\nu_F^{\sigma'}\underline{k}^2\right\}\left\{1+\mathcal{O}\left(\frac{\epsilon}{\epsilon_F},\frac{u}{\epsilon_F}\right)\right\}.\end{aligned}\tag{A.182}$$

At this point, it is useful to consider

$$\left(\vec{k};\mathfrak{C}_0^{(a)}\circ\vec{k}\right)=\left(\vec{k};\mathfrak{C}_0^{(a)}\circ\vec{k}\right)^{(')}+\left(\vec{k};\mathfrak{C}_0^{(a)}\circ\vec{k}\right)^{('')}\tag{A.183}$$

as the sum of

$$
\begin{aligned}
&\big(\vec{k}; \mathfrak{C}_0^{(a)} \circ \vec{k}\big)^{(')} \\
&= \frac{u_0 v_0}{2\pi^2 N_F D} \int \frac{d\epsilon}{\pi} w(\epsilon) \int_0^\infty \frac{du}{\pi} \sum_\sigma \big\{ n(u) + f(u - \sigma\epsilon) \big\} \underline{k} f^{(')}_{\sigma\sigma'}(\epsilon, \epsilon - \sigma u; \underline{k}),
\end{aligned}
\tag{A.184}
$$

and

$$
\begin{aligned}
&\big(\vec{k}; \mathfrak{C}_0^{(a)} \circ \vec{k}\big)^{('')} \\
&= \frac{u_0 v_0}{2\pi^2 N_F D} \int \frac{d\epsilon}{\pi} w(\epsilon) \int_0^\infty \frac{du}{\pi} \sum_\sigma \big\{ n(u) + f(u - \sigma\epsilon) \big\} \underline{k} f^{('')}_{\sigma\sigma'}(\epsilon, \epsilon - \sigma u; \underline{k}),
\end{aligned}
\tag{A.185}
$$

whereupon we have (to the same approximation) both

$$
\begin{aligned}
&\big(\vec{k}; \mathfrak{C}_0^{(a)} \circ \vec{k}\big)^{('')} \\
&\approx \mathcal{V}_0 \frac{\pi \gamma \sqrt{T_1}}{2\epsilon_F} \int_{T_{\min}}^\infty du \frac{\sqrt{u}}{4T \sinh^2 \frac{u}{2T}} \sum_\sigma \left\{ \left[\frac{k_F^\sigma}{k_F} \right]^2 - \frac{T_1}{u} \left[\frac{T_0}{T_1} - \frac{u^2}{T_1^2} \right] \right\},
\end{aligned}
\tag{A.186}
$$

and

$$
\big(\vec{k}; \mathfrak{C}_0^{(a)} \circ \vec{k}\big)^{(')} \approx \mathcal{V}_0 \frac{\gamma T_1^{3/2}}{2\pi\epsilon_F} \int_0^\infty \frac{du}{\sqrt{u}} \frac{1}{4T \sinh^2 \frac{u}{2T}} \sum_\sigma h^\sigma(u, T_0/T_1),
\tag{A.187}
$$

with

$$
\begin{aligned}
&h^\sigma(u, T_0/T_1) \\
&= \Bigg\{ \left[\left(\frac{k_F^\sigma}{k_F} \right)^2 \frac{u}{T_1} - \frac{T_0}{T_1} + \frac{u^2}{T_1^2} \right] \\
&\qquad \times \log \left| \frac{\sigma\sqrt{\frac{T_0}{T_1}} + \frac{u}{T_1} - \frac{k_F^\sigma}{k_F}\sqrt{\frac{u}{T_1}}}{\sigma\sqrt{\frac{T_0}{T_1}} + \frac{u}{T_1} + \frac{k_F^\sigma}{k_F}\sqrt{\frac{u}{T_1}}} \right| \log \left| \frac{\sigma\sqrt{\frac{T_0}{T_1}} - \frac{u}{T_1} + \frac{k_F^{\sigma'}}{k_F}\sqrt{\frac{u}{T_1}}}{\sigma\sqrt{\frac{T_0}{T_1}} - \frac{u}{T_1} - \frac{k_F^{\sigma'}}{k_F}\sqrt{\frac{u}{T_1}}} \right| \\
&\qquad + 2\frac{k_F^{\sigma'}}{k_F}\sqrt{\frac{u}{T_1}} \left[\sigma\sqrt{\frac{T_0}{T_1}} + \frac{u}{T_1} \right] \log \left| \frac{\sigma\sqrt{\frac{T_0}{T_1}} + \frac{u}{T_1} - \frac{k_F^\sigma}{k_F}\sqrt{\frac{u}{T_1}}}{\sigma\sqrt{\frac{T_0}{T_1}} + \frac{u}{T_1} + \frac{k_F^\sigma}{k_F}\sqrt{\frac{u}{T_1}}} \right| \\
&\qquad - 2\frac{k_F^\sigma}{k_F}\sqrt{\frac{u}{T_1}} \left[\sigma\sqrt{\frac{T_0}{T_1}} - \frac{u}{T_1} \right] \log \left| \frac{\sigma\sqrt{\frac{T_0}{T_1}} - \frac{u}{T_1} + \frac{k_F^{\sigma'}}{k_F}\sqrt{\frac{u}{T_1}}}{\sigma\sqrt{\frac{T_0}{T_1}} - \frac{u}{T_1} - \frac{k_F^{\sigma'}}{k_F}\sqrt{\frac{u}{T_1}}} \right| \\
&\qquad\qquad + 4\frac{k_F^\sigma k_F^{\sigma'}}{k_F^2} \frac{u}{T_1} \Bigg\}.
\end{aligned}
\tag{A.188}
$$

From the two integrals of Equation A.186 and Equation A.187, we can extract an asymptotically exact expression for the resistivity when T is in two of the four temperature regimes allowed for by the hierarchy of scales $T_0 \ll T_1 \ll \epsilon_F$.

$T \ll T_0 \ll T_1$: In this case, $\left(\vec{k}; \mathfrak{C}_0^{(a)} \circ \vec{k}\right)^{('')}$ is exponentially suppressed; hence, we need only

$$\begin{aligned}
&\left(\vec{k}; \mathfrak{C}_0^{(a)} \circ \vec{k}\right)^{(')} \\
&\approx \mathcal{V}_0 \frac{\gamma T_1^{3/2}}{2\pi\epsilon_F\sqrt{T}} \int_0^\infty \frac{du}{\sqrt{u}} \frac{1}{4\sinh^2\frac{u}{2T}} \sum_\sigma h^\sigma(uT, T_0/T_1) \\
&\approx -\mathcal{V}_0 \frac{\gamma\sqrt{T_1}T^{3/2}}{\pi\epsilon_F T_0} \int_0^\infty du \frac{u^{3/2}}{\sinh^2\frac{u}{2}} \Big\{1 + \mathcal{O}(T/T_0) + \mathcal{O}(T_0/T_1)\Big\}.
\end{aligned} \tag{A.189}$$

$T_0 \ll T \ll T_1$: Here,

$$\begin{aligned}
&\left(\vec{k}; \mathfrak{C}_0^{(a)} \circ \vec{k}\right)^{('')} \\
&\approx \mathcal{V}_0 \frac{\pi\gamma}{\epsilon_F} \left\{ \frac{4}{3}\sqrt{\frac{T_1}{T_{min}}}T + \left[\int_0^\infty \frac{u^{3/2}}{4\sinh^2\frac{u}{2}}\right] \frac{T^{3/2}}{\sqrt{T_1}} \right\} \Big[1 + \mathcal{O}(T_0/T)\Big]
\end{aligned} \tag{A.190}$$

and

$$\begin{aligned}
&\left(\vec{k}; \mathfrak{C}_0^{(a)} \circ \vec{k}\right)^{(')} \\
&\approx \mathcal{V}_0 \frac{\gamma T_1^{3/2}\sqrt{T_{min}}}{2\pi\epsilon_F T} \int_0^\infty \frac{du}{\sqrt{u}} \frac{1}{4\sinh^2\frac{uT_{min}}{2T}} \sum_\sigma h^\sigma(uT_{min}, T_0/T_1) \\
&\approx \mathcal{V}_0 \frac{\gamma T_1^{3/2} T}{2\pi\epsilon_F T_{min}^{3/2}} \int_0^\infty \frac{du}{u^{5/2}} \sum_\sigma h^\sigma(uT_{min}, T_0/T_1) \Big\{1 + \mathcal{O}(T_0/T)\Big\} \\
&\approx -\mathcal{V}_0 \frac{4\pi\gamma}{3\epsilon_F} \sqrt{\frac{T_0}{T_1}} T \Big\{1 + \mathcal{O}(T_0/T) + \mathcal{O}(T_0/T_1)\Big\},
\end{aligned} \tag{A.191}$$

where

$$\begin{aligned}
&\frac{1}{2}\sum_\sigma h^\sigma(uT_{min}, T_0/T_1) \\
&\approx \frac{T_0}{T_1} \left\{ 4u + \left(4\sqrt{u} + \frac{1-u}{2}\right) \log\left|\frac{\sqrt{u}-1}{1+\sqrt{u}}\right| \right\} \Big[1 + \mathcal{O}(T_0/T_1)\Big].
\end{aligned} \tag{A.192}$$

APPENDIX B

MANY-BODY FORMALISM

In order to establish notational conventions, this chapter presents aspects of the well known mathematical structure upon which our research is based.

B.1 Second quantization

First, we recite the derivation given by Belitz and Kirkpatrick [56] of the following functional integral form for the partition function[1]

$$\mathcal{Z} = \int_{\bar{\psi},\psi} e^{S\left[\bar{\psi},\psi\right]}, \tag{B.1}$$

where $\bar{\psi}, \psi$ are Fermionic[2] auxilary fields on the spacetime manifold.

To this end, we apply the method of second quantization by using classical considerations to write the Hamiltonian for a system of interacting particles as a function of their coordinates, x and p, which are then canonically quantized and expressed in terms of the creation and annihilation operators, $a_\alpha^\dagger$ and a_α, i.e. (for a one body A)

$$\hat{A} = \sum_{\alpha\beta} a_\alpha^\dagger \langle \alpha | \hat{A} | \beta \rangle a_\beta. \tag{B.2}$$

Indeed, Berezin [43] has shown that such a (normal form) representation exists whenever $\hat{A}$ is bounded.

B.1.1 Coherent states. As the Fermionic field operators obey anti-commutation relations

$$a_\alpha^\dagger a_\beta + a_\beta a_\alpha^\dagger = \delta_{\alpha\beta}, \tag{B.3}$$

[1] We are also aided by Negele and Orland [63].

[2] For the Bosonic case, we reference Casher, Lurie, and Revzen [64].

it is required for the notion of an eigenfunction $|\psi\rangle$ (with eigenvalue ψ) of the destruction operator a, i.e.

$$a|\psi\rangle = \psi|\psi\rangle, \tag{B.4}$$

that ψ be Grassmann valued. That is, we want to generate a graded algebra (over $\mathbb{C}$) of anti-commuting scalars

$$\psi_\alpha\psi_\beta + \psi_\beta\psi_\alpha = 0, \tag{B.5}$$

with a ψ_α for each a_α (and a $\bar{\psi}_\alpha$ for each $a_\alpha^\dagger$) obeying

$$\left\{\psi_\alpha, a_\beta\right\} = \left\{\psi_\alpha, a_\beta^\dagger\right\} = \left\{\bar{\psi}_\alpha, a_\beta\right\} = \left\{\bar{\psi}_\alpha, a_\beta^\dagger\right\} = 0. \tag{B.6}$$

Then, the vector

$$|\psi\rangle = \exp\Big[-\sum_\alpha \psi_\alpha a_\alpha^\dagger\Big]|0\rangle, \tag{B.7}$$

with $|0\rangle$ the vacuum state, exists in a well defined space and satisfies (by construction)

$$a_\alpha|\psi\rangle = a_\alpha \prod_\beta \left[1 - \psi_\beta a_\beta^\dagger\right]|0\rangle = \psi_\alpha|\psi\rangle. \tag{B.8}$$

Note that

$$\langle\psi| = \langle 0| \exp\Big[-\sum_\alpha a_\alpha\bar{\psi}_\alpha\Big] \tag{B.9}$$

obeys

$$\langle\psi|a_\alpha^\dagger = \langle\psi|\bar{\psi}_\alpha. \tag{B.10}$$

Thus, after determining the overlap

$$\langle\psi|\psi'\rangle = \exp\Big[\sum_\alpha \bar{\psi}_\alpha\psi'_\alpha\Big], \tag{B.11}$$

one achieves the completeness relation

$$1 = \int \prod_\alpha d\bar{\psi}_\alpha d\psi_\alpha e^{-\sum_\beta \bar{\psi}_\beta\psi_\beta}|\psi\rangle\langle\psi|, \tag{B.12}$$

which can be used to trace over coherent states

$$\begin{aligned}\operatorname{tr}\hat{A} &= \sum_n \langle n|\hat{A}|n\rangle \\ &= \int \prod_\alpha d\bar{\psi}_\alpha d\psi_\alpha e^{-\sum_\beta \bar{\psi}_\beta \psi_\beta} \langle -\psi|\hat{A}|\psi\rangle.\end{aligned} \tag{B.13}$$

B.1.2 Partition function. The strategy here is to foliate the second quantized density matrix (for $N \to \infty$)

$$\begin{aligned}&\langle\psi(N)|e^{-\beta K}|\psi(0)\rangle \\ &= \int \prod_{n'=1}^{N-1} \prod_{\alpha'} d\bar{\psi}_{\alpha'}(n') d\psi_{\alpha'}(n') \\ &\quad \times \exp \sum_\alpha \Bigg\{ -\epsilon \sum_{n=0}^{N-1} \Bigg(\bar{\psi}_\alpha(n+1) \left[\frac{\psi_\alpha(n+1) - \psi_\alpha(n)}{\epsilon} - \mu\psi_\alpha(n) \right] \\ &\qquad + H\big[\bar{\psi}_\alpha(n+1), \psi_\alpha(n)\big] \Bigg) - \bar{\psi}(0)\psi(N) \Bigg\},\end{aligned} \tag{B.14}$$

with $K = H - \mu N$ and $\epsilon = \beta/N$. The term $\bar{\psi}(0)\psi(N)$, which appears on account of the boundary condition

$$\langle\psi(N)| \stackrel{!}{=} \langle -\psi(0)|, \tag{B.15}$$

is suppressed in the customary continuum notation

$$S[\bar{\psi}, \psi] = \int_0^\beta d\tau \Bigg\{ \sum_\alpha \bar{\psi}_\alpha(\tau) \left[\frac{-d}{d\tau} + \mu \right] \psi_\alpha(\tau) - H\big[\bar{\psi}_\alpha(\tau), \psi_\alpha(\tau)\big] \Bigg\}, \tag{B.16}$$

where

$$\psi_\alpha(0) = -\psi_\alpha(\beta); \tag{B.17}$$

when α labels the position basis, we write

$$S[\bar{\psi}, \psi] = \int dx \Bigg\{ \bar{\psi}(x) \left[\frac{-d}{d\tau} + \mu \right] \psi(x) - H\big[\bar{\psi}(x), \psi(x)\big] \Bigg\}. \tag{B.18}$$

Our Fourier transform convention is

$$\psi(x) = \sqrt{\frac{T}{V}} \sum_{p} e^{ipx} \psi(p), \quad \bar{\psi}(x) = \sqrt{\frac{T}{V}} \sum_{p} e^{-ipx} \bar{\psi}(p), \tag{B.19}$$

with shorthand $px = \vec{p} \cdot \vec{x} - \omega_p \tau$. The system volume is V and the inverse temperature $\beta = 1/T$ gives the linear extent of imaginary time. In the limit $V \to \infty$, periodic boundary conditions in real space yield a theory of propagation in the material bulk.

The free electron action

$$S_0[\bar{\psi}, \psi] = \sum_{p} \bar{\psi}(p) G_0^{-1}(p) \psi(p) \tag{B.20}$$

is determined by the bare Green function

$$G_0(p) = -\langle \psi(p) \psi^\dagger(p) \rangle_0 = \frac{1}{i\omega - \xi_{\vec{p}}}, \quad \xi_{\vec{p}} = \frac{p^2}{2m} - \epsilon_F, \tag{B.21}$$

and the full Green function $\mathfrak{G}$ is defined by

$$\mathfrak{G}(p) = \frac{1}{\mathcal{Z}} \int_{\bar{\psi}, \psi} \bar{\psi}(p) \psi(p) e^{S[\bar{\psi}, \psi]}. \tag{B.22}$$

B.2 Kubo formula

The response to a mechanical (as opposed to thermal) disturbance may be ascertained from microscopic principles; in seeking the electrical conductivity, we consider the Hamiltonian [65]

$$H_{\text{Maxwell}} = \frac{1}{2m} \left[\vec{p} - e\vec{A}(\vec{x}, t) \right]^2 + e\phi(\vec{x}, t), \tag{B.23}$$

for a single electron of mass m, charge e, momentum $\vec{p}$, and position $\vec{x}$, in the presence of a coherent gauge field $A^\mu = (\phi, \vec{A})$ applied at time t_0. In the regime of non-relativistic quantum statistical mechanics, it is natural to use units with $c = \hbar = k_B = 1$, where c is the speed of light, $\hbar$ is the reduced Planck constant, and k_B is the Boltzmann constant.

Incorporating techniques of many-body field theory [33], we represent the second-quantized energy

$$H^A(t) = \int d\vec{x}\,\left[J^0(\vec{x})\phi(\vec{x},t) - \frac{e}{2m}J^0(\vec{x})A^2(\vec{x},t) - \vec{J}(\vec{x},t)\cdot\vec{A}(\vec{x},t)\right], \tag{B.24}$$

where the components of the electrical current density operator[3]

$$\begin{aligned} J^0(\vec{x}) &= en(\vec{x}), \\ \vec{J}(\vec{x},t) &= e\vec{j}(\vec{x}) - \frac{e^2}{m}n(\vec{x})\vec{A}(\vec{x},t), \end{aligned} \tag{B.27}$$

with

$$\begin{aligned} n(\vec{x}) &= \psi^\dagger(\vec{x})\psi(\vec{x}), \\ \vec{j}(\vec{x}) &= \frac{-i}{2m}\left\{\psi^\dagger(\vec{x})\vec{\nabla}\psi(\vec{x}) - \left[\vec{\nabla}\psi^\dagger(\vec{x})\right]\psi(\vec{x})\right\}, \end{aligned} \tag{B.28}$$

obey, in the Heisenberg picture with respect to a total Hamiltonian[4]

$$\mathcal{H}(t) = \int d\vec{x}\,\left\{\psi^\dagger(\vec{x})\frac{-\Delta}{2m}\psi(\vec{x})\right\} + H^{(1)} + H^A(t), \tag{B.30}$$

an equation of continuity

$$\frac{d}{dt}J^0_{\mathcal{H}}(\vec{x},t) = i\left[\mathcal{H}(t), J^0_{\mathcal{H}}(\vec{x},t)\right] = -\vec{\nabla}\cdot\vec{J}_{\mathcal{H}}(\vec{x},t), \tag{B.31}$$

as can be checked by applying the canonical equal-time commutation relations

$$\psi^\dagger(\vec{x})\psi(\vec{y}) + \psi(\vec{y})\psi^\dagger(\vec{x}) = \delta(\vec{x}-\vec{y}), \quad \psi(\vec{x})\psi(\vec{y}) + \psi(\vec{y})\psi(\vec{x}) = 0. \tag{B.32}$$

---

[3]Strictly speaking, the term

$$\frac{e}{2m}\vec{\nabla}\times\vec{n}_s(\vec{x}), \tag{B.25}$$

where $\vec{n}_s$ is the electronic spin-density, should contribute to $\vec{J}_S$; Lévy-Leblond [66] has shown that it is not necessary to invoke relativistic effects in order to demonstrate this. Note that Equation B.25 produces the Zeeman coupling

$$\frac{-e}{2m}\int d\vec{x}\,\vec{n}_s(\vec{x})\cdot\vec{B}(\vec{x},t). \tag{B.26}$$

[4]We've introduced $H^{(1)}$ to portray all underlying interactions, which must be such that

$$\left[H^{(1)}_S, n(\vec{x})\right] = 0. \tag{B.29}$$

Now, from the equation of motion[5]

$$\begin{aligned}\frac{m}{e}\frac{d}{dt}J^i_{\mathcal{H}}(\vec{x},t) &= -\partial^l T^{il}_{\mathcal{H}}(\vec{x},t) + i\big[H^{(1)}_{\mathcal{H}}(t), J^i_{\mathcal{H}}(\vec{x},t)\big] \\ &\quad + J^0_{\mathcal{H}}(\vec{x},t)E^i(\vec{x},t) + \big[\vec{J}_{\mathcal{H}}\times\vec{B}\big]^i(\vec{x},t),\end{aligned} \tag{B.34}$$

where

$$\begin{aligned}-4mT^{il}_{\mathcal{H}}(\vec{x},t) &= \partial_i\partial_l\big[\psi^\dagger_{\mathcal{H}}(\vec{x},t)\psi_{\mathcal{H}}(\vec{x},t)\big] \\ &\quad + 2\big[\big(-i\partial_i + eA_i\big)\psi^\dagger_{\mathcal{H}}(\vec{x},t)\big]\big[\big(-i\partial_l - eA_l\big)\psi_{\mathcal{H}}(\vec{x},t)\big] \\ &\quad + 2\big[\big(-i\partial_l + eA_l\big)\psi^\dagger_{\mathcal{H}}(\vec{x},t)\big]\big[\big(-i\partial_i - eA_i\big)\psi_{\mathcal{H}}(\vec{x},t)\big]\end{aligned} \tag{B.35}$$

is the stress tensor, we see that electronic momentum is sourced by both the electric and magnetic fields

$$\begin{aligned}\vec{E}(\vec{x},t) &= -\vec{\nabla}\phi(\vec{x},t) - \partial_t\vec{A}(\vec{x},t), \\ \vec{B}(\vec{x},t) &= \vec{\nabla}\times\vec{A}(\vec{x},t),\end{aligned} \tag{B.36}$$

and simultaneously dissipated by a sink $H^{(1)}$ that contains, e.g. a resistive bath of phonons, magnons, and impurities. When the inhomogeneities (without which the electronic momentum would be conserved) in Equation B.34 are sufficiently weak,[6] the dominant relaxation mechanism of the microscopic quantity j^μ is local transport; this process is slow on the macroscopic scale. It is therefore expected that we will observe the ensemble average

$$\big\langle\vec{J}(\vec{x},t)\big\rangle^t = \mathrm{Tr}\big\{\rho_S(t)\vec{J}_S(\vec{x},t)\big\}, \tag{B.37}$$

where the Schrödinger picture density matrix ρ_S is assumed to take the form

$$\rho_S(t_0) = \sum_i w_i|i;t_0\rangle\langle i;t_0| \tag{B.38}$$

[5]For reference, we give

$$\begin{aligned}m\big[j^i(\vec{k}), n(\vec{q})\big] &= q^i n(\vec{q}+\vec{k}), \\ m\big[j^i(\vec{q}), j^l(\vec{k})\big] &= k^i j^l(\vec{q}+\vec{k}) - q^l j^i(\vec{q}+\vec{k}).\end{aligned} \tag{B.33}$$

[6]Boson exchange is frozen out at zero temperature, so our analysis pertains to reasonably clean, low temperature, metals.

at time $t_0 = -|t_0|$ (very far) in the past. The state vectors $|i; t_0\rangle$ (instantaneously) span the equilibrium Hilbert space; w_i is the fractional population [36], e.g. the Boltzmann weight, of the i'th equilibrium configuration.

If the external conditions enact adiabatic evolution, then the kets for later times can be written (in continuous connection with the initial data)

$$|i; t \geqslant t_0\rangle = \mathcal{U}_{\mathcal{H}}(t, t_0)|i; t_0\rangle, \tag{B.39}$$

where the wave-operator

$$\mathcal{U}_{\mathcal{H}}(t, t_0) = \mathrm{T} \exp\left[-i \int_{t_0}^{t} dt' \, \mathcal{H}(t') \right] \tag{B.40}$$

is time ordered along the $t_0 \;\rightarrow\; t$ contour. This so-called T-product is an instruction to arrange terms in chronological order, with the earliest time appearing on the right. Adopting the same one-parameter "switching-on" ideology for interactions, one has

$$|i; t\rangle = \mathcal{U}_{\mathcal{H}}\big(t, -\infty\big)|i\rangle_0, \tag{B.41}$$

where $|i\rangle_0$ is of the free theory and can be realized, e.g. by occupation number states. It follows that the statistical operator obeys von Neumann's equation of motion[7]

$$i\frac{d}{dt}\rho_S(t) = \big[\mathcal{H}_S(t), \rho_S(t)\big]; \tag{B.43}$$

on converting to Dirac's unitarily equivalent picture, we define (for any Schrödinger operator Q_S)

$$Q_D(t) = \mathcal{U}_H(t_0, t) Q_S(t) \mathcal{U}_H(t, t_0), \tag{B.44}$$

[7]Note that ρ_S inherets state-like time evolution from the Schrödinger equation

$$i\frac{d}{dt}\big|\Psi(t)\big\rangle = \mathcal{H}(t)\big|\Psi(t)\big\rangle. \tag{B.42}$$

where

$$H(t) = \int d\vec{x}\, \{\psi^{\dagger}_{\mathcal{H}}(\vec{x},t)\frac{-\Delta}{2m}\psi(\vec{x})\} + H^{(1)} + \int d\vec{x}\, \{J^0_S(\vec{x})\phi(\vec{x},t) - \frac{e}{2m}J^0_S(\vec{x})A^2(\vec{x},t)\}, \tag{B.45}$$

so that

$$i\frac{d}{dt}\rho_D(t) = -\int d\vec{x}\big[J^l_D(\vec{x},t), \rho_D(t)\big]A^l(\vec{x},t), \tag{B.46}$$

from which we find Kubo's formula

$$\big\langle J^i(\vec{x},t)\big\rangle_t = \mathrm{Tr}\big\{\rho_D(t_0)J^i_D(\vec{x},t)\big\} + \int dt' d\vec{x}'\, \Pi^{il}_R\big[\vec{x},t;\vec{x}',t'\big]A^l(\vec{x}',t'), \tag{B.47}$$

with response function

$$\Pi^{il}_R\big[\vec{x},t;\vec{x}',t'\big] = -i\theta\big[t>t'\big]\ \mathrm{Tr}\big\{\rho_D(t)\big[J^i_D(\vec{x},t), J^l_D(\vec{x}',t')\big]\big\} \tag{B.48}$$

that accounts for after-effects in the relaxation process by causally propagating information.

B.2.1 Linear response. In this approximation,[8] the susceptibility is determined entirely by equilibrium (in the absence of A^{μ}) fluctuations, i.e.

$$\Pi^{il}_R\big[x;x'\big] \to -ie^2\theta\big[t>t'\big]\ \mathrm{Tr}\big\{\rho_{EQ}\big[j^i_H(x), j^l_H(x')\big]\big\}, \tag{B.49}$$

where

$$\rho_{EQ} = \frac{e^{-\beta K}}{\mathrm{Tr}\ e^{-\beta K}}, \tag{B.50}$$

with $K = H - uN$ the Gibbs free energy, and

$$\vec{j}_H(\vec{x},t) = e^{iKt}\vec{j}(\vec{x})e^{-iKt} \tag{B.51}$$

is now understood in Heisenberg's picture with respect to K. Note that $\vec{j}$ commutes with N, the total particle number operator. As a Lagrange multiplier, the chemical potential μ maintains the experimental electron number density

[8] We've sent $t_0 \to -\infty$ and (unjustifiably) assumed that there exists a long-time regime in which the transient dynamics (arising from initial correlations) are sufficiently weak.

$\langle N \rangle = N_{exp}$ and serves as an energy scale (characteristic of the material) in addition to β, the inverse temperature.

The electric current response is readily obtained via Weyl's gauge fixing condition $A^0 = 0$, which implies $\vec{E} = -\partial_t \vec{A}$. On taking the applied field

$$\vec{E}_{\vec{k},\Omega}(\vec{x},t) = \vec{E}_0 \cos\left[\vec{k}\cdot\vec{x} - \Omega t\right], \tag{B.52}$$

to be monochromatic and of definite wavevector, it behooves us to introduce Fourier transforms

$$\Pi^{il}_R\left[x;x'\right] = \int_{p,p'} e^{ipx - ip'x'}\Pi^{il}_R\left[p;p'\right] \tag{B.53}$$

and

$$\vec{E}_k(q) = \frac{1}{2}\vec{E}_0(2\pi)^4\left\{\delta\left[q+k\right] + \delta\left[q-k\right]\right\}, \tag{B.54}$$

i.e.

$$\vec{E}_{\vec{k},\Omega}(\vec{x},t) = \int_q e^{iqx}\vec{E}_{\vec{k},\Omega}(q). \tag{B.55}$$

We now have

$$\begin{aligned}
\left\langle \delta J^i(x)\right\rangle_t &= -\int_{p,q}\Pi^{il}_R\left[p;q\right]A^l(q)e^{ipx} \\
&= -\int_{p,q}\Pi^{il}_R\left[p;q\right]\frac{E^l(q)}{i\omega_q}e^{ipx} \\
&= \frac{-1}{2i\Omega}\int_p e^{ipx}\left\{\Pi^{il}_R\left[p,k\right] - \Pi^{il}_R\left[p,-k\right]\right\}E^l_0.
\end{aligned} \tag{B.56}$$

After performing the disorder average,

$$e\left\{\left\langle \delta J^i(x)\right\rangle_t\right\}_{\text{dis}} = \frac{-1}{\Omega}E^l_0 \,\text{Im}\left[\Pi^{il}_R(k)e^{ikx}\right] \tag{B.57}$$

reveals the DC conductivity tensor

$$\sigma_{ij} = \lim_{\Omega\to 0}\lim_{k\to 0}\left[\frac{-1}{\Omega}\text{Im}\,\Pi^{ij}_R(\vec{k},\Omega)\right], \tag{B.58}$$

which is obtained in the hydrodynamic limit; conventional current is induced by the driving electric field E according to Ohm's law. Here,

$$\left\{\Pi_R^{il}\left[p;p'\right]\right\}_{\text{dis}} = (2\pi)^4\delta\left[p-p'\right]\Pi_R^{il}(p) \tag{B.59}$$

follows by (restored) translation invariance and

$$\left\langle\delta J^i(x)\right\rangle_t = \left\langle J^i(x)\right\rangle_t - \left\langle J^i(x)\right\rangle_{t_0} \tag{B.60}$$

lacks the diamagnetic current.

B.2.1.1 Symmetry requirements. The density response function

$$\Pi_R\left[x;x'\right] = -i\theta\left[t>t'\right]\ \text{Tr}\left\{\rho_{EQ}\left[\delta n_H(x),\delta n_H(x')\right]\right\}, \tag{B.61}$$

must conserve total electron number (in the statistical sense). Thus, if an external (scalar) potential Φ is applied to the system, then

$$0 = \int d\vec{x}\,\left\langle\delta n(\vec{x},t)\right\rangle = \int d\vec{x}\int dx'\,\Pi_R\left[x;x'\right]\Phi(\vec{x}',t'). \tag{B.62}$$

Indeed,

$$\frac{d}{dt}\left\langle n(x)\right\rangle = -\vec{\nabla}\cdot\left\langle\vec{j}(x)\right\rangle \tag{B.63}$$

is equivalent to

$$i\Omega\int dx'\,\Pi_R\left[k;x'\right]\Phi(x') = i\vec{k}\cdot\left\langle\vec{j}(-k)\right\rangle, \tag{B.64}$$

which, in the $\vec{k}\rightarrow 0$ limit, implies Equation B.62. Additionally, there is an identity relating the density-density susceptibility with the (longitudinal) current-current susceptibility that is derivable from the equations of motion [45, 67]

$$\begin{aligned} &i\frac{d}{dt}\left\langle\left\langle B_H(t);C_H(t')\right\rangle\right\rangle \\ &= \delta\left[t-t'\right]\ \text{Tr}\left\{\rho_{EQ}\left[B_H(t),C_H(t')\right]\right\} + \left\langle\left\langle i\frac{d}{dt}B_H(t);C_H(t')\right\rangle\right\rangle, \end{aligned} \tag{B.65}$$

and

$$
\begin{aligned}
&i\frac{d}{dt'}\langle\langle B_H(t); C_H(t')\rangle\rangle \\
&= -\delta[t-t']\ \mathrm{Tr}\{\rho_{EQ}[B_H(t), C_H(t')]\} + \Big\langle\Big\langle B_H(t); i\frac{d}{dt'}C_H(t')\Big\rangle\Big\rangle,
\end{aligned}
\tag{B.66}
$$

which apply to the double-time Green function

$$
\langle\langle B_H(t); C_H(t')\rangle\rangle = -i\theta[t > t']\ \mathrm{Tr}\{\rho_{EQ}[B_H(t), C_H(t')]\}, \tag{B.67}
$$

where both B and C are Bosonic second quantized operators with their spatial arguments suppressed. It follows that

$$
\begin{aligned}
&i\frac{d}{dt'}i\frac{d}{dt}\langle\langle n_H(x); n_H(x')\rangle\rangle \\
&= \frac{-1}{m}\delta[t-t']\vec{\nabla}\cdot\vec{\nabla}'\ \mathrm{Tr}\Big\{\rho_{EQ}\Big[\delta(\vec{x}-\vec{x}')n_H(x)\Big]\Big\} \\
&\quad - \langle\langle \vec{\nabla}\cdot\vec{j}_H(x); \vec{\nabla}'\cdot\vec{j}_H(x')\rangle\rangle,
\end{aligned}
\tag{B.68}
$$

or,

$$
\Omega^2\Pi_R(k) = \frac{n}{m}k^2 + \frac{k^i k^l}{e^2}\Pi_R^{il}(k). \tag{B.69}
$$

Lastly, with the definition

$$
\Pi''[x; x'] = \frac{-1}{2}\langle[n_H(x), n_H(x')]\rangle, \tag{B.70}
$$

we consider

$$
\frac{d}{dt}\Pi''[x; x'] = \frac{1}{2}\langle[\vec{\nabla}\cdot\vec{j}_H(x), n_H(x')]\rangle \tag{B.71}
$$

in the limit $t' \to t^+$, which yields the sum rule

$$
\int\frac{d\omega}{\pi}\omega\Pi''(\vec{k},\omega) = \frac{-n}{m}k^2. \tag{B.72}
$$

B.2.1.2 Zubarev's method. While mechanical perturbations can be introduced at the level of the Hamiltonian, one must generalize the Gibbs weight of equilibrium statistical mechanics in order to describe a system that, due to the influence of applied reservoirs, presents with spatiotemporal fluctuations of its thermodynamic parameters. Such a nonequilibrium statistical operator can be constructed as the solution to the Liouville equation with broken time reversal symmetry [38]

$$\partial_t \rho - i\big[\rho, \mathcal{H}(t)\big] = -\epsilon\big\{\rho - \rho_{rel}\big\}, \tag{B.73}$$

where the inhomogeneity enforces the boundary condition

$$\rho(t \to -\infty) \to \rho_{rel}, \tag{B.74}$$

with ρ_{rel} the relevant distribution, which is to be determined as follows. One seeks a grand canonical ensemble for the state of local equilibrium wherein the averages $\langle \mathcal{H} \rangle$ and $\langle N \rangle$ vary only on the macroscopic scale; if these quantities are taken as given, then [68]

$$\rho_{rel}(t) = \exp\Big\{ - \Phi(t) - \int d\vec{x}\, \beta(\vec{x}, t)\Big[\mathcal{H}(\vec{x}, t) - \mu(\vec{x}, t) n(\vec{x})\Big]\Big\} \tag{B.75}$$

is the distribution that maximizes $-\big\langle \log \rho_{rel} \big\rangle$, the information entropy. Here

$$\Phi(t) = \mathrm{Tr} \exp\Big\{ - \int d\vec{x}\, \beta(\vec{x}, t)\Big[\mathcal{H}(\vec{x}, t) - \mu(\vec{x}, t) n(\vec{x})\Big]\Big\} \tag{B.76}$$

is the Massieu-Planck functional, which enforces the normalization $\mathrm{Tr}\, \rho_{rel} = 1$, while the Lagrange multipliers β and μ ensure the self-consistency conditions

$$\mathrm{Tr}\big[\rho_{rel} \mathcal{H}\big] = \mathrm{Tr}\big[\rho \mathcal{H}\big], \quad \mathrm{Tr}\big[\rho_{rel} N\big] = \mathrm{Tr}\big[\rho N\big]. \tag{B.77}$$

It follows that

$$\rho(t) = \rho_{rel}(t) - \int_{-\infty}^{t} dt' e^{-\epsilon(t-t')} U(t, t') \Big[\frac{d}{dt'} \rho_{rel}(t')\Big] U^\dagger(t, t'), \tag{B.78}$$

where

$$i\partial_t U(t, t') = \mathcal{H}(t) U(t, t'), \quad \lim_{t' \to t^+} U(t, t') = 1. \tag{B.79}$$

Now then, for the purpose of studying thermoelectric effects, we take

$$\mathcal{H}(t) = H + e \int d\vec{x}\, n(\vec{x}) \phi(\vec{x}, t), \tag{B.80}$$

where $\vec{E}(\vec{x},t) = -\vec{\nabla}\phi(\vec{x},t)$ is an electric field, in addition to allowing for internal temperature inhomogeneities

$$\beta(\vec{x},t) = \beta_0 + \delta\beta(\vec{x},t), \quad \mu(\vec{x},t) = \mu_0 + \delta\mu(\vec{x},t), \tag{B.81}$$

such that, to linear order in the departure from the unperturbed state,

$$\begin{aligned} &\rho(t) - \rho_{\text{rel}}(t) \\ &\approx \frac{1}{\beta_0}\int_{-\infty}^{0} dt' e^{\epsilon t'} \int_0^{\beta_0} d\tau \int d\vec{x} \Big\{ \delta\beta(\vec{x},t'+t)\Big[\dot{H} - \mu_0\dot{n}\Big](\vec{x},t'+i\tau) \\ &\qquad\qquad - \beta_0\delta\tilde{\mu}(\vec{x},t'+t)\dot{n}(t'+i\tau)\Big\}\rho_{EQ}. \end{aligned} \tag{B.82}$$

Thus, on using the continuity equations for energy and particle number

$$\dot{n} = -\vec{\nabla}\cdot\vec{j}_n, \quad \dot{H} = -\vec{\nabla}\cdot\vec{j}_\varepsilon \tag{B.83}$$

to define the associated fluxes $\vec{j}$, Equation B.82 becomes

$$\begin{aligned} &\rho(t) - \rho_{\text{rel}}(t) \\ &\approx -\int_{-\infty}^{0} dt' e^{\epsilon t'} \int_0^{\beta_0} d\tau \int d\vec{x} \Big\{ \vec{j}_s(\vec{x},t'+i\tau)\cdot\vec{\nabla}T(\vec{x},t'+t) \\ &\qquad\qquad + \vec{j}_n(\vec{x},t'+i\tau)\cdot\vec{\nabla}\tilde{\mu}(\vec{x},t'+t)\Big\}\rho_{EQ}, \end{aligned} \tag{B.84}$$

where

$$\vec{j}_s = \beta_0\Big[\vec{j}_\varepsilon - \mu_0\vec{j}_n\Big], \tag{B.85}$$

is the entropy current density, with both $\beta = 1/T$ the inverse temperature and $\tilde{\mu} = \mu + e\phi$ the electrochemical potential. As a result, we arrive at the Onsager transport equations

$$\begin{aligned} \left\langle j_n^i(x)\right\rangle_t &= \int dx' \big\{ C_{ns}^{il}(x-x')\partial_l T(x') + C_{nn}^{il}(x-x')\partial_l\tilde{\mu}(x')\big\}, \\ \left\langle j_s^i(x)\right\rangle_t &= \int dx' \big\{ C_{ss}^{il}(x-x')\partial_l T(x') + C_{sn}^{il}(x-x')\partial_l\tilde{\mu}(x')\big\}, \end{aligned} \tag{B.86}$$

which predict, for the steady state currents that develop when both the disturbances $\vec{\nabla}T$ and $\vec{\nabla}\tilde{\mu}$ are both uniform and static,

$$\begin{aligned} \langle j^i_n \rangle &\approx C^{il}_{ns}(k=0)\partial_l T + C^{il}_{nn}(k=0)\partial_l \tilde{\mu} \\ \langle j^i_s \rangle &\approx C^{il}_{ss}(k=0)\partial_l T + C^{il}_{sn}(k=0)\partial_l \tilde{\mu} \end{aligned} \tag{B.87}$$

where the Fourier component

$$C(\vec{k}, \Omega) = \lim_{\epsilon \to 0} C(\vec{k}, \Omega + i\epsilon) \tag{B.88}$$

of the retarded correlation function

$$C^{il}(x - x') = -\theta(t - t' > 0)e^{-\epsilon(t-t')} \int_0^{\beta_0} d\tau \,\mathrm{Tr}\left\{ j^i(x) j^l(x' + i\tau)\rho_{EQ} \right\} \tag{B.89}$$

may be obtained from the limiting value (as the complex frequency z approaches the real axis from above) of the causal function

$$C(z) = \int \frac{d\omega}{\pi i} \frac{\Pi''(\omega)}{\omega(\omega - z)}, \tag{B.90}$$

which involves the spectrum of

$$\frac{-1}{2} \left\langle \left[j^i_H(t), j^l_H(t') \right] \right\rangle = \int \frac{d\omega}{2\pi} \Pi''_{il}(\omega) e^{-i\omega(t-t')}; \tag{B.91}$$

Π'' defines the Green function

$$\Pi(z) = \int \frac{d\omega}{\pi} \frac{\Pi''(\omega)}{\omega - z} \tag{B.92}$$

and facilitates the expression

$$C(z) = \frac{1}{iz} \left\{ \Pi(z) - \Pi(i0) \right\}, \tag{B.93}$$

for the transport coefficient. The term

$$-\Pi(i0) = \frac{n}{3m} \tag{B.94}$$

is known exactly from Equation B.69; in section B.3, we will review the means of computing the dynamical susceptibility Π.

Note that Zubarev's formalism handles both mechanical and thermal perturbations in a unified manner. Furthermore, by this method, one does not rely on Boltzmann's phenomenological prescription for introducing broken time reversal symmetry.

B.3 Matsubara technique

In this section, the diagrammatic expansion for the retarded two-time correlator Π_R is developed through its relation to the time ordered temperature Green function[9]

$$\pi^{il}\left[\vec{k},\tau;\vec{k}',\tau'\right] = -e^2\left\langle T_\tau\left[j^i_H(-\vec{k},\tau)j^l_H(\vec{k}',\tau')\right]\right\rangle, \tag{B.96}$$

which obeys the boundary conditions[10]

$$\begin{aligned}\pi^{il}\left[\vec{k},0;\vec{k}',\tau'\right] &= \pi^{il}\left[\vec{k},\beta;\vec{k}',\tau'\right],\\ \pi^{il}\left[\vec{k},\tau;\vec{k}',0\right] &= \pi^{il}\left[\vec{k},\tau;\vec{k}',\beta\right],\end{aligned} \tag{B.97}$$

and can therefore be decomposed into its Fourier components

$$\pi^{il}\left[\vec{k},\tau;\vec{k}',\tau'\right] = \frac{1}{\beta^2}\sum_{\Omega_n,\Omega_n'}\pi^{il}\left[\vec{k},i\Omega_n;\vec{k}';i\Omega_n'\right]e^{-i\Omega_n\tau+i\Omega_n'\tau'}, \tag{B.98}$$

with $\Omega_n,\Omega_n' \in \frac{2\pi}{\beta}\mathbb{Z}$.

B.3.1 Lehmann representation. Now, it is possible to infer $\Pi_R(\Omega)$ by analytic continuation of $\pi(i\Omega_n)$ to the [29] complex function which has no

---

[9] We've extended the domain of definition of the time evolution operator to include the complex plane, which allows for

$$j_H(\vec{k},\tau) \equiv j_H(\vec{k},t=-i\tau) = e^{H\tau}j_S(\vec{k})e^{-H\tau}. \tag{B.95}$$

[10] Equation B.97 is easily derived when $0<\tau,\tau'<\beta$ and it follows that π^{il} is periodic, so we need only consider $0\leqslant\tau,\tau'<\beta$.

essential singularity at $z = \infty$. Indeed, the Hilbert-Stieltjes forms [69]

$$\begin{aligned}\pi^{il}\left[\vec{k},\vec{k}';i\Omega_n\right] &= \int \frac{d\Omega'}{\pi}\frac{\Pi''_{il}\left[\vec{k},\vec{k}';\Omega'\right]}{\Omega' - i\Omega_n},\\ \Pi^{il}_R\left[\vec{k},\vec{k}';\Omega\right] &= \int \frac{d\Omega'}{\pi}\frac{\Pi''_{il}\left[\vec{k},\vec{k}';\Omega'\right]}{\Omega' - \Omega - i0}\end{aligned} \tag{B.99}$$

are determined by the same spectral weight [45]

$$\Pi''_{il}\left[x;x'\right] = \frac{-1}{2}\left\langle\left[\delta j^i(x), \delta j^l(x')\right]\right\rangle. \tag{B.100}$$

By similar reasoning,

$$\mathfrak{G}\left[x;x'\right] = -\left\langle T_\tau\left[\psi_H(x)\psi^\dagger_H(x')\right]\right\rangle \tag{B.101}$$

admits the spectral decomposition

$$\mathfrak{G}\left[\vec{k},\vec{k}';i\omega_n\right] = \int \frac{d\omega}{\pi}\frac{\mathfrak{G}''\left[\vec{k},\vec{k}';\omega\right]}{\omega - i\omega_n}, \tag{B.102}$$

with $\omega_n \in \frac{2\pi}{\beta}\left(\mathbb{Z} + 1/2\right)$, since

$$\begin{aligned}\mathfrak{G}\left[\vec{x},0;\vec{x}',\tau'\right] &= -\mathfrak{G}\left[\vec{x},\beta;\vec{x}',\tau'\right],\\ \mathfrak{G}\left[\vec{x},\tau;\vec{x}',0\right] &= -\mathfrak{G}\left[\vec{x},\tau;\vec{x}',\beta\right],\end{aligned} \tag{B.103}$$

when $0 < \tau, \tau' < \beta$; the real-space propagator [70]

$$\mathfrak{g}_R\left[\vec{k},t;\vec{k}',t'\right] = -i\theta\left[t > t'\right]\left\langle\left\{\psi(\vec{k},t), \psi^\dagger(\vec{k}',t')\right\}\right\rangle \tag{B.104}$$

is obtained on analytic continuation

$$\mathfrak{g}_R\left[\vec{k},\vec{k}';\omega\right] = \mathfrak{G}\left[\vec{k},\vec{k}';\omega + i0\right]. \tag{B.105}$$

We will often work with the retarded and advanced Green functions

$$\begin{aligned}\mathfrak{G}_R(\omega) &= \mathfrak{G}(\omega + i0) = \mathfrak{G}'(\omega) + i\mathfrak{G}''(\omega),\\ \mathfrak{G}_A(\omega) &= \mathfrak{G}(\omega - i0) = \mathfrak{G}'(\omega) - i\mathfrak{G}''(\omega),\end{aligned} \tag{B.106}$$

where

$$\mathfrak{G}''(\omega) = \frac{1}{2i}\left[\mathfrak{G}(\omega + i0) - \mathfrak{G}(\omega - i0)\right] \tag{B.107}$$

is the "jump across the cut" and the reactive part

$$\mathfrak{G}'(\omega) = \frac{1}{2}\left[\mathfrak{G}(\omega + i0) + \mathfrak{G}(\omega - i0)\right] \tag{B.108}$$

may be obtained from the principal value integral

$$\mathfrak{G}'(\omega) = \text{P.V.}\int \frac{d\nu}{\pi}\frac{\mathfrak{G}''(\nu)}{\nu - \omega}. \tag{B.109}$$

Furthermore,

$$\pi^{il}\left[2\vec{k}, \tau; 2\vec{k}', \tau'\right] = \frac{-e^2}{m^2}\frac{1}{V^2}\sum_{\vec{p},\vec{p}'} p_i p_l' \mathfrak{G}^{II}\left[\vec{p}_+, \tau\middle|\vec{p}'_-, \tau'\middle|\vec{p}'_+, \tau'\middle|\vec{p}_-, \tau\right], \tag{B.110}$$

where $\vec{p}_\pm = \vec{p} \pm \vec{k}$ and $\vec{p}'_\pm = \vec{p}' \pm \vec{k}'$, involves a special case of the four-point function

$$\mathfrak{G}^{II}\left[x_1, x_2; x_3; x_4\right] = \left\langle T_\tau\left\{\psi(x_1)\psi(x_2)\psi^\dagger(x_3)\psi^\dagger(x_4)\right\}\right\rangle, \tag{B.111}$$

which obeys [39]

$$\begin{aligned} \mathfrak{G}^{II}\left[0, \tau_2; \tau_3, \tau_4\right] &= -\mathfrak{G}^{II}\left[\beta, \tau_2; \tau_3, \tau_4\right], \\ \mathfrak{G}^{II}\left[\tau_1, 0; \tau_3, \tau_4\right] &= -\mathfrak{G}^{II}\left[\tau_1, \beta; \tau_3, \tau_4\right], \\ \mathfrak{G}^{II}\left[\tau_1, \tau_2; 0, \tau_4\right] &= -\mathfrak{G}^{II}\left[\tau_1, \tau_2; \beta, \tau_4\right], \\ \mathfrak{G}^{II}\left[\tau_1, \tau_2; \tau_3, 0\right] &= -\mathfrak{G}^{II}\left[\tau_1, \tau_2; \tau_3, \beta\right], \end{aligned} \tag{B.112}$$

when $0 < \tau_1, \tau_2, \tau_3, \tau_4 < \beta$. Thus, we can write

$$\pi^{il}\left[2k; 2k'\right] = \frac{-e^2}{m^2}\frac{T^2}{V^2}\sum_{p,p'} p_i p_l' \mathfrak{G}^{II}\left[p_+, p'_-; p'_+, p_-\right], \tag{B.113}$$

where[11]

$$\begin{aligned} p &= (\vec{p}, \omega_n), \qquad p_\pm = p \pm k, \quad k = (\vec{k}, \Omega_n), \\ p' &= (\vec{p}', \omega_n'), \qquad p'_\pm = p' \pm k', \;\; k' = (\vec{k}', \Omega_n'). \end{aligned} \tag{B.114}$$

B.3.1.1 Replica trick. After disorder averaging

$$\left\{\pi^{il}\left[\vec{k}, \vec{k}'; i\Omega_n\right]\right\}_{\text{dis}} = (2\pi)^3\delta\left[\vec{k} - \vec{k}'\right]\pi^{il}(k), \tag{B.115}$$

[11] Had we used the coherent states to represent the thermodynamic trace, then we could replace $\psi^\dagger, \psi$ with $\bar{\psi}, \psi$ inside the brackets $\langle\cdot\rangle$ and invoke Equation B.17.

we have [34]

$$\pi^{il}(k) = -e^2\left\{\left\langle j^i(-k)j^l(k)\right\rangle\right\}_{dis}, \tag{B.116}$$

where

$$\vec{j}(2k) = \frac{T}{V}\sum_p \bar{\psi}(p_+)\frac{\vec{p}}{m}\psi(p_-), \tag{B.117}$$

with

$$\vec{j}(x) = \frac{T}{V}\sum_k e^{-ikx}\vec{j}(k). \tag{B.118}$$

For the purpose of evaluating Equation B.116, we employ the replica trick [46].

To this end, first define a generating function [56]

$$Z = \int_{\bar{\psi},\psi} \exp\left\{S\left[\bar{\psi},\psi,u\right] + e\int_k \vec{j}(k)\cdot\vec{A}(\vec{k})\right\}, \tag{B.119}$$

which facilitates the expression

$$\Pi^{il}(k) = \frac{-\delta^2}{\delta A^i(-k)\delta A^l(k)}\bigg|_{A=0}\log Z. \tag{B.120}$$

Then, use the formal identity

$$\log Z = \lim_{N\to 0}\frac{1}{N}\left[Z^N - 1\right], \tag{B.121}$$

and introduce N replicas of the system

$$Z^N = \int_{\{\bar{\psi},\psi\}} \exp\sum_{\zeta=1}^{N}\left\{S_\zeta\left[\bar{\psi}_\zeta,\psi_\zeta\right] + e\int_k \vec{j}_\zeta(k)\cdot\vec{A}(k)\right\}. \tag{B.122}$$

In this way

$$\begin{aligned}\pi^{il}(k) &= \left\{\frac{-\delta^2}{\delta A^i(-k)\delta A^l(k)}\bigg|_{A=0}\log Z\right\}_{dis} \\ &= \lim_{N\to 0}\frac{-e^2}{N}\int_{\{\bar{\psi},\psi\}}\left[\sum_{\zeta,\zeta'=1}^{N} j^i_\zeta(-k)j^l_{\zeta'}(k)\right]\exp\left(\tilde{S}\left[\{\bar{\psi},\psi\}\right]\right) \\ &= \lim_{N\to 0}\frac{-e^2}{N}\sum_{\zeta,\zeta'=1}^{N}\left\langle j^i_\zeta(-k)j^l_{\zeta'}(k)\right\rangle_{\tilde{S}},\end{aligned} \tag{B.123}$$

since

$$\lim_{N\to 0} \tilde{Z} = 1, \tag{B.124}$$

where

$$\tilde{S} = \log \tilde{Z} = \log \left\{ \exp \sum_{\zeta=1}^{N} S_\zeta \right\}_{\text{dis}} . \tag{B.125}$$

Thus, by making use of Wick's theorem and the usual Dyson argument for reorganizing the perturbative series, we may compute π^{ij} as a sum of polarization diagrams

$$\pi_{ij}(2k) = \frac{e^2}{m^2} \lim_{N\to 0} \frac{1}{N} \sum_{\zeta=1}^{N} \frac{T}{V} \sum_{p} p_i \mathfrak{G}_\zeta(p_+)\mathfrak{G}_\zeta(p_-)\gamma^j_\zeta\left[p_+, p_-; 2k\right] \tag{B.126}$$

where

$$\gamma^j_\zeta\left[p_+, p_-; 2k\right] = p_j + \frac{T}{V} \sum_{p'} \sum_{\zeta'=1}^{N} p'_j \Lambda_{\zeta\zeta'}\left[p, p'; 2k\right] \mathfrak{G}_{\zeta'}(p'_+)\mathfrak{G}_{\zeta'}(p'_-) \tag{B.127}$$

parameterizes the current vertex in terms of the scattering amplitude Λ which is defined by

$$\begin{aligned} &\mathfrak{G}^{II}\left[x_1, x_2; x_3, x_4\right] \\ &= \mathfrak{G}\left[x_1; x_4\right]\mathfrak{G}\left[x_2; x_3\right] - \mathfrak{G}\left[x_1; x_3\right]\mathfrak{G}\left[x_2; x_4\right] \\ &\quad - \int_{x'} \mathfrak{G}\left[x_1; x'_1\right]\mathfrak{G}\left[x_2; x'_2\right]\Lambda\left[x'_1, x'_2; x'_3, x'_4\right]\mathfrak{G}\left[x'_3; x_3\right]\mathfrak{G}\left[x'_4; x_4\right]; \end{aligned} \tag{B.128}$$

here we've introduced the notation

$$\begin{aligned} &\tilde{\Lambda}\left[x_1, x_2; x_3, x_4\right] \\ &= \int_{q,p,k} \tilde{\Lambda}\left[q, p; 2k\right] e^{iq(x_1-x_4)+ip(x_2-x_3)+ik(x_1+x_4-x_2-x_3)}. \end{aligned} \tag{B.129}$$

Clearly, π^{ij} describes the propagation of two-particle states; a hydrodynamic description is contained in the particle-hole terms, as these paired excitations are endowed with a long lifetime.

B.3.2 Analytic continuation. One can transform a Matsubara sum into an integral along the real line if the analytic structure of the summand is known. For example, if we wish to exchange summation for integration in Equation B.126, then the desired information about $\mathfrak{G}$ and $\mathfrak{G}^{II}$ may be gleaned from their Lehmann expansions. Indeed, Equation B.102 implies that $\mathfrak{G}$ has a single branch cut on the real line; Éliashberg [54] has given the corresponding expression that locates the cuts for $\mathfrak{G}^{II}$, thereby demonstrating that $\Lambda[q, p; 2k]$ defines 32 functions,[12] each of which is analytic on one of the regions bounded by[13]

$$\omega_q = \pm\Omega_k, \quad \omega_p = \pm\Omega_k, \tag{B.131a}$$

$$\omega_p = \pm\omega_q, \quad \Omega_k = 0. \tag{B.131b}$$

Therefore, $\gamma_j[q_+, q_-; 2k]$ has cuts at

$$\omega_q = \pm\Omega, \quad \Omega_k = 0; \tag{B.132}$$

the sum over ω_p may be generated by drawing a contour around the (simple) poles of

$$f(\epsilon) = \frac{1}{e^{\beta\epsilon} + 1} \tag{B.133}$$

which lie precisely at the Fermionic frequencies and each have residue equal to $-1/\beta$.

---

[12]That Λ inherets the analytic structure of $\mathfrak{G}^{II}$ follows from

$$\begin{aligned} &\mathfrak{G}^{II}[p_1, p_2|p_3] \\ &= \mathfrak{G}(p_1)\mathfrak{G}(p_2) \times \beta(2\pi)^3\{\delta[p_1 - p_4]\delta[p_2 - p_3] - \delta[p_1 - p_3]\delta[p_2 - p_4]\} \\ &\quad - \mathfrak{G}(p_1)\mathfrak{G}(p_2)\Lambda[p_1, p_2; p_3 p_4]\mathfrak{G}(p_3)\mathfrak{G}(p_4); \end{aligned} \tag{B.130}$$

one can check that Λ contains the single particle resonances of the external Green functions, Equation B.131a, in addition to bound excitations, Equation B.131b, formed by two interacting particles [51].

[13]When we want to emphasize the discrete nature of the Matsubara frequencies, we'll write $i\omega_n$ or $i\Omega_n$. Otherwise, we sometimes prefer to use the notation $p = (\vec{p}, i\omega_p)$ or $k = (\vec{k}, i\Omega_k)$.

Now then, since the DC conductivity requires only the zero wavenumber susceptibility, we know that

$$\gamma_j\left[q_+, q_-; 2k\right] = q_j \gamma\left[q_+, q_-; 2i\Omega\right], \quad q_\pm = q \pm i\Omega, \tag{B.134}$$

which implies

$$\begin{aligned}\sigma = \frac{e^2}{6m^2} \int \frac{d\epsilon}{\pi} w(\epsilon) \frac{1}{V} \sum_{\vec{q}} \\ \times \operatorname{Re}\left\{\mathfrak{G}_R(\vec{q}, \epsilon) \vec{q} \cdot \vec{\gamma}_{RA}(\vec{q}, \epsilon) \mathfrak{G}_A(\vec{q}, \epsilon) - \left[\mathfrak{G}_R(\vec{q}, \epsilon)\right]^2 \vec{q} \cdot \vec{\gamma}_{RR}(\vec{q}, \epsilon)\right\},\end{aligned} \tag{B.135}$$

where

$$w(\epsilon) = \frac{-\partial f}{\partial \epsilon}(\epsilon) = \frac{1}{4T} \cosh^{-2} \frac{\epsilon}{2T}, \tag{B.136}$$

and

$$\begin{aligned}\vec{\gamma}_{RA}(\vec{q}, \epsilon) &= \lim_{i\Omega \to i0} \lim_{iq \to \epsilon + i0} \vec{\gamma}\left[q_+, q_-; 2i\Omega\right], \\ \vec{\gamma}_{RR}(\vec{q}, \epsilon) &= \lim_{iq \to \epsilon + i0} \lim_{i\Omega \to i0} \vec{\gamma}\left[q_+, q_-; 2i\Omega\right], \\ \vec{\gamma}_{AA}(\vec{q}, \epsilon) &= \lim_{iq \to \epsilon - i0} \lim_{i\Omega \to i0} \vec{\gamma}\left[q_+, q_-; 2i\Omega\right],\end{aligned} \tag{B.137}$$

with $\gamma_{RR} = \gamma^*_{AA}$, since γ is symmetric in its arguments.

We still need to transform the $\omega_{p'}$ sum in Equation B.127, but rather than doing this in the general case, we will consider only the cuts mandated by the subset of diagrams that we select for Λ.

B.3.2.1 Quasi-particle weight. The Dyson equation [63]

$$\mathfrak{G}(p) = G_0(p) + G_0(p)\Sigma(p)\mathfrak{G}(p), \tag{B.138}$$

expresses the full Green function $\mathfrak{G}$ in terms of the bare Green function G_0 dressed by the proper self-energy Σ; if its spectrum

$$\mathfrak{G}''(\vec{p}, \epsilon) = \frac{\Sigma''(\vec{p}, \epsilon)}{\left[\epsilon - \xi(\vec{p}) - \Sigma'(\vec{p}, \epsilon)\right]^2 + \left[\Sigma''(\vec{p}, \epsilon)\right]^2}, \tag{B.139}$$

peaks (sharply) in $\vec{p}$ space on the shell $p_0[\epsilon]$ obeying

$$\epsilon - \xi(p_0[\epsilon]) - \Sigma'(p_0[\epsilon], \epsilon) = 0, \tag{B.140}$$

where we have made use of isotropy, then (to the extent that Σ'' damping is small) it is plausible that the physics is dominated by this (coherent) single-particle excitation. Such a quasi-particle pole comes with a reduced field strength, which can be determined as follows. First, change variables

$$\Sigma'(\varepsilon_p; \epsilon) = \Sigma'(p = \sqrt{2m\varepsilon_p}, \epsilon) \tag{B.141}$$

and denote by $\mu[\epsilon]$ the value of ε_p that satisfies

$$\epsilon - \mu[\epsilon] + \epsilon_F - \Sigma'(\mu[\epsilon]; \epsilon) = 0. \tag{B.142}$$

Next, recall that

$$\xi(\vec{p}) = \varepsilon_p - \epsilon_F \tag{B.143}$$

and rewrite

$$\epsilon - \xi(p) - \Sigma'(\varepsilon_p; \epsilon) = \mu[\epsilon] - \varepsilon_p + \Sigma'(\mu[\epsilon]; \epsilon) - \Sigma'(\varepsilon_p; \epsilon). \tag{B.144}$$

Thus, in the vicinity of the peak

$$\epsilon - \xi(p) - \Sigma'(\varepsilon_p; \epsilon) \approx Z^{-1}[\epsilon](\mu[\epsilon] - \varepsilon_p), \tag{B.145}$$

where the inverse quasi-particle weight

$$Z^{-1}[\epsilon] = 1 + \left.\frac{\partial}{\partial \varepsilon_p}\right|_{\mu[\epsilon]} \Sigma'(\varepsilon_p; \epsilon), \tag{B.146}$$

renormalizes

$$-\mathfrak{G}''(\vec{p}, \epsilon) \sim \frac{-\Sigma''(\vec{p}, \epsilon)}{Z^{-2}[\epsilon]\big(\mu[\epsilon] - \varepsilon_p\big)^2 + \big[\Sigma''(\vec{p}, \epsilon)\big]^2} \to Z[\epsilon]\pi\delta\big(\mu[\epsilon] - \varepsilon_p\big). \tag{B.147}$$

To determine μ, we can separate

$$\mu[\epsilon] = \epsilon_F + \epsilon + \delta\mu[\epsilon] \tag{B.148}$$

and then compute the first correction

$$-\delta\mu[\epsilon] = \Sigma'(\epsilon_F + \delta\mu[\epsilon]; \epsilon) \approx \Sigma'(\epsilon_F + \epsilon; \epsilon). \tag{B.149}$$

B.4 Lindhard function

Of particular importance is the density-density susceptibility

$$\Pi(k) = -\langle \delta n(-k)\delta n(k)\rangle; \tag{B.150}$$

in the absence of vertex corrections

$$\begin{aligned} \Pi_0(\vec{k}, i\Omega) &= \frac{T}{V}\sum_{\vec{q}, i\omega} G_0(\vec{q}, i\omega)G_0(\vec{q}-\vec{k}, i\omega - i\Omega) \\ &= \int \frac{d\epsilon\, du/\pi^2}{u-\epsilon+i\Omega}\frac{1}{V}\sum_{\vec{q}} G_0''(\vec{q},\epsilon)G_0''(\vec{q}-\vec{k}, u)\big[f(u)-f(\epsilon)\big], \end{aligned} \tag{B.151}$$

which has poles (wherever $i\Omega_k$ is equal to the difference of any two energy eigenvalues) with residues determined by the spectral weight

$$\begin{aligned} \Pi_0''(\vec{k}, \Omega) &= -\int \frac{d\epsilon}{\pi}\frac{1}{V}\sum_{\vec{q}} G_0''(\vec{q},\epsilon)G_0''(\vec{q}-\vec{k}, \epsilon-\Omega)\big[f(\epsilon-\Omega)-f(\epsilon)\big] \\ &= \frac{1}{V}\sum_{\vec{q}}\big\{f(\xi_q)-f(\xi_q-\Omega)\big\}\pi\delta\big[-\Omega+\vec{q}\cdot\vec{k}/m-\varepsilon_k\big]. \end{aligned} \tag{B.152}$$

As the imaginary part of the response function, Π'' specifies the time arrowed dissipation that irreversibly draws the system to equilibrium.

On performing the integral, we find

$$\Pi_0''(\vec{k}, \Omega) = \frac{m^2 T}{4\pi k}\log\frac{1+\exp\big\{\beta\epsilon_F\big[1-\zeta\big(k,\Omega\big)\big]\big\}}{1+\exp\big\{\beta\epsilon_F\big[1-\zeta(k,-\Omega)\big]\big\}}, \tag{B.153}$$

where

$$\zeta(k,\Omega) = \left[\frac{k}{2k_F}+\frac{m\Omega}{kk_F}\right]^2. \tag{B.154}$$

At zero temperature

$$
\begin{aligned}
&-\Pi_0''(k < 2k_F, \Omega) \\
&= \operatorname{sgn}(\Omega)\frac{m^2\epsilon_F}{4\pi k} \times \begin{cases} |\Omega|/\epsilon_F & |\Omega| < \frac{kk_F}{m} - \frac{k^2}{2m} \\ 1 - \zeta(k, -|\Omega|) & \frac{kk_F}{m} - \frac{k^2}{2m} < |\Omega| < \frac{kk_F}{m} + \frac{k^2}{2m} \\ 0 & |\Omega| > \frac{kk_F}{m} + \frac{k^2}{2m} \end{cases}
\end{aligned} \tag{B.155}
$$

and

$$
\begin{aligned}
&-\Pi_0''(k > 2k_F, \Omega) \\
&= \operatorname{sgn}(\Omega)\frac{m^2\epsilon_F}{4\pi k} \times \begin{cases} 0 & |\Omega| < \frac{k^2}{2m} - \frac{kk_F}{m} \\ 1 - \zeta(k, -|\Omega|) & \frac{k^2}{2m} - \frac{kk_F}{m} < |\Omega| < \frac{kk_F}{m} + \frac{k^2}{2m} \\ 0 & |\Omega| > \frac{kk_F}{m} + \frac{k^2}{2m} \end{cases}
\end{aligned} \tag{B.156}
$$

Both of these expressions exhaust the sum rule that is Equation B.72.

The sharpness of the Fermi surface is evident in the abrupt change of behavior at $k = 2k_F$, where the frequency spectrum becomes gapped [71]; only for $|\Omega|$ large enough is there an allowed (particle number conserving) transition between energy eigenstates (mediated by a density fluctuation). This limits the ability of electrons to scatter across the Fermi sphere and is a consequence of there being no hole states that can singly compensate a particle momentum of magnitude greater than $2k_F$.

As a result, we find the Lindhard function [72]

$$
\begin{aligned}
&-\Pi_0(\vec{k}, z) \\
&= \frac{N_F}{2}\Bigg\{1 + \frac{k_F}{2k}\left[1 - \left(\frac{k}{2k_F} + \frac{mz}{kk_F}\right)^2\right] \\
&\qquad\qquad \times \Big(\log\left[k^2 + 2kk_F + 2mz\right] - \log\left[k^2 - 2kk_F + 2mz\right]\Big) \\
&\qquad + \frac{k_F}{2k}\left[1 - \left(\frac{k}{2k_F} - \frac{mz}{kk_F}\right)^2\right] \\
&\qquad\qquad \times \Big(\log\left[k^2 + 2kk_F - 2mz\right] - \log\left[k^2 - 2kk_F - 2mz\right]\Big)\Bigg\}.
\end{aligned} \tag{B.157}
$$

In the static limit, the logarithmic singularity of the slope at $k = 2k_F$

$$
-\Pi_0^R(\vec{k}, 0) = \frac{N_F}{2}\left\{1 + \frac{k_F}{k}\left[1 - \frac{k^2}{4k_F^2}\right]\log\left|\frac{k^2 + 2kk_F}{k^2 - 2kk_F}\right|\right\}, \tag{B.158}
$$

gives rise to Friedel oscillations: when attempting to put a complete screen on an impurity charge, the electron field configures in a density wave; such a seemingly nonequilibrium arrangement is the best available [34] when a sharp Fermi surface denies scattering of screening charges into momentum states of magnitude less than k_F.

B.5 Effective action

We find it worthwhile to state the action functionals (that exhibit explicitly only information pertaining to electrons) for the two-body interactions that we've considered.

B.5.1 Phonons. From the Fröhlich [35] interaction energy[14]

$$
H_1 = g\int d\vec{x}\,\psi^\dagger(\vec{x})\psi(\vec{x})\phi(\vec{x}), \tag{B.159}
$$

[14]Midgal [41] has shown that all corrections to the bare electron-phonon vertex g are reduced in strength by factors of c/v_F.

where ϕ is the ion displacement operator, one finds the action[15]

$$
\begin{aligned}
& S\left[\bar{\psi}, \psi, \phi^*, \phi\right] \\
& \quad = \sum_p \bar{\psi}(p) G_0^{-1}(p) \psi(p) + \sum_k \phi^*(k) D_0^{-1}(k) \phi(k) - g\sqrt{\frac{T}{V}} \sum_k n(k) \phi(k),
\end{aligned}
\tag{B.161}
$$

with both $\phi(k) = \phi^*(-k)$ and $n(k) = n^*(-k)$. We're using Debye's isotropic model of longtitudinal sound, which considers only wavelengths much larger than any atomic length scale and ignores umklapp processes [73]; the structure of the unit cell is washed out.

The free phonon propagator

$$
D_0(k) = -\left\langle \phi(k) \phi^*(k) \right\rangle_0 = \frac{\omega_0^2(\vec{k})}{(i\Omega)^2 - \omega_0^2(\vec{k})},
\tag{B.162}
$$

has spectrum

$$
D_0''(\vec{k}, u) = -u^2 \mathrm{sgn}(u) \pi \delta\left[u^2 - \omega_0^2(\vec{k})\right]
\tag{B.163}
$$

which neglects interaction induced damping, while the full phonon

$$
\mathfrak{D}\left[\vec{x}, \tau; \vec{x}', \tau'\right] = -\left\langle \phi(\vec{x}, \tau) \phi(\vec{x}', \tau') \right\rangle
\tag{B.164}
$$

obeys the Schwinger-Dyson equation [34]

$$
\left[\nabla_x^2 + \frac{1}{c^2} \frac{d^2}{d\tau^2}\right] \mathfrak{D}\left[x; x'\right] = -\nabla_x^2 \delta\left[x - x'\right] + g \nabla_x^2 \left\langle \psi^\dagger(x) \psi(x) \phi(x') \right\rangle,
\tag{B.165}
$$

i.e. $\mathfrak{D}$ propagates compression waves (as enabled by the restoring Coulomb force). Now, if we use [33]

$$
\mathfrak{G}\left[x_1, x_2; x_3\right] = \left\langle \psi(x_1) \psi^\dagger(x_2) \phi(x_3) \right\rangle
\tag{B.166}
$$

[15] For completeness, note

$$
\phi(x) = \sqrt{\frac{T}{V}} \sum_q e^{iqx} \phi(q).
\tag{B.160}
$$

to define $\mathcal{T}$, the total vertex part of the electron-phonon interaction, by

$$\mathfrak{G}\big[p, p'; k\big] = \mathfrak{G}(p)\mathfrak{G}(p')\mathfrak{D}(k)\mathcal{T}\big[p, p'; k\big], \tag{B.167}$$

then, given that any operator involving an odd number of ϕ fields vanishes on average, we have

$$\mathfrak{D}(k) = D_0(k) + D_0(k)\Pi(k)\mathfrak{D}(k), \tag{B.168}$$

with the proper polarization

$$\Pi(2k) = g\frac{T}{V}\sum_p \mathfrak{G}(p_-)\mathcal{T}\big[p_-\big|p_+\big]\mathfrak{G}(p_+), \tag{B.169}$$

where

$$\mathcal{T}\big[p, p'; k\big] = \beta(2\pi)^3\delta(p - p' + k)\mathcal{T}\big[p\big|p'\big]. \tag{B.170}$$

Thus, the effective action

$$S_{\text{eff}}\big[\bar{\psi}, \psi\big] = \log\int_{\phi^*,\phi} \exp\Big\{S\big[\bar{\psi}, \psi, \phi^*, \phi\big]\Big\} \tag{B.171}$$

may be expressed in terms of the dynamical potential

$$\mathcal{V}(k) = -g^2\mathfrak{D}(k), \tag{B.172}$$

as

$$\begin{aligned} S_{\text{eff}} &= \sum_p \bar{\psi}(p)G_0^{-1}(p)\psi(p) - \frac{T}{2V}\sum_k n(k)g^2\mathfrak{D}(k)n(-k) \\ &= \int dx\,\bar{\psi}(x)\left[\frac{-d}{d\tau} + \frac{\Delta}{2m} + \epsilon_F\right]\psi(x) + \frac{1}{2}\int dx dy\,\delta n(x)\mathcal{V}(x-y)\delta n(y). \end{aligned} \tag{B.173}$$

Such a frequency dependent interaction is a consequence of causality [27]; the screening response of the ions is delayed due to their (relatively) sluggish motion.

On similar manipulation of Heisenberg's equation, one comes by

$$\left[\frac{-d}{d\tau} + \frac{\Delta}{2m} + \epsilon_F\right]\mathfrak{G}\big[x; x'\big] = \delta\big[x - x'\big] - \big\langle\psi(x)\phi(x)\psi^\dagger(x')\big\rangle, \tag{B.174}$$

from which we extract the self-energy of the quasi-electron

$$-\Sigma(p) = g\frac{T}{V}\sum_k \mathfrak{G}(p-k)\mathfrak{D}(k)\mathcal{T}\big[p-k\big|p\big]. \tag{B.175}$$

Phonon drag. Because the equations of motion for $\mathfrak{G}$ and $\mathfrak{D}$ are coupled, as the charges experience dynamic feedback while arranging in the most mutually pleasing state, it is plausible (due to the electron-ion mass ratio, which effects a separation of time scales between their typical velocities) that the phonon background may be taken thermalized.[16] To investigate this assertion, we consider the leading correction[17]

$$\mathfrak{D}^{-1}(k) \to D_0^{-1}(k) - g^2\Pi_0(k), \tag{B.176}$$

which reflects the momentum imparted on the phonons by the electrons and vice versa; insofar as the phonon lifetime is large, $\mathfrak{D}$ has a particle resonance on frequencies Ω_k obeying the transcendental equation

$$\Omega_k^2 = \omega_0^2(k)\Big[1 + g^2\Pi'(k,\Omega_k)\Big], \tag{B.177}$$

which we solve by assuming the existence of a (truncated) series solution in powers of g^2, i.e.

$$\Omega_k^2\big[g^2\big] \approx \omega_0^2(k)\Big[1 + g^2\Pi'(k,ck)\Big]. \tag{B.178}$$

Note that Π' is even in its frequency argument. As $k \to 0$,

$$-\Pi'(k,\Omega_k[0]) \approx N_F\Big[1-(c/v_F)^2\Big]. \tag{B.179}$$

So, [34]

$$\Omega_k^2 = \omega_0^2(k)\left[1 - g^2N_F\left(1-\frac{c^2}{v_F^2}\right)\right] \tag{B.180}$$

---

[16]This is sometimes called the Bloch hypothesis [8].

[17]Strictly speaking, one should also inspect the Aslamazov-Larkin diagrams of Figure B.3.

simply rescales speed of sound. It follows that $\mathfrak{D}''$ has the structure of D_0'' so long as

$$\omega_0^2(\vec{k})\Pi_0''(\vec{k},\Omega) = (ck)^2 \frac{m^2}{4\pi k}\Omega \tag{B.181}$$

is small near $\Omega = \Omega_k$, which holds in the sense that $\mathfrak{D}''$ will inevitably appear in an integrand containing Bose and Fermi distribution functions in such a way that $\Omega \sim T$.

B.5.2 Ferromagnons. We follow Bharadwaj, Belitz, and Kirkpatrick [3] in subjecting conduction electrons to a magnetization field $\vec{M}$ by introducing (to the free theory) a spin-triplet interaction

$$S\left[\bar{\psi}, \psi; \vec{M}\right] = S_0\left[\bar{\psi}, \psi\right] + \frac{\Gamma_t}{2}\int dx\, \vec{n}_s(x) \cdot \vec{M}(x); \tag{B.182}$$

Γ_t favors a ferromagnetic ground state. In the mean field approximation,

$$M^i(x) \to \delta_{i3}\frac{\lambda}{\Gamma_t}, \tag{B.183}$$

and we are reduced to

$$\begin{aligned} S_\lambda\left[\bar{\psi}, \psi\right] &= S_0\left[\bar{\psi}, \psi\right] + \lambda \int dx\, n_s^3(x) \\ &= \sum_p \left\{ \bar{\psi}_+(p)\left[G_0^{-1}(p) + \lambda\right]\psi_+(p) + \bar{\psi}_-(p)\left[G_0^{-1}(p) - \lambda\right]\psi_-(p) \right\}, \end{aligned} \tag{B.184}$$

which presents a spin degeneracy breaking Zeeman term. Here,

$$n_s^i(x) = \sum_{\alpha\beta} \bar{\psi}_\alpha(x)\sigma^i_{\alpha\beta}\psi_\beta(x) = \sum_{\alpha\beta} \sigma^i_{\alpha\beta} n_{\alpha\beta}(x), \tag{B.185}$$

with $\sigma^i_{\alpha\beta}$ the Pauli matrices $(i \in \{1,2,3\})$, and

$$G_0^\sigma(p) = \langle \psi_\sigma^\dagger(p)\psi_\sigma(p)\rangle_{S_\lambda} = \frac{1}{G_0^{-1}(p) + \sigma\lambda}. \tag{B.186}$$

While λ is yet undetermined, imposing the self-consistency requirement

$$\begin{aligned}\lambda &= \Gamma_t \big\langle n_s^3(x) \big\rangle_{S_\lambda} \\ &= \Gamma_t \frac{T}{V} \sum_p \Big[G_0(i\omega, \xi_p - \lambda) - G_0(i\omega, \xi_p + \lambda) \Big] \\ &= \Gamma_t N_F \frac{2\epsilon_F}{3} \Big[\big(1 + \lambda/\epsilon_F\big)^{3/2} - \big(1 - \lambda/\epsilon_F\big)^{3/2} \Big],\end{aligned} \tag{B.187}$$

yields the Stoner theory equation of state, which relates the thermodynamic variable (magnetization) to the physical conditions subjected on the system (electron density, interaction strength), and reveals a necessary condition

$$1 - 2N_F\Gamma_t < 0 \tag{B.188}$$

for the existence of a solution with nonzero λ.

When magnetization fluctuations $\delta\vec{M}$ are included,

$$S \to S_\lambda + \Gamma_t \int dx\, \delta\vec{M}(x) \cdot \vec{n}_s(x) - \frac{1}{2} \int dx dy\, \delta M_i(x) \chi_{ij}^{-1}(x, y) \delta M_j(y), \tag{B.189}$$

with χ the magnetic susceptibility. Having appended Gaussian fluctuations onto an otherwise mean field treatment of the magnetization, it is possible to integrate out $\delta\vec{M}$; this yields the effective interaction

$$S_{\text{eff}} = S_\lambda + \frac{\Gamma_t^2}{2} \int dx dy\, n_s^i(x) \chi_{ij}(x, y) n_s^j(y). \tag{B.190}$$

Henceforth, we will neglect the longitudinal sector ($i = 3$ or $j = 3$) of χ because only the transverse components ($i, j \in \{1, 2\}$) contain magnons, the Goldstone modes associated with long range ferromagnetic order.

Now then, $\chi_\perp$ is isotropic and therefore has two degrees of freedom; furthermore, the diagonal and off-diagonal components are of even and odd parity, respectfully, under time reversal. Thus, [3]

$$\chi_\perp(k) = \frac{1}{2}\begin{pmatrix} \chi_+(k)+\chi_-(k) & -i\big[\chi_+(k)-\chi_-(k)\big] \\ i\big[\chi_+(k)-\chi_-(k)\big] & \chi_+(k)+\chi_-(k) \end{pmatrix}, \tag{B.191}$$

where the circularly polarized magnons χ_σ obey $\chi_\sigma(k) = \chi_{-\sigma}(-k)$. In fact, $\chi_\perp(k)$ must be singular whenever either of the energy-momentum relations $i\Omega_\sigma = -\sigma Dk^2$, with D the stiffness coefficient, is satisfied; this yields

$$\chi_\sigma(k) = \frac{\sigma}{i\Omega + \sigma Dk^2}. \tag{B.192}$$

As polarized Bosonic spin-waves, the χ_σ facilitate interband transitions, which is to say[18]

$$\begin{aligned} &\frac{1}{2}\int dx dy\, n_s^i(x)\chi_\perp^{ij}(x,y)n_s^j(y) \\ &= \frac{T}{V}\sum_k n_{\alpha\beta}(k)\sigma^i_{\alpha\beta}\chi_\perp^{ij}(k)\sigma^j_{\gamma\delta}n_{\gamma\delta}(-k) \\ &= \sum_{\sigma\sigma'}\big[1-\delta_{\sigma\sigma'}\big]\frac{T}{V}\sum_k n_{\sigma'\sigma}(k)\chi_\sigma(k)n_{\sigma\sigma'}(-k); \end{aligned} \tag{B.196}$$

[18] Our Fourier convention for the electron density,

$$n(x) = \frac{T}{V}\sum_k e^{-ikx}n(k), \tag{B.193}$$

comes with an unconventional sign; we prefer to remember

$$n(2k) = \sum_p \bar\psi(p+k)\psi(p-k). \tag{B.194}$$

For the susceptibility

$$\chi_{ij}(x) = \frac{T}{V}\sum_k e^{ikx}\chi_{ij}(k), \tag{B.195}$$

we maintain the traditional decomposition into right moving waves.

from this position, it is not necessary to specify the origin of the magnetization in order prescribe a phenomenological (magnon mediated) effective potential

$$\mathcal{V}_{\sigma\sigma'}(k) = 2\Gamma_t^2\left[1-\delta_{\sigma\sigma'}\right]\chi_\sigma(k) = \left[1-\delta_{\sigma\sigma'}\right]\frac{\nu_0}{\omega_0(\vec{k})+\sigma(i\Omega)} \tag{B.197}$$

with resonance frequency

$$\omega_0(\vec{k}) = -\sigma D(\lambda)k^2, \quad D(\lambda\to 0)\to 0 \tag{B.198}$$

proportional to the spin stiffness $D(\lambda)$; ν_0 is a coupling constant. The resulting interacting action is

$$S_{int} = \frac{T}{2V}\sum_k\sum_{\sigma\sigma'} n_{\sigma'\sigma}(k)\mathcal{V}_{\sigma\sigma'}(k)n_{\sigma\sigma'}(-k); \tag{B.199}$$

that $\mathcal{V}$ does not provide a channel for intraband transitions results in an exchange gap.

B.5.2.1 A model calculation. The structure of $\mathcal{V}$ put forth above is found, for example, in the work of Moriya and Ueda [4], who analyzed the Hubbard model (in the random phase approximation) to realize the s-d exchange interaction with definite expressions for both ν_0 and $D(\lambda)$. Similarly, one may determine the (ordered phase) magnetic susceptibility of free electrons [74]

$$\chi_{ij}^{-1}(x,y) = \left\langle \delta n_s^i(x)\delta n_s^j(y)\right\rangle_{S_\lambda}^{-1} - \delta_{ij}\Gamma_t\delta(x-y), \tag{B.200}$$

by considering the action of Equation B.182 with $\vec{M} \to \vec{n}_s$. It follows that we need to compute the zero eigenvalue resonances of

$$\chi_\perp^{-1}(k) = \frac{1}{f_1^2(k)+f_2^2(k)}\begin{bmatrix}\tilde{f}_1(k) & -f_2(k)\\ f_2(k) & \tilde{f}_1(k)\end{bmatrix}, \tag{B.201}$$

where

$$\tilde{f}_1(k) = f_1(k) - \left[f_1^2(k)+f_2^2(k)\right]\Gamma_t, \tag{B.202}$$

and

$$-f_1(k) = \Pi_0^{+-}(k) + \Pi_0^{-+}(k),$$
$$-if_2(k) = \Pi_0^{+-}(k) - \Pi_0^{-+}(k), \tag{B.203}$$

are known in terms of

$$\Pi_0^{\sigma\sigma'}(k) = \frac{T}{V}\sum_p G_0^{\sigma}(p)G_0^{\sigma'}(p-k), \tag{B.204}$$

which can be evaluated exactly. On inverting $\chi_\perp^{-1}$, we find

$$\chi_\perp(k) = \frac{f_1^2(k)+f_2^2(k)}{\tilde{f}_1^2(k)+f_2^2(k)}\begin{bmatrix}\tilde{f}_1(k) & f_2(k)\\ -f_2(k) & \tilde{f}_1(k)\end{bmatrix}, \tag{B.205}$$

or

$$\chi_\sigma(k) = \frac{f_1^2(k)+f_2^2(k)}{\tilde{f}_1(k)-i\sigma f_2(k)} = \frac{-2\Pi_0^{\sigma}(k)}{1+2\Gamma_t\Pi_0^{\sigma}(k)}, \tag{B.206}$$

where we've introduced the notation

$$\Pi_0^{\sigma}(k) = \sum_{\sigma'}\left[1-\delta_{\sigma\sigma'}\right]\Pi_0^{\sigma\sigma'}(k). \tag{B.207}$$

Evidently, $\chi_\perp$ is singular for frequencies $\omega_0^{\sigma}[k]$ satisfying

$$1+2\Gamma_t\left[\Pi_0\right]'_{\sigma}(\vec{k},\omega_0^{\sigma}[k]) = 0, \quad \left[\Pi_0\right]''_{\sigma}(\vec{k},\omega_0^{\sigma}[k]) = 0. \tag{B.208}$$

Note that [55]

$$1+2\Gamma_t\Pi'_{\sigma}(k\to 0,0) = 0 \tag{B.209}$$

holds true when λ obeys Equation B.187. Therefore, the spin-wave energy must admit

$$\Pi'_{\sigma}(k\to 0,0) = \Pi'_{\sigma}(k,\omega_0^{\sigma}[k]), \tag{B.210}$$

which has no explicit Γ_t dependence. By assuming a power series for $\omega_0^{\sigma}(k)$, expanding $\Pi'_{\sigma}\left[k,\omega_0^{\sigma}(k)\right]$ for small k, and equating coefficients, we find

$$\omega_0^{\sigma}(k) = -\sigma D(\lambda)k^2 + o(k^2), \tag{B.211}$$

with

$$D(\lambda) = \frac{\lambda}{6k_F^2} + o(\lambda). \tag{B.212}$$

To this approximation, $k < k_0 = 2\lambda/v_F$ defines the region (of undamped excitations) where there is a singular contribution to the spectrum

$$\chi''_\sigma(\vec{k}, \Omega) = -4\sigma N_F \lambda\pi \times \delta\big[\Omega + \sigma D(\lambda)k^2\big], \tag{B.213}$$

as anticipated.

Modified Lindhard function In determining

$$\begin{aligned} \Pi_0^\sigma(k) &= \frac{T}{V}\sum_p \frac{1}{\big[G_0\big]^{-1}_{-\sigma}(p-k) - \big[G_0\big]^{-1}_{\sigma}(p)}\Big[G_0^\sigma(p) - G_0^{-\sigma}(p-k)\Big] \\ &= \frac{1}{V}\sum_{\vec{p}} \frac{1}{-ik - \xi_{\vec{p}-\vec{k}} + \xi_{\vec{p}} - 2\sigma\lambda} \\ &\qquad \times \int \frac{d\epsilon}{\pi} f(\epsilon)\Big\{\big[G_0\big]''_{-\sigma}(\vec{p}-\vec{k}, \epsilon) - \big[G_0\big]''_{\sigma}(\vec{p}, \epsilon)\Big\}, \end{aligned} \tag{B.214}$$

it is wise to first assess the integral

$$\begin{aligned} &\big[\Pi_0\big]''_\sigma(\vec{k}, \Omega) \\ &= \frac{1}{V}\sum_{\vec{p}} \Big\{f(\xi_p - \sigma\lambda) - f(\xi_p - \Omega - \sigma\lambda)\Big\}\pi\delta\big[-\Omega + \vec{p}\cdot\vec{k}/m - \varepsilon_k - 2\sigma\lambda\big], \end{aligned} \tag{B.215}$$

which leads to

$$\big[\Pi_0\big]''_\sigma(k, \Omega) = \frac{m^2 T}{4\pi k}\log\frac{1 + \exp\left\{\beta\left[\epsilon_F^\sigma - \frac{1}{2m}\left(\frac{\Omega + 2\sigma\lambda}{k/m} + \frac{k}{2}\right)^2\right]\right\}}{1 + \exp\left\{\beta\left[\epsilon_F^\sigma + \Omega - \frac{1}{2m}\left(\frac{\Omega + 2\sigma\lambda}{k/m} + \frac{k}{2}\right)^2\right]\right\}}, \tag{B.216}$$

or, in the notation of Equation B.153,

$$\big[\Pi_0\big]''_\sigma(k, \Omega) = \frac{m^2 T}{4\pi k}\log\frac{1 + \exp\left\{\beta\epsilon_F\left[1 - \zeta_\sigma(\Omega)\right]\right\}}{1 + \exp\left\{\beta\epsilon_F\left[1 - \zeta_{-\sigma}(-\Omega)\right]\right\}}, \tag{B.217}$$

with

$$\zeta_\sigma(\Omega) = \zeta(k, \Omega) + \varpi_\sigma(\Omega), \tag{B.218}$$

where the symmetry $\Pi''_\sigma(k,\Omega) = -\Pi''_{-\sigma}(k,-\Omega)$ is upheld by

$$\varpi_\sigma(\Omega) = \frac{2m\sigma\lambda(\sigma\lambda+\Omega)}{k^2\epsilon_F} = \varpi_{-\sigma}(-\Omega). \tag{B.219}$$

On discarding explicit temperature dependencies, it follows that

$$\begin{aligned} &-\left[\Pi_0\right]''_\sigma(k,\Omega) \\ &= \frac{m^2\epsilon_F}{4\pi k} \times \begin{cases} \Omega/\epsilon_F & \zeta_{-\sigma}(-\Omega) < 1,\ \zeta_\sigma(\Omega) < 1 \\ 1-\zeta(k,-\Omega)-\varpi_{-\sigma}(-\Omega) & \zeta_{-\sigma}(-\Omega) < 1,\ \zeta_\sigma(\Omega) > 1 \\ -1+\zeta(k,\Omega)+\varpi_\sigma(\Omega) & \zeta_{-\sigma}(-\Omega) > 1,\ \zeta_\sigma(\Omega) < 1 \\ 0 & \zeta_{-\sigma}(-\Omega) > 1,\ \zeta_\sigma(\Omega) > 1 \end{cases} \end{aligned} \tag{B.220}$$

which defines the hyperfunction

$$\begin{aligned} &-\Pi_0^\sigma(k,z) \\ &= \frac{N_F}{2}\Bigg\{\frac{k_F^\sigma+k_F^{-\sigma}}{2k_F} + \frac{k_F^\sigma-k_F^{-\sigma}}{k}\frac{z+2\sigma\lambda}{kv_F} \\ &\qquad + \frac{k_F}{2k}\left[1-\left(\frac{k}{2k_F}+\frac{mz}{kk_F}\right)^2 - \frac{2m\sigma\lambda}{k^2\epsilon_F}(\sigma\lambda+z)\right] \\ &\qquad\qquad \times\Bigg(\log\left[k^2+2kk_F^\sigma+2m\big(z+2\sigma\lambda\big)\right] \\ &\qquad\qquad\qquad -\log\left[k^2-2kk_F^\sigma+2m\big(z+2\sigma\lambda\big)\right]\Bigg) \\ &\qquad + \frac{k_F}{2k}\left[1-\left(\frac{k}{2k_F}-\frac{mz}{kk_F}\right)^2 - \frac{2m\sigma\lambda}{k^2\epsilon_F}(\sigma\lambda+z)\right] \\ &\qquad\qquad \times\Bigg(\log\left[k^2+2kk_F^{-\sigma}-2m\big(z+2\sigma\lambda\big)\right] \\ &\qquad\qquad\qquad -\log\left[k^2-2kk_F^{-\sigma}-2m\big(z+2\sigma\lambda\big)\right]\Bigg)\Bigg\}. \end{aligned} \tag{B.221}$$

B.6 Ward-Takahashi identities

Toyoda has shown [14] that for each conservation law involving only one-body operators, there exists a generalized (finite temperature) Ward-Takahashi identity that relates the self-energy to the proper (particle-hole) vertex; this provides a (naively) nonperturbative relation involving the two-point function that ensures the associated continuity equation is maintained in any particular approximation for the (four-point) scattering amplitude.

Here, we'll derive the identities corresponding to particle number conservation, momentum balance, and energy flow for the theory described by the field equations

$$\begin{aligned} \left[\frac{-d}{d\tau} - \xi(\vec{p})\right]\psi(x) &= -\int dy\, n(y)(y)\mathcal{V}(x-y)\psi(x), \\ \left[\frac{d}{d\tau} - \xi(-\vec{p})\right]\psi^{\dagger}(x) &= -\int dy\, \psi^{\dagger}(x)\mathcal{V}(x-y)n(y), \end{aligned} \tag{B.222}$$

with both $\xi(\vec{p}) = \frac{-\Delta}{2m} - \epsilon_F$ and $\mathcal{V}$ static, which lead to[19]

$$\begin{aligned} \int G_0^{-1}[1,\bar{1}]\mathfrak{G}[\bar{1};4] &= \delta(1-4) + \int \mathcal{V}(1-\bar{2})\mathfrak{G}^{II}[1,\bar{2};\bar{2}^{+},4], \\ \int \mathfrak{G}[1;\bar{4}]G_0^{-1}[\bar{4},4] &= \delta(1-4) + \int \mathcal{V}(\bar{2}-4)\mathfrak{G}^{II}[1,\bar{2};\bar{2}^{+},4], \end{aligned} \tag{B.223}$$

where

$$\begin{aligned} G_0^{-1}[1,\bar{1}] &= \delta(1-\bar{1})\left\{\frac{-d}{d\tau_{\bar{1}}} - \xi(\vec{p}_{\bar{1}})\right\} = \delta(1-\bar{1})G_0^{-1}(1), \\ G_0^{-1}[\bar{4},4] &= \left\{\frac{-d}{d\tau_{\bar{4}}} - \xi(\vec{p}_{\bar{4}})\right\}\delta(\bar{4}-4) = G_0^{-1}(\bar{4})\delta(\bar{4}-4). \end{aligned} \tag{B.224}$$

B.6.1 Particle number conservation. To find a necessary condition, begin with the continuity equation

$$\frac{d}{d\tau}n(x) = \psi^{\dagger}(x)\frac{\Delta}{2m}\psi(x) - \left[\frac{\Delta}{2m}\psi^{\dagger}(x)\right]\psi(x) = -\vec{\nabla}\cdot\vec{j}(x), \tag{B.225}$$

where

$$\vec{j}(x) = \frac{-1}{2m}\left\{\psi^{\dagger}(x)\vec{\nabla}\psi(x) - \left[\vec{\nabla}\psi^{\dagger}(x)\right]\psi(x)\right\} \tag{B.226}$$

[19] To avoid a proliferation of symbols, we're using the standard shorthand $\psi(1) \equiv \psi(x_1)$ while omitting both the spin and replica indices. Futhermore, barred numbers $\bar{1}$ are implicitly understood to be integration variables througout.

is the number current. Then, consider the equality

$$\begin{aligned}\frac{d}{d\tau}T_\tau\{n(x)\psi(1)\psi^\dagger(2)\} = \delta(\tau-\tau_1)&T_\tau\{[n(x),\psi(1)]\psi^\dagger(2)\} \\ &+\delta(\tau-\tau_2)T_\tau\{\psi(1)[n(x),\psi^\dagger(2)]\} \\ &+T_\tau\left\{\frac{d}{d\tau}n(x)\psi(1)\psi^\dagger(x)\right\},\end{aligned} \tag{B.227}$$

which, with

$$\begin{aligned}\delta(\tau-\tau_1)[n(x),\psi(1)] &= -\psi(x)\delta(x-1), \\ \delta(\tau-\tau_2)[n(x),\psi^\dagger(2)] &= \psi^\dagger(x)\delta(x-2),\end{aligned} \tag{B.228}$$

implies

$$[\delta(1-2)-\delta(1-3)]\mathfrak{G}[2;3] = \lim_{4\to 1^+}\{G_0^{-1}(1)-G_0^{-1}(-4)\}\mathfrak{G}^{\mathrm{II}}[1,2;3;4]. \tag{B.229}$$

Next, write

$$\begin{aligned}G_0^{-1}(1)\mathfrak{G}[1;4] &= \delta(1-4)+[\Sigma*\mathfrak{G}](1;4), \\ G_0^{-1}(-4)\mathfrak{G}[1;4] &= \delta(1-4)+[\mathfrak{G}*\Sigma](1;4),\end{aligned} \tag{B.230}$$

with

$$\begin{aligned}[\Sigma*\mathfrak{G}](1;4) &= \int \Sigma[1;\bar{2}]\mathfrak{G}[\bar{2};4], \\ [\mathfrak{G}*\Sigma](1;4) &= \int \mathfrak{G}[1;\bar{2}]\Sigma[\bar{2};4],\end{aligned} \tag{B.231}$$

which yields both

$$\begin{aligned}&G_0^{-1}(1)\mathfrak{G}_F^{\mathrm{II}}[1,2;3,4] \\ &= \mathfrak{G}[2;3]\{\delta(1-4)+[\Sigma*\mathfrak{G}](1;4)\} - \mathfrak{G}[2;4]\{\delta(1-3)+[\Sigma*\mathfrak{G}](1;3)\}\end{aligned} \tag{B.232}$$

and

$$\begin{aligned}&G_0^{-1}(-4)\mathfrak{G}_F^{\mathrm{II}}[1,2;3,4] \\ &= \mathfrak{G}[2;3]\{\delta(1-4)+[\mathfrak{G}*\Sigma](1;4)\} - \mathfrak{G}[2;4]\{\delta(1-3)+[\mathfrak{G}*\Sigma](1;3)\},\end{aligned} \tag{B.233}$$

where

$$\mathfrak{G}_F^{\mathrm{II}}[1,2;3,4] = \mathfrak{G}[1;4]\mathfrak{G}[2;3]-\mathfrak{G}[1;3]\mathfrak{G}[2;4]. \tag{B.234}$$

Hence,

$$\lim_{4\to1^+}\left\{G_0^{-1}(1)-G_0^{-1}(-4)\right\}\mathfrak{G}_F^{II}(1,2;3,4)$$
$$=\left[\delta(1-2)-\delta(1-3)\right]\mathfrak{G}\left[2;3\right]+\left[\mathfrak{G}*\Sigma\right](2;1)\mathfrak{G}\left[1;3\right]-\left[\Sigma*\mathfrak{G}\right](1;3)\mathfrak{G}\left[2;1\right]. \tag{B.235}$$

From here, with

$$\mathfrak{G}_B^{II}\left[1,2;3,4\right]=-\int\mathfrak{G}\left[1;\bar{1}\right]\mathfrak{G}\left[2;\bar{2}\right]\Lambda\left[\bar{1},\bar{2};\bar{3},\bar{4}\right]\mathfrak{G}\left[\bar{3};3\right]\mathfrak{G}\left[\bar{4};4\right], \tag{B.236}$$

we have

$$\begin{aligned}
&\left[\Sigma*\mathfrak{G}\right](1;3)\mathfrak{G}\left[2;1\right]-\left[\mathfrak{G}*\Sigma\right](2;1)\mathfrak{G}\left[1;3\right]\\
&=\lim_{4\to1^+}\left\{G_0^{-1}(1)-G_0^{-1}(-4)\right\}\mathfrak{G}_B^{II}\left[1,2;3,4\right]\\
&=-\lim_{4\to1^+}\int\mathfrak{G}\left[2;\bar{2}\right]\Lambda\left[\bar{1},\bar{2};\bar{3},\bar{4}\right]\mathfrak{G}\left[\bar{3};3\right]\\
&\qquad\times\Big(\mathfrak{G}\left[\bar{4};4\right]\left\{\delta(1-\bar{1})+\left[\Sigma*\mathfrak{G}\right](1;\bar{1})\right\}\\
&\qquad\quad-\mathfrak{G}\left[1;\bar{1}\right]\left\{\delta(4-\bar{4})+\left[\mathfrak{G}*\Sigma\right](\bar{4};4)\right\}\Big)\\
&=-\int\left[\mathfrak{G}\Lambda\mathfrak{G}\right](\bar{1},2;3,\bar{4})\\
&\qquad\times\Big\{\mathfrak{G}\left[\bar{4};\bar{1}\right]\left[\delta(1-\bar{1})-\delta(1-\bar{4})\right]\\
&\qquad\quad+\mathfrak{G}\left[\bar{4};1\right]\left[\Sigma*\mathfrak{G}\right](1;\bar{1})-\left[\mathfrak{G}*\Sigma\right](\bar{4};1)\mathfrak{G}\left[1;\bar{1}\right]\Big\},
\end{aligned} \tag{B.237}$$

which is equivalent to

$$\begin{aligned}
F\left[2,1,3\right]&=\left[\Sigma*\mathfrak{G}\right](1;3)\mathfrak{G}\left[2;1\right]-\left[\mathfrak{G}*\Sigma\right](2;1)\mathfrak{G}\left[1;3\right]\\
&=\int\left[\mathfrak{G}\Lambda\mathfrak{G}\right](\bar{1},2;3,\bar{4})\left\{\mathfrak{G}\left[\bar{4};\bar{1}\right]\left[\delta(1-\bar{4})-\delta(1-\bar{1})\right]-F\left[\bar{4},1,\bar{1}\right]\right\};
\end{aligned} \tag{B.238}$$

wherefrom

$$F\left[2,1,3\right]=\int\left[\mathfrak{G}\tilde{\Lambda}\mathfrak{G}\right](\bar{1},2;3,\bar{4})\mathfrak{G}\left[\bar{4};\bar{1}\right]\left\{\delta(1-\bar{4})-\delta(1-\bar{1})\right\} \tag{B.239}$$

may be deduced by iterating the Bethe-Salpeter equation. Finally,

$$\begin{aligned}\{\delta(1-2)-\delta(1-3)\}\Sigma[2;3] &= \int \mathfrak{G}^{-1}[2;\bar{2}]F[\bar{2},1,\bar{3}]\mathfrak{G}^{-1}[\bar{3};3] \\ &= \int \tilde{\Lambda}[\bar{1},2;3,\bar{4}]\mathfrak{G}[\bar{4},\bar{1}]\{\delta(1-\bar{4})-\delta(1-\bar{1})\}\end{aligned} \tag{B.240}$$

yields

$$\Sigma(q_+)-\Sigma(q_-) = \int \left[\mathfrak{G}(\bar{p}_+)-\mathfrak{G}(\bar{p}_-)\right]\tilde{\Lambda}[\bar{p},q;2k] \tag{B.241}$$

after making use of spacetime translation invariance.

B.6.1.1 Langer's identity. Both γ_{RR} and γ_{AA} can be expressed in a form that is independent of γ_{RA}. To achieve this, first define

$$\begin{aligned}W[q_+,q_-;2k] &= \Sigma(q_+)-\Sigma(q_-)+\xi(\vec{q}_+)-\xi(\vec{q}_-)-2ik \\ &= \frac{2\vec{k}}{m}\cdot\vec{W}[q_+,q_-;2k]-2ik.\end{aligned} \tag{B.242}$$

Then, use Equation B.241 to write [26, 57]

$$\begin{aligned}W[q_+,q_-;2k] = {} & \xi(\vec{q}_+)-\xi(\vec{q}_-)-2ik \\ & + \int \tilde{\Lambda}[\bar{p},q;2k]\mathfrak{G}(\bar{p}_+)\mathfrak{G}(\bar{p}_-)W[\bar{p}_+,\bar{p}_-;2k],\end{aligned} \tag{B.243}$$

which implies

$$\begin{aligned}&\vec{W}_{RR}(\vec{q},\epsilon) \\ &= \vec{q} + \lim_{iq\to\epsilon+i0}\lim_{k\to i0}\int \tilde{\Lambda}[\bar{p},q;2k]\mathfrak{G}(\bar{p}_+)\mathfrak{G}(\bar{p}_-)\vec{W}[\bar{p}_+,\bar{p}_-;2k].\end{aligned} \tag{B.244}$$

Evidently, $\vec{\gamma}_{RR}(\vec{q},\epsilon)$ and $\vec{W}_{RR}(\vec{q},\epsilon)$ obey the same integral equation; we conclude that

$$\vec{\gamma}_{RR}(\vec{q},\epsilon) = \vec{W}_{RR}(\vec{q},\epsilon) = \vec{q} + m\frac{\partial}{\partial\vec{q}}\Sigma^R(\vec{q},\epsilon), \tag{B.245}$$

which demonstrates that $\vec{\gamma}_{RR}$ is regular in the limit of vanishing temperature. Thus, the hydrodynamic approximation

$$\sigma \sim \frac{e^2}{6m^2}\frac{1}{V}\sum_{\vec{q}}\int\frac{d\epsilon}{\pi}w(\epsilon)\mathfrak{G}_R(\vec{q},\epsilon)\vec{q}\cdot\vec{\gamma}_{RA}(\vec{q},\epsilon)\mathfrak{G}_A(\vec{q},\epsilon), \tag{B.246}$$

contains all singular contributions to the conductivity; now,

$$\vec{\varphi}(\vec{q},\epsilon) = \frac{\vec{\gamma}_{RA}(\vec{q},\epsilon)}{-\Sigma''(\vec{q},\epsilon)} \tag{B.247}$$

obeys an integral equation of the form

$$-\Sigma''(\vec{q},\epsilon)\vec{\varphi}(\vec{q},\epsilon) = \vec{q} + \left[\mathfrak{K}_0 \circ \vec{\varphi}\right](\vec{q},\epsilon), \tag{B.248}$$

where

$$-\Sigma''(\vec{q},\epsilon) = \left[\mathfrak{K}_0 \circ 1\right](\vec{q},\epsilon) \tag{B.249}$$

is implied by Equation B.241.

B.6.2 Momentum balance.

Even when electronic momentum is not conserved, a useful identity may be extracted from [75]

$$m\frac{d}{d\tau}\int d\vec{x}\, j^i(x) = \int d\vec{x}\, n(x)\left[-\partial_i \int dy\, \mathcal{V}(x-y)n(y)\right]; \tag{B.250}$$

manuevering as before, we find

$$\begin{aligned}&\left\{\delta(\tau_1-\tau_2)\vec{\nabla}_2 + \delta(\tau_1-\tau_3)\vec{\nabla}_3\right\}\mathfrak{G}[2;3]\\ &= \vec{Q}_{\tau_1}[2,3] - \frac{1}{2}\int d\vec{x_1} \lim_{4\to 1^+}\left\{\vec{\nabla}_1 - \vec{\nabla}_4\right\}\left\{\frac{d}{d\tau_1}+\frac{d}{d\tau_4}\right\}\mathfrak{G}^{II}[1,2;3,4],\end{aligned} \tag{B.251}$$

where

$$\vec{Q}_{\tau_x}[2,3] = \int d\vec{x_1}\left\langle T_\tau\left\{n(x)\left[-\partial_x^i \int dy\, \mathcal{V}(x-y)n(y)\right]\psi(2)\psi^\dagger(3)\right\}\right\rangle \tag{B.252}$$

is nonzero iff global momentum is conserved. It follows that

$$\begin{aligned}&\vec{F}_{\tau_1}[2,3]\\ &= \vec{Q}_{\tau_1}[2,3] - \frac{1}{2}\int d\vec{x}_1 \lim_{4\to 1^+}\left\{\vec{\nabla}_1 - \vec{\nabla}_4\right\}\left\{\frac{d}{d\tau_1}+\frac{d}{d\tau_4}\right\}\mathfrak{G}^{II}_B[1,2;3,4],\end{aligned} \tag{B.253}$$

where

$$\vec{F}_{\tau_1}[2,3] = \int d\vec{x}_1\left\{\mathfrak{G}[2;1]\vec{\nabla}_1\left[\Sigma * \mathfrak{G}\right](1;3) + \mathfrak{G}[1;3]\vec{\nabla}_1\left[\mathfrak{G}*\Sigma\right](2;1)\right\}; \tag{B.254}$$

further manipulation leads to

$$\begin{aligned}\vec{F}_{\tau_1}[2,3] &= \vec{Q}_{\tau_1}[2,3] - \int [\mathfrak{G}\Lambda\mathfrak{G}](\bar{1},2;3,\bar{4})\vec{F}_{\tau_1}[\bar{4},\bar{1}] \\ &\quad + \int [\mathfrak{G}\Lambda\mathfrak{G}](\bar{1},2;3,\bar{4})\{\delta(\tau_1-\tau_{\bar{1}})\vec{\nabla}_1 + \delta(\tau_{\bar{4}}-\tau_1)\vec{\nabla}_{\bar{4}}\}\mathfrak{G}[\bar{4},\bar{1}],\end{aligned} \tag{B.255}$$

which is solved by

$$\begin{aligned}&\vec{F}_{\tau_1}[2,3] - \vec{Q}_{\tau_1}[2,3] \\ &= \int [\mathfrak{G}\tilde{\Lambda}\mathfrak{G}](\bar{1},2;3,\bar{4})\{[\delta(\tau_1-\tau_{\bar{1}})\vec{\nabla}_1 + \delta(\tau_{\bar{4}}-\tau_1)\vec{\nabla}_{\bar{4}}]\mathfrak{G}[\bar{4};\bar{1}] - \vec{Q}_{\tau_1}[\bar{4},\bar{1}]\},\end{aligned} \tag{B.256}$$

and therefore

$$\begin{aligned}&\vec{q}[\Sigma(q+i\Omega) - \Sigma(q-i\Omega)] \\ &= \int \vec{p}[\mathfrak{G}(\bar{p}+i\Omega) - \mathfrak{G}(\bar{p}-i\Omega)]\tilde{\Lambda}[\bar{p},q;2i\Omega] + \vec{P}_{2i\Omega}(q),\end{aligned} \tag{B.257}$$

where

$$\begin{aligned}&\vec{P}_{2i\Omega}(q) \\ &= -\mathfrak{G}^{-1}(q+i\Omega)\vec{Q}_{2i\Omega}(q)\mathfrak{G}^{-1}(q-i\Omega) + \int \tilde{\Lambda}[\bar{p},q;2i\Omega]\vec{Q}_{2i\Omega}(\bar{p}).\end{aligned} \tag{B.258}$$

Here,

$$\vec{Q}_{\tau_1}[x_2,x_3] = \int \vec{Q}_{2i\bar{\Omega}}(\bar{q})e^{-2i\bar{\Omega}\tau_1 - i\bar{k}(x_2+x_3) + i\bar{q}(x_2-x_3)}. \tag{B.259}$$

From Equation B.257 we have

$$-\Sigma''(\vec{q},\epsilon) = P(\vec{q},\epsilon) + [\mathfrak{K}\circ 1](\vec{q},\epsilon), \tag{B.260}$$

where

$$P(\vec{q},\epsilon) = \lim_{i\Omega\to i0}\lim_{i\omega_q\to\epsilon+i0}\frac{\vec{q}\cdot\vec{P}_{2i\Omega}(q)}{q^2}. \tag{B.261}$$

Thus, if Equation B.260 is to be consistent with Equation B.249, it must be true that

$$P(\vec{q},\epsilon) = [\{\mathfrak{K}_0 - \mathfrak{K}\}\circ 1](\vec{q},\epsilon). \tag{B.262}$$

B.6.3 Energy conservation. While the identity associated with energy conservation may be derived along the same lines as above, the task is notoriously more laborious. For this reason, our presentation is brief, and considers only the case where

$$\begin{aligned} 0 &= \frac{d}{d\tau_1}\varepsilon_{\tau_1} \\ &= \frac{1}{4}\frac{d}{d\tau_1}\int d\vec{x}_1 \lim_{4\to 1^+}\left\{\frac{d}{d\tau_4} - \frac{d}{d\tau_1} + \xi(-\vec{p}_4) + \xi(\vec{p}_1)\right\}\psi^\dagger(4)\psi(1); \end{aligned} \tag{B.263}$$

at last [42]

$$\begin{aligned} &2i\omega_q\Big\{\Sigma(q+i\Omega) - \Sigma(q-i\Omega)\Big\} - i\Omega\Big\{\Sigma(q+i\Omega) + \Sigma(q-i\Omega)\Big\} \\ &= \int \tilde{\Lambda}(\bar{p},q,2i\Omega)\Big\{2i\omega_{\bar{p}}\Big[\mathfrak{G}(\bar{p}+i\Omega) - \mathfrak{G}(\bar{p}-i\Omega)\Big] \\ &\qquad\qquad + i\Omega\Big[\mathfrak{G}(\bar{p}+i\Omega) + \mathfrak{G}(\bar{p}-i\Omega)\Big]\Big\}. \end{aligned} \tag{B.264}$$

B.7 Conserving approximations

Having demonstrated (in section B.6) that it is necessary for Σ and $\tilde{\Lambda}$ to simultaneously satisify a trio of integral equations if one reorganizes the perturbation series such that both $\mathfrak{G}$ and $\mathfrak{G}^{II}$ are defined (self-consistently) through the one-particle and two-particle irreducible vertices, we are now in a position to state a single condition that suffices to preserve the conservation laws even when one truncates the diagrammatic expansion for $\tilde{\Lambda}$. To this end, it is useful to introduce to the action an external potential

$$S_U(t) = -\int dx dy\, \psi^\dagger(x) U\big[x;y\big]\psi(y), \tag{B.265}$$

which facilitates generating the linear response function L according to

$$L\big[1,2;3,4\big] = \frac{-\delta\mathfrak{G}_U\big[1;4\big]}{\delta U\big[3;2\big]}\Bigg|_{U\to 0}. \tag{B.266}$$

Indeed, L is immediately related to the expected variation of one-particle quantities such as the electron density

$$\delta\langle n(x)\rangle_U \approx \lim_{y\to x^+}\int L\big[x,\bar{2};\bar{3},y\big]\delta U\big[\bar{3};\bar{2}\big] = U_0 \mathrm{Re}\Big[e^{ikx}\Pi^R_{nn}(\vec{k},\Omega)\Big], \tag{B.267}$$

where we have set

$$\delta U[x;y] = \delta(x-y) U_0 \cos\left[\vec{k}\cdot\vec{x} - \Omega t\right] e^{\epsilon t}, \quad \epsilon \to 0^+ \tag{B.268}$$

to be a plane wave that is adiabatically switched on, introduced the density-density susceptibility

$$\Pi_{nn}(k) = -\int_{q,p} L(q,p;k), \tag{B.269}$$

and used time reversal symmetry

$$L(p,q;-k) = L(q,p;k), \tag{B.270}$$

which implies that

$$\Pi^R_{nn}(-\Omega) = \Pi_{nn}(-\Omega + i0) = \Pi_{nn}(\Omega - i0) = \left[\Pi^R_{nn}(\Omega)\right]^*. \tag{B.271}$$

Thus, in order to determine transport coefficient, one must consider the Bethe-Salpeter equation

$$L(q,p;2k) = \mathfrak{G}(q_+)\mathfrak{G}(q_-)\left\{-\delta(p-q) + \int_Q \tilde{\Lambda}(q,Q;2k) L(Q,p;2k)\right\}, \tag{B.272}$$

where

$$\tilde{\Lambda}[1,2;3,4] = \frac{\delta\Sigma[1;4]}{\delta\mathfrak{G}[3;2]} \tag{B.273}$$

is the proper particle-hole vertex. Crucially, if the differential Ward-Takahashi identity Equation B.273 is satisfied, then one can show [42] that the three integral relations

$$\Sigma(q_+) - \Sigma(q_-) = \int_p \left[\mathfrak{G}(p_+) - \mathfrak{G}(p_-)\right]\tilde{\Lambda}[p,q;2k], \tag{B.274a}$$

$$\vec{q}\left[\Sigma(q_+) - \Sigma(q_-)\right] = \int_p \vec{p}\left[\mathfrak{G}(p_+) - \mathfrak{G}(p_-)\right]\tilde{\Lambda}[p,q;2i\Omega], \tag{B.274b}$$

$$i\omega_q\left[\Sigma(q_+) - \Sigma(q_-)\right] = \int_p i\omega_p\left[\mathfrak{G}(p_+) - \mathfrak{G}(p_-)\right]\tilde{\Lambda}(p,q,2i0) \tag{B.274c}$$

are obeyed, no matter the approximation, whenever the bare interaction of the equilibrium ensemble is both conserving and static; evidently, this construction (originally due to Kadanoff and Baym [24, 39]) is of paramount importance, for while the unsullied theory admits[20]

$$\begin{aligned}
&\frac{d}{d\tau_1}\langle n(1)\rangle_U + \vec{\nabla}_1 \cdot \langle \vec{j}(1)\rangle_U \\
&= \Big(\Big\{\big[G_0^\dagger\big]^{-1}(4) - G_0^{-1}(1)\Big\}\mathfrak{G}_U[1;4]\Big)_{4\to 1^+} \\
&= \int \mathfrak{G}_U^{II}\big[1,\bar{2};\bar{2}^+,1^+\big]\big\{\mathcal{V}_0(1-\bar{2}) - \mathcal{V}_0(2-1^+)\big\} \\
&= 0
\end{aligned} \tag{B.276}$$

so long as $\mathcal{V}_0$ is number conserving, it is nontrivial that the act of closing the Schwinger–Dyson equation for $\mathfrak{G}$ by the replacement[21] of $\mathfrak{G}^{II}[\mathcal{V}_0]\mathcal{V}_0$ with $\Sigma[\mathfrak{G},\mathcal{V}]\mathfrak{G}[\Sigma]$ does not result in a violation of the conservation laws if only a portion of the contributions to Σ are retained, as such a diagrammatic expansion is not in powers of the coupling.

Now then, that Equation B.273 is sufficient to maintain the conservation laws follows on writing

$$L(q,p;2k) = -\big[\mathfrak{G}(q_+) - \mathfrak{G}(q_-)\big]\varphi(q,p;2k), \tag{B.277}$$

which reveals the linearized quantum Boltzmann equation

$$\delta(p-q) = \big\{-i\Omega + \vec{k}\cdot\vec{q}/m\big\}\varphi(q,p;k) - \big[I\circ\varphi\big](q,p;k), \tag{B.278}$$

[20] Recall that

$$\langle n(x)\rangle_U = \mathfrak{G}_U[x;x^+], \quad \langle \vec{j}(x)\rangle_U = \frac{-1}{2m}\{(\vec{\nabla}-\vec{\nabla}')\mathfrak{G}_U[x,x']\}_{x'\to x^+}. \tag{B.275}$$

[21] Bogoliubov's principle of weakening of initial correlations constitutes a sufficient condition for the two-particle problem to admit a solution involving only one-particle quantities. In this way, the stronger assumption of molecular chaos, introduced by Bloch, is not necessary for the existence of a transport equation.

where the collision operator[22]

$$\begin{aligned}&\left[I \circ \varphi\right](q,p;2k)\\&= \int_Q \tilde{\Lambda}(q,Q;2k)\left[\mathfrak{G}(Q_+) - \mathfrak{G}(Q_-)\right]\left\{\varphi(Q,p;2k) - \varphi(q,p;2k)\right\}\end{aligned} \tag{B.279}$$

admits an invariant for each conserved quantity[23]

$$\begin{aligned}\left[I \circ \mathfrak{e}_n\right](q,p;k) &= 0,\\ \left[I \circ \mathfrak{e}_j\right](q,p;i\Omega) &= 0,\\ \left[I \circ \mathfrak{e}_\varepsilon\right](q,p;i0) &= 0,\end{aligned} \tag{B.280}$$

with the zero modes

$$\begin{aligned}&\mathfrak{e}_n(q,p) = 1, \quad \text{(Local number conservation)}\\ &\mathfrak{e}_j(q,p) = \vec{q} \otimes \vec{p}, \quad \text{(Global momentum conservation)}\\ &\mathfrak{e}_\varepsilon(q,p) = i\omega_q i\omega_p; \quad \text{(Global energy conservation)}\end{aligned} \tag{B.281}$$

the vanishing eigenvalues impart upon each of the susceptibilities

$$\begin{aligned}\Pi_{nn}(2k) &= \int_p \phi_n(p;2k),\\ \Pi^{il}_{jj}(2k) &= \int_p \frac{p^i}{m}\phi^l_j(p;2k),\\ \Pi_{\varepsilon\varepsilon}(2k) &= \frac{1}{2}\int_p \left\{i\omega_p + \xi_p + \varepsilon_k\right\}\phi_\varepsilon(p;2k),\end{aligned} \tag{B.282}$$

where we have introduced the relaxation functions

$$\begin{aligned}\phi_n(p;2k) &= -\int_q L(q,p;2k),\\ \vec{\phi}_j(p;2k) &= -\int_q \frac{\vec{q}}{m}L(q,p;2k),\\ \phi_\varepsilon(p;2k) &= \frac{-1}{2}\int_q \left\{i\omega_q + \xi_q + \varepsilon_k\right\}L(q,p;2k),\end{aligned} \tag{B.283}$$

[22] We have already used Equation B.274a in arriving at Equation B.279.

[23] The identity associated with $\mathfrak{e}_\varepsilon$ requires zero external frequency or else the initial and final states of a collision will not share the same energy.

a divergence on approaching the point ($\vec{k} = 0, \Omega \to 0$). In fact, Equation B.273 leads to[24]

$$\begin{aligned}
&-2i\Omega\phi_n(p;2k) + 2\vec{k}\cdot\vec{\phi}_j(p;2k) = \left[\mathfrak{G}(p_+) - \mathfrak{G}(p_-)\right],\\
&-2i\Omega\vec{\phi}_j(p;2i\Omega) = \frac{\vec{p}}{m}\left[\mathfrak{G}(p_+) - \mathfrak{G}(p_-)\right],\\
&-2i\Omega\phi_\varepsilon(p;2i\Omega) = i\omega_p\left[\mathfrak{G}(p_+) - \mathfrak{G}(p_-)\right] + \frac{i\Omega}{2}\left[\mathfrak{G}(p_+) + \mathfrak{G}(p_-)\right].
\end{aligned} \tag{B.284}$$

B.7.1 Screened Coulomb. Since long range Coulomb forces strongly encourage the conduction electrons to cooperatively screen charge fluctuations, spatiotemporal variations of the local displacement field are rectified on a time scale that is significantly shorter than is characteristic of the slowly relaxing dissipative modes that render the transport coefficients finite, one expects that the jellium model[25] is dependable in its ability to interpret experiments that target the electron-lattice coupling. Nevertheless, we sometimes implicitly include the electromagnetic interactions between electrons because the associated collision operator provides a reference about which the nonconservative effects can be treated perturbatively; more precisely, we account for the instantaneous Coulomb potential

$$S_{ee} = \frac{1}{2}\int dx dy n(x)\mathcal{V}_0(x-y)n(y) \tag{B.285}$$

[24]Notice that integrating Equation B.284 on p yields the continuity equation.

[25]That is to say, one takes the electron liquid to share space with a uniformly charged positive background such that the system is neutral in addition to using the notion of quasi-particles to parameterize the low-lying excitations, each of which behaves like a nearly free electron.

in an approximation that takes the effective action[26]

$$\begin{aligned} S_{\text{eff}}[\mathfrak{G}] &= \log \mathcal{Z}[\mathfrak{G}] \\ &= \operatorname{Tr}\log\{-\mathfrak{G}^{-1}\} + \operatorname{Tr}\{\mathfrak{G}\cdot\Sigma[\mathfrak{G}]\} + \Phi[\mathfrak{G}] \end{aligned} \tag{B.288}$$

to be defined by the Luttinger-Ward functional Φ of Figure B.1; Φ is the sum of all skeleton diagrams that are closed, completely connected, and two-particle irreducible. To define the electronic self-energy, one can extend the exact relation

$$\Sigma = \frac{-\delta\Phi}{\delta\mathfrak{G}} \tag{B.289}$$

to the truncated theory, which yields

$$\Sigma[1;1'] = \delta(1-1')\int \mathcal{V}_0(1-\bar{2})\mathcal{G}[\bar{2},\bar{2}^+] - \mathcal{V}_s[1;1']\mathcal{G}[1;1'], \tag{B.290}$$

where screened potential [see Figure B.2]

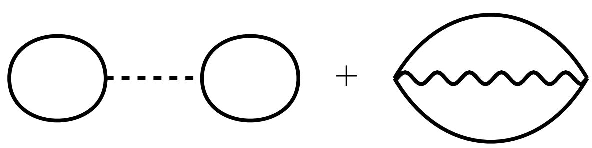

Figure B.1. Diagrammatic representation of the Luttinger-Ward functional Φ in the screened potential approximation.

$$\mathcal{V}_s[1;1'] = \mathcal{V}_0(1-1') - \int \mathcal{V}_0(1-\bar{2})\Pi_0[\bar{2};\bar{2}']\mathcal{V}_s[\bar{2}';1'] \tag{B.291}$$

exhibits collective modes due to the zero wavenumber divergence of

$$\mathcal{V}_0(q) = \frac{4\pi e^2}{q^2}. \tag{B.292}$$

[26]As the presentation of this expression is convention dependent, we are compelled to define our functional determinant

$$\operatorname{Det}[-\mathfrak{G}^{-1}] = \int d\bar{\psi}\, d\psi\, e^{\bar{\psi}\cdot\mathfrak{G}^{-1}\cdot\psi} \tag{B.286}$$

in obvious shorthand. The point is that one has a choice of sign in the measure; either way

$$\mathfrak{G} = \frac{-\delta S_{\text{eff}}}{\delta \mathfrak{U}}. \tag{B.287}$$

Here, the Lindhard bubble

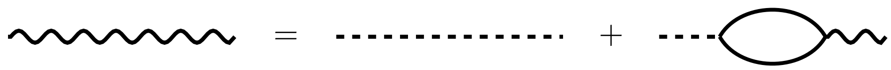

Figure B.2. The renormalized Coulomb propagator of the screened potential approximation.

$$\Pi_0[2;2'] = \mathcal{G}[2;2']\mathcal{G}[2';2] \tag{B.293}$$

leads to [24, 39]

$$\frac{-\delta\mathcal{V}_s[1;1']}{\delta\mathcal{G}[2';2]} = \{\mathcal{V}_s[1;2']\mathcal{V}_s[2;1'] + \mathcal{V}_s[1;2]\mathcal{V}_s[2';1']\}\mathcal{G}[2;2'], \tag{B.294}$$

which implies [see Figure B.3]

$$\begin{aligned}\tilde{\Lambda}[1,2;2',1'] &= \delta(1-1')\mathcal{V}_0(1'-2')\delta(2'-2) - \delta(1-2')\mathcal{V}_s[1;1']\delta(2-1') \\ &\quad + \mathcal{G}[1;1']\{\mathcal{V}_s[1;2']\mathcal{V}_s[2;1'] + \mathcal{V}_s[1;2]\mathcal{V}_s[2';1']\}\mathcal{G}[2;2'].\end{aligned} \tag{B.295}$$

It follows that the precursor equation

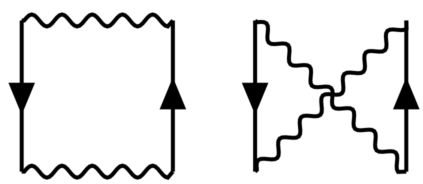

Figure B.3. The Aslamazov-Larkin diagrams that result from the dependence of $\mathcal{V}_s$ on $\mathcal{G}$.

$$\begin{aligned}&L[1,2;2',1'] \\ &= -\mathcal{G}[1;2']\mathcal{G}[2;1'] + \int \mathcal{G}[1;\bar{1}]\tilde{\Lambda}[\bar{1},\bar{2};\bar{2}',\bar{1}']\mathcal{G}[\bar{1}',1']L[\bar{2}',2;\bar{2},2']\end{aligned} \tag{B.296}$$

involves a conserving kernel. Evidently, Coulomb scattering does not contribute to the resistivity directly [12]; in order for the conductivity to be finite there must be a sink that siphons electron momentum into the pool of non-charge carrying degrees of freedom.

B.8 Diffuson

Here we obtain a result of Vollhardt and Wölfle [76] on the zero temperature properties of free electrons in the presence of a random (one-body, static) potential field $u(\vec{x})$ that models an imperfect lattice.

Since physical reasoning [77] suggests that large specimens with a truly random impurity distribution ought to obey a "self-averaging" principle whenever the electrons are not so encumbered that they are depraved of the requisite mobility for sampling many defect patterns over a time period commensurate with the measurement interval, it is plausible that the Hamiltonian

$$H = \int d\vec{x}\, \psi^\dagger(\vec{x}) \left[\frac{-\Delta}{2m} - \epsilon_F + u(\vec{x}) \right] \psi(\vec{x}) \tag{B.297}$$

will provide a valid course grained description in the absence of strong localization effects [78]; if the positioning of scattering sources is uncorrelated, then their statistics are realized by a Gaussian ensemble[27]

$$\left\{ u(\vec{k}) \right\}_{\text{dis}} = 0, \quad \left\{ u(\vec{k}) u(\vec{k}') \right\}_{\text{dis}} = u_0 \delta(\vec{k} + \vec{k}'), \tag{B.298}$$

where

$$u_0 = \frac{1}{2\pi N_F \tau}, \tag{B.299}$$

is the random potential strength, N_F is the density of states on the Fermi surface, and the phenomenological lifetime τ is representative of damping in the dirty medium.

Even though a given $u(\vec{x})$ destroys spatial isotropy, an effective translation invariance is restored upon marginalizing over disorder configurations; the

[27] The $k = 0$ component of u can be absorbed into the definition of the Fermi level ϵ_F [79].

Green function

$$\mathcal{G}(x - x') \equiv \mathcal{G}[x; x'] = -\{\psi(x)\psi^\dagger(x')\}_{\text{dis}} \tag{B.300}$$

then admits a single variable Fourier transform. With this simplification, the Kubo function reads

$$\Pi(2k) = \frac{T}{V}\sum_{p} \gamma[p_+, p_-; 2k]\,\mathcal{G}(p_+)\mathcal{G}(p_-), \tag{B.301}$$

where, in the conserving ladder approximation,

$$\begin{aligned} &\gamma[p_+, p_-; 2k] \\ &= 1 + u_0 \frac{1}{V}\sum_{\vec{q}} \gamma[p_+ + \vec{q}, p_- + \vec{q}; 2k]\,\mathcal{G}(p_+ + \vec{q})\mathcal{G}(p_- + \vec{q}), \end{aligned} \tag{B.302}$$

which shows that γ does not depend on $\vec{p}$, as recognized by Belitz and Kirkpatrick [56]. Therefore

$$\gamma[i\omega + i\Omega, i\omega - i\Omega, 2k] = [1 - u_0 I(i\omega + i\Omega, i\omega - i\Omega; 2\vec{k})]^{-1}, \tag{B.303}$$

where

$$I(i\omega + i\Omega, i\omega - i\Omega; 2k) = \frac{1}{V}\sum_{\vec{q}} \mathcal{G}(\vec{q} + \vec{k}, i\omega + i\Omega)\mathcal{G}(\vec{q} - \vec{k}, i\omega - i\Omega). \tag{B.304}$$

B.8.0.1 Self-energy. From the Ward-Takahashi identity of Equation B.241, we have

$$-\Sigma''(\epsilon) = -u_0 \frac{1}{V}\sum_{\vec{p}} \mathcal{G}''(\vec{p}, \epsilon) \approx \pi N_0(\epsilon) u_0 \tag{B.305}$$

when the disorder is weak ($1 \ll \epsilon_F \tau$). Causality then requires

$$-\Sigma'(\omega) = -\Sigma(0) \approx \text{P.V.}\int d\epsilon \frac{u_0 N_0(\epsilon)}{\epsilon}, \tag{B.306}$$

which vanishes in the limit $\epsilon_F \to \infty$; we'll ignore this term. Here,

$$N_0(\epsilon) = \frac{1}{V}\sum_{\vec{p}} \delta[\epsilon - \xi_p] = \theta[\epsilon > -\epsilon_F]\frac{m k_0(\epsilon)}{2\pi^2} \tag{B.307}$$

is the density of states displaced by ϵ from the Fermi surface, and

$$k_0(\epsilon) = k_F\sqrt{1+\epsilon/\epsilon_F} \tag{B.308}$$

is a thermally modified Fermi wavenumber. As usual,

$$\epsilon_F = k_F^2/2m, \quad \xi_{\vec{p}} = p^2/2m - \epsilon_F. \tag{B.309}$$

Hence, summing an indefinite succession of independent scattering events leads to a renormalized quasi-electron

$$\mathcal{G}_0^{-1}(\vec{p}, i\omega) = i\omega - \xi_{\vec{p}} + i\gamma \operatorname{sgn} \operatorname{Im}(i\omega) \tag{B.310}$$

that suffers an elastic rate $\gamma = 1/2\tau$ and has spectrum

$$\mathcal{G}_0''(\vec{p}, \epsilon) = -\pi\delta_\gamma\left[\epsilon - \xi_{\vec{p}}\right] \tag{B.311}$$

with Lorentzian shape

$$\delta_\gamma\left[\epsilon - \xi_{\vec{p}}\right] = \frac{\gamma/\pi}{\left[\epsilon - \xi_{\vec{p}}\right]^2 + \gamma^2}, \tag{B.312}$$

which, in the clean (distribution) limit $\gamma \to 0$, is a Dirac delta that requires on shell propagation. While the theory is no longer that of true particles, as the energy-momentum resonance is now lifetime broadened, meaning excitations will spontaneously decay [80], we expect it, at the very least, to describe ballistic transport as exhibited in relatively clean conditions.[28]

[28] Strictly speaking, the full Ward-Takahashi identity of Equation B.241, which constitutes the necessary and sufficient condition for particle number conservation fails beyond a critical value of the disorder strength at which point one can no longer ignore the fact that each Hilbert space corresponding to a particular disorder realization is rife with localized states that the configurationally averaged theory is unaware of [81].

B.8.0.2 Vertex. Now then, Equation B.301 is equivalent to

$$
\begin{aligned}
\Pi(2k) &= T\sum_{i\omega} F(i\omega+i\Omega, i\omega-i\Omega; 2\vec{k}) \\
&= -\int \frac{d\epsilon}{2\pi i} f(\epsilon)\Big\{F(\epsilon+2i\Omega, \epsilon+i0; 2\vec{k}) - F(\epsilon+2i\Omega, \epsilon-i0; 2\vec{k}) \\
&\qquad + F(\epsilon+i0, \epsilon-2i\Omega; 2\vec{k}) - F(\epsilon-i0, \epsilon-2i\Omega; 2\vec{k})\Big\},
\end{aligned} \tag{B.313}
$$

where

$$
F(i\omega+i\Omega, i\omega-i\Omega; 2\vec{k}) = \frac{I(i\omega+i\Omega, i\omega-i\Omega; 2\vec{k})}{1-u_0 I(i\omega+i\Omega, i\omega-i\Omega; 2\vec{k})}. \tag{B.314}
$$

For the purpose of hunting singularities on the physical sheet where $\operatorname{sgn}\operatorname{Im}(i\Omega) > 0$, it is convenient to decompose the density response function into a hydrodynamic piece

$$
\begin{aligned}
&\phi(2\vec{k}, 2i\Omega) \\
&= \int \frac{d\epsilon}{2\pi i} f(\epsilon)\Big\{F(\epsilon+2i\Omega, \epsilon-i0; 2\vec{k}) - F(\epsilon+i0, \epsilon-2i\Omega; 2\vec{k})\Big\},
\end{aligned} \tag{B.315}
$$

and the remaining part

$$
\begin{aligned}
&\chi(2\vec{k}, 2i\Omega) \\
&= \int \frac{d\epsilon}{2\pi i} f(\epsilon)\Big\{F(\epsilon-i0, \epsilon-2i\Omega; 2\vec{k}) - F(\epsilon+2i\Omega, \epsilon+i0; 2\vec{k})\Big\}.
\end{aligned} \tag{B.316}
$$

In order to determine

$$
\begin{aligned}
\phi_R(2\vec{k}, 2\Omega) &= \int \frac{d\epsilon}{2\pi i}\big[f(\epsilon-\Omega) - f(\epsilon+i\Omega)\big]F(\epsilon+\Omega+i0, \epsilon-\Omega-i0; 2\vec{k}) \\
&\approx \frac{2\Omega}{2\pi i} F(+\Omega+i0, -\Omega-i0; 2\vec{k}),
\end{aligned} \tag{B.317}
$$

where $f(\epsilon-\Omega) - f(\epsilon+\Omega) \approx 2\Omega\delta(\epsilon)$ for $\Omega \to 0$ at $T = 0$, we first evaluate[29]

$$
\begin{aligned}
&I(i\omega+i\Omega, i\omega-i\Omega; 2\vec{k}) \\
&= \frac{im^2}{4\pi k}\Big\{\log\big[2k + Q(i\omega+i\Omega) + Q(i\omega-i\Omega)\big] \\
&\qquad - \log\big[-2k + Q(i\omega+i\Omega) + Q(i\omega-i\Omega)\big]\Big\},
\end{aligned} \tag{B.318}
$$

[29] When the meaning is clear from context, we denote $k = |\vec{k}|$.

with

$$\begin{aligned} &Q(i\omega \pm i\Omega) \\ &= \operatorname{sgn} \operatorname{Im}(i\omega \pm i\Omega) \times k_F \sqrt{1 + \big[i\omega \pm i\Omega + i\gamma \operatorname{sgn} \operatorname{Im}(i\omega \pm i\Omega)\big]/\epsilon_F}; \end{aligned} \tag{B.319}$$

the principal branch is chosen for both the logarithm and the square root. Note that

$$Q(\epsilon + \Omega + i0) + Q(\epsilon - \Omega - i0) \approx \frac{1}{v_F}\big[2\Omega + 2i\gamma\big], \tag{B.320a}$$

$$Q(\epsilon + \Omega - i0) + Q(\epsilon - \Omega - i0) \approx \frac{-1}{v_F}\big[4\epsilon_F + 2\epsilon - 2i\gamma\big] \tag{B.320b}$$

when $\omega, \Omega, \gamma \ll \epsilon_F$. Thus, writing

$$\begin{aligned} &\chi_R(2\vec{k}, 2\Omega) \\ &= \int \frac{d\epsilon}{2\pi i}\big[f(\epsilon + \Omega) - f(\epsilon - \Omega)\big] \operatorname{Re} F(\epsilon + \Omega - i0, \epsilon - \Omega - i0; 2\vec{k}) \\ &\quad - \int \frac{d\epsilon}{2\pi}\big[f(\epsilon + \Omega) + f(\epsilon - \Omega)\big] \operatorname{Im} F(\epsilon + \Omega - i0, \epsilon - \Omega - i0; 2\vec{k}), \end{aligned} \tag{B.321}$$

reveals that

$$\begin{aligned} \chi_R(2\vec{k}, 2\Omega) &\approx -\int \frac{d\epsilon}{\pi} f(\epsilon) \operatorname{Im} F(\epsilon + i0, \epsilon + i0; \vec{0}) + \mathcal{O}(k^2, \Omega^2) \\ &\approx -\int_{-\infty}^{0} \frac{d\epsilon}{\pi} \operatorname{Im} \frac{1}{V} \sum_{\vec{q}} \mathcal{G}_0^R(\vec{q}, \epsilon) \mathcal{G}_0^R(\vec{q}, \epsilon) \\ &\approx \int_{-\infty}^{0} \frac{d\epsilon}{\pi} \operatorname{Im} \frac{1}{V} \sum_{\vec{q}} \partial_\epsilon G_0^R(\vec{q}, \epsilon) \\ &= \frac{1}{V} \sum_{\vec{q}} \frac{1}{\pi} G_0''(\vec{q}, 0) \\ &= -N_F \end{aligned} \tag{B.322}$$

is regular; χ_R determines the electronic compressibility. So

$$F^{-1}(\epsilon + \Omega + i0, \epsilon - \Omega - i0; 2\vec{k}) \approx \frac{1}{2\pi N_F}\left\{ -2i\Omega + \frac{D(2k)^2}{1 - i\frac{2\Omega}{2\gamma}} + \mathcal{O}(k^4) \right\}, \tag{B.323}$$

implies

$$\phi_R(\vec{k}, \Omega) \approx N_F \frac{1 - i\Omega\tau}{1 - i\Omega\tau + iDk^2/\Omega}, \tag{B.324}$$

which produces

$$\Pi_R(\vec{k},\Omega) = N_F \frac{-iDk^2/\Omega}{1 - i\Omega\tau + iDk^2/\Omega}, \tag{B.325}$$

where

$$D = \frac{v_F^2}{6\gamma} \tag{B.326}$$

is the semi-classical diffusion constant. Notice that Equation B.325 maintains

$$0 = \Pi_R(\vec{0}, i\Omega), \tag{B.327}$$

as required by particle conservation. When $\Omega\tau \ll 1$, the response has a diffusive resonance, i.e.

$$\Pi_R(\vec{k},\Omega) \approx N_F \frac{-iDk^2}{\Omega + iDk^2}; \tag{B.328}$$

this essential change to the spectrum is due to the fact that the relaxation of a long wavelength density fluctuation requires that particles wishing to traverse the region of imbalance must overcome their tendency to undergo a random walk when amidst an array of static scattering sources.

B.8.1 Dynamical conductivity. Finally, from [see Equation B.69 and Equation B.57]

$$\sigma(\vec{k},\Omega) = e^2 \frac{i\Omega}{k^2} \Pi_R(\vec{k},\Omega) \tag{B.329}$$

we find

$$\sigma(\vec{k},\Omega) = \frac{e^2 N_F D}{1 - i\Omega\tau + iDk^2/\Omega}, \tag{B.330}$$

which is the standard weak-coupling result [71]. In the hydrodynamic limit,

$$\sigma(\vec{k},\Omega) \to \frac{ne^2}{2m\gamma} \frac{1}{1 - iDk^2/\Omega}, \quad \Omega \to 0 \text{ with } Dk^2/\Omega \text{ fixed} \tag{B.331}$$

applies to the stage of relaxation wherein global equilibrium is approached from a state of local equilibrium through the transport of conserved quantities, which occurs over a characteristic time scale that is much larger than the typical time between interactions [38]; with the kinetics coursened to such a degree, the equilibriating collisions are effectively incessant.

B.8.2 Spectral method. Our approach is quite useful whenever one is interested in susceptibilities associated with conserved (or nearly conserved) quantities; that its applicability is not limited to statics allows for an economical derivation of the diffusion coefficient. To this end, consider the density-density function[30]

$$\Pi_R(k) \approx -N_F\{1 + i\Omega\langle 1, \mathcal{L}^{-1} \circ 1\rangle^k\}, \tag{B.334}$$

where the Liouvillian

$$i\mathcal{L} = \frac{d}{dt} - iC_0 \tag{B.335}$$

contains the streaming term d/dt in addition to a collision operator C_0, which admits[31]

$$C_0 \circ \mathfrak{e}_n = 0, \quad \mathfrak{e}_n(q;k) = 1 \tag{B.336}$$

on account of number conservation. It follows that there exists a vector u_n obeying

$$\mathcal{L} \circ u_n = \nu_n u_n, \tag{B.337}$$

[30]The inner product

$$\langle \psi, \varphi \rangle = \int \frac{d\epsilon}{\pi} w(\epsilon) \frac{1}{N_F V} \sum_{\vec{q}} \left[-G''(q) \right] \psi(q) \varphi(q) \tag{B.332}$$

facilitates self-adjointness

$$\langle \psi, \mathcal{L} \circ \varphi \rangle = \langle \mathcal{L} \circ \psi, \varphi \rangle. \tag{B.333}$$

[31]For simplicity of presentation, C_0 is taken at $k = 0$; one can check that the leading small k behavior is still faithfully captured.

with

$$\nu_n(k) = \mathcal{O}(k), \quad u_n(q;k) = \mathfrak{e}_n(q;k) + \mathcal{O}(k). \tag{B.338}$$

Thus, we have

$$\left\langle 1, \mathcal{L}^{-1} \circ 1 \right\rangle \approx \nu_n^{-1}, \tag{B.339}$$

which can be determined by successive approximation. To first order,

$$-C_0 \circ u_n^{(1)} - i\frac{d}{dt} \circ \mathfrak{e}_n = \nu_n^{(1)} \mathfrak{e}_n \tag{B.340}$$

implies both

$$\nu_n^{(1)}(k) = i\Omega \tag{B.341}$$

and

$$u_n^{(1)} = -C_0^{-1} \circ \left\{ \nu_n^{(1)} \mathfrak{e}_n + i\frac{d}{dt} \circ \mathfrak{e}_n \right\}, \tag{B.342}$$

which yields

$$\nu_n^{(2)} \left\langle \mathfrak{e}_n, \mathfrak{e}_n \right\rangle = -\left\langle \mathfrak{e}_p^{(0)}, C_0^{-1} \circ \mathfrak{e}_p^{(0)} \right\rangle \tag{B.343}$$

from

$$C_0 \circ u_n^{(2)} - i\frac{d}{dt} \circ u_n^{(1)} = \nu_n^{(2)} \mathfrak{e}_n + \nu_n^{(1)} u_n^{(1)}; \tag{B.344}$$

here,

$$\left[\frac{d}{dt} \circ \mathfrak{e}_n \right](q;k) = -\Omega \mathfrak{e}_n(q;k) + \mathfrak{e}_p^{(0)}(q;k), \tag{B.345}$$

with

$$\mathfrak{e}_p^{(0)}(q;k) = \frac{\vec{k} \cdot \vec{q}}{m} \tag{B.346}$$

the bare momentum mode of C_0. Therefore,

$$\nu_n^{(2)} \approx -\left\langle \mathfrak{e}_p^{(0)}, \mathfrak{e}_p^{(0)} \right\rangle \lambda_p^{-1}, \tag{B.347}$$

where[32]

$$C_0 \circ \mathfrak{e}_p = \lambda_p \mathfrak{e}_p. \tag{B.348}$$

As a result,

$$\Pi_R(k) \approx -N_F \frac{iDk^2}{\Omega + iDk^2}, \tag{B.349}$$

where

$$D \approx \frac{v_F^2}{3\lambda_p(k=0)}; \tag{B.350}$$

if we take C_0 to contain only the simplest impurity vertex, then

$$\lambda_p(k=0) = 2\gamma \tag{B.351}$$

implies

$$D = \frac{v_F^2}{6\gamma}, \tag{B.352}$$

which agrees with Equation B.326.

B.8.2.1 Wiedemann–Franz law. An analogous sequence of manipulations yields for the energy-energy function

$$\Pi^R_{\varepsilon\varepsilon}(k) \approx -TC_V \frac{iD_\varepsilon k^2}{\Omega + iD_\varepsilon k^2}, \tag{B.353}$$

where the electronic specific heat $C_V = \pi^2 N_F T/3$, and

$$D_\varepsilon k^2 \approx \left\langle \mathfrak{e}_p^{(0)} \mathfrak{e}_\varepsilon, C_0^{-1} \circ \mathfrak{e}_p^{(0)} \mathfrak{e}_\varepsilon \right\rangle / \left\langle \mathfrak{e}_\varepsilon, \mathfrak{e}_\varepsilon \right\rangle, \tag{B.354}$$

which does not involve the action of C_0^{-1} on any of the hydrodynamic modes; however, since

$$K_0 \circ \mathfrak{e}_p^{(0)} \mathfrak{e}_\varepsilon = 0, \tag{B.355}$$

[32]In the limit of global conservation of electronic momentum, $\mathfrak{e}_p \to \mathfrak{e}_p^{(0)}$ and $\lambda_p \to 0$.

thanks to the spherically symmetric scattering centers, it follows that the heat conductivity

$$\sigma_h = \lim_{\Omega \to 0} \lim_{\vec{k} \to 0} \frac{i\Omega}{k^2 T} \Pi_{\varepsilon\varepsilon}(k) = C_V D; \tag{B.356}$$

here

$$\mathfrak{e}_\varepsilon(q;k) = \epsilon \tag{B.357}$$

and

$$C_0(q,p) = 2\gamma\delta(p-q) - K_0(q,p). \tag{B.358}$$

Therefore,

$$\frac{e^2\sigma_h}{\sigma} = \frac{\pi^2 T}{3} \tag{B.359}$$

is independent of γ. Because the contents of this subsubsection are well known, there is a wealth of relevant literature; we referenced the papers [82–86], as accomplished by Castellani et al. and Langer.

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