EFFECTIVE SOFT-MODE THEORY OF STRONGLY INTERACTING FERMIONS IN DIRAC SEMIMETALS

by

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Title: Effective Soft-Mode Theory of Strongly Interacting Fermions in Dirac Semimetals

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DISSERTATION ABSTRACT

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We present an effective field theory for interacting electrons in clean semimetals (both three dimensional Dirac semimetals and graphene) in terms of their soft or massless bosonic degrees of freedom. We show, by means of a Ward identity, that the intrinsic semimetal ground state breaks the $\mathrm{Sp}(4M)$ symmetry of the theory. In Fermi liquids this enables one to identify the massive, non-Goldstone modes of the theory and integrate them out. Due to the vanishing density of states in semimetals, unlike in Fermi liquids, both Goldstone and non-Goldstone modes are equally soft, and so all two-particle correlations need to be kept. The resulting theory is not perturbative with respect to the electron-electron interaction; rather, it is controlled by means of a systematic loop expansion and allows one to determine the exact asymptotic form of observables in the limits of small frequencies and/or wave vectors. Equivalently, it provides a mechanism of determining the long time-tail and long wavelength behavior of observables and excitations. As a representative application, we use the theory to compute the zero-bias anomaly for the density of states for both short and long-range interactions in two and three dimensions. We find that the leading non-analyticity in

semimetals with a long-ranged interaction appears at the same order in frequency as the one in Fermi liquids, since the effects of the vanishing density of states at the Fermi level are offset by the breakdown of screening. Consequently, we are able to provide a logical scheme to determine the leading non-analytical behavior of observables in semimetals using knowledge of the corresponding non-analyticities in a Fermi liquid.

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For Mom and Dad,

your support and love was unyielding.

For Grandfather,

who showed that curiosity is a lifelong affliction.

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CHAPTER I

INTRODUCTION

There is no shortage of fascinating phases of matter to study, and this dissertation will be devoted to the understanding of just one class, Dirac semimetals in two and three dimensions. More specifically, we will theoretically investigate the nature of interacting electrons in Dirac semimetals.

Dirac semimetals are a class of solid state system with a storied development. The conditions for their existence were first laid out in 1937 by Conyers Herring, who proved that electronic band structures in crystals can cross each other linearly at positions of high symmetry in momentum space [1]. Their name is derived from the fact that the band crossings can be viewed as the linear version of a regular zero-gap semiconductor, also known as a semimetal, and the energy-momentum dispersion of electrons near the band crossings can be described by a Dirac Hamiltonian [2]. In 1947 Philip Wallace first theoretically derived the Dirac spectrum in the two dimensional material graphene [3]. In the 1970s, theoretical studies of the zero-gap semiconductor α-Sn showed that under suitable strain, electrons near the Fermi energy behave according to a three dimensional version of the Dirac Hamiltonian [4–6]. Graphene was first experimentally realized by Novosolev *et al.* in 2004, and thus two dimensional Dirac semimetals became a reality [7, 8]. The first realization of the three dimensional case was in Cd_3As_2 by Neupane *et al.* in 2014. While strained α-Sn was the first material theoretically predicted to be a Dirac semimetal, growth of the appropriately strained state was not achieved until 2017 by Xu *et al.* [9–11]

This dissertation will focus on deriving an effective quantum field theory to understand the nature of strongly interacting electrons in a clean Dirac semimetal.

By clean we mean that the Dirac semimetal crystal is devoid of impurities, and strongly interacting implies that the Coulomb interaction between electrons is not perturbatively weak, even after accounting for any renormalizations. The strength of the Coulomb interaction will be quantitatively discussed in Chapter II. The construction of the effective field theory will involve integrating out the electron (fermionic) degrees of freedom from the action of the quantum partition function in order to obtain an action in terms of soft, bosonic degrees of freedom. The analogous program in a Fermi liquid has proven phenomenally successful at determining the long time-tail and wavelength behavior of observables and correlation functions that couple to the soft bosonic modes [12–16]. Mathematically the long time-tail and and wavelength behaviors respectively manifest as algebraic decays of quantities in time or position. This is equivalent to a non-analytic dependence on vanishing frequency and momentum when the quantity is Fourier transformed to frequency-momentum space. The concept of long time-tail and wavelength phenomena, and non-analyticities will be elaborated on in Chapter III. The immediate application of the effective field theory in the case of the Dirac semimetal is determining non-analytical properties of observables, and comparing the results to their Fermi liquid counterparts.

In Chapter II, using α-Sn as an example, we give an in depth introduction to Dirac semimetals, and their time reversal symmetry broken state known as Weyl semimetals. In Chapter III we briefly discuss the Goldstone Theorem, the nature of soft modes in a Fermi gas, and the impact of soft modes on observables in condensed matter systems. Chapters IV and V are the novel parts of this dissertation. In Chapter IV we derive the effective field theory in terms of soft bosonic degrees of freedom for interacting electrons in clean Dirac semimetal systems, and determine the nature of the soft bosonic modes. In Chapter V we employ the field theory to

calculate non-analytic corrections to observables, ultimately comparing the results to those for Fermi liquids.

CHAPTER II

INTRODUCTION TO DIRAC AND WEYL SEMIMETALS

Basics of $\boldsymbol{k}\cdot\boldsymbol{p}$ Theory

The most crucial ingredient for investigating any condensed matter system is the Hamiltonian. In this chapter we will discuss the $\boldsymbol{k}\cdot\boldsymbol{p}$ Kane-Hamiltonian for diamond lattice crystals[1] [17], and explicitly show how it gives rise to Dirac and Weyl semimetal Hamiltonians under different conditions [5, 6]. $\boldsymbol{k}\cdot\boldsymbol{p}$ Theory is a deep subject and we will only summarize some of its key ideas here, referring to the literature for further details [18–20].

Recall the Pauli equation for the two component, spinful electron wavefunction $\psi(\boldsymbol{r})$, in a crystal with Hamiltonian $\hat{H}$, is given to order $\mathcal{O}(v/c)^2$ by:

$$\hat{H}\psi(\boldsymbol{r}) \equiv \left(\frac{\hat{\boldsymbol{p}}^2}{2m_e} + V_0(\boldsymbol{r}) - \hat{H}_R\right)\psi(\boldsymbol{r}) = E\psi(\boldsymbol{r}) \ , \tag{2.1a}$$

$$\hat{H}_R \equiv \frac{e\hbar\boldsymbol{\sigma}\cdot\hat{\boldsymbol{p}}\times\boldsymbol{\mathcal{E}}}{4m_e^2c^2} + \frac{e\hbar^2}{8m_e^2c^2}\boldsymbol{\nabla}\cdot\boldsymbol{\mathcal{E}} + \frac{p^4}{8m_e^3c^2} \ , \tag{2.1b}$$

with $\hat{\boldsymbol{p}}$ the momentum operator, m_e the electron mass, $v \equiv \langle\hat{\boldsymbol{p}}\rangle/m_e$, and E the eigenenergy. $V_0(\boldsymbol{r})$ is the static, equilibrium potential of the crystal lattice and has the same periodicity as the crystal, that is $V_0(\boldsymbol{r}) = V_0(\boldsymbol{r}+\boldsymbol{R})$ for any Bravais lattice vector $\boldsymbol{R}$. $\boldsymbol{\mathcal{E}} = (1/e)\boldsymbol{\nabla}V_0(\boldsymbol{r})$ is the electric field of the crystal. The operator $\hat{H}_R$ contains the relativistic corrections to the Schrödinger Equation that appear from taking the non-relativistic expansion of the Dirac equation. The first term in Equation (2.1b) is known as the Pauli spin orbit coupling term, the second is known as the Darwin term,

[1] A diamond lattice is formed by two inter-penetrating face centered cubic (fcc) lattices of the same element.

and the third is a correction to the non-relativistic kinetic energy. We will ignore the last term in $\hat{H}_R$ for the current discussion, since it will principally serve to complicate expressions. Bloch's theorem states the solutions to Equation (2.1a) with $\hat{H}_R$ set to zero can be written as

$$\psi^{\sigma}_{n,\boldsymbol{k}}(\boldsymbol{r}) = e^{i\boldsymbol{k}\cdot\boldsymbol{r}} u^{\sigma}_{n,\boldsymbol{k}}(\boldsymbol{r}), \tag{2.2}$$

where $n \in \mathbb{N}$ is the discrete band index, and $u^{\sigma}_{n,\boldsymbol{k}}(\boldsymbol{r})$ is the Bloch function with the same periodicity as the crystal lattice. We have also explicitly included the spin index $\sigma = \uparrow, \downarrow$. $\boldsymbol{k}$ is the crystal momentum wavevector, assuming periodic boundary conditions on a crystal with side lengths L, it takes values $\boldsymbol{k} = \frac{2\pi}{L}(N_x, N_y, N_z)$ for $N_{i=x,y,z} = 0, \pm 1, \pm 2 \ldots \pm L/a$. This set of $\boldsymbol{k}$ values defines the Brillouin Zone (BZ). It is important to understand the two components of the solution $\psi^{\sigma}_{n,\boldsymbol{k}}(\boldsymbol{r})$: $e^{i\boldsymbol{k}\cdot\boldsymbol{r}}$ can be viewed as a free-electron type envelope wavefunction extended across the entirety of the crystal, while $u^{\sigma}_{n,\boldsymbol{k}}(\boldsymbol{r})$ contains the details of an electron's wavefunction within a single unit cell of the crystal and thus carries important local information[2]. The Bloch functions are unit cell (u.c.) normalized, and form an orthonormal basis of integrable functions we can use as a basis to represent the Hamiltonian operator:

$$\int_{\text{u.c.}} d\boldsymbol{r}\ \bar{u}^{\sigma}_{n,\boldsymbol{k}}(\boldsymbol{r}) u^{\sigma'}_{n',\boldsymbol{k}}(\boldsymbol{r}) = \delta^{\sigma\sigma'}_{n,n'}\ . \tag{2.3}$$

[2]The 'nearly free' nature of electrons in a metal can be ascribed to the fact that $u^{\sigma}_{n,\boldsymbol{k}}(\boldsymbol{r}) \approx const$ in the conduction band of a metal, and can safely be ignored [21].

Inserting Equation (2.2) into (2.1) we carry out the action of $\hat{\boldsymbol{p}}$ on the plane wave $e^{i\boldsymbol{k}\cdot\boldsymbol{r}}$ to obtain the equation:

$$\left(\frac{\hat{\boldsymbol{p}}^2}{2m_e} + V_0(\boldsymbol{r}) + \frac{\hbar^2\boldsymbol{k}^2}{2m_e} + \frac{\hbar\boldsymbol{k}\cdot\boldsymbol{\pi}}{m_e} + \frac{e\hbar\hat{\boldsymbol{p}}\cdot\boldsymbol{\sigma}\times\boldsymbol{\mathcal{E}}}{4m_e^2c^2} - \frac{e\hbar^2\boldsymbol{\nabla}\cdot\boldsymbol{\mathcal{E}}}{8m_e^2c^2}\right) u_{n,\boldsymbol{k}}(\boldsymbol{r}) = E_n(\boldsymbol{k})u_{n,\boldsymbol{k}}(\boldsymbol{r}) \ , \tag{2.4a}$$

$$\boldsymbol{\pi} := \hat{\boldsymbol{p}} + \frac{e\hbar\boldsymbol{\sigma}\times\boldsymbol{\mathcal{E}}}{4m_e^2c^2} \ . \tag{2.4b}$$

In general we will want to know the behavior of $E_n(\boldsymbol{k})$ around some momentum $\boldsymbol{k}_0$ where $E_n(\boldsymbol{k}_0)$ is an extremum. Usually this is at the BZ center known as the Γ point where $\boldsymbol{k}_0 = 0$, or a corner of the BZ where $(\boldsymbol{k}_0)_i = \pm\pi/a$ for at least one of $i = x, y, z$. We can use Equation (2.3) to expand $u_{n,\boldsymbol{k}}(\boldsymbol{r})$ a basis of functions defined at $\boldsymbol{k}_0$:

$$u_{n,\boldsymbol{k}}(\boldsymbol{r}) = \sum_{\nu'} c_{n\nu'}(\boldsymbol{k})u_{n,\boldsymbol{k}_0}(\boldsymbol{r}) \tag{2.5}$$

Inserting this into Equation (2.4), multiplying from the left by $u^*_{\nu,\boldsymbol{k}_0}(\boldsymbol{r})$ and integrating over real space, we obtain an algebraic eigenvalue problem for the dispersion $E_n(\boldsymbol{k})$ in terms of only $\boldsymbol{k}$ and material constants:

$$\sum_{\nu',\sigma'}\left[\left(E_{\nu'}(\boldsymbol{k}_0) + \frac{\hbar^2k^2}{2m_e}\right)\delta_{\nu\nu'} + \frac{\hbar}{m_e}\boldsymbol{k}\cdot\boldsymbol{P}^{\sigma,\sigma'}_{\nu\nu'} + \Delta^{\sigma\sigma'}_{\nu\nu'} - D^{\sigma\sigma'}_{\nu\nu'}\right] c^{\sigma'}_{n\nu'}(\boldsymbol{k}) \quad = \quad E_n(\boldsymbol{k})c^{\sigma}_{n\nu}(\boldsymbol{k}) \ , \tag{2.6}$$

where

$$\boldsymbol{P}^{\sigma,\sigma'}_{\nu\nu'} = \int d\boldsymbol{r}\ \bar{u}^{\sigma}_{\nu\boldsymbol{k}_0}(\boldsymbol{r})\boldsymbol{\pi}u^{\sigma'}_{\nu'\boldsymbol{k}_0}(\boldsymbol{r})\ , \tag{2.7a}$$

$$\Delta^{\sigma,\sigma'}_{\nu\nu'} = \int d\boldsymbol{r}\ \bar{u}^{\sigma}_{\nu\boldsymbol{k}_0}(\boldsymbol{r})\frac{e\hbar\hat{\boldsymbol{p}}\cdot\boldsymbol{\sigma}\times\boldsymbol{\mathcal{E}}}{4m_e^2c^2}u^{\sigma'}_{\nu'\boldsymbol{k}_0}(\boldsymbol{r})\ , \tag{2.7b}$$

$$D_{\nu\nu'} = \delta^{\sigma\sigma'}\int d\boldsymbol{r}\ \bar{u}^{\sigma}_{\nu\boldsymbol{k}_0}(\boldsymbol{r})\frac{e\hbar^2\boldsymbol{\nabla}\cdot\boldsymbol{\mathcal{E}}}{8m_e^2c^2}u^{\sigma}_{\nu'\boldsymbol{k}_0}(\boldsymbol{r})\ . \tag{2.7c}$$

We have now arrived at what looks like the momentum space Schrödinger equation for free electrons coupled by the momentum dependent $\boldsymbol{k}\cdot\boldsymbol{P}_{\nu\nu'}$ term. The spin orbit coupling and Darwin terms, respectively Δ and D, can provide constant couplings between bands, but also the terms principally responsible for band inversions, which we will discuss later. Equation (2.6) is an equation for an infinite matrix, but often one is only interested in the physics of a few bands, or just one in the case of a Fermi liquid. Suppose the bands we are interested in are given by $n \in S = \{n_1, \ldots, n_N\}$, then by means of the technique known as Löwdin perturbation theory, one can calculate a Hamiltonian that is block diagonal for the states $n \in S$ and $m \notin S$ [22]. Technically, this means one uses perturbation theory to account for the coupling between the n and m states, allowing one to focus solely on the $n \in S$ states, which now have renormalized couplings amongst themselves. We call the reduced matrix acting on the set of bands S the $\boldsymbol{k}\cdot\boldsymbol{p}$ Hamiltonian for S. In the case of a Fermi liquid, we are generally just interested in the conduction band c. Usually spin orbit coupling is negligible in such a situation, and we can set $\Delta = 0$, and $\boldsymbol{\pi} = \hat{\boldsymbol{p}}$ in Equation (2.7). To second order in perturbation theory the Hamiltonian describing electrons in the conduction band relative to the zone center $\boldsymbol{k} = 0$, is given by:

$$E_c(\boldsymbol{k}) = E_c(0) + \frac{\hbar^2k^2}{2m_c^*}, \tag{2.8}$$

where

$$\frac{m_e}{m_c^*} = 1 + \frac{2}{m_e} \sum_{\nu} \frac{|P_{c\nu}|^2}{E_c(0) - E_\nu(0)}. \tag{2.9}$$

There is no linear term because $E_c(\boldsymbol{k}_0)$ is an extremum, which implies that $P_{cc} = 0$. We have thus arrived at the familiar, simple kinetic Hamiltonian that is the basis of Fermi liquid theory. It is a good approximation to use when $E_c(0) - E_\nu(0)$ is large for all $\nu \neq c$.

When bands lie close to each other in energy, or are degenerate, it is necessary to incorporate multiple bands in the Löwdin perturbation theory for an accurate description of the $E_n(\boldsymbol{k})$ in a small neighborhood of $\boldsymbol{k}_0$. The determination of $\boldsymbol{k} \cdot \boldsymbol{p}$ Hamiltonians is made easier by exploiting the discrete symmetries of the underlying crystal. Symmetries dictate several properties of $u_{n,\boldsymbol{k}}(\boldsymbol{r})$, and specifically what elements $P_{nn'}$ can be non-zero. Exact forms of $\boldsymbol{k} \cdot \boldsymbol{p}$ Hamiltonians for any reduced set of bands can thus be constructed to arbitrary order in $\boldsymbol{k}$, and one then relies on perturbation theory or numerical methods to determine the values of $P_{nn'}$ and effective masses [23, 24].

One last important point is that the inclusion of spin orbit coupling in $\boldsymbol{k} \cdot \boldsymbol{p}$ theory generically renders spin a bad quantum number. When describing electrons at the Γ point, the Bloch functions can be chosen to be eigenstates of the total angular momentum operator J^2 and its projection J_z, similar to the case of the Hydrogen atom when spin orbit coupling is included. A heuristic way to understand this is that near any individual atom in a crystal, the potential is approximately radial, allowing one to write the Pauli spin orbit coupling term as a quantity proportional to $\boldsymbol{\sigma} \cdot \boldsymbol{l}$, for $\boldsymbol{l}$ the orbital angular momentum operator. A Bloch wave function can constructed

as a superposition of atomic orbitals, allowing one to adapt the J^2, J_z basis to the Bloch functions. We will see such a basis employed in the next section.

$\boldsymbol{k}\cdot\boldsymbol{p}$ Theory of α-Sn

In this dissertation we are interested in the physics of Dirac and Weyl semimetals. A crystal structure these electronic systems can occur in is the cubic diamond lattice allotrope of tin known as α-Sn. We will need the $\boldsymbol{k}\cdot\boldsymbol{p}$ Hamiltonian of the conduction, valence and sub-valence bands of α-Sn around the Γ point. These bands consist of the p-like heavy and light-hole states carrying total angular momentum $J = 3/2$, and s-like states with $J = 1/2$ coming only from spin. At the Γ point, the heavy-hole states form the valence band, and are described by the basis functions $|\Gamma^+_{8,v}, \pm 3/2\rangle$ with angular momentum projection numbers $J_z = \pm 3/2$. The light-hole states form the conduction band and are given by $|\Gamma^+_{8,c}, \pm 1/2\rangle$ with $J_z = \pm 1/2$. The s-like states form the sub-valence band, they are given by $|\Gamma^-_7, \pm 1/2\rangle$. We define an ordered basis at the Γ point:

$$\begin{aligned}\big\{|1\rangle = |\Gamma^-_7, 1/2\rangle\,, |2\rangle = |\Gamma^-_7, -1/2\rangle\,, |3\rangle = |\Gamma^+_{8,v}, +3/2\rangle\,,\\ |4\rangle = |\Gamma^+_{8,c}, +1/2\rangle\,, |5\rangle = |\Gamma^+_{8,c}, -1/2\rangle\,, |6\rangle = |\Gamma^+_{8,v}, -3/2\rangle\big\}\,.\end{aligned} \tag{2.10}$$

The $\boldsymbol{k}\cdot\boldsymbol{p}$ Hamiltonian acting on these states is given by [25]:

$$H(\boldsymbol{k}) = \begin{pmatrix} E_{\Gamma_7^-}+T & 0 & -\frac{1}{\sqrt{2}}Pk_+ & \sqrt{\frac{2}{3}}Pk_z & \frac{1}{\sqrt{6}}Pk_- & 0 \\ 0 & E_{\Gamma_7^-}+T & 0 & -\frac{1}{\sqrt{6}}Pk_+ & \sqrt{\frac{2}{3}}Pk_z & \frac{1}{\sqrt{2}}Pk_- \\ -\frac{1}{\sqrt{2}}Pk_- & 0 & E_{\Gamma_8^+}+U+V & -S_- & R & 0 \\ \sqrt{\frac{2}{3}}Pk_z & -\frac{1}{\sqrt{6}}Pk_- & -S_-^\dagger & E_{\Gamma_8^+}+U-V & 0 & R \\ \frac{1}{\sqrt{6}}Pk_+ & \sqrt{\frac{2}{3}}Pk_z & R^\dagger & 0 & E_{\Gamma_8^+}+U-V & S_+^\dagger \\ 0 & \frac{1}{\sqrt{2}}Pk_+ & 0 & R^\dagger & S_+ & E_{\Gamma_8^+}+U+V \end{pmatrix}, \tag{2.11}$$

where P is the momentum matrix element, and

$$k_\parallel^2 = k_x^2 + k_y^2,\ k_\pm = k_x \pm ik_y,\ k^2 = k_\parallel^2 + k_z^2,$$

$$\epsilon = \operatorname{tr}\tilde{\epsilon} = \epsilon_{xx} + \epsilon_{yy} + \epsilon_{zz},$$

$$T = \tfrac{\hbar^2}{2m_e}(\gamma_0 k_\parallel^2 + k_z\gamma_0 k_z) + a'\epsilon,$$

$$U = -\tfrac{\hbar^2}{2m_e}\gamma_1 k^2 - a\epsilon, \quad V = -\tfrac{\hbar^2}{2m_e}\gamma_2(k_\parallel^2 - 2k_z^2) + b(\epsilon_{xx} - \epsilon_{zz}),$$

$$S_\pm = -\tfrac{\hbar^2}{m_e}\sqrt{3}\gamma_3 k_\pm k_z, \quad R = -\tfrac{\hbar^2}{2m_e}\sqrt{3}(\mu k_+^2 - \bar{\gamma}k_-^2).$$

In this Hamiltonian, the x, y, and z axes are aligned with the (100), (010), and (001) axes of the crystal respectively. The coefficients $\gamma_{i=1,2,3}$ are the *modified* Luttinger parameters appropriate to the Kane model, and a, a', and b are the deformation potentials. For notational brevity we have defined $\mu = (\gamma_3 - \gamma_2)/2$, $\bar{\gamma} = (\gamma_3 + \gamma_2)/2$. The effects of (001) epitaxial strain have been included in Equation (2.11) according to the generalized methods of Bir and Pikus [19, 26]. The matrix $\tilde{\epsilon}$ is the strain tensor, and for (001) epitaxial strain its components are given by $\epsilon_{xx} = \epsilon_{yy} = \epsilon_\parallel$

and $\epsilon_{zz} = -2c_{12}\epsilon_{\|}/c_{11} = -0.85\epsilon_{\|}$. Assuming pseudomorphic growth of α-Sn on a zincblende substrate, the epitaxial strain is given by $\epsilon_{\|} = a_l^{\text{sub}}/a_l^{Sn} - 1$, for $a_l^{\text{sub/Sn}}$ the lattice constant of the substrate/α-Sn. The values of the parameters in Equation (2.11) can be found in Table 1. The exact values of $E_{\Gamma_7^-}$ and $E_{\Gamma_8^+}$ are arbitrary for the current discussion, what is important is their energy bandgap:

$$E_g \;=\; E_{\Gamma_7^-} - E_{\Gamma_8^+} \; . \tag{2.12}$$

Notice in Table 1 that $E_g < 0$ for α-Sn, this is because it has an *inverted* band structure, directly resulting from the large values of the spin orbit coupling $\Delta_{\nu\nu'}^{\sigma\sigma'}$, and Darwin term $D_{\nu\nu'}^{\sigma\sigma'}$, which we discussed earlier, and have now absorbed into the definitions of $E_{\Gamma_7^-}$ and $E_{\Gamma_8^+}$ [27, 28]. Typically in diamond[3] lattices the s-like Γ_7^- band is the conduction band, and the p-like $\Gamma_{8,v}^+$ bands are the sub-valence bands [29]. In Figure 1 we have plotted the energy levels along the k_z-axis $E(0,0,k_z)$, of α-Sn subject to a (001) epitaxial strain of $\epsilon_{\|} = -0.0012$, which can be achieved by pseudomorphic growth of α-Sn on CdTe substrates [30, 31]. In 2017, Xu *et al.* achieved the Dirac semimetal state in α-Sn by pseudomorphic growth on InSb(111), a different zincblende crystal with a smaller lattice constant than α-Sn [11]. In Figure 2 we have made the sign of E_g positive to schematically demonstrate the dispersion for a crystal with *regular* band ordering. In both cases the $|\Gamma_7^-, \pm 1/2\rangle$ and $|\Gamma_{8,c}^+, \pm 1/2\rangle$ bands strongly repel each other due to the k_z dependent coupling manifest in the terms $H_{1,4}(\boldsymbol{k})$ and $H_{2,5}(\boldsymbol{k})$ in Equation (2.11). Importantly, as we see for α-Sn in

[3]Some examples are Si, InSb and CdTe. The latter two aren't diamond lattice crystals but zincblende. A diamond lattice is a zincblende lattice in which both basis atoms are the same element.

TABLE 1. Table of bare parameters for α-Sn $\boldsymbol{k}\cdot\boldsymbol{p}$ Hamiltonian. Values are taken from [5, 32–35] Note: $E_p = 2m_eP^2/\hbar^2$, and $\hbar^2/(2m_e) = 3.809982$ eV Å^2.

γ_0	γ_1	γ_2	γ_3	κ	P (eV· Å)	E_P (eV)	E_g (eV)
1	4.19	-1.73	1.64	-2.18	9.55	23.93	-0.413
a (eV)	b (eV)	d (eV)	a' (eV)	c_{11} (GPa)	c_{12} (GPa)	c_{44} (GPa)	a_l^{Sn} (Å)
7.77	-2.4	-4.1	-14.81	69.0	29.3	36.2	6.4892

Figure 1, this repulsion imbues the $|\Gamma^+_{8,c}, \pm 1/2\rangle$ band with opposite curvature to the $|\Gamma^+_{8,v}, \pm 3/2\rangle$ band along the k_z-axis, resulting in a band crossing for any $\epsilon_\parallel < 0$.

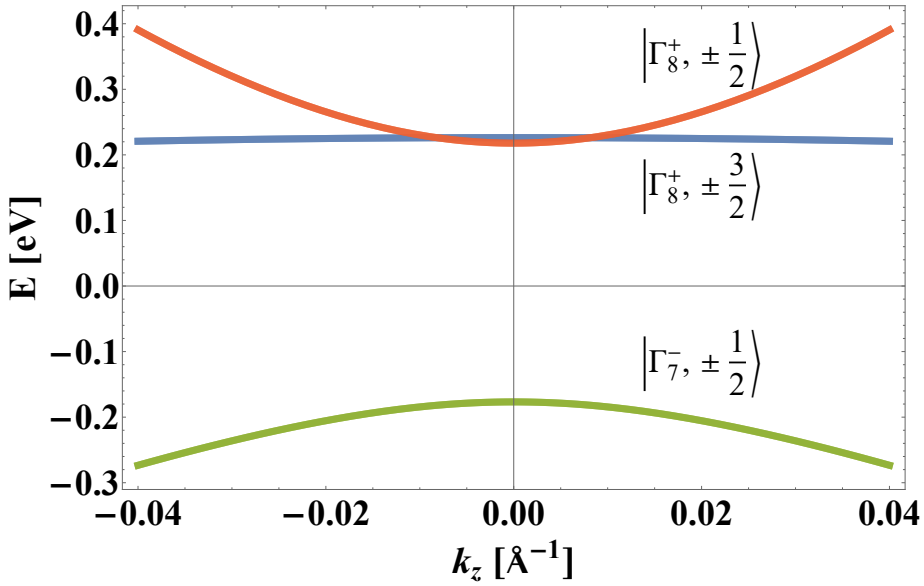

FIGURE 1. Inverted band structure for the $\Gamma^+_{8,c/v}$ and Γ_7^- bands of α-Sn near the Γ point of the BZ. An epitaxial strain of $\epsilon_\parallel = -0.0012$ is incorporated in the calculation to generate a band crossing between the $\Gamma^+_{8,c/v}$ bands.

The Dirac semimetal state of α-Sn

We will now examine the $\Gamma^+_{8,c/v}$ band crossing for general $\epsilon_\parallel < 0$. Since we have established that the bands we are interested in are $\Gamma^+_{8,c/v}$ we can use Löwdin perturbation theory to perturbatively decouple the $\Gamma^+_{8,c/v}$ and Γ_7^- bands. To $\mathcal{O}(k^2)$

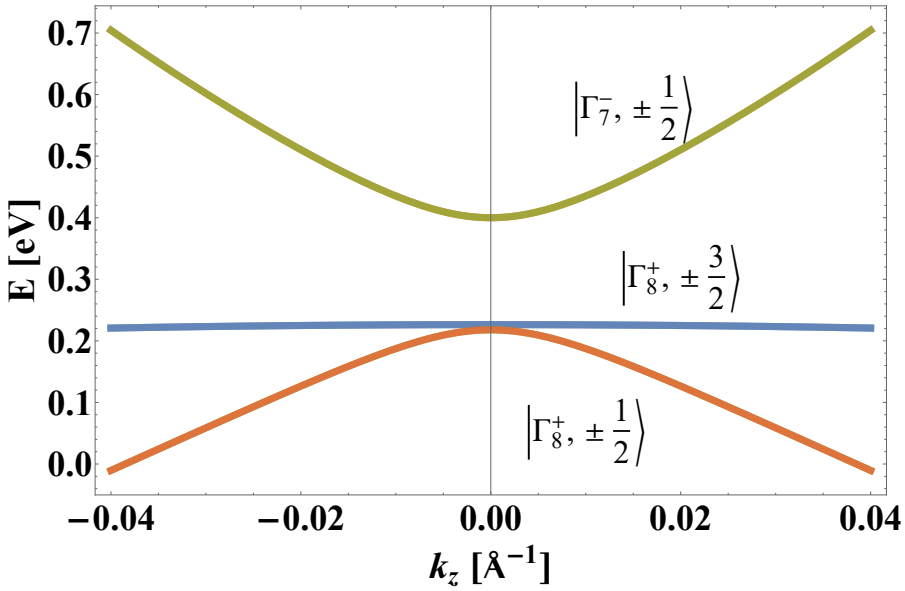

FIGURE 2. Schematic typical band structure about the Γ point of a cubic diamond lattice crystal. The values corresponds to the band structure of α-Sn if E_g were positive in Fig. 1. There are no longer any band crossings despite the presence of epitaxial strain.

this renormalizes the *modified* Luttinger parameters, now simply called the Luttinger parameters $\tilde{\gamma}_i$:

$$\gamma_1 \rightarrow \tilde{\gamma}_1 = \gamma_1 + \frac{2|P|^2}{3m_e E_g} = -15.1 \tag{2.13a}$$

$$\gamma_2 \rightarrow \tilde{\gamma}_2 = \gamma_2 + \frac{2|P|^2}{6m_e E_g} = -11.4 \tag{2.13b}$$

$$\gamma_3 \rightarrow \tilde{\gamma}_3 = \gamma_3 + \frac{2|P|^2}{6m_e E_g} = -8.01 \tag{2.13c}$$

The energies $E_{c/v}(0,0,k_z)$ can be read off the diagonal entries of $H(0,0,k_z)$:

$$\begin{aligned} E_{c/v}(0,0,k_z) &= -a\epsilon - \frac{\hbar^2}{2m_e}\tilde{\gamma}_1 k_z^2 \pm \left(b(\epsilon_{zz} - \epsilon_{xx}) - \frac{\hbar^2}{m_e}\tilde{\gamma}_2 k_z^2\right) \\ &= -a\epsilon - \frac{\hbar^2}{2m_e}\tilde{\gamma}_1 k_z^2 \pm \left(-1.85 b\epsilon_{\parallel} - \frac{\hbar^2}{m_e}\tilde{\gamma}_2 k_z^2\right) . \end{aligned} \tag{2.14}$$

Due to the fact $b, \epsilon_\parallel, \tilde{\gamma}_2 < 0$, the momenta $k_z = \pm k_0 = \pm\sqrt{-1.85 b \epsilon_\parallel m_e/\tilde{\gamma}_2 \hbar^2}$ are real valued solutions of $E_c(0,0,\pm k_0) = E_v(0,0,\pm k_0)$. Taylor expanding to first order the reduced $\boldsymbol{k}\cdot\boldsymbol{p}$ Hamiltonian around $\pm k_0 \hat{z}$ for small $\boldsymbol{k}$ we obtain:

$$H(\boldsymbol{k} \pm k_0\hat{z}) \equiv H^{\pm}_{\alpha\text{-Sn}}(\boldsymbol{k}) = E_D \mp v_0\hbar\tilde{\gamma}_1 k_z \pm v_0\hbar\, \tau_z \otimes \begin{pmatrix} 2\tilde{\gamma}_2 k_z & \sqrt{3}\tilde{\gamma}_3 k_- \\ \sqrt{3}\tilde{\gamma}_3 k_+ & -2\tilde{\gamma}_2 k_z \end{pmatrix}, \tag{2.15a}$$

$$E_D = E_{\Gamma_8^+} - a\epsilon - v_0\hbar\frac{k_0}{2}\ , \tag{2.15b}$$

$$v_0 = \frac{\hbar}{m_e}k_0 \approx 8.74\times 10^{-4}\ c \times \sqrt{|\epsilon_\parallel|}\ . \tag{2.15c}$$

We have defined the Dirac energy E_D, bare Dirac velocity[4] v_0 and additional Pauli matrices $\tau_{x,y,z}$ acting on the space of $\operatorname{sgn} J_z = \pm 1$, and $\sigma_{x,y,z}$ acting on the space of $|J_z| = 3/2, 1/2$. There are many important properties of $H^{\pm}_{\alpha\text{-Sn}}(\boldsymbol{k})$ one should understand. First, it is anisotropic and tilted along the k_z axis due to the $\tilde{\gamma}_1 k_z$ term. If we ignore the global energy shift, anisotropies and tilt and fix the Dirac velocity to some v_D for all directions, $H^{\pm}_{\alpha\text{-Sn}}(\boldsymbol{k})$ becomes:

$$H^{\pm}_D(\boldsymbol{k}) = \pm\hbar v_D \boldsymbol{k}\cdot(\tau_z \otimes \boldsymbol{\sigma})\ . \tag{2.16}$$

This new Hamiltonian looks exactly like the massless Dirac Hamiltonian from high energy physics [36]. For the idealized H_D, the electron energy dispersion forms perfect Dirac cones around the Dirac nodes $\pm k_0\hat{z}$. There are four copies of the Dirac cones, two at each Dirac node. If we focus on the σ space, the Equation (2.16) can also be

[4]In the literature what we call the Dirac velocity is often called the Fermi velocity. We distinguish between the two to emphasize that the Dirac velocity is specific to the Dirac point, and may differ from the Fermi velocity if the chemical potential is not located at the Dirac point.

written as:

$$H_D(\boldsymbol{k})_i \;\; = \;\; \lambda_i v_D \boldsymbol{k} \cdot \boldsymbol{\sigma} \; , \tag{2.17}$$

for $\lambda_{1,4} = +1$ and $\lambda_{2,3} = -1$. λ_i is called the chirality of the Dirac cone, and it is essential that $\sum_i \lambda_i = 0$. If the sum rule weren't obeyed, $H_D(\boldsymbol{k})$ would be unphysical due to the chiral anomaly, which manifests as a lack of charge conservation in the presence of parallel external electric and magnetic fields [37, 38]. The form of the Hamiltonian $\boldsymbol{k} \cdot \boldsymbol{\sigma}$ is known as the Weyl Hamiltonian, originally used to describe relativistic, massless spin-1/2 particles [36].

The Dirac velocity v_0, of the Dirac semimetal state in α-Sn is proportional to $\sqrt{\epsilon_\parallel}$, which is necessarily perturbatively small for the methods of Bir and Pikus to apply[5]. A reasonable upper limit for v_0 therefore comes from setting $|\epsilon_\parallel| = 0.01$. Then in any given direction, due to $\tilde{\gamma}_i \approx \mathcal{O}(10)$, electrons in the Dirac cone spectrum will experience a velocity on the order $\mathcal{O}(10^{-4}c)$. Other Dirac semimetals, such as Na_3Bi and Cd_3As_2, have strain independent Fermi velocities on the order of $\mathcal{O}(10^{-4}c)$ [39]. These Fermi velocities are small compared to the typical velocities in metals, ranging from $\mathcal{O}(10^{-3}c)$ to $\mathcal{O}(10^{-2}c)$, and in graphene where $v \approx \mathcal{O}(10^{-2}c)$ [8, 40]. The strength of the Coulomb interaction in quantum electrodynamics is set by the fine structure constant α, as we are dealing with Dirac electrons with some velocity $v_0 < c$, the strength of the Coulomb interaction between electrons near the Dirac point is governed by the modified structure constant

$$\alpha_{v_0} = \frac{1}{4\pi\epsilon}\frac{e^2}{\hbar v_0} = \alpha\frac{c}{v_0} = \frac{1}{\epsilon_r 137}\frac{c}{v_0} \; , \tag{2.18}$$

[5]In addition to the breakdown of perturbation theory, a very large $\epsilon_\parallel$ tends to induce crystals to assume new equilibrium structures, further invalidating the $\boldsymbol{k} \cdot \boldsymbol{p}$ Hamiltonian

where $\epsilon = \epsilon_0\epsilon_r$ is the permittivity of the material, ϵ_0 is the permittivity of free space and ϵ_r is the dielectric constant of the material. In vacuum $\epsilon_r = 1$, and for materials in general $\epsilon_r > 1$. From the above equation we can then conclude that in graphene $\alpha_v \approx 100/\epsilon_r 137 < 1$, and the Coulomb interaction between electrons can be considered slightly weak. However, in α-Sn we have $\alpha_{v_0} \approx 10^4/(\epsilon_r 137) \approx 73/\epsilon_r$, which is very large if $\epsilon_r \approx 1$. Single crystals of unstrained α-Sn have been measured to have dielectric constants as high as $\epsilon_r = 24$ at room temperature [41], so we can expect ϵ_r in strained α-Sn to bring α_{v_0} down to some $\mathcal{O}(1)$ to $\mathcal{O}(10)$ number, which is the strong coupling regime quantum field theory techniques are well suited to treat. Compare this with the case of electrons in a metal, for which the strength of the Coulomb interaction is set by the dimensionless density parameter r_s, which is the ratio of the average bare Coulomb energy[6] per particle to its kinetic energy. r_s can be expressed in the following ways :

$$r_s = \frac{r_0}{a_0} = \frac{1}{k_F a_0} = 2\alpha\frac{c}{v_{\mathrm{F}}} \ , \tag{2.19}$$

where a_0 is the Bohr radius, $r_0 = 1/k_F$ is the average wavelength of an energy at the Fermi surface, k_F is the Fermi wave vector, and v_{F} the Fermi velocity. The factor of 2 relating r_s to $\alpha_{v_{\mathrm{F}}}$ is a consequence of the quadratic kinetic energy $p^2/2m$ in a metal, compared to the linear case of a Dirac semimetal. While in most metals the Fermi velocity is of the order $v_{\mathrm{F}} = \mathcal{O}(10^{-3}c$, with $2 < r_s < 6$, it is well understood that Landau Fermi liquid theory and quantum field theoretic methods accurately describe physics in this coupling regime [40, 42].

[6]By bare we mean setting $\epsilon_r = 1$.

The Weyl semimetal state of α-Sn

When subjected to an external magnetic field $\boldsymbol{B}$, time reversal symmetry is broken in the Dirac semimetal. Due to the Zeeman coupling of electrons' spin to magnetic fields, the Dirac semimetal state is broken into a so called Weyl semimetal state. To see this consider the Zeeman coupling term:

$$H_Z = \frac{\mu_B g}{\hbar} \boldsymbol{J} \cdot \boldsymbol{B} \ . \tag{2.20}$$

The Landé g-factor is a material dependent quantity, μ_B is the Bohr magneton. The matrices $\boldsymbol{J}$ are the generators of the 4 dimensional representation of SU(2) and are given by:

$$J_x = \begin{pmatrix} 0 & \frac{\sqrt{3}}{2} & 0 & 0 \\ \frac{\sqrt{3}}{2} & 0 & 1 & 0 \\ 0 & 1 & 0 & \frac{\sqrt{3}}{2} \\ 0 & 0 & \frac{\sqrt{3}}{2} & 0 \end{pmatrix}, \qquad J_y = \begin{pmatrix} 0 & -\frac{\sqrt{3}}{2}i & 0 & 0 \\ \frac{\sqrt{3}}{2}i & 0 & -i & 0 \\ 0 & i & 0 & -\frac{\sqrt{3}}{2}i \\ 0 & 0 & \frac{\sqrt{3}}{2}i & 0 \end{pmatrix}$$

$$J_z = \begin{pmatrix} \frac{3}{2} & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & -\frac{3}{2} \end{pmatrix} . \tag{2.21}$$

For simplicity, we restrict ourselves to the simplified Dirac Hamiltonian of α-Sn in Equation (2.15). When the Zeeman term is added to $H_D^{\pm}(\boldsymbol{k})$, each set of Dirac nodes split in momentum space into two non-degenerate Weyl nodes, as demonstrated in Figure 3. There are two distinct types of splitting for α-Sn. The first case is $B_x = B_y = 0$ and $B_z \neq 0$ and is demonstrated in Figure 3b. The magnetic field

coupled Hamiltonian takes the form:

$$\begin{aligned} H_D^{\pm}(\boldsymbol{k}) + H_Z &= \frac{\mu_B g}{\hbar} B_z \left(\tau_z \otimes \sigma_0\right) \pm \hbar v_D \boldsymbol{k}_{\parallel} \cdot \left(\tau_z \otimes \boldsymbol{\sigma}_{\parallel}\right) \\ &\quad \pm \left(\hbar v_D k_z \pm \frac{\mu_B g}{\hbar} B_z\right) \left(\tau_z \otimes \sigma_z\right) \ . \end{aligned} \tag{2.22}$$

From the Dirac cones originally at $\boldsymbol{k} = (0, 0, \pm k_0)$, we obtain two non-degenerate Weyl cones at $\boldsymbol{k} = (0, 0, \pm k_0 \pm \frac{\mu_B g}{v_D \hbar^2} B_z)$, offset in energy. Ignoring the energy offsets, for momenta close to the Weyl nodes the Hamiltonian is given by:

$$H_{W1}(\boldsymbol{k}) = \lambda \hbar v_D \boldsymbol{k} \cdot \boldsymbol{\sigma} \ , \tag{2.23}$$

for $\lambda = \pm 1$. As can be seen from Figure 3b, the Weyl nodes are isolated from other bands making the reduced two band Hamiltonian in Equation (2.23) an accurate description of the physics near the nodes. The Weyl Hamiltonian is very special because it is highly robust against being gapped by perturbations. Any perturbative term P, involving the two bands can be decomposed into a linear combination of Pauli matrices

$$P = p_0 \sigma_0 + \boldsymbol{p} \cdot \boldsymbol{\sigma} \ . \tag{2.24}$$

For constant $p_{\mu=0,1,2,3}$ such a term can only shift the energy/momentum location of a Weyl node, but it cannot gap the Weyl cones. The original Dirac Hamiltonian in Equation (2.16) did not share this robustness to arbitrary small perturbations, because many Hermitian terms that are block off-diagonal compared to $\boldsymbol{k} \cdot (\tau_z \otimes \boldsymbol{\sigma})$ can be written down that would open up a gap in the Dirac spectrum. For Dirac semimetals, various symmetries are necessary to forbid the existence of such gapping

terms. The other band crossings we see in Figure 3b actually lie on a closed ellipse of band degeneracies in $\boldsymbol{k} - E$ space, forming a 'line-node' degeneracy. They do not share the Weyl Hamiltonian's simple two band form, and are not highly protected against perturbation induced gapping.

The case in which the magnetic field components $B_x, B_y \neq 0$ and $B_z = 0$ is more complicated. Without loss of generality we can fix $B_y = 0$ by a rotation. Under such a field the new band structure is shown in Figure 3c. Unlike the $B_z \neq 0$ case, there are only two band crossings, both being Weyl nodes with the same energy. A non-trivial unitary transformation of the Hamiltonian around these Weyl nodes results in a Hamiltonian:

$$H_{W2}(\boldsymbol{k}) \;\; = \;\; \sum_{i=x,y,z} \hbar u_i k_i \sigma_i + \hbar u_0 k_z \sigma_0 \; . \tag{2.25}$$

Here the Pauli matrices σ_μ act on a complicated linear combination of all the original bands. The velocities u_i depend on B_x and have the property $u_x = u_y$. There is also an additional tilt along the k_z axis through the u_0 term. In $H_{W2}(\boldsymbol{k})$, the chirality of a Weyl node is given by $\mathrm{sgn}\,(u_x u_y u_z)$. The case of a general $\boldsymbol{B}$ will be a more complicated combination of what has already been discussed. The most general linear, two-band Hamiltonian one can write down describing electrons in the vicinity of a Weyl node is:

$$H_{W3}(\boldsymbol{k}) \;\; = \;\; \sum_{i=x,y,z} \hbar \boldsymbol{u}_i \cdot \boldsymbol{k}_i \sigma_i + \hbar \boldsymbol{u}_0 \cdot \boldsymbol{k} \sigma_0 \; . \tag{2.26}$$

The chirality of a specific Weyl node in this Hamiltonian is given by $\mathrm{sgn}\,(\boldsymbol{u}_x \cdot \boldsymbol{u}_y \times \boldsymbol{u}_z)$.

We have so far discussed Dirac semimetals that become Weyl semimetals upon the breaking of time reversal symmetry. There also exists a class of Dirac semimetals

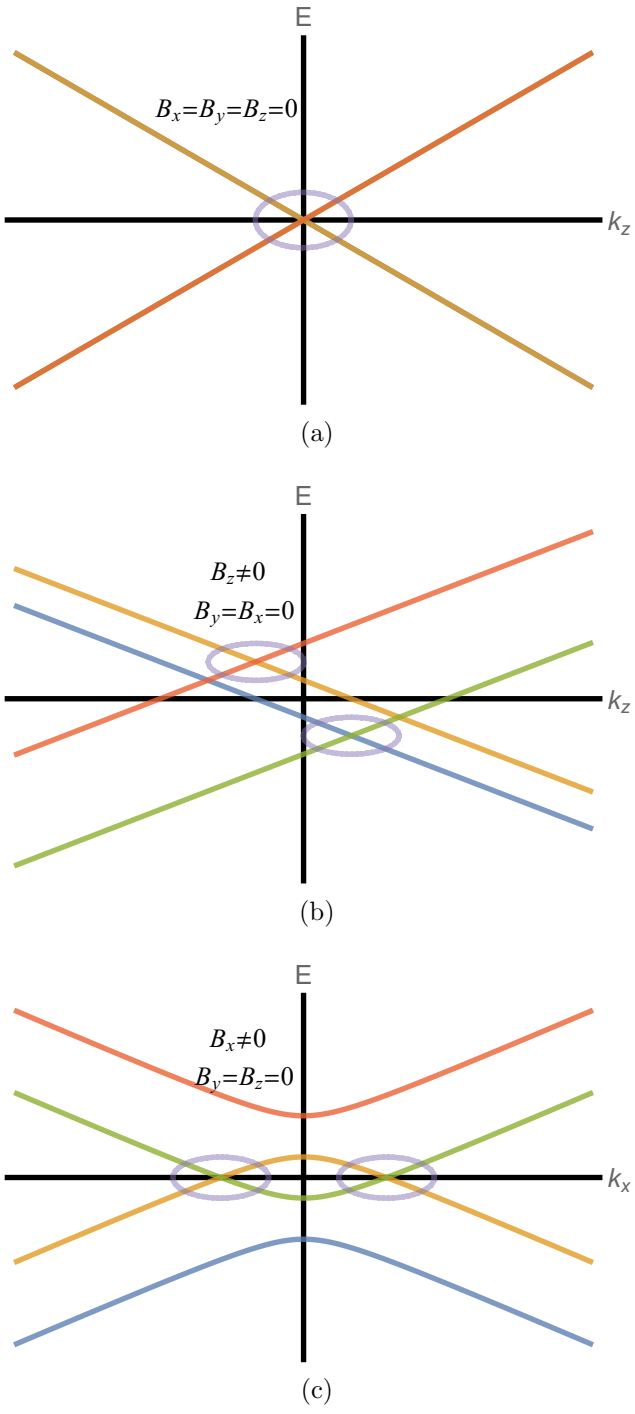

FIGURE 3. (a) Schematic plot of idealized Dirac dispersion with Dirac node circled by purple. Plots (b) and (c) respectively show the result of applying a magnetic field in the z and x direction to the dispersion in (a). The Weyl nodes are circled by purple, all other crossings cannot be described by the Weyl Hamiltonian. The gapped bands in (c) are a consequence of the inter Dirac cone coupling elements in J_x.

that form a Weyl semimetal upon breaking *inversion* symmetry [43]. This is not the case for α-Sn, and so cannot be demonstrated in the current $\boldsymbol{k} \cdot \boldsymbol{p}$ Hamiltonian example. As already stated, α-Sn possesses a diamond lattice, which is formed by two inter-penetrating fcc lattices of the same element. A zincblende lattice is the inversion asymmetric case of diamond, in which the two fcc lattices are formed by different elements. HgTe is an example of a zincblende lattice with the same type of band inversion as α-Sn, and thus is adiabatically equivalent to an inversion symmetry broken form of α-Sn[7]. It has been shown that HgTe is not a Weyl semimetal, and thus breaking inversion symmetry in strained α-Sn cannot realize a Weyl semimetal state [44].

Suitable Minimal Model for Dirac and Weyl Semimetals

There are multiple real material systems in which Dirac and Weyl semimetals arise, each with their own unique Hamiltonians and Zeeman coupling terms. We have also seen that applying a magnetic field can dramatically complicate any analysis due to the non-trivial band couplings in the Zeeman Hamiltonian of Equation (2.20). For phenomenological studies, it is therefore desirable to work with a minimal model [45].

For the purposes of this dissertation, we will be interested in Dirac semimetals that reduce to Weyl semimetals upon the breaking of time reversal symmetry, so that we can eventually use our work to study magnetic effects in Dirac and Weyl semimetals. A suitable minimal model for this problem looks similar to Equation

[7]By adiabatically equivalent, we mean that upon adding inversion symmetry breaking terms to the α-Sn $\boldsymbol{k} \cdot \boldsymbol{p}$ Hamiltonian, one can continuously deform it into the Hamiltonian for HgTe without crossing any bands.

(2.16) [46]:

$$H_D(\boldsymbol{k}) \;=\; \hbar v_D \boldsymbol{k} \cdot (\tau_z \otimes \boldsymbol{s}) \; . \tag{2.27}$$

This Hamiltonian describes a single Dirac node, in which $s_\mu = \sigma_\mu$ are the Pauli matrices acting on the spin-1/2 degrees of freedom, and τ_μ act on the chirality degrees of freedom. The Zeeman coupling term is given by:

$$H_Z \;=\; \mu \boldsymbol{B} \cdot \boldsymbol{s} \; . \tag{2.28}$$

All coefficients have been absorbed into a single coupling constant μ. The simplicity of this model is immediately obvious. Any magnetic field $\boldsymbol{B}$, will split the Dirac nodes into Weyl nodes at the same energy, and not mix any quantum numbers. It will be the model we use in the rest of this work, and therefore expect our results to qualitatively describe phenomena in general Dirac and time reversal symmetry broken Weyl semimetals.

CHAPTER III

INTRODUCTION TO SOFT MODES

The Goldstone Theorem

The Goldstone theorem (also known as the Nambu-Goldstone theorem) is a powerful concept that is the foundation of much of the physics explored in this dissertation. In its high energy incarnation, the theorem is straightforward and easy to prove [47–49]. Within the realm of condensed matter physics, the theorem becomes far more complicated as the Lorentz invariance of a relativistic theory is abandoned by fixing a frame for the problem, and allowing the Coulomb interaction to be instantaneous. A full statement and proof of the theorem in this context did not exist until 2012 [50, 51]. In this section we will state the theorem as it applies to fermionic Lagrangians, for a rigorous proof of the statement and the bosonic case, we refer to Reference [52].

Spontaneous Symmetry Breaking

Let $\mathcal{L}_F$ be a fermion Lagrangian density for fermionic (Grassman valued) fields ψ and $\bar{\psi}$,

$$\mathcal{L}_F(\bar{\psi}, \psi) \;=\; \bar{\psi} L \psi + V(\bar{\psi}, \psi) \;, \tag{3.1}$$

where L is the quadratic level Lagrangian operator and V is the interaction term of order $\mathcal{O}(\psi^4)$. We say $\mathcal{L}_F$ has a continuous symmetry if it is invariant under the action

of some Lie group G.[1] Let the Lie algebra associated to G be denoted by $\mathfrak{g}$, and its generators $\{g_a\}$. Then for any $g_a \in \{g_a\}$, and $\theta \in \mathbf{R}$, it must be that:

$$\mathcal{L}_F(e^{i\theta g_a}\bar{\psi}, e^{-i\theta g_a}\psi) = \mathcal{L}_F(\bar{\psi}, \psi) \ . \tag{3.2}$$

We can now introduce a symmetry breaking term $\mathcal{L}_{\rm SB}$ that is not invariant under the action of G

$$\mathcal{L}_{\rm SB} \ = \ \lambda \bar{\psi} M \psi \ , \tag{3.3}$$

where λ is an infinitesimal real number, and M is a hermitian matrix, linearly composed of the generators g_a, such that for some $g_b \in \{g_a\}$ we have $[g_b, M] \neq 0$. In a ferromagnet $\mathcal{L}_{\rm SB}$ would represent the Zeeman coupling to an external magnetic field of strength λ. We now assume that through adding $\mathcal{L}_{\rm SB}$ we have eliminated enough symmetries from the Hamiltonian H_T, associated to $\mathcal{L}_F + \mathcal{L}_{\rm SB}$, such that H_T has a unique ground state. The order parameter Δ, of the action is then defined as the the expectation of $\mathcal{L}_{SB}/\lambda$ with respect to the ground state:

$$\Delta = \langle \bar{\psi} M \psi \rangle \ . \tag{3.4}$$

The symmetry of $\mathcal{L}_F$ generated by the g_b such that $[g_b, M] \neq 0$ is defined to be *spontaneously broken* if $\lim_{\lambda \to 0} \Delta \neq 0$. An equivalent statement is to say the symmetry is spontaneously broken when the homogeneous Δ-susceptibility χ_Δ, diverges in the

[1] It is important that the measure term of the partition function is also invariant under the action of G, otherwise one arrives at an anomaly, see for example Reference [53].

same limit:

$$\lim_{\lambda \to 0} \chi_\Delta = \lim_{\lambda \to 0} \frac{\Delta}{\lambda} = \infty \ . \tag{3.5}$$

The Goldstone Bosons

For the generators g_a that don't commute with M, we can construct the bosonic field π_a, from the fermion biproduct term

$$\pi_a = i\bar{\psi}[g_a, M]\psi \ . \tag{3.6}$$

The fields π_a are called the Goldstone bosons of the $\lambda = 0$ ground state, and their correlation functions (called Goldstone modes) are soft (massless). More specifically, the retarded real space Goldstone mode is given by

$$D^R_{aa}(t, \boldsymbol{r}) \ = \ -\theta(t) \left\langle [\pi_a(t, \boldsymbol{r}), \pi_a(0,0)] \right\rangle \ . \tag{3.7}$$

The frequency momentum-space version of the Goldstone mode is given by

$$D_{aa}(\omega, \boldsymbol{q}) \ = \ \frac{Q(\omega, \boldsymbol{q})}{\alpha\lambda + P(\omega, \boldsymbol{q})} \ , \tag{3.8}$$

for which in the limit $\omega, \boldsymbol{q} \to 0$, the function $P(\omega, \boldsymbol{q}) \to 0$, and $Q(\omega, \boldsymbol{q})$ and α are constant. The order parameter expectation value is related to D_{aa} by the relation:

$$\Delta \propto \lambda D_{aa}(\omega = 0, q = 0) \ . \tag{3.9}$$

The Goldstone mode $D_{aa}(\omega, \boldsymbol{q})$ contains important information about the excitations. Since $P(\omega, \boldsymbol{q})$ goes to zero at $\omega = 0$, $\boldsymbol{q} = 0$, one obtains the frequency-momentum dispersion $\omega(\boldsymbol{q})$ satisfying $\omega(\boldsymbol{q}) \to 0$ as $\boldsymbol{q} \to 0$. The dispersion can be written in the general form

$$\omega(\boldsymbol{q}) \propto |\boldsymbol{q}|^n \ , \tag{3.10}$$

and is broken into two types, those for n-odd (Type I) and n-even (Type II). It has been proven in References [50, 51] that the number of type I and II Goldstone modes, n_I and n_{II} respectively, satisfy the following equations:

$$n_I + 2n_{II} \ \geq \ n_{\mathrm{BG}} \ , \tag{3.11a}$$

$$n_{\mathrm{BG}} - (n_I + n_{II}) \ = \ \frac{1}{2}\mathrm{rank}\langle [g_a, g_b] \rangle \ . \tag{3.11b}$$

where n_{BG} is the number of generators broken by M, and $g_a, g_b \in \{g_a | \, [g_a, M] \neq 0\}$. We see only if $\mathrm{rank}\langle [g_a, g_b] \rangle = 0$ does the number of Goldstone modes equal the number of broken generators. For actions that are Lorentz invariant, the only type of Goldstone mode that can occur has the dispersion $\omega(q) = q$, and $n_{\mathrm{BG}} = n_I$. In condensed matter problems, Lorentz invariance is generally not present, so one has to be careful in identifying all the Goldstone modes. For example, in the case of a Heisenberg ferromagnet with magnetization in the z direction, the symmetries corresponding to two of the three generators of $SU(2)$, σ_x and σ_y, are broken, but there is only one soft spin-wave excitation.

Soft Modes in Fermi Gases and Semimetals

The Goldstone modes discussed in the previous chapter were just one class of soft modes that can appear in a physical system. A generalized soft mode can be defined as a frequency and momentum space correlation function $C(\omega, q)$, that can be expressed as the ratio of two functions $Q(\omega, q)$ and $P(\omega, q)$, such that $P(\omega, q)$ has real valued roots $\omega_0,\ q_0$ and $Q(\omega_0, q_0)$ is finite:

$$C(\omega, q) \;=\; \frac{Q(\omega, q)}{P(\omega, q)} \; . \tag{3.12a}$$

There several phenomena in addition to spontaneous symmetry breaking that give rise to soft modes, for in depth discussions see References [13] and [54]. For the purposes of this dissertation we will be concerned with the cases of soft single particle excitations and Goldstone modes. To understand sfot single particle excitations, consider non-interacting electrons with an energy-momentum dispersion relation $\epsilon(\boldsymbol{k})$, spin σ, and chemical potential μ, described by fermion (Grassmann-valued) fields $\bar{\psi}_{n\sigma}(\boldsymbol{k})$ and $\psi_{n\sigma}(\boldsymbol{k})$ that depend on a wave vector $\boldsymbol{k}$ and a fermionic Matsubara frequency $\omega_{n=0,\pm1,\ldots} = 2\pi T(n + 1/2)$. In terms of these fields, the quantum partition function $\mathcal{Z}$ is given by [21]:

$$\mathcal{Z} = \int D[\bar{\psi}, \psi] \; e^{S_0[\bar{\psi},\psi]} \; , \tag{3.13}$$

for the finite temperature action S_0:

$$S_0 = \sum_{\boldsymbol{k},n} \sum_{\sigma} \bar{\psi}_{n,\sigma}(\boldsymbol{k}) \; [i\omega_n + \mu - \epsilon(\boldsymbol{k})] \; \psi_{n,\sigma}(\boldsymbol{k}) \; . \tag{3.14}$$

Defining $\xi(\boldsymbol{k}) = \epsilon(\boldsymbol{k}) - \mu$, the single particle Green function is given by:

$$G_n(\boldsymbol{k}) \;\; = \;\; \langle \psi_{n,\sigma}(\boldsymbol{k}) \bar{\psi}_{n,\sigma}(\boldsymbol{k}) \rangle = \frac{1}{i\omega_n - \xi(\boldsymbol{k})} \; . \tag{3.15}$$

The Green function is divergent whenever the wave vector satisfies $\epsilon(\boldsymbol{k}) = \mu$ and $\omega_n = 0$. In the case of a d dimensional Fermi gas, where $\epsilon(\boldsymbol{k}) = \boldsymbol{k}^2/2m_e$, the Green function is soft for an entire $d-1$ dimensional manifold of wave vectors, i.e. the Fermi surface. The same is true for a Dirac or Weyl semimetal for a chemical potential tuned away from the Dirac/Weyl points, but when the chemical potential lies specifically at the Dirac/Weyl points, and does not intersect any other band, then the Green function is divergent at only finitely many wave vectors.

The simple non-interacting partition function also possesses a Goldstone mode. Consider subjecting the fermion fields to a frequency dependent rotation:

$$\begin{aligned} \psi_{n,\sigma}(\boldsymbol{k}) \;\; &\rightarrow \;\; \bar{\psi}_{n+\alpha,\sigma}(\boldsymbol{k}) \; , \\ \bar{\psi}_{n,\sigma}(\boldsymbol{k}) \;\; &\rightarrow \;\; \bar{\psi}_{n+\alpha,\sigma}(\boldsymbol{k}) \; , \end{aligned} \tag{3.16a}$$

here α is a real constant, not necessarily an integer. In the imaginary time τ representation of the fields $\psi_\sigma(\tau, \boldsymbol{k})$, this is equivalent to performing the $U(1)$ gauge transformation:

$$\begin{aligned} \psi_n(\tau, \boldsymbol{k}) \;\; &\rightarrow \;\; e^{-i2\pi\alpha T\tau} \psi_\sigma(\tau, \boldsymbol{k}) \; , \\ \bar{\psi}_\sigma(\tau, \boldsymbol{k}) \;\; &\rightarrow \;\; e^{i2\pi\alpha T\tau} \bar{\psi}_\sigma(\tau, \boldsymbol{k}) \; . \end{aligned} \tag{3.16b}$$

We can define this operation as the action under some operator $\hat{T}_\alpha$, which in the imaginary time representation clearly leaves the measure term in $\mathcal{Z}$ invariant. Under

the action of $\hat{T}_\alpha$, while the second and third terms of Equation (3.14) are invariant as they are carry no frequency index information, the first term is not. The action becomes:

$$S_0 \to \sum_{\boldsymbol{k},n} \sum_{\sigma} \bar{\psi}_{n,\sigma}(\boldsymbol{k}) \left[i\omega_{n+\alpha} + i\mu - \epsilon(\boldsymbol{k}) \right] \psi_{n,\sigma}(\boldsymbol{k}) \ . \tag{3.17}$$

For transformations fluctuating slowly in time, i.e. $\alpha \ll 1$, the action of $\hat{T}_\alpha$ only weakly modifies the action S_0, and in the limit $\alpha \to 0$, $\hat{T}_\alpha$ becomes an exact symmetry of S_0. Consider the action of $\hat{T}_\alpha$ on the Green function of particles near the Fermi surface:

$$\begin{aligned} \hat{T}_\alpha \lim_{\omega_n \to 0} G_n(\boldsymbol{k}) &= \hat{T}_\alpha \left[-i\pi \mathrm{sgn}\left(\omega_n\right) \delta(\xi(\boldsymbol{p})) - P(1/\xi(\boldsymbol{p})) \right] \\ &= -i\pi \mathrm{sgn}\left(\omega_\alpha\right) \delta(\xi(\boldsymbol{p})) - P(1/\xi(\boldsymbol{p})) \ , \end{aligned} \tag{3.18}$$

where by P we denote the Cauchy principal value operator, and $\mathrm{sgn}\left(\omega_n\right)$ reflects the Fermi surface being approached from above or below, corresponding respectively to advanced (hole) or retarded (particle) degrees of freedom. We emphasize that this can only be discussed at zero temperature ($T = 0$), at which point the Matsubara frequencies become a continuum and their limiting behavior can be discussed. The last line follows from the fact that as the limit $\omega_n \to 0$ has been taken, the action of $\hat{T}_\alpha$ is entirely responsible for the sign of the frequency. If we now take the limit of $\alpha \to 0$, the last line remains unchanged. We can then conclude that the quantity

$$\lim_{\omega_n \to 0} G_n(\boldsymbol{k}) \ , \tag{3.19}$$

is invariant under the action of $\hat{T}_\alpha$ only for $\operatorname{sgn}\alpha = \operatorname{sgn} n$, but for $\operatorname{sgn}\alpha \neq \operatorname{sgn} n$ the expectation value is changed even in the $\alpha \to 0$ limit. This is a case of spontaneous symmetry breaking; we have found an ground state expectation value that is not invariant under the action of a symmetry of S_0. More specifically it is the imaginary part $\operatorname{Im} \lim_{\omega_n \to 0} G_n(\boldsymbol{k})$ that spontaneously breaks the symmetry, as $\operatorname{Re} \lim_{\omega_n \to 0} G_n(\boldsymbol{k})$ is insensitive to the action of $\hat{T}_\alpha$. The imaginary part of $G_n(\boldsymbol{k})$ also known as the spectral density function [21], and it is the order parameter of this broken symmetry.

In order to apply the concepts developed in Section 3.1 to the symmetry breaking we just found, we need to expand the group under consideration, as $U(1)$ turns out to be only a subgroup of the larger symmetry group. Let us rewrite the action S_0 in frequency-position space:

$$S_0 = \sum_n \int d\boldsymbol{x}\ \bar{\psi}_{n,\sigma}(\boldsymbol{x})\ [i\omega_n + \mu - \epsilon(-i\boldsymbol{\nabla})]\ \psi_{n,\sigma}(\boldsymbol{x})\ . \tag{3.20}$$

Clearly any transformation $\hat{T}$ that leaves $\sum_n \bar{\psi}_n \psi_n$ invariant is a symmetry of the action, except for the $i\omega_n$ term. We can think of $\hat{T}$ as rotations in Matsubara frequency space, and for a model with $2M$ Matsubara frequencies, $\{\hat{T}\}$ comprise the group $\mathrm{Sp}(2M)$, which follows from the Grassman nature of the $\psi(\boldsymbol{x})$ fields [55]. We can express general rotations in frequency space that mix the frequencies n_1 and n_2 as:

$$\hat{T}^{\pm}_{nm} = \delta_{nm}[1 + (\delta_{nn_1} + \delta_{nn_2})(\cos\theta - 1)] + (\delta_{nn_1}\delta_{mn_2} \pm \delta_{nn_2}\delta_{mn_1}) \sin\theta\ . \tag{3.21}$$

The action of $\hat{T}^{\pm}$ on the Matsubara vector $\psi(\boldsymbol{x})$ mixes the n_1 and n_2 components of $\psi(\boldsymbol{x})$ with a mixing angle θ. The generator of this transformation $\hat{t}^{\pm}$, can be found

by taking the infinitesimal θ limit:

$$\hat{t}^{\pm}_{nm} = (\delta_{nn_1}\delta_{mn_2} \pm \delta_{nn_2}\delta_{mn_1}) \ . \tag{3.22}$$

It is important to observe that we are in different a situation than the one posed in the Goldstone theorem. In this case, the symmetry breaking term $\mathcal{L}_{\rm SB} \equiv \sum_n \int d\boldsymbol{x} \ \bar{\psi}_{n,\sigma}(\boldsymbol{x})\psi_{n,\sigma}(\boldsymbol{x})i\omega_n$ is always present in the Lagrangian, and the symmetry breaking field $\lambda \equiv |\omega_n|$ is an internal degree of freedom.[2] We can rewrite the $i\omega_n$ term in S_0 as:

$$\begin{aligned} \sum_n i\omega_n\bar{\psi}_n(\boldsymbol{x})\psi_n(\boldsymbol{x}) &= i\sum_{nm}\bar{\psi}_n(\boldsymbol{x})\omega_n\delta_{nm}\psi_m(\boldsymbol{x}) \\ &= i\sum_{nm}|\omega_n|\bar{\psi}_n(\boldsymbol{x})\mathrm{sgn}\,(\omega_n)\delta_{nm}\psi_m(\boldsymbol{x}) \ . \end{aligned} \tag{3.23}$$

Treating ω_n as an external field, the intuitive choice for the term analogous to $\bar{\psi}M\psi$ in the Goldstone theorem is in this case

$$i\sum_{nm}\bar{\psi}_n(\boldsymbol{x})\mathrm{sgn}\,(\omega_n)\delta_{nm}\psi_m(\boldsymbol{x}) \ . \tag{3.24}$$

To determine the Goldstone boson, we need to know the commutator of $\hat{t}^{\pm}$ and the $\mathrm{sgn}\,(\hat{\omega})\mathbb{1}$ operator, by explicit computation one obtains

$$[\mathrm{sgn}\,(\hat{\omega})\mathbb{1}, t^{\pm}]_{nm} = [\mathrm{sgn}\,(\omega_{n_1}) - \mathrm{sgn}\,(\omega_{n_2})]\,[\delta_{nn_1}\delta_{mn_2} \mp \delta_{nn_2}\delta_{mn_1}] \ . \tag{3.25}$$

[2]This result is similar to the case of the action of a classical Heisenberg ferromagnet coupled to an external magnetic field $\boldsymbol{h}$. The presence of $\boldsymbol{h}$ violates the rotational symmetry of the action, and imbues what would be the transverse Goldstone modes of the $h = 0$ problem with an h dependent mass. Suppose that the magnetic field was in fact an internal degree of freedom, as would be the case if we were to consider a spin orbit coupling term which scales as $|\boldsymbol{k}|$. The ferromagnetic action would not be generally rotationally invariant except for $h \to 0$, and the original Goldstone modes of the problem would still be soft.

We see that the commutator is only non-zero for $\omega_{n_1}\omega_{n_2} < 0$, reflecting the fact that $\mathrm{sgn}\,(\omega_n)$ is invariant under frequencies rotations that preserve the sign of ω_n. We will now assume $\omega_{n_1}\omega_{n_2} < 0$ for the rest of discussion, so that we can make the simplification $\mathrm{sgn}\,(\omega_{n_1}) - \mathrm{sgn}\,(\omega_{n_2}) = \mathrm{sgn}\,(\omega_{n_1} - \omega_{n_2})$. The Goldstone boson is then given by

$$\begin{aligned} Q^{\pm}_{n_1 n_2}(\boldsymbol{x}) &= \sum_{nm} \bar{\psi}_n(\boldsymbol{x})[\mathrm{sgn}\,(\hat{\omega})\mathbb{1}, t^{\pm}]_{nm}\psi_m(\boldsymbol{x}) \\ &= \mathrm{sgn}\,(\omega_{n_1} - \omega_{n_2})\left(\bar{\psi}_{n_1}(\boldsymbol{x})\psi_{n_2}(\boldsymbol{x}) \mp \bar{\psi}_{n_2}(\boldsymbol{x})\psi_{n_1}(\boldsymbol{x})\right) \ . \end{aligned} \tag{3.26}$$

Fourier transforming we obtain the momentum space description of the field,

$$\begin{aligned} & Q^{\pm}_{n_1 n_2}(\boldsymbol{q}) = \\ & \mathrm{sgn}\,(\omega_{n_1} - \omega_{n_2}) \sum_{\boldsymbol{k}} \left(\bar{\psi}_{n_1}(\boldsymbol{k}+\boldsymbol{q}/2)\psi_{n_2}(\boldsymbol{k}-\boldsymbol{q}/2) \mp \bar{\psi}_{n_2}(\boldsymbol{k}+\boldsymbol{q}/2)\psi_{n_1}(\boldsymbol{k}-\boldsymbol{q}/2)\right) \\ \equiv \ & \sum_{\boldsymbol{k}} Q^{\pm}_{n_1 n_2}(\boldsymbol{k};\boldsymbol{q}) \ . \end{aligned} \tag{3.27}$$

The derivation of this Goldstone boson has been slightly ad-hoc as we had to guess what exactly to write down for the symmetry breaking term. In Section 4.6 we will rigorously derive the Goldstone bosons of a Fermi liquid, semiconductor and Dirac semimetal by means of a Ward identity, which is the same technique used to show π_a are the Goldstone bosons in conventional spontaneous symmetry breaking. For now, let us examine the momentum space correlation functions $\langle Q^{\pm}_{n_1 n_2}(\boldsymbol{k};\boldsymbol{q}) Q^{\pm}_{n_1 n_2}(\boldsymbol{k};-\boldsymbol{q})\rangle$, which must be summed over to obtain the real space Goldstone mode $\langle Q^{\pm}_{n_1 n_2}(\boldsymbol{x}) Q^{\pm}_{n_1 n_2}(0)\rangle$. The correlation functions will contain four point

functions of the form:

$$\begin{aligned} D_{nm}(\boldsymbol{k},\boldsymbol{q}) &= \langle \bar{\psi}_{n,\sigma}(\boldsymbol{k}_+)\psi_{m,\sigma}(\boldsymbol{k}_-)\bar{\psi}_{m,\sigma}(\boldsymbol{k}_-)\psi_{n,\sigma}(\boldsymbol{k}_+)\rangle \ , \\ &= -G_n(\boldsymbol{k}_+)G_m(\boldsymbol{k}_-) \ , \\ &= \frac{G_n(\boldsymbol{k}_+) - G_m(\boldsymbol{k}_-)}{i\Omega_{n-m} - \epsilon(\boldsymbol{k}_+) + \epsilon(\boldsymbol{k}_-)} \end{aligned} \tag{3.28}$$

where $\boldsymbol{k}_\pm = \boldsymbol{k} \pm \boldsymbol{q}/2$. The second line follows from Wick's theorem assuming $n \neq m$, and the third by algebraic manipulation using Equation (3.15) and the formula $-ab = (a-b)/(1/a - 1/b)$. When taking the limit $\omega_n, \omega_m \to 0$, the numerator is only non-zero provided $\omega_n\omega_m < 0$, which follows directly from the first line in Equation (3.18), and is precisely the condition imposed by requiring $[\mathrm{sgn}\,(\hat{\omega})\mathbb{1}, t^\pm]_{nm} \neq 0$. Upon analytic continuation to real frequencies according to $\Omega_{1-2} \to \Omega + i0^+$, Equation (3.28) becomes

$$D(\boldsymbol{k}, \boldsymbol{q} \to 0, \Omega \to 0) = \frac{i\mathrm{Im}\, G(\boldsymbol{k}, \Omega = 0)}{\Omega - \epsilon(\boldsymbol{k}_+) + \epsilon(\boldsymbol{k}_-)} \ . \tag{3.29}$$

This correlation function is soft, and diverges as $\boldsymbol{q}, \Omega \to 0$ provided $\mathrm{Im}\, G(\boldsymbol{k}, \Omega = 0)$ is non-zero, which is equivalent to stating that the d dimensional non-interacting system possesses a $d-1$ dimensional Fermi surface. Any lower dimensional Fermi surface results in $\mathrm{Im}\, G(\boldsymbol{k} \pm \boldsymbol{q}, \Omega = 0)$ vanishing as $\boldsymbol{q}, \Omega \to 0$, and will be the principal topic of discussion in Section 4.6. We will also show that all moments of $D_{nm}(\boldsymbol{k}, \boldsymbol{q})$ with respect to $|\boldsymbol{k}|$ are soft, in particular $\sum_{\boldsymbol{k}} D_{nm}(\boldsymbol{k}, \boldsymbol{q})$, which is necessary to show that the fluctuations of $Q^\pm_{n_1 n_2}(\boldsymbol{q})$ are soft.

The Importance of Soft Modes

In Chapter IV we will follow a program to map the strongly interacting version of the fermionic field theory for Dirac semimetals in Equation (3.13) to a bosonic field theory emphasizing the effects of the Goldstone and other soft bosonic modes of the action. The ultimate usefulness of this theory will be its ability to exactly capture the non-analytic behaviors of observables and correlation functions. In this section we will discuss some of the many profound effects soft modes can have.

The first thing to note is a soft correlation implies long time tails or long wavelength phenomena. Consider the imaginary time representation of the Green function

$$\begin{aligned} G(\tau, \boldsymbol{k}) &= \frac{1}{\beta} \sum_n \frac{1}{i\omega_n - \xi(\boldsymbol{k})} e^{-i\omega_n \tau} \\ &= \begin{cases} -e^{-\xi(\boldsymbol{k})\tau} \left[1 - n_F(\xi(\boldsymbol{k}))\right] & \text{for } \tau > 0 \\ e^{-\xi(\boldsymbol{k})\tau} \left[n_F(\xi(\boldsymbol{k}))\right] & \text{for } \tau < 0 \end{cases} \\ &\xrightarrow[T\to 0]{} \begin{cases} e^{-\xi(\boldsymbol{k})\tau} \theta(\xi(\boldsymbol{k})) & \text{for } \tau > 0 \\ e^{-\xi(\boldsymbol{k})\tau} \theta(-\xi(\boldsymbol{k})) & \text{for } \tau < 0 \end{cases}, \end{aligned} \tag{3.30}$$

where $\theta(x)$ denotes the Heaviside step function. The different limits for $\operatorname{sgn}\tau = \mp 1$ respectively mark the difference between advanced and retarded modes. We see at $T = 0$ the green function exponentially decays in imaginary time, with a half life $\tau_0 = 1/\xi(\boldsymbol{k})$, which diverges for particles at the Fermi surface $\xi(\boldsymbol{k_f}) = 0$. A real time analog of this example is found in the correlation function for diffusive processes, for example fluctuations in the magnetization of a fluid of uncharged, spin 1/2 particles

[54]. In such a case, the correlation function of magnetization $C(t, \boldsymbol{k})$ behaves as

$$C(t, \boldsymbol{k}) \propto e^{-Dk^2 t} \ , \tag{3.31}$$

for a diffusion coefficient D. The lifetime of this correlation function $\tau_0 = 1/Dk^2$, diverges for long wavelength fluctuations $k \to 0$. We can calculate the local real time correlation function

$$C(t, \boldsymbol{x} = 0) \frac{1}{V} \sum_{\boldsymbol{k}} C(t, \boldsymbol{k}) \propto \frac{1}{t^{d/2}} \ , \tag{3.32}$$

and see that local correlations decay algebraically in time. This is a quintessential property of soft correlations. We can further Laplace transform $C(t, \boldsymbol{x} = 0)$ to the complex frequency z-domain with $\mathrm{Im}\, z > 0$ to

$$C(z) = i \int_0^\infty dt \ e^{izt} C(t, \boldsymbol{x} = 0) \propto z^{\frac{d}{2}-1} \ , \tag{3.33}$$

which holds for when $(d/2)$ is not an even integer. $C(z)$ can now be seen to possess non-analytic dependence on the complex frequency z as $z \to 0$. If an observable couples to a soft mode, it can be expressed in terms of integrals of soft modes[3], and found to also have non-analytic dependences on frequency and/or wave vectors. A paradigmatic example is the density of states non-analytic dependence on frequency about the Fermi energy, also known as the zero bias anomaly. The non-analyticity can be directly observed in the dependence of the conductance on the applied voltage in scanning tunneling spectroscopy measurements used to probe the density of states [56]. In disordered metals and strongly doped semiconductors, Altshuler and Aronov

[3]Often it's a convolution of soft *and* massive modes.

showed in Reference [16] that

$$N(\mu+\omega) - N_F \propto |\omega|^{(d-2)/2} \ . \tag{3.34}$$

At zero temperature this result holds for frequencies near the Fermi energy, and at non-zero temperature it is true for frequencies that are large with respect to temperature. It follows from the coupling of massive modes in a disordered metal to the Goldstone modes, known as the diffusons. Non-analyticities in the density of states are especially important because many other observables, or susceptibilities are proportional to the density of states (e.g. conductivity) or inversely so (e.g. resitivity), these quantities then inherit the density of states non-analyticity, in addition to any other non-analyticities due to their particular structure.

The presence of interaction terms in the action, i.e. terms quartic and higher the fields, allow soft and massive modes to couple. In extreme cases, a mode that is massive at the level of a quadratic saddle point expansion can become soft. For example, consider a classical Heisenberg ferromagnet with the Hamiltonian

$$H = -J \sum_{\langle i,j \rangle} \boldsymbol{s}_i \cdot \boldsymbol{s}_j \ , \tag{3.35}$$

where J is the coupling constant, $\boldsymbol{s}_i$ are spins at lattice site i, and the sum is over nearest neighbors. Below the Curie temperature the system is ferromagnetic, with some magnetization $m_0\hat{z}$. Then at the saddle point level, the longitudinal magnetic susceptibility $\chi_L(\boldsymbol{k})$ is massive and the transverse susceptibility $\chi_T(\boldsymbol{k})$ is massless,

and are described by

$$\chi_L(\boldsymbol{k}) \propto \frac{1}{t + k^2} \ ,$$
$$\chi_T(\boldsymbol{k}) \propto \frac{1}{k^2} \ , \tag{3.36a}$$

with $t \propto (T_C - T)/T$ the dimensionless distance from the critical point. If higher order corrections about the saddle point are taken into account, the coupling of the longitudinal and transverse modes makes $\chi_L(\boldsymbol{k})$ massless in momentum space for dimensions $d < 4$, and algebraically decaying in real space for $d > 2$:

$$\chi_L(\boldsymbol{k} \to 0) \propto k^{d-4} \ , \tag{3.37}$$

$$\chi_L(\boldsymbol{r} \to \infty) \propto 1/r^{2d-4} \ . \tag{3.38}$$

In the presence of a magnetic field, $\chi_L(\boldsymbol{k})$ is once again massive, but the homogeneous longitudinal susceptibility diverges with vanishing field in $d < 4$ as a result of the mode coupling:

$$\chi_L = \partial m/\partial h \propto h^{(d-4)/2} \ . \tag{3.39}$$

These discoveries were first made by Vaks, Larkin and Pikin in 1968, and Brezin and Wallace in 1973 [57, 58], and showcase some of the profound consequences of the presence of soft modes on an entire phase.

One final consequence of soft modes we will mention is the modification of phase transitions. As an example, Belitz, Kirkpatrick and Vojta found in 1997 that the static spin susceptibility of a Fermi liquid exhibits a $T = 0$ non-analyticity of the form $\chi_s(\boldsymbol{q}) \propto q^2 \log q$ in $d = 3$ as a result of the soft modes in the system [59]. In

metallic ferromagnets with small Curie temperatures, this non-analytic behavior has the profound effect of modifying the nature of the second order ferromagnetic phase transition to be first order, see Reference [13] for a detailed review.

Until now, we have not discussed any non-analytic properties resulting directly from the soft, single particle Green function in Equation (3.15). It is indeed responsible for non-analytic functions appearing in physical observables, for example in a non-interacting clean Fermi gas in $d = 3$ dimensions with particle density n, the optical conductivity is

$$\sigma(\omega) = \frac{ne^2}{m_e}\frac{i}{\omega} \ , \tag{3.40}$$

reflecting the ballistic motion of the electron current in the presence of an electric force field. Now once again consider the density of states, which in d dimensions and for an energy momentum dispersion $\epsilon(\boldsymbol{k}) \propto k^p$ goes as

$$N(\omega) \sim (\mu + \omega)^{\frac{d}{p}-1} \ , \tag{3.41}$$

with ω measured from the Fermi surface. While $(\mu+\omega)^{\frac{d}{p}-1}$ is a non-analytic function for non-integer values of d/p at $\omega + \mu = 0$, it is analytic for $\omega \to 0$. Without interactions or disorder there is no non-analyticity in the density of states about the Fermi surface (unless $\mu = 0$). As stated at the beginning of this subsection, the program of the next chapter will be to integrate out the fermionic degrees of freedom of the partition function for interacting electrons in Dirac semimetals and graphene. One might be concerned about the fate of any non-analyticities due to non-interacting electrons, and whether such a program accounts for them. As the non-interacting fermion action is Gaussian (quadratic) in the fermionic fields, the

partition function can be solved exactly. Mathematically speaking, this means $\mathcal{Z}[J]$ can be computed exactly for any source functional J, that couples to terms linear or quadratic in $\psi_n(\boldsymbol{k})$, and thus all thermodynamic quantities and response functions can be computed. When integrating out the fermionic fields, the exact, non-interacting solution will be encoded in the saddle point of the new bosonic field theory theory, and thus so will any non-interacting fermionic induced non-analyticities.

CHAPTER IV

DERIVING THE EFFECTIVE FIELD THEORY

Fermionic Action

The quantum partition function is defined as [60–62]:

$$\mathcal{Z} = \int D[\bar{\psi}, \psi]\ e^{S[\bar{\psi},\psi]}, \tag{4.1}$$

where the action S for a semimetal is given by:

$$S = \int dx \sum_{\sigma'\sigma\alpha\alpha'} \bar{\psi}^{\alpha'}_{\sigma'}(x)\ \left[-\partial_\tau \delta^{\alpha'\alpha}_{\sigma'\sigma} - \epsilon^{\alpha'\alpha}_{\sigma'\sigma}(-i\boldsymbol{\nabla})\right]\ \psi^{\alpha}_{\sigma}(x) + S_{\text{int}}. \tag{4.2}$$

We have used space and imaginary time coordinates $x = (\boldsymbol{x}, \tau)$, $\int dx = \int_V d\boldsymbol{x} \int_0^{1/T} d\tau$, and the fields $\psi, \bar{\psi}$ are fermionic (Grassmann valued) fields describing the electrons in the system. We have adopted a unit system where $\hbar = 1$. In this work, we will be concerned with two types of semimetals: the $d = 2$ dimensional Dirac semimetal graphene, and $d = 3$ Dirac semimetals. We will always refer to the $d = 2$ case as graphene, and the $d = 3$ case as DSMs, for any discussion including both cases, we will speak of semimetals. For the DSM case we are letting σ be a genuine spin index, while α runs over the cone index[63]. For simplicity we limit ourselves to 2 cones, one can generalize to more cones if desirable. In graphene σ runs over the sub-lattices A and B of the 2D Carbon lattice (pseudo-spin), and α runs over the two non-degenerate

Dirac cones that appear in the Brillouin Zone. [1] For now we will be concerned with developing the theory for DSMs, an analogous program can be followed for graphene.

The explicit dispersion relation for DSMs as discussed in Chapter II is given by[4]:

$$\epsilon(\boldsymbol{\nabla})^{\alpha'\alpha}_{\sigma'\sigma} = -iv_D\,(\sigma_z \otimes \boldsymbol{\nabla}\cdot\boldsymbol{\sigma})^{\alpha'\alpha}_{\sigma'\sigma} \equiv -iv_D\sigma_z^{\alpha'\alpha}\boldsymbol{\nabla}\cdot\boldsymbol{\sigma}_{\sigma'\sigma'}\ , \tag{4.3}$$

and the interaction part S_{int} is given by:

$$S_{\text{int}} \;=\; -\frac{1}{2}\sum_{\sigma'\sigma\alpha'\alpha}\int dx_1 dx_2\ v(x_1-x_2)\bar{\psi}^{\alpha}_{\sigma}(x_1)\bar{\psi}^{\alpha'}_{\sigma'}(x_2)\psi^{\alpha'}_{\sigma'}(x_2)\psi^{\alpha}_{\sigma}(x_1)\ . \tag{4.4}$$

For a short ranged interaction in d-dimensions, $v(\boldsymbol{x}) \equiv \Gamma\delta^{(d)}(\boldsymbol{x})$, and a long ranged interaction takes the form $v(\boldsymbol{x}) = \Gamma/|\boldsymbol{x}|$. Switching to momentum space via Fourier transformation, defining four-momentum $k = (i\omega_n, \boldsymbol{k})$, we have the relations

$$\bar{\psi}(k) \;\equiv\; \bar{\psi}_n(\boldsymbol{k}) = \sqrt{T/V}\int dx\ e^{-ikx}\,\bar{\psi}(x)\ , \tag{4.5a}$$

$$\psi(k) \;\equiv\; \psi_n(\boldsymbol{k}) = \sqrt{T/V}\int dx\ e^{ikx}\,\psi(x)\ , \tag{4.5b}$$

$$\psi(x) \;\equiv\; \psi_n(\boldsymbol{k}) = \sqrt{T/V}\sum_{n,\boldsymbol{k}} e^{-ikx}\,\psi(k)\ . \tag{4.5c}$$

[1] To make the action (4.2) apply to graphene, one would need to include an additional resolution of identity acting on the real spin indices, two account for the spin degeneracy of graphene's Dirac cones.

Then in momentum space the action reads:

$$
\begin{aligned}
S &= S_0 + S_{\text{int}} \ , && \text{(4.6a)}\\
S_0 &= \sum_k \boldsymbol{\psi}^\dagger(k)\left(i\omega_n \mathbb{1} - v_D(\sigma_z \otimes \boldsymbol{k}\cdot\boldsymbol{\sigma})\right)\boldsymbol{\psi}(k)\\
&\equiv \sum_k \boldsymbol{\psi}^\dagger(k) G_0^{-1}(\omega_n, \boldsymbol{k})\boldsymbol{\psi}(k) \ , && \text{(4.6b)}\\
S_{\text{int}} &= -\frac{T}{2V}\sum_{\substack{\sigma\sigma'\\ \alpha\alpha'}}\sum_{\{k_i\}} \delta_{k_1+k_2,k_3+k_4} v(k_1-k_4)\bar{\psi}^\alpha_\sigma(k_1)\bar{\psi}^{\alpha'}_{\sigma'}(k_2)\psi^{\alpha'}_{\sigma'}(k_3)\psi^\alpha_\sigma(k_4) \ . && \text{(4.6c)}
\end{aligned}
$$

After Fourier transforming to momentum space, the short ranged interaction scales as a constant $v(\boldsymbol{k}) \propto \Gamma$, and the long ranged interaction scales as $v(\boldsymbol{k}) \propto k^{1-d}$ for $d = 2, 3$. Now, let $\alpha = \pm 1$, then the non-interacting Green function $\bar{G}_0$, in this basis is a diagonal matrix composed of the of the Green function for each Dirac cone G_0,

$$
\begin{aligned}
\bar{G}_0(i\omega_n, \boldsymbol{k}) &= -\frac{\omega_n \mathbb{1} + v_D(\sigma_z \otimes \boldsymbol{k}\cdot\boldsymbol{\sigma})}{\omega^2 + v_D^2 k^2}\\
&\equiv (G_0(i\omega_n, +\boldsymbol{k}), G_0(i\omega_n, -\boldsymbol{k})\\
&\equiv \delta^{\alpha\beta} G_0(i\omega_n, \alpha\boldsymbol{k}) \ . && \text{(4.7)}
\end{aligned}
$$

We are interested in long wavelength effects, so it will be convenient to decompose the interaction term into the direct, exchange and cooper interaction channels[21]. These are the different interaction processes that can take place involving small momentum transfer between quasiparticles. This decomposition is not exact, there are left over interactions that involve the exchange of large momentum, which will be unimportant to the EFT, so they are discarded. For a rigorous derivation see the Appendix A. Note that since we lack a Fermi surface to project the interaction term onto, a further angular momentum decomposition of the interaction is not possible, contrary to the

Fermi liquid case in Reference [64].

$$S_{\rm int} \approx S^{\rm d}_{\rm int} + S^{\rm e}_{\rm int} + S^{\rm c}_{\rm int} \ , \tag{4.8a}$$

where the three channels, direct, exchange and cooper are respectively given by:

$$S^{\rm d}_{\rm int} = \\ -\frac{T}{2V} \sum_{\alpha,\beta=\pm} \sum_{k,p} {\sum_{q}}' \Gamma^d(\boldsymbol{q})(\psi^\alpha(k), s_0\psi^\alpha(k+q))(\psi^\beta(p+q), s_0\psi^\beta(p)) \ , \tag{4.8b}$$

$$S^{\rm e}_{\rm int} = \\ -\frac{T}{2V} \sum_{\alpha,\beta=\pm} \sum_{i=0}^{3} \sum_{k,p} {\sum_{q}}' \Gamma^e_i(\boldsymbol{p}-\boldsymbol{k})(\psi^\alpha(k), s_i\psi^\alpha(k+q))(\psi(p+q), s_i\psi(p)) \ , \tag{4.8c}$$

$$S^{\rm c}_{\rm int} = \\ -\frac{T}{2V} \sum_{\substack{\alpha,\beta \\ \sigma,\sigma'}} \sum_{k,p} {\sum_{q}}' v(\boldsymbol{p}+\boldsymbol{k})\Theta(|\boldsymbol{p}+\boldsymbol{k}| > \lambda)\bar{\psi}^\alpha_\sigma(k)\bar{\psi}^\beta_{\sigma'}(-k+q)\psi^\beta_{\sigma'}(p+q)\psi^\alpha_\sigma(-p) \ . \tag{4.8d}$$

Here the inner product is the usual complex one, $(\psi, \phi) = \psi^\dagger\phi$, the variable $q = (i\Omega_n, \boldsymbol{q})$ comprises a wave vector $\boldsymbol{q}$ and a bosonic Matsubara frequency $\Omega_n = 2\pi T n$. ${\sum_q}'$ denotes a sum over wave vectors that is restricted to $\boldsymbol{q} < \Lambda$ with cutoff wave number Λ. The long-wavelength properties we are interested in do not depend on Λ. Note that without this cutoff all interaction channels would be the same. The channel specific interactions are given by $\Gamma^d(\boldsymbol{q}) = v(\boldsymbol{q})$ and $\Gamma^e_i(\boldsymbol{p}-\boldsymbol{k}) = \frac{1}{2}v(\boldsymbol{p}-\boldsymbol{k})\Theta(|\boldsymbol{p}-\boldsymbol{k}| > \Lambda)(-,+,+,+)_i$.

Converting the Action to a Bosonic Formulation

In order to map the fermionic action onto a bosonic one, we follow the procedure of Reference [65] of first defining bispinor fields η, which encompass the particle-hole degrees of freedom of the fermions:

$$\eta_n^\alpha(\boldsymbol{k}) = \frac{1}{\sqrt{2}}(\bar{\psi}_{n,\uparrow}^\alpha(-\boldsymbol{k}), \bar{\psi}_{n,\downarrow}^\alpha(-\boldsymbol{k}), \psi_{n,\downarrow}^\alpha(\boldsymbol{k}), -\psi_{n,\uparrow}^\alpha(\boldsymbol{k}))^T \ , \tag{4.9}$$

$$\eta_n^{\alpha+}(\boldsymbol{k}) = C\eta_n^\alpha(-\boldsymbol{k})\frac{i}{\sqrt{2}}(-\psi_{n,\uparrow}^\alpha(-\boldsymbol{k}), -\psi_{n,\downarrow}^\alpha(-\boldsymbol{k}), \bar{\psi}_{n,\downarrow}^\alpha(\boldsymbol{k}), -\bar{\psi}_{n,\uparrow}^\alpha(\boldsymbol{k})) \ , \tag{4.10}$$

where we have defined the charge conjugation matrix $C = i(\tau_1 \otimes s_2)$ in the spin-quaternion basis spanned by $\tau_i \otimes s_j$, $(i, j = 0, 1, 2, 3)$, with $\tau_j = -s_j = -i\sigma_j$, and σ_0 the 2×2 identity matrix. The quaternion matrices act on the particle-hole components of the bispinors, while the spin matrices act on the spin components. This can be seen explicitly by examining the action of $\tau_i \otimes s_0$ and $\tau_0 \otimes s_i$ on $\eta_n(\boldsymbol{k})$. Let us now write the action in the language of the bispinors. We construct a Green function for the spin-quaternion space by requiring:

$$S_0 = \sum_{\boldsymbol{k}} \psi^\dagger(k)\bar{G}_0^{-1}(k)\psi(k) = -i\sum_k \eta_n^+(\boldsymbol{k})\tilde{G}_0^{-1}(i\omega_n, -\boldsymbol{k})\eta_n(\boldsymbol{k}) \ , \tag{4.11}$$

where a contraction over all suppressed indices is implied. Using the notation $(G_0)^{-1}_{ij} \equiv l_{ij}$, we show in the appendix that the Green function must take the form:

$$
\begin{aligned}
&(\tilde{G}_0)^{-1}(i\omega_n, -\boldsymbol{k}) = \\
&\delta^{\alpha\beta} \begin{pmatrix} l_{11}(\omega_n, -\alpha\boldsymbol{k}) & l_{21}(\omega_n, -\alpha\boldsymbol{k}) & 0 & 0 \\ l_{12}(\omega_n, -\alpha\boldsymbol{k}) & l_{22}(\omega_n, -\alpha\boldsymbol{k}) & 0 & 0 \\ 0 & 0 & l_{22}(\omega_n, \alpha\boldsymbol{k}) & -l_{21}(\omega_n, \alpha\boldsymbol{k}) \\ 0 & 0 & -l_{12}(\omega_n, \alpha\boldsymbol{k}) & l_{11}(\omega_n, \alpha\boldsymbol{k}) \end{pmatrix} .
\end{aligned} \tag{4.12}
$$

For the DSM this reads explicitly:

$$
\begin{aligned}
&(\tilde{G}_0^{\alpha})^{-1}(i\omega_n, -\boldsymbol{k}) = \\
&= \alpha v_D \begin{pmatrix} i\omega_n/\alpha v_D + k_z & k_x + ik_y & & \\ k_x - ik_y & i\omega_n/\alpha v_D - kz & & \\ & & i\omega_n/\alpha v_D + k_z & k_x + ik_y \\ & & k_x - ik_y & i\omega_n/\alpha v_D - k_z \end{pmatrix} \\
&= \begin{pmatrix} (G_0^{-1}(i\omega_n, -\alpha\boldsymbol{k}))^T & \\ & (G_0^{-1}(i\omega_n, -\alpha\boldsymbol{k}))^T \end{pmatrix} .
\end{aligned}
$$

We now define a bilinear tensor product

$$
\begin{aligned}
&B^{\alpha\beta}_{nm}(\boldsymbol{x},\boldsymbol{y}) = \\
&= \eta^{\alpha+}_{n}(\boldsymbol{x}) \otimes \eta^{\beta}_{m}(\boldsymbol{y}) \\
&= \frac{i}{2}\begin{pmatrix}
-\psi_{n\uparrow}(\boldsymbol{x})\bar{\psi}_{m\uparrow}(\boldsymbol{y}) & -\psi_{n\uparrow}(\boldsymbol{x})\bar{\psi}_{m\downarrow}(\boldsymbol{y}) & -\psi_{n\uparrow}(\boldsymbol{x})\psi_{m\downarrow}(\boldsymbol{y}) & \psi_{n\uparrow}(\boldsymbol{x})\psi_{m\uparrow}(\boldsymbol{y}) \\
-\psi_{n\downarrow}(\boldsymbol{x})\bar{\psi}_{m\uparrow}(\boldsymbol{y}) & -\psi_{n\downarrow}(\boldsymbol{x})\bar{\psi}_{m\downarrow}(\boldsymbol{y}) & -\psi_{n\downarrow}(\boldsymbol{x})\psi_{m\downarrow}(\boldsymbol{y}) & \psi_{n\downarrow}(\boldsymbol{x})\psi_{m\uparrow}(\boldsymbol{y}) \\
\bar{\psi}_{n\downarrow}(\boldsymbol{x})\bar{\psi}_{m\uparrow}(\boldsymbol{y}) & \bar{\psi}_{n\downarrow}(\boldsymbol{x})\bar{\psi}_{m\downarrow}(\boldsymbol{y}) & \bar{\psi}_{n\downarrow}(\boldsymbol{x})\psi_{m\downarrow}(\boldsymbol{y}) & -\bar{\psi}_{n\downarrow}(\boldsymbol{x})\psi_{m\uparrow}(\boldsymbol{y}) \\
-\bar{\psi}_{n\uparrow}(\boldsymbol{x})\bar{\psi}_{m\uparrow}(\boldsymbol{y}) & -\bar{\psi}_{n\uparrow}(\boldsymbol{x})\bar{\psi}_{m\downarrow}(\boldsymbol{y}) & -\bar{\psi}_{n\uparrow}(\boldsymbol{x})\psi_{m\downarrow}(\boldsymbol{y}) & \bar{\psi}_{n\uparrow}(\boldsymbol{x})\psi_{m\uparrow}(\boldsymbol{y})
\end{pmatrix},
\end{aligned}
\tag{4.13}
$$

and its Fourier transform

$$
B^{\alpha\beta}_{nm}(\boldsymbol{k},\boldsymbol{p}) = \frac{1}{V}\int d\boldsymbol{x}\, d\boldsymbol{y}\; e^{-i\boldsymbol{k}\cdot\boldsymbol{x}+i\boldsymbol{p}\cdot\boldsymbol{y}} B^{\alpha\beta}_{nm}(\boldsymbol{x},\boldsymbol{y}) \;, \tag{4.14a}
$$

$$
B^{\alpha\beta}_{nm}(\boldsymbol{x},\boldsymbol{y}) = \frac{1}{V}\sum_{\boldsymbol{k},\boldsymbol{p}} e^{i\boldsymbol{k}\cdot\boldsymbol{x}-i\boldsymbol{p}\cdot\boldsymbol{y}} B^{\alpha\beta}_{nm}(\boldsymbol{k},\boldsymbol{p}) \;. \tag{4.14b}
$$

The 4×4 matrix $B^{\alpha\beta}_{nm}(\boldsymbol{k},\boldsymbol{p})$ can be expanded in the spin-quaternion basis defined above,

$$
B^{\alpha\beta}_{nm}(\boldsymbol{k},\boldsymbol{p}) \;=\; \sum_{i,r=0}^{3} {}^{i}_{r}B^{\alpha\beta}_{nm}(\boldsymbol{k},\boldsymbol{p})\,(\tau_r\otimes s_i) \;, \tag{4.15a}
$$

$$
{}^{i}_{r}B^{\alpha\beta}_{nm}(\boldsymbol{k},\boldsymbol{p}) \;\equiv\; \frac{1}{4}\mathrm{tr}\,\left[(\tau_r\otimes s_i)^{\dagger} B^{\alpha\beta}_{nm}(\boldsymbol{k},\boldsymbol{p})\right] \;. \tag{4.15b}
$$

It is further useful to define

$$
B^{\alpha\beta}_{nm}(\boldsymbol{k};\boldsymbol{q}) = B^{\alpha\beta}_{nm}(\boldsymbol{k}+\boldsymbol{q}/2,\boldsymbol{k}-\boldsymbol{q}/2) \;, \tag{4.16}
$$

with analogous definitions for other objects that depend on two wavevectors.

Q-matrix Field Theory

Our next step is to constrain the matrices B in the interaction terms to a classical matrix field Q by means of a Lagrange multiplier field $\tilde{\Lambda}$. The fermion fields then enter the action only bilinearly and can be integrated out exactly. This way we obtain an effective action $\mathcal{A}$ that depends on Q and $\tilde{\Lambda}$ according to

$$\begin{aligned}
\mathcal{Z} &= \int D[\bar{\psi},\psi] e^{S[\bar{\psi},\psi]} = \int D[\eta] e^{S[\eta]} \\
&= \int D[\eta] e^{S_0[\eta]+S_{\rm int}[B]} \int D[Q,\tilde{\Lambda}] e^{{\rm Tr}\,[\tilde{\Lambda}^T(Q-B)]} \\
&= \int D[\eta] e^{S_0[\eta]-{\rm Tr}\,[\tilde{\Lambda}^T B]} \int D[Q,\tilde{\Lambda}] e^{{\rm Tr}\,[\tilde{\Lambda}^T Q]+S_{\rm int}[Q]} \\
&= \int D[Q,\tilde{\Lambda}] e^{A[Q,\tilde{\Lambda}]} \ . \qquad (4.17)
\end{aligned}$$

Here and in what follows Tr denotes a trace over all degrees of freedom, including the continuous position in real space, while by tr we will denote a trace over all discrete degrees of freedom that are not explicitly shown. Note the components of $\eta_n(\boldsymbol{k})$ are not independent and $\det \tilde{G}_0 = (\det \bar{G}_0)^2$. Hence the following equality:

$$\int D[\bar{\psi},\psi] e^{{\rm Tr}\,\bar{\psi} G_0^{-1} \psi} = \int D[\eta] e^{-i{\rm Tr}\,\eta^+ \tilde{G}_0^{-1}\eta} = \det \bar{G}_0^{-1} = (\det \tilde{G}_0^{-1})^{1/2} = e^{\frac{1}{2}{\rm Tr}\,\ln \tilde{G}_0^{-1}} \ . \qquad (4.18)$$

Since ${\rm Tr}\,[\tilde{\Lambda}^T B] = \eta^+ \tilde{\Lambda} \eta$, it gets lumped in the with the log term. The new action A is:

$$A[Q,\tilde{\Lambda}] = A_0 + {\rm Tr}\,[\tilde{\Lambda}^T Q] + A_{\rm int}[Q] \ . \qquad (4.19)$$

Following the discussion above,

$$A_0 = \frac{1}{2}\mathrm{Tr}\,\ln(\tilde{G}_0^{-1} - i\tilde{\Lambda}) \equiv \frac{1}{2}\mathrm{Tr}\,\ln G^{-1}, \tag{4.20}$$

where

$$(\tilde{G}_0^{-1})^{\alpha\beta}_{nm}(\boldsymbol{k},\boldsymbol{p}) = \delta_{\boldsymbol{k},\boldsymbol{p}}\delta^{\alpha\beta}_{nm}(\tau_0 \otimes G_0^{-1}(i\omega_n, -\alpha\boldsymbol{k}))^T \ . \tag{4.21}$$

Next we use the fact that for the standard inner product on $\mathbb{R}^n$, $(\ ,\ )$, the following holds: $(\boldsymbol{a}, A\boldsymbol{b}) = a_i A_{ij} b_j = \mathrm{Tr}\,(A^T a \otimes b)$, where $\boldsymbol{a},\boldsymbol{b} \in \mathbb{R}^n$ and $A \in Mat_{n\times n}(\mathbb{C})$. This yields for the interaction terms:

$$A_{\mathrm{int}} = A^{\mathrm{d}}_{\mathrm{int}} + A^{\mathrm{e}}_{\mathrm{int}} + A^{\mathrm{c}}_{\mathrm{int}} \ , \tag{4.22a}$$

with the direct, exchange and cooper channels having the explicit forms

$$\begin{aligned} A^{\mathrm{d}}_{\mathrm{int}} \ &= \ \frac{T}{2V}\sum_{r=0,3}(-1)^r \sum_{\substack{n_1,n_2\\ n_3,n_4}} \delta_{n_1-n_2,n_4-n_3} \sum_{\boldsymbol{k},\boldsymbol{p}} \sum_{q\ll\Lambda} \Gamma^d(\boldsymbol{q}) \times \\ &\quad \times \mathrm{tr}\,((\tau_r \otimes s_0)^T Q^{\alpha\alpha}_{n_1 n_2}(\boldsymbol{k},\boldsymbol{k}+\boldsymbol{q}))\mathrm{tr}\,((\tau_r \otimes s_0)^T Q^{\beta\beta}_{n_3 n_4}(\boldsymbol{p}+\boldsymbol{q},\boldsymbol{p}))\ , \end{aligned} \tag{4.22b}$$

$$\begin{aligned} A^{\mathrm{e}}_{\mathrm{int}} \ &= \ \frac{T}{2V}\sum_{r=0,3}(-1)^r \sum_{i=1}^{3}\sum_{\substack{n_1,n_2\\ n_3,n_4}} \delta_{n_1-n_2,n_4-n_3} \sum_{\boldsymbol{k},\boldsymbol{p}} \sum_{q\ll\Lambda} \Gamma^e_i(\boldsymbol{p}-\boldsymbol{k}) \times \\ &\quad \times \mathrm{tr}\,((\tau_r \otimes s_i)^T Q^{\alpha\beta}_{n_1 n_2}(\boldsymbol{k},\boldsymbol{k}+\boldsymbol{q}))\mathrm{tr}\,((\tau_r \otimes s_i)^T Q^{\beta\alpha}_{n_3 n_4}(\boldsymbol{p}+\boldsymbol{q},\boldsymbol{p}))\ , \end{aligned} \tag{4.22c}$$

$$A_{\rm int}^{{\rm c}(s)} = \frac{T}{4V}\sum_{r=1,2}\sum_{\substack{n_1,n_2\\ m}}\sum_{\boldsymbol{k},\boldsymbol{p}}\sum_{q\ll\Lambda}\Gamma^{c(s)}{}_{\boldsymbol{k},\boldsymbol{p}}^{\alpha\beta\gamma\delta}(\boldsymbol{q})\times$$
$$\times{\rm tr}\left((\tau_r\otimes s_0)^T Q^{\alpha\beta}_{n_1,-n_1+m}(-\boldsymbol{k},\boldsymbol{k}+\boldsymbol{q})\right){\rm tr}\left((\tau_r\otimes s_0)^T Q^{\gamma\delta}_{-n_2,n_2+m}(-\boldsymbol{p},-\boldsymbol{p}-\boldsymbol{q})\right)\,, \tag{4.22d}$$

$$A_{\rm int}^{{\rm c}(t)} = \frac{T}{4V}\sum_{r=1,2}\sum_{i=1}^{3}\sum_{\substack{n_1,n_2\\ m}}\sum_{\boldsymbol{k},\boldsymbol{p}}\sum_{q\ll\Lambda}\Gamma^{c(t)}{}_{\boldsymbol{k},\boldsymbol{p}}^{\alpha\beta\gamma\delta}(\boldsymbol{q})\times$$
$$\times{\rm tr}\left((\tau_r\otimes s_i)^T Q^{\alpha\beta}_{n_1,-n_1+m}(-\boldsymbol{k},\boldsymbol{k}+\boldsymbol{q})\right){\rm tr}\left((\tau_r\otimes s_i)^T Q^{\gamma\delta}_{-n_2,n_2+m}(-\boldsymbol{p},-\boldsymbol{p}-\boldsymbol{q})\right)\,. \tag{4.22e}$$

We have further decomposed the Cooper channel into its singlet and triplet components. In this case the interaction coefficients in the cooper channel are given in Appendix A.

Symmetry Properties, and Representation of Observables

We now provide some useful symmetry properties of the Q-matrices derived in Reference [14]. B as defined in Equation (4.13) is self-adjoint under the operation defined in Equation (4.10). Q inherits this property, so it holds that

$$Q^+ = C^T\,Q^T\,C = Q\,. \tag{4.23a}$$

In the spin-quaternion basis defined in Eq.(4.15) this implies

$$ {}^{i}_{r}Q^{\alpha\beta}_{nm}(\boldsymbol{x},\boldsymbol{y}) = \begin{pmatrix} + \\ + \\ + \\ - \end{pmatrix}_r \begin{pmatrix} + \\ - \\ - \\ - \end{pmatrix}_i {}^{i}_{r}Q^{\beta\alpha}_{mn}(\boldsymbol{y},\boldsymbol{x}) \ , \tag{4.23b} $$

$$ {}^{i}_{r}Q^{\alpha\beta}_{nm}(\boldsymbol{k},\boldsymbol{p}) = \begin{pmatrix} + \\ + \\ + \\ - \end{pmatrix}_r \begin{pmatrix} + \\ - \\ - \\ - \end{pmatrix}_i {}^{i}_{r}Q^{\beta\alpha}_{mn}(-\boldsymbol{p},-\boldsymbol{k}) \ , \tag{4.23c} $$

$$ {}^{i}_{r}Q^{\alpha\beta}_{nm}(\boldsymbol{k};\boldsymbol{q}) = \begin{pmatrix} + \\ + \\ + \\ - \end{pmatrix}_r \begin{pmatrix} + \\ - \\ - \\ - \end{pmatrix}_i {}^{i}_{r}Q^{\beta\alpha}_{mn}(-\boldsymbol{k};\boldsymbol{q}) \ . \tag{4.23d} $$

Here the symbols $\begin{pmatrix} + \\ + \\ + \\ - \end{pmatrix}_r$ etc. denote a factor of $+1$ for $r = 0, 1, 2$, and a factor of -1 for $r = 3$, and analogously for i. These relations imply that all of the Q-matrix elements are not independent. In a model with N Matsubara frequencies, only $N(N+1)/2$ matrix elements are independent. We will later choose these to be the ones with $n \geq m$. Physical correlation functions can be easily derived using the source functional formalism [21, 66]. By maintaining the appropriate source terms in the fermionic action while transforming to the Bosonic one, one can derive the relevant correlation functions in terms of Q-matrices. For the purposes of this dissertation, we will be most interested in the density of states, which is given by:

$$ N(\omega) = -\frac{1}{\pi} \mathrm{Im}\, \mathrm{Tr}\, G^{+}(\omega) \ . \tag{4.24} $$

Here $G^{+}(\omega)$ is the fermionic retarded Green function, and we have suppressed all additional indices that are traced over by Tr . It is obtained by analytic continuation

of the Matsubara Green function defined in Equation (4.7)

$$G^{+}(\omega) \equiv G(i\omega_n \to \omega + i0^{+}) \ . \tag{4.25}$$

To compute $N(\omega)$ we need the diagonal components of $G(i\omega_n, \boldsymbol{x} - \boldsymbol{y})$, which in the Q-matrix formalism are given by:

$$G^{\alpha}_{\sigma\sigma}(i\omega_n, \boldsymbol{x} - \boldsymbol{y}) = \frac{i}{2}\mathrm{tr}\ \left[(\tau_0 + i\tau_3) \otimes (s_0 - \sigma i s_3)\left\langle Q^{\alpha\alpha}_{nn}(\boldsymbol{x}, \boldsymbol{y})\right\rangle\right] \ , \tag{4.26}$$

for $\sigma = +1, -1$ corresponding to spin $\uparrow, \downarrow$ respectively. Using $\mathrm{Im}\, iQ = \mathrm{Re}\, Q$, the density of states is calculated in the Q-matrix formalism as:

$$N(\omega) \ = \ \frac{4}{\pi}\mathrm{Re}\,\mathrm{tr}\ \left\langle {}^{0}_{0}Q^{\alpha\alpha}_{nn}(\boldsymbol{x}, \boldsymbol{x})\right\rangle\Big|_{i\omega_n \to \omega + i0^{+}} \ . \tag{4.27}$$

Saddle Points

We minimize the action A with respect to $Q^{\alpha\beta}_{mn}(\boldsymbol{k},\boldsymbol{p})$ and $\tilde{\Lambda}^{\alpha\beta}_{mn}(\boldsymbol{k},\boldsymbol{p})$. This is a straightforward functional derivative problem when using the following two properties. First in momentum space the $\mathrm{Tr}\,(\tilde{\Lambda}^T Q)$ term takes the form:

$$\begin{aligned}\mathrm{Tr}\,(\tilde{\Lambda}^T Q) &= \mathrm{tr}\int d\boldsymbol{x}d\boldsymbol{y}(\tilde{\Lambda}^T)^{\alpha\beta}_{nm}(\boldsymbol{x},\boldsymbol{y})Q^{\alpha\beta}_{nm}(\boldsymbol{x},\boldsymbol{y}) \\ &= \mathrm{tr}\,(\tilde{\Lambda}^T)^{\alpha\beta}_{nm}(\boldsymbol{k},\boldsymbol{p})Q^{\alpha\beta}_{nm}(-\boldsymbol{k},-\boldsymbol{p}),\end{aligned} \tag{4.28}$$

where the transpose acts on the spin-quaternion basis in the last equality, and repeated indices are summed over. Secondly, for any function $f(\hat{A})$ of matrices $\hat{A}$, the derivative acting on the trace of $f(\hat{A})$ has the property

$$\partial_X \mathrm{tr}\,(f(\hat{A})) = \mathrm{tr}\,(f'(\hat{A})\partial_X \hat{A})\ . \tag{4.29}$$

The saddle point equations can then be expressed as

$$0 = -\frac{i}{2}(G^T)^{\beta\alpha}_{nm}(\boldsymbol{p},\boldsymbol{k}) + (Q^{\mathrm{sp}})^{\alpha\beta}_{mn}(-\boldsymbol{k},-\boldsymbol{p}), \tag{4.30a}$$

$$0 = \tilde{\Lambda}^{\mathrm{sp}}_{mn}(\boldsymbol{k},\boldsymbol{p}) + \left.\frac{\delta}{\delta Q_{mn}(\boldsymbol{k},\boldsymbol{p})}\right|_{Q^{\mathrm{sp}}} A_{\mathrm{int}}[Q], \tag{4.30b}$$

with $G = (\tilde{G}_0^{-1} - i\tilde{\Lambda}^{\mathrm{sp}})^{-1}$ the saddle point Green function. Using the decomposition of A_{int}, we can write out the last term in the equation above more explicitly using $\partial_M \mathrm{tr}\,(A^T M) \equiv A$:

$$\frac{\delta A_{\mathrm{int}}}{\delta Q^{\alpha\beta}_{mn}(\boldsymbol{k},\boldsymbol{p})} = \underbrace{\frac{\delta A^{\mathrm{d}}_{\mathrm{int}}}{\delta Q^{\alpha\beta}_{mn}(\boldsymbol{k},\boldsymbol{p})}}_{\text{(i)}} + \underbrace{\frac{\delta A^{\mathrm{e}}_{\mathrm{int}}}{\delta Q^{\alpha\beta}_{mn}(\boldsymbol{k},\boldsymbol{p})}}_{\text{(ii)}} + \underbrace{\frac{\delta A^{\mathrm{c}}_{\mathrm{int}}}{\delta Q^{\alpha\beta}_{mn}(\boldsymbol{k},\boldsymbol{p})}}_{\text{(iii)}}\ , \tag{4.31a}$$

$$
\begin{aligned}
\text{(i)} &= \frac{T}{V}\sum_{r=0,3}(-1)^r(\tau_r\otimes s_0)\sum_{\substack{n_1,n_2\\ \boldsymbol{s}}}{\sum_{\boldsymbol{q}}}'\delta_{n_1-n_2,n-m}\delta^{\alpha\beta}\Gamma^d(\boldsymbol{q})\times \\
&\quad\times\delta_{\boldsymbol{k}+\boldsymbol{q},\boldsymbol{p}}\mathrm{tr}\,\left((\tau_r\otimes s_0)^T Q^{\beta\beta}_{n_1n_2}(\boldsymbol{s}+\boldsymbol{q},\boldsymbol{s})\right)\ , \qquad (4.31b)\\
\text{(ii)} &= \frac{T}{2V}\sum_{r=0,3}(-1)^r\sum_{i=0}^{3}(\tau_r\otimes s_i)\sum_{\substack{n_1,n_2\\ \boldsymbol{s}}}{\sum_{\boldsymbol{q}}}'\delta_{n_1-n_2,n-m}(\tau_r\otimes s_i)^T\times \\
&\quad\times\left[\delta_{\boldsymbol{p},\boldsymbol{k}+\boldsymbol{q}}\Gamma^e_i(\boldsymbol{k}-\boldsymbol{s})Q^{\beta\alpha}_{n_1n_2}(\boldsymbol{s}+\boldsymbol{q},\boldsymbol{s})+\delta_{\boldsymbol{k},\boldsymbol{p}+\boldsymbol{q}}\Gamma^e_i(\boldsymbol{s}-\boldsymbol{p})Q^{\beta\alpha}_{n_1n_2}(\boldsymbol{s},\boldsymbol{s}+\boldsymbol{q})\right]\ . \qquad (4.31c)
\end{aligned}
$$

We have ignored the Cooper channel (iii), because its effects will not be included in field theory we will build. If one's aim were to build an effective field theory to explore s and p-wave superconductivity in semimetals, the singlet and triplet Cooper channels, respectively, would need to be accounted for.

Dirac Semimetal Ansatz

There are multiple solutions to the saddle point equation, the one we will consider maintains the Dirac semimetal spin structure, which can be determined by computing $\mathrm{tr}\,((\tau_r\otimes s_i)^\dagger\times l)$, for all r and i, and l the operator of the non-interacting action in bispinor space. The ansatz for the DSM solution then takes the form:

$$
\begin{aligned}
(Q^{\mathrm{sp}})^{\alpha\beta}_{mn}(\boldsymbol{k},\boldsymbol{p}) &= \delta_{\boldsymbol{k},\boldsymbol{p}}\delta^{\alpha\beta}_{mn}\tau_0\otimes\left[{}^0Q^\alpha_n(\boldsymbol{p})s_0+{}^1Q^\alpha_n(\boldsymbol{p})s_1+{}^2Q^\alpha_n(\boldsymbol{p})s_2+{}^3Q^\alpha_n(\boldsymbol{p})s_3\right]\\
&\equiv Q^\alpha_n(\boldsymbol{p})\delta_{\boldsymbol{k},\boldsymbol{p}}\delta^{\alpha\beta}_{mn}\ , \qquad (4.32)\\
(\tilde{\Lambda}^{\mathrm{sp}})^{\alpha\beta}_{mn}(\boldsymbol{k},\boldsymbol{p}) &= \delta_{\boldsymbol{k},\boldsymbol{p}}\delta^{\alpha\beta}_{mn}\tau_0\otimes\left[-{}^0\tilde{\Lambda}^\alpha_n(\boldsymbol{p})s_0+{}^1\tilde{\Lambda}^\alpha_n(\boldsymbol{p})s_1-{}^2\tilde{\Lambda}^\alpha_n(\boldsymbol{p})s_2+{}^3\tilde{\Lambda}^\alpha_n(\boldsymbol{p})s_3\right]\\
&\equiv \tilde{\Lambda}^\alpha_n(\boldsymbol{p})\delta_{\boldsymbol{k},\boldsymbol{p}}\delta^{\alpha\beta}_{mn}\ . \qquad (4.33)
\end{aligned}
$$

Note that the alternating signs of the $\tilde{\Lambda}$ ansatz are just in place to give $-i\tilde{\Lambda}^{\rm sp}_{mn}(\boldsymbol{k},\boldsymbol{p})$ the explicit form:

$$-i(\tilde{\Lambda}^{\rm sp})^{\alpha\beta}_{mn}(\boldsymbol{k},\boldsymbol{p}) = \delta_{\boldsymbol{k},\boldsymbol{p}}\delta^{\alpha\beta}_{mn}\begin{pmatrix} \Lambda_{11} & 0 \\ 0 & \Lambda_{22} \end{pmatrix} , \tag{4.34}$$

$$\begin{aligned}
\Lambda_{11} &= \begin{pmatrix} i\ {}^0\tilde{\Lambda}^{\alpha}_n(\boldsymbol{p}) + {}^3\tilde{\Lambda}^{\alpha}_n(\boldsymbol{p}) & {}^1\tilde{\Lambda}^{\alpha}_n(\boldsymbol{p}) + i\ {}^2\tilde{\Lambda}^{\alpha}_n(\boldsymbol{p}) \\ {}^1\tilde{\Lambda}^{\alpha}_n(\boldsymbol{p}) - i\ {}^2\tilde{\Lambda}^{\alpha}_n(\boldsymbol{p}) & i\ {}^0\tilde{\Lambda}^{\alpha}_n(\boldsymbol{p}) - {}^3\tilde{\Lambda}^{\alpha}_n(\boldsymbol{p}) \end{pmatrix} , \\
\Lambda_{22} &= \begin{pmatrix} i\ {}^0\tilde{\Lambda}^{\alpha}_n(\boldsymbol{p}) + {}^3\tilde{\Lambda}^{\alpha}_n(\boldsymbol{p}) & {}^1\tilde{\Lambda}^{\alpha}_n(\boldsymbol{p}) + i\ {}^2\tilde{\Lambda}^{\alpha}_n(\boldsymbol{p}) \\ {}^1\tilde{\Lambda}^{\alpha}_n(\boldsymbol{p}) - i\ {}^2\tilde{\Lambda}^{\alpha}_n(\boldsymbol{p}) & i\ {}^0\tilde{\Lambda}^{\alpha}_n(\boldsymbol{p}) - {}^3\tilde{\Lambda}^{\alpha}_n(\boldsymbol{p}) \end{pmatrix} .
\end{aligned}$$

Defining $\tilde{\boldsymbol{\Lambda}} = ({}^1\tilde{\Lambda}, {}^2\tilde{\Lambda}, {}^3\tilde{\Lambda})$, in turn allows us to write

$$-i(\tilde{\Lambda}^{\rm sp})^{\alpha\beta}_{mn}(\boldsymbol{k},\boldsymbol{p}) \equiv \delta_{\boldsymbol{k},\boldsymbol{p}}\delta^{\alpha\beta}_{mn}\tilde{\Lambda}^{\rm sp}_n(\boldsymbol{p}) = \delta_{\boldsymbol{k},\boldsymbol{p}}\delta^{\alpha\beta}_{mn}\tilde{G}_0^{-1}(i\ {}^0\tilde{\Lambda}^{\alpha}_n(\boldsymbol{p}), \tilde{\boldsymbol{\Lambda}}^{\alpha}_n(\boldsymbol{p})/v_D) , \tag{4.35}$$

Thus, by the linearity of the Hamiltonian the saddle point Green function reads:

$$G^{\alpha\beta}_{nm}(\boldsymbol{p},\boldsymbol{k}) = \delta_{\boldsymbol{k},\boldsymbol{p}}\delta^{\alpha\beta}_{mn}\tilde{G}_0(i\omega_n + i\ {}^0\tilde{\Lambda}^{\alpha}_n(\boldsymbol{p}), \alpha\boldsymbol{p} + \tilde{\boldsymbol{\Lambda}}^{\alpha}_n(\boldsymbol{p})/v_D) \tag{4.36}$$

If we assume charge neutrality, then $\Gamma(\boldsymbol{q}=0)=0$ as it is just the integral of the potential over all space.[2] Then by the saddle point equations (4.30) we have:

$$\tilde{\Lambda}^{\alpha}_{n}(\boldsymbol{p})\delta_{\boldsymbol{k},\boldsymbol{p}}\delta^{\alpha\beta}_{mn} \;=\; -\delta_{\boldsymbol{p},\boldsymbol{k}}\delta^{\alpha\beta}_{nm}\frac{T}{V}\sum_{\boldsymbol{s},n_1}(\tau_0\otimes s_i)\Gamma^{e}_{i}(\boldsymbol{p}-\boldsymbol{s})\mathrm{tr}\,\left[(\tau_0\otimes s_i)^{T}Q^{\alpha}_{n_1}(\boldsymbol{s})\right]. \tag{4.37}$$

Using $(\tau_0\otimes s_i)^T = \begin{pmatrix} + \\ - \\ + \\ - \end{pmatrix}_i (\tau_0\otimes s_i)^{\dagger}$, the forms of Q and $\tilde{\Lambda}$ and the saddle point equation for Q we get:

$${}^{i}\tilde{\Lambda}^{\alpha}_{n}(\boldsymbol{p}) = \frac{2T}{V}\sum_{m,\boldsymbol{s}}\frac{\Gamma^{e}_{i}(\boldsymbol{p}-\boldsymbol{s})}{\tilde{\omega}^{2}_{m}+v_D\tilde{s}^{2}}\begin{pmatrix} \tilde{\omega}_m \\ \alpha\tilde{s}_x v_D \\ \alpha\tilde{s}_y v_D \\ \alpha\tilde{s}_z v_D \end{pmatrix}_i, \tag{4.38}$$

where we have defined $\tilde{\omega}_m \equiv \omega_m + {}^{0}\tilde{\Lambda}^{\alpha}_{m}(\boldsymbol{s})$ and $\alpha v_D\tilde{s}_i = \alpha v_D s_i + {}^{i}\tilde{\Lambda}^{\alpha}_{m}(\boldsymbol{s})$. Equation (4.38) can then be solved iteratively. Note that this is equivalent to computing the electron self energy. We obtain

$${}^{0}\tilde{\Lambda}^{\alpha}_{n}(\boldsymbol{p}) \;=\; 0\,, \tag{4.39a}$$

$${}^{i}\tilde{\Lambda}^{\alpha}_{n}(\boldsymbol{p}) \;=\; \frac{T}{V}\sum_{m}\sum_{|\boldsymbol{p}-\boldsymbol{s}|>\Lambda} v(\boldsymbol{p}-\boldsymbol{s})e^{i\omega_m 0}\frac{v_D s_i}{\omega^{2}_{m}+v^{2}_{D}s^{2}}\,. \tag{4.39b}$$

In the first line we used the oddness of the ω_m sum, assuming ${}^{0}\tilde{\Lambda}^{\alpha}_{m}(\boldsymbol{s}) \equiv f^{\alpha}(\boldsymbol{s})$ is some function independent of m, see Appendix D. In the second line, we set ${}^{i}\tilde{\Lambda}^{\alpha}_{n}(\boldsymbol{s})\equiv 0$ and plugged this into (4.38). It is easy to see Eq. (4.39b) is UV divergent. Here it is important to recall that the Hamiltonians we are dealing with are linear approximations about a Dirac point, therefore it is necessary to impose a UV cutoff Λ' that reflects the transition from linear to quadratic or higher power momentum

[2]This follows from the Jellium model, see Reference [61].

dispersion relations (Crystals have smooth cutoffs [67]). Remember there was freedom in how we chose our cutoff Λ for the interaction terms, so the result of the integral Eq. (4.39b) ultimately depends on the relative values of Λ and Λ'. If they are the same, the integral is zero, if they are different we just get ${}^{i}\tilde{\Lambda}^{\alpha}_{n}(\boldsymbol{p}) \propto \Gamma p_i$ in all interaction cases and dimensions. This is ultimately a renormalization of the Dirac velocity and can be absorbed into the definition of v_D. We can write our saddle point Q solution as:

$$\begin{aligned} Q^{\alpha\beta}_{mn}(\boldsymbol{k},\boldsymbol{p}) &= \delta_{\boldsymbol{k},\boldsymbol{p}}\delta^{\alpha\beta}_{mn}Q^{\alpha}_{n}(\boldsymbol{p}) \\ &= \frac{i}{2}\delta_{\boldsymbol{k},\boldsymbol{p}}\delta^{\alpha\beta}_{mn}\tilde{G}_0(i\omega_n,-\alpha\boldsymbol{p})^T \\ &= -\frac{i}{2}\delta_{\boldsymbol{k},\boldsymbol{p}}\delta^{\alpha\beta}_{mn}\frac{\tau_0\otimes(i\omega_n\sigma_0-\alpha v_D\boldsymbol{p}\cdot\boldsymbol{\sigma})}{\omega_n^2+p^2v_D^2}\,. \end{aligned} \tag{4.40}$$

Now observe that if we consider the small frequency limit of the Q saddle point:

$$\lim_{n\to 0} Q^{\alpha}_{n}(\boldsymbol{p}) = \tau_0\otimes\left(\frac{\pi}{2}\operatorname{sgn}\omega_n\delta(v_Dp)\sigma_0+\mathrm{P}\;\frac{i}{2}\frac{\hat{\boldsymbol{p}}\cdot\boldsymbol{\sigma}}{v_Dp}\right), \tag{4.41}$$

where P denotes the Cauchy principal value. We can see that this saddle point is not invariant under rotations in ω_n-space between frequencies of different signs (i.e. mixing of retarded and advanced degrees of freedom), however, in the limit of zero frequency, the action A is invariant. Following our discussion in Section 3.2, by the Goldstone theorem we know that soft modes must be present. At this point, one might draw on earlier works [14, 64, 68], and immediately integrate out the non-Goldstone modes. We will show this would pose issues due to the fact that the non-Goldstone modes are still soft, but not for symmetry breaking reasons.

Ward Identity

Generalized bosonic basis for all two band systems

We would like to develop a Ward Identity to determine the soft modes of the system. To do this, we note that the bare action can be viewed in another way. Let us restrict ourselves to discussing the physics of a single Dirac cone, and diagonalize the Hamiltonian. In this section the index σ is a band index, such that $\sigma = 0, 1$ corresponds to the conduction and valence band respectively. This new basis is equivalent to thinking of spinless fermions now with a band index determined by σ, and will prove to be the best formalism to directly compare the Fermi liquid, semiconductor and semimetal Ward identities. In the band basis, the bare action of the system is given by:

$$S_0 = \sum_{n,\boldsymbol{k}} \sum_{\sigma} \bar{\psi}^{\sigma}(k) \left[i\omega_n - \epsilon^{\sigma}(\boldsymbol{k}) + \mu \right] \psi^{\sigma}(k) \ . \tag{4.42}$$

	Semimetal	Fermi Liquid	Semiconductor
$\xi^0(\boldsymbol{k})$	$v_D\lvert\boldsymbol{k}\rvert$	$k^2/2m + E_g - \mu$ $\equiv k^2/2m - \mu_{eff}$	$k^2/2m + E_g - \mu$ $\equiv k^2/2m + \mu_{eff}$
$\xi^1(\boldsymbol{k})$	$-v_D\lvert\boldsymbol{k}\rvert$	$-k^2/2m - \mu$	$-k^2/2m - \mu$

TABLE 2. Forms of $H - \mu N$ for semimetals, metals and semiconductors.
$H - \mu N$ for semimetals, metals and semiconductors. E_g and μ_{eff} are always positive, but for metals $\mu > E_g$ (the chemical potential is in the conduction band), and for semiconductors $\mu < E_g$ (the chemical potential lies in the gap). One could also consider a quadratic semimetal, which is the gapless case of a semiconductor with $E_g = \mu = 0$.

We want to allow this action to describe semimetals, metals and insulators. Therefore ϵ^{σ} is general, and we include a chemical potential μ. Then defining $\xi^{\sigma}(\boldsymbol{k}) = \epsilon^{\sigma}(\boldsymbol{k}) - \mu$, we list the possible cases in Table 2. In the case of semimetals,

the transformation matrix $U(\boldsymbol{k})$ to bring $\hat{H}$ to this basis is momentum dependent. Therefore the interaction term S_{int} acquires additional momentum structure, making this basis ill suited for formulating the effective field theory. It is, however, the ideal basis for computing the Ward identity, as will see. We redefine the spinors to be given by:

$$\eta_n^\sigma(\boldsymbol{k}) = \frac{1}{\sqrt{2}}(\psi^\sigma(\boldsymbol{k}), -\bar{\psi}_n^\sigma(-\boldsymbol{k}))^T \ , \tag{4.43a}$$

$$\eta_n^{+\sigma}(\boldsymbol{k}) = (\tau_2\eta_n^\sigma(-\boldsymbol{k}))^T = \frac{1}{\sqrt{2}}(\bar{\psi}^\sigma(\boldsymbol{k}), \psi_n^\sigma(-\boldsymbol{k})) \ . \tag{4.43b}$$

Then $B = \eta^+ \otimes \eta$ as usual, except now we decompose B and Q matrices as $Q_{nm}^{\sigma_1\sigma_2}(\boldsymbol{p},\boldsymbol{q}) = \sum_{r=0}^{3} {}_rQ_{nm}^{\sigma_1\sigma_2}(\boldsymbol{p},\boldsymbol{q})\tau_r$, etc. For $[Q_{nm}^{\sigma_1\sigma_2}(\boldsymbol{x},\boldsymbol{y})]^T = (Q^T)_{mn}^{\sigma_2\sigma_1}(\boldsymbol{y},\boldsymbol{x})$, with (Q^T) denoting a transpose acting on the τ basis, we have that

$$Q^+ \equiv \tau_2^T Q^T \tau_2 = Q \ . \tag{4.44a}$$

This results in the following properties:

$${}_rQ_{nm}^{\sigma_1\sigma_2}(\boldsymbol{x},\boldsymbol{y}) = \begin{pmatrix} + \\ - \\ - \\ - \end{pmatrix}_r {}_rQ_{mn}^{\sigma_2\sigma_1}(\boldsymbol{y},\boldsymbol{x}) \ , \tag{4.44b}$$

$${}_rQ_{nm}^{\sigma_1\sigma_2}(\boldsymbol{p},\boldsymbol{q}) = \begin{pmatrix} + \\ - \\ - \\ - \end{pmatrix}_r {}_rQ_{mn}^{\sigma_2\sigma_1}(-\boldsymbol{q},-\boldsymbol{p}) \ . \tag{4.44c}$$

The diagonal basis makes all the dispersion relations even in momentum, so the Green function takes on a simple general form. The action can once again be mapped to a

bosonic one using the techniques of Section 4.3. We get:

$$A[Q,\tilde{\Lambda}] = A_0 + \mathrm{Tr}\,[\tilde{\Lambda}^T Q] + A_{\mathrm{int}}[Q]\ , \tag{4.45a}$$

where

$$A_0 = \frac{1}{2}\mathrm{Tr}\,\ln\left(G_0^{-1} - i\tilde{\Lambda}\right)\ , \tag{4.45b}$$

$$(G_0^{-1})^{\sigma_1\sigma_2}_{nm}(\boldsymbol{k},\boldsymbol{p}) = \tau_0 \frac{\delta^{\sigma_1\sigma_2}\delta_{nm}\delta_{\boldsymbol{k},\boldsymbol{p}}}{i\omega_n - \xi^{\sigma_1}(p)}\ . \tag{4.45c}$$

Ward Identity for Noninteracting Electrons

Consider transformations of the bispinors given by:

$$\eta_n(\boldsymbol{x}) \to \int d\boldsymbol{y}\ \hat{T}^{(\pm)}_{nm}(\boldsymbol{x},\boldsymbol{y})\eta_m(\boldsymbol{y})\ , \tag{4.46a}$$

where the operator $\hat{T}^{(\pm)}$ defines non-local infinitesimal rotations in frequency-band space:

$$\hat{T}^{(\pm)}_{nm}(\boldsymbol{x},\boldsymbol{y}) = \tau_0 (t^{(\pm)})^{\alpha\beta}_{nm}(\boldsymbol{x},\boldsymbol{y})\ , \tag{4.46b}$$

$$(t^{(\pm)})^{\alpha\beta}_{nm}(\boldsymbol{x},\boldsymbol{y}) = \delta_{nm}\delta^{\alpha\beta}\delta(\boldsymbol{x}-\boldsymbol{y}) + \left[\delta_{n1}\delta_{m2}\delta^{\alpha\sigma}\delta^{\beta\tau} \mp \delta_{n2}\delta_{m1}\delta^{\alpha\tau}\delta^{\beta\sigma}\right]\varphi^{(\pm)}(\boldsymbol{x},\boldsymbol{y})\ . \tag{4.46c}$$

Here the indices $1, 2$ correspond to fixed Matsubara frequency indices n_1 and n_2 respectively, not actual numbers. The function $\varphi^{(\pm)}$ is given by

$$\varphi^{(\pm)}(\boldsymbol{x}, \boldsymbol{y}) = \frac{1}{2}\left[\phi(\boldsymbol{x}, \boldsymbol{y}) \pm \phi(\boldsymbol{y}, \boldsymbol{x})\right] = \pm\varphi^{(\pm)}(\boldsymbol{y}, \boldsymbol{x}) \ . \tag{4.46d}$$

It is easy to check $T^{(\pm)}$ is symmetric, and since it acts as identity in τ space, it leaves τ_2 invariant, $(T^{(\pm)})^T \tau_2 T^{(\pm)} = \tau_2$ (see Appendix E). The condition $(T^{(\pm)})^T \tau_2 T^{(\pm)} = \tau_2$ implies the set of all $\{T^{\pm}\}$ forms the group $\mathrm{Sp}(4M)$, where $2M$ is the number of Matsubara frequencies, and the additional factor of 2 comes from the number of bands. The action of T on τ_2 implies that for $\eta \to T\eta$, it follows $Q \equiv \tau_2 \eta \otimes \eta \to TQT^T$. Q is not invariant under the action of T, it transforms as $Q^{\alpha\beta}_{nm}(\boldsymbol{x}, \boldsymbol{y}) \to Q^{\alpha\beta}_{nm}(\boldsymbol{x}, \boldsymbol{y}) + \delta Q^{\alpha\beta}_{nm}(\boldsymbol{x}, \boldsymbol{y})$ with

$$\begin{aligned} \delta Q^{\alpha\beta}_{nm}(\boldsymbol{x}, \boldsymbol{y}) \ &= \\ &\int d\boldsymbol{y_1} \varphi^{(\pm)}(\boldsymbol{y}, \boldsymbol{y_1}) \left(\delta^{\beta\sigma}_{m1} Q^{\alpha\tau}_{n2}(\boldsymbol{x}, \boldsymbol{y_1}) \mp \delta^{\beta\tau}_{m2} Q^{\alpha\sigma}_{n1}(\boldsymbol{x}, \boldsymbol{y_1})\right) + \\ &\int d\boldsymbol{x_1} \varphi^{(\pm)}(\boldsymbol{x}, \boldsymbol{x_1}) \left(\delta^{\alpha\sigma}_{n1} Q^{\tau\beta}_{2m}(\boldsymbol{x_1}, \boldsymbol{y}) \mp \delta^{\alpha\tau}_{n2} Q^{\sigma\beta}_{1m}(\boldsymbol{x_1}, \boldsymbol{y})\right) \ , \end{aligned} \tag{4.47}$$

where we have defined $\delta^{\alpha\beta}_{nm} \equiv \delta^{\alpha\beta}\delta_{nm}$. The Lagrange multiplier field $\tilde{\Lambda}$ transforms as Q does, on account of the bilinear coupling between the two. Of the three terms in the action in Equation (4.45), the second one is invariant under these transformations, but $\mathcal{A}_0$ and $\mathcal{A}_{\mathrm{int}}$ are not. Focusing on noninteracting electrons for the time being, we

find in Appendix F, $\mathcal{A}_0 \to \mathcal{A}_0 + \delta\mathcal{A}_0$ with

$$\delta\mathcal{A}_0 \;=\; \frac{1}{2}\sum_{\boldsymbol{p},\boldsymbol{k}} \varphi^{(\pm)}(\boldsymbol{k},\boldsymbol{p}) \mathrm{tr}\,\left[i\Omega_{1-2} + (-\xi^{\sigma}(\boldsymbol{k}) + \xi^{\tau}(\boldsymbol{p}))\right] (G^{\tau\sigma}_{21}(\boldsymbol{p},\boldsymbol{k}) + G^{\sigma\tau}_{12}(-\boldsymbol{k},-\boldsymbol{p})) \;. \tag{4.48}$$

A Ward identity can now be derived by the techniques of Reference [69]. We introduce a matrix source field J for Q in the partition function to obtain the generating functional

$$\mathcal{Z}[J] = \int D[Q,\tilde{\Lambda}]\; e^{\mathcal{A}_0 + \mathrm{Tr}\,(\tilde{\Lambda}^T Q) + \mathrm{Tr}\,(JQ)} \;. \tag{4.49}$$

Then by performing the infinitesimal rotation defined by $\varphi^{(\pm)}$, differentiating with respect to $J^{\tau\sigma}_{n_2 n_1}(\boldsymbol{y},\boldsymbol{x})$, and putting $J = 0$, we obtain a Ward identity:

$$\left\langle \delta\mathcal{A}_0\, Q^{\sigma\tau}_{n_1 n_2}(\boldsymbol{x},\boldsymbol{y}) \right\rangle_{\mathcal{A}_0} + \left\langle \delta Q^{\sigma\tau}_{n_1 n_2}(\boldsymbol{x},\boldsymbol{y}) \right\rangle_{\mathcal{A}_0} = 0 \;. \tag{4.50}$$

From Equations (4.48) and (4.50) we see that this relates correlation functions of the structure $\langle \mathrm{tr}\, G\, Q \rangle$ to $\langle Q \rangle$. The former can be rewritten in terms of $\langle Q\, Q \rangle$ by generalizing the generating functional given in Equation (4.49). Since the Q are isomorphic to B, we can write the source term $JQ = xJQ + (1-x)JB$ with an arbitrary real number x. Putting this source term in the original $S[\eta,\tilde{\Lambda},Q]$ action before integrating out η makes the generating functional

$$\mathcal{Z}[J] = \int D[Q,\tilde{\Lambda}]\; e^{\frac{1}{2}\,\mathrm{Tr}\,\ln[G^{-1} + i(1-x)J^T] + x\,\mathrm{Tr}\,(JQ) + \mathrm{Tr}\,(\tilde{\Lambda}^T Q)} \;. \tag{4.51}$$

Note that this is independent of x, and that by choosing $x = 1$ we recover Equation (4.49). By differentiating with respect to J, choosing $x = 0$ and $x = 1$, respectively,

and putting $J = 0$ we obtain an identity

$$\langle G^{\tau\sigma}_{n_2 n_1}(\boldsymbol{x}_2, \boldsymbol{x}_1)\rangle = -2i\,\langle Q^{\sigma\tau}_{n_1 n_2}(\boldsymbol{x}_1, \boldsymbol{x}_2)\rangle\ . \tag{4.52a}$$

Then differentiating $\langle Q_{34}\rangle_{\mathcal{Z}[J]}$ with respect to J_{12} (for 1-4 being general indices) we find

$$\langle G^{\sigma_1\sigma_2}_{n_1 n_2}(\boldsymbol{x}_1, \boldsymbol{x}_2) Q^{\sigma_3\sigma_4}_{n_3 n_4}(\boldsymbol{x}_3, \boldsymbol{x}_4)\rangle = -2i\,\langle Q^{\sigma_1\sigma_2}_{n_1 n_2}(\boldsymbol{x}_1, \boldsymbol{x}_2) Q^{\sigma_3\sigma_4}_{n_3 n_4}(\boldsymbol{x}_3, \boldsymbol{x}_4)\rangle\ . \tag{4.52b}$$

Equation (4.52b) allows us to rewrite the $\operatorname{tr} G$ as a $\operatorname{tr} Q$ in Equation (4.48) when we insert $\delta\mathcal{A}_0$ into Equation (4.50). We are then able to use the symmetry properties of Q, further simplifying the expression. Equation (4.50) now reads explicitly as[3]

$$\begin{aligned}
&\langle \delta\mathcal{A}_0\, Q^{\sigma_3\sigma_4}_{n_3 n_4}\rangle = \\
&= -4i \sum_{\boldsymbol{p},\boldsymbol{k}} \varphi^{(\pm)}(\boldsymbol{k},\boldsymbol{p})\left[i\Omega_{1-2} + (-\xi^{\sigma_1}(\boldsymbol{k}) + \xi^{\sigma_2}(\boldsymbol{p}))\right]\langle {}_0Q^{\sigma_2\sigma_1}_{n_2 n_1}(\boldsymbol{p},\boldsymbol{k})\, Q^{\sigma_3\sigma_4}_{n_3 n_4}(\boldsymbol{p_3},\boldsymbol{p_4})\rangle \\
&\equiv -\varphi^{(\pm)}(\boldsymbol{p}_3,\boldsymbol{p}_4)\left[\pm\delta^{\sigma_4\sigma_1}_{n_4 n_1}\delta^{\sigma_3\sigma_2}_{n_3 n_2}\left(\langle Q^{\sigma_2\sigma_2}_{n_2 n_2}(\boldsymbol{p}_3,\boldsymbol{p}_3)\rangle - \langle Q^{\sigma_1\sigma_1}_{n_1 n_1}(\boldsymbol{p}_4,\boldsymbol{p}_4)\rangle\right) + \right. \\
&\qquad\qquad \left. + \delta^{\sigma_3\sigma_1}_{n_3 n_1}\delta^{\sigma_2\sigma_4}_{n_2 n_4}\left(\langle Q^{\sigma_2\sigma_2}_{n_2 n_2}(\boldsymbol{p}_4,\boldsymbol{p}_4)\rangle - \langle Q^{\sigma_1\sigma_1}_{n_1 n_1}(\boldsymbol{p}_3,\boldsymbol{p}_3)\rangle\right)\right] \\
&= -\langle \delta Q^{\sigma_3\sigma_4}_{n_3 n_4}(\boldsymbol{p}_3,\boldsymbol{p}_4)\rangle\ .
\end{aligned} \tag{4.53}$$

[3]The total factor of $-4i$ comes from a factor of 2 from the trace, 2 from summing the two G's using the Q symmetry properties and -2i from the above identity.

Differentiating (4.53) with respect to ϕ yields two identities, one for $\varphi^{(+)}$ and one for $\varphi^{(-)}$. Adding them gives us the Ward identity:

$$\begin{aligned} D^{\sigma_1\sigma_2\sigma_3\sigma_4}_{n_1n_2,n_3n_4}(\boldsymbol{k},\boldsymbol{p};\boldsymbol{q}) \;\equiv\; \left\langle {}_0Q^{\sigma_1\sigma_2}_{n_1n_2}(\boldsymbol{k};\boldsymbol{q})\, {}_0Q^{\sigma_3\sigma_4}_{n_3n_4}(\boldsymbol{p};-\boldsymbol{q})\right\rangle = \\ \frac{i}{4}\,\delta_{\boldsymbol{k},-\boldsymbol{p}}\delta^{\sigma_1\sigma_3}_{n_1n_3}\delta^{\sigma_2\sigma_4}_{n_2n_4}\frac{\langle {}_0Q^{\sigma_1\sigma_1}_{n_1n_1}(\boldsymbol{p}-\boldsymbol{q}/2)\rangle - \langle {}_0Q^{\sigma_2\sigma_2}_{n_2n_2}(\boldsymbol{p}+\boldsymbol{q}/2)\rangle}{i\Omega_{1-2}+\xi^{\sigma_2}(\boldsymbol{k}-\boldsymbol{q}/2)-\xi^{\sigma_1}(\boldsymbol{k}+\boldsymbol{q}/2)} \\ +\frac{i}{4}\,\delta_{\boldsymbol{k},\boldsymbol{p}}\delta^{\sigma_1\sigma_4}_{n_1n_4}\delta^{\sigma_2\sigma_3}_{n_2n_3}\frac{\langle {}_0Q^{\sigma_1\sigma_1}_{n_1n_1}(\boldsymbol{p}+\boldsymbol{q}/2)\rangle - \langle {}_0Q^{\sigma_2\sigma_2}_{n_2n_2}(\boldsymbol{p}-\boldsymbol{q}/2)\rangle}{i\Omega_{1-2}+\xi^{\sigma_2}(\boldsymbol{k}-\boldsymbol{q}/2)-\xi^{\sigma_1}(\boldsymbol{k}+\boldsymbol{q}/2)} \end{aligned} \tag{4.54}$$

Let us now define $J_{1234} = \delta^{\sigma_1\sigma_3}_{n_1n_3}\delta^{\sigma_2\sigma_4}_{n_2n_4}$, then upon restricting ourselves to Q_{nm} where $n \geq m$, and noting $n_1 \neq n_2$, we arrive at the Ward identity in its final form:

$$\begin{aligned} D^{\sigma_1\sigma_2\sigma_3\sigma_4}_{n_1n_2,n_3n_4}(\boldsymbol{k},\boldsymbol{p};\boldsymbol{q}) \;\equiv\; \left\langle {}_0Q^{\sigma_1\sigma_2}_{n_1n_2}(\boldsymbol{k};\boldsymbol{q})\, {}_0Q^{\sigma_3\sigma_4}_{n_3n_4}(-\boldsymbol{p};-\boldsymbol{q})\right\rangle = \\ \frac{i}{4}\,\delta_{\boldsymbol{k},\boldsymbol{p}}J_{1234}\frac{\langle {}_0Q^{\sigma_1\sigma_1}_{n_1n_1}(\boldsymbol{k}+\boldsymbol{q}/2)\rangle - \langle {}_0Q^{\sigma_2\sigma_2}_{n_2n_2}(\boldsymbol{k}-\boldsymbol{q}/2)\rangle}{i\Omega_{1-2}+\xi^{\sigma_2}(\boldsymbol{k}-\boldsymbol{q}/2)-\xi^{\sigma_1}(\boldsymbol{k}+\boldsymbol{q}/2)}\ . \end{aligned} \tag{4.55}$$

Soft Modes in Noninteracting Electron Systems

Let us first discuss the Ward identity for noninteracting systems. We will see that it reveals a family of soft modes for each system, corresponding to the symmetry breaking Goldstone modes identified in the saddle point expansion. Consider the limiting behavior of (4.55) for $\omega_1, \omega_2, q \to 0$. Using equation (4.52a), we see this is determined by:

$$-2i\left\langle {}_0Q^{\sigma\sigma}_{n_1n_1}(\boldsymbol{p}+\boldsymbol{q/2})\right\rangle = \frac{1}{i\omega_{n_1}-\xi^{\sigma}(\boldsymbol{p}+\boldsymbol{q/2})}\ , \tag{4.56}$$

which in the limit $\omega_{n_1} \to 0$ goes to

$$-i\pi \mathrm{sgn}\,(\omega_{n_1})\delta(\xi^{\sigma}(\boldsymbol{p}+\boldsymbol{q}/\boldsymbol{2})) - P\left(\frac{i\omega_{n_1}}{\xi^{\sigma}(\boldsymbol{p}+\boldsymbol{q}/\boldsymbol{2})^2} + \frac{1}{\xi^{\sigma}(\boldsymbol{p}+\boldsymbol{q}/\boldsymbol{2})}\right) \ , \quad (4.57)$$

where P denotes the Cauchy principal value. We then see the limiting behavior of Equation (4.55) for small frequencies is:

$$\begin{aligned} &D^{\sigma_1\sigma_2\sigma_3\sigma_4}_{n_1n_2,n_3n_4}(\boldsymbol{k},\boldsymbol{p};\boldsymbol{q}) \to \frac{i}{4}\,\delta_{\boldsymbol{k},\boldsymbol{p}} J_{1234} \frac{1}{i\Omega_{1-2} + \xi^{\sigma_2}(\boldsymbol{p}-\boldsymbol{q}/2) - \xi^{\sigma_1}(\boldsymbol{p}+\boldsymbol{q}/2)} \times \\ &[-i\pi\left(\mathrm{sgn}\,(\omega_{n_1})\delta(\xi^{\sigma_1}(\boldsymbol{p}+\boldsymbol{q}/\boldsymbol{2})) - \mathrm{sgn}\,(\omega_{n_2})\delta(\xi^{\sigma_2}(\boldsymbol{p}-\boldsymbol{q}/\boldsymbol{2}))\right) - \\ &\quad - P\left(\frac{i\omega_{n_1} + \xi^{\sigma_1}(\boldsymbol{p}+\boldsymbol{q}/\boldsymbol{2})}{\xi^{\sigma_1}(\boldsymbol{p}+\boldsymbol{q}/\boldsymbol{2})^2} - \frac{i\omega_{n_2} + \xi^{\sigma_2}(\boldsymbol{p}-\boldsymbol{q}/\boldsymbol{2})}{\xi^{\sigma_2}(\boldsymbol{p}-\boldsymbol{q}/\boldsymbol{2})^2}\right)\Bigg] \ . \end{aligned} \quad (4.58)$$

For simplicity assume $\sigma_1 = \sigma_2$, then in the small $\boldsymbol{q}$ limit we obtain

$$\begin{aligned} &D^{\sigma_1\sigma_1\sigma_3\sigma_4}_{n_1n_2,n_3n_4}(\boldsymbol{k},\boldsymbol{p};\boldsymbol{q}) \to \frac{i}{4}\,\delta_{\boldsymbol{k},\boldsymbol{p}} J_{1234} \frac{1}{i\Omega_{1-2} + \xi^{\sigma_1}(\boldsymbol{p}-\boldsymbol{q}/2) - \xi^{\sigma_1}(\boldsymbol{p}+\boldsymbol{q}/2)} \times \\ &[i\pi(\mathrm{sgn}\,(\omega_{n_2}) - \mathrm{sgn}\,(\omega_{n_1}))\delta(\xi^{\sigma_1}(\boldsymbol{p})) - \\ &\quad - P\left(\frac{i\Omega_{1-2} + \xi^{\sigma_1}(\boldsymbol{p}-\boldsymbol{q}/2) - \xi^{\sigma_1}(\boldsymbol{p}+\boldsymbol{q}/2)}{\xi^{\sigma_1}(\boldsymbol{p})^2}\right)\Bigg] \ . \end{aligned} \quad (4.59)$$

This is an extension of the argument by Belitz and Kirkpatrick in Reference [14]. We see that in Fermi liquids, for $n_1 n_2 < 0$, the numerator approaches something finite, while the denominator goes to zero, whereas for $n_1 n_2 > 0$, the numerator and denominator cancel, leaving a finite number. This in turn implies that there is an infinite number of soft modes in the conduction band of Fermi liquids (i.e. $\sigma_1 = \sigma_2 = 0$) that can be obtained by taking all possible moments of Equation (4.55)

with respect to the center-of-mass wave vector $\boldsymbol{p}$, provided $Q_{n_1 n_2}$ satisfies $\omega_{n_1}\omega_{n_2} < 0$. These soft modes are the Goldstone bosons of the system, previously identified by the saddle point $Q^{sp}_{n_1 n_2}$ not being invariant under mixing between retarded and advanced frequencies. By equivalently inspecting D for $\sigma_1 \neq \sigma_2$ in Fermi liquids, and all choices of (σ_1, σ_2) in semiconductors, one can easily see there are no soft-modes for those cases. This is because either the delta functions cannot be satisfied by real valued momenta for interband excitations (due to filled/empty bands) or the denominator in Equation (4.57) picks up a "mass" determined by the energy gap E_g that prevents it from vanishing in intraband excitations.

We would like to now extend this argument to semimetals to understand the nature of the soft modes such systems. However, the conclusions above do not immediately apply to semimetals due to their vanishing density of states (DOS). Due to the lack of a Fermi surface, $\xi^\sigma(p) = 0$ requires $p = 0$ in DSMs. Thus, if $\boldsymbol{q} = 0$, the delta functions in Equation (4.59) yield zero for any moment of the center-of-mass wave vector $\boldsymbol{p}$, i.e.

$$\int dp\ p^{d-1+m} f(p)\delta(\xi^\sigma(p)) = 0, \tag{4.60}$$

for any function $f(p)$ that is finite at $p = 0$. Thus we need to use the full function Equation (4.58). Let us focus first on the real part of this equation. Calculating the

mth $\boldsymbol{p}$ moment of $\mathrm{Re}\, D^{\sigma_1\sigma_2\sigma_1\sigma_2}_{n_1n_2n_1n_2}(\boldsymbol{k},\boldsymbol{p},\boldsymbol{q})$ we find

$$\begin{aligned}
&\sum_{\boldsymbol{k},\boldsymbol{p}} p^m \mathrm{Re}\, D^{\sigma_1\sigma_2\sigma_1\sigma_2}_{n_1n_2n_1n_2}(\boldsymbol{k},\boldsymbol{p},\boldsymbol{q}) \propto \\
\propto\ & \frac{\pi}{4}\int d\boldsymbol{p}\; p^m \frac{\mathrm{sgn}\,(\omega_{n_1})\delta(\xi^{\sigma_1}(\boldsymbol{p}+\boldsymbol{q}/2)) - \mathrm{sgn}\,(\omega_{n_2})\delta(\xi^{\sigma_2}(\boldsymbol{p}-\boldsymbol{q}/2))}{i\Omega_{1-2} + \xi^{\sigma_2}(\boldsymbol{p}-\boldsymbol{q}/2) - \xi^{\sigma_1}(\boldsymbol{p}+\boldsymbol{q}/2)} \\
\approx\ & \frac{\pi}{4}\int d\Omega \int dp\; \delta(\xi^{\sigma_1}(p)) \left[\frac{|\boldsymbol{p}-\boldsymbol{q}/2|^{d-1+m}\mathrm{sgn}\,(\omega_{n_1})}{i\Omega_{1-2} + \xi^{\sigma_2}(\boldsymbol{p}-\boldsymbol{q}) - \xi^{\sigma_1}(\boldsymbol{p})} - \right. \\
& \qquad\qquad \left. - \frac{|\boldsymbol{p}+\boldsymbol{q}/2|^{d-1+m}\mathrm{sgn}\,(\omega_{n_2})}{i\Omega_{1-2} + \xi^{\sigma_2}(\boldsymbol{p}) - \xi^{\sigma_1}(\boldsymbol{p}+\boldsymbol{q})}\right] \\
=\ & \frac{\pi}{4} S_{d-1} \frac{q^{d-1+m}}{v_D}\left[\frac{\mathrm{sgn}\,(\omega_{n_1})}{i\Omega_{1-2} + \xi^{\sigma_2}(q)} - \frac{\mathrm{sgn}\,(\omega_{n_2})}{i\Omega_{1-2} - \xi^{\sigma_1}(q)}\right] , \qquad (4.61)
\end{aligned}$$

where in the second line we have used that for semimetals $\delta(\xi^1(p)) = \delta(\xi^0(p))$, and neglected corrections to the angular integration limits due to the $\boldsymbol{q}/2$ shifts.[4] In the third line S_{d-1} is the area of the $d-1$ sphere. In the analogous calculation for a Fermi liquid, the delta function would contribute to leading order in $\boldsymbol{q}$ a factor of the density of states at the Fermi level N_F, to the final expression, for semimetals we see N_F is replaced by the wave number dependent DOS of semimetals q^{d-1}/v_D.

At this level of approximation, we see that $n_1 n_2 > 0$ yields a non-zero result provided $\sigma_1 \neq \sigma_2$. This agrees with the decomposition made by Abrikosov and Beneslavskii in Reference [4]. One could be tempted at this moment to claim modes that have $n_1 = n_2$ and $\sigma_1 \neq \sigma_2$ are of less importance to the field theory, and can be integrated out. However, the calculations done thus far are in the $qv_D \gg \Omega_{1-2}$ limit. Dimensional analysis, or a rigorous calculation show that if we move away from this limit, Equation (4.61) will pick up a $\omega_{n_i}^{d-1+m}$ dependence. We have therefore

[4]The trick of Taylor expanding the δ-function has the unfortunate consequence of eliminating any log divergent $\log(1/q)$ type terms from the final result. These will be present in later sections of the work when the integrals are performed exactly.

demonstrated that all channels of $D^{\sigma_1\sigma_2\sigma_1\sigma_2}_{n_1n_2n_1n_2}(\boldsymbol{k},\boldsymbol{p},\boldsymbol{q})$ are equally important[5] in non-interacting semimetals. Moreover, Equation (4.61) suggests that the generalized particle-hole excitations in semimetals behave like those in Fermi liquids, only with any instance of N_F replaced by q and ω_{n_i} dependent terms that scale as the the semimetal DOS $N(\omega) \sim |\omega|^{d-1} \sim q^{d-1}$. This is indeed the case, and the claim can be made more rigorous by an exact calculation. Equation (4.57) allows us to write the zeroth moment of the Ward identity, Equation (4.55), in the following way:

$$\begin{aligned}\mathcal{D}^{\sigma_1\sigma_2}_{n_1n_2}(\boldsymbol{q}) \;&\equiv\; \frac{1}{V}\sum_{\boldsymbol{k},\boldsymbol{p}} D^{\sigma_1\sigma_2\sigma_1\sigma_2}_{n_1n_2n_1n_2}(\boldsymbol{k},\boldsymbol{p};\boldsymbol{q}) = \\ &\frac{1}{8}\frac{1}{V}\sum_{\boldsymbol{p}} \frac{1}{i\omega_{n_1} - \xi^{\sigma_1}(\boldsymbol{p}+\boldsymbol{q}/\boldsymbol{2})}\frac{1}{i\omega_{n_2} - \xi^{\sigma_2}(\boldsymbol{p}-\boldsymbol{q}/\boldsymbol{2})},\end{aligned} \tag{4.62}$$

We will first consider $\mathcal{D}^{\sigma_1\sigma_2}_{n_1n_2}(0)$, as this will already demonstrate the softness/massiveness of each mode. These results are summarized in Tables 3 and 4, in which we can easily see the softness/massiveness of the various modes. It is immediately obvious that all DSM modes are equally soft, but with a vanishing weight that scales as the density states times a log divergent term. What we would like to now demonstrate, is that modulo logarithmic divergences, the particle-hole modes of semimetals can be viewed as those for Fermi liquids in the semimetal limit. By this we mean taking $\mu, N_F, m_{\rm e} \to 0$, while keeping $v_{\rm F} = k_F/m_e$ fixed. This is of interest because it has implications for the difference in the scaling behavior of observables for the two

[5]We have not addressed what the imaginary part of D does in the text. Due to the lack of a chemical potential, the Cauchy principal value will vanish at 0 momentum, so it cannot provide any $\mathcal{O}(1)$ terms as in the Fermi liquid case. Moreover it ignores the frequency signs, so it cannot elevate any subset of modes to greater importance.

systems. To demonstrate the limiting behavior, we will need the non-zero momentum dependence of $\mathcal{D}^{\sigma_1\sigma_2}_{nm}(\boldsymbol{q})$ for the two systems.

It is sufficient to compute the particle-hole excitations in the conduction band for Fermi liquids $\mathcal{D}^{00}_{nm}(\boldsymbol{q})$, as that is where the soft modes occur. For semimetals, all channels are soft, so we must compute all channels of $\mathcal{D}^{\sigma_1\sigma_2}_{nm}(\boldsymbol{q})$. We note here that from Equation (4.62) it is easy to see that the quantity $\operatorname{Im}\mathcal{D}^{\sigma_1\sigma_2}_{(n+m)n}(\boldsymbol{q})$ is odd in ω_n, and thus any contraction over ω_n will eliminate the imaginary part. Table 3 already demonstrates that $\operatorname{Im}\mathcal{D}^{\sigma_1\sigma_2}_{(n+m)n}(\boldsymbol{q})$ is soft, with a faster vanishing weight than the real part. We can thus restrict ourselves to evaluating $\operatorname{Re}\mathcal{D}^{\sigma_1\sigma_2}_{(n+m)n}(\boldsymbol{q})$ for semimetals. If explicit knowledge of $\operatorname{Im}\mathcal{D}$ is desired, it can be computed from $\operatorname{Re}\mathcal{D}$ using the Kramers-Kronig relationship [61]. Defining:

$$\varphi(q,\omega_n,\omega_m) \;=\; \begin{cases} i\log\left[\dfrac{-iqv_D-|\omega_n+\omega_m|}{iqv_D-|\omega_n+\omega_m|}\right] & \text{for } \omega_n\omega_m>0 \\ i\log\left[\dfrac{-iqv_D-|\Omega_{n-m}|}{iqv_D-|\Omega_{n-m}|}\right] & \text{for } \omega_n\omega_m<0 \end{cases}, \tag{4.63a}$$

in the limit of small q we obtain for DSMs ($d=3$):

$$\operatorname{Re}\mathcal{D}^{\sigma_1\sigma_2}_{nm}(\boldsymbol{q}) \;=\; \frac{1}{\pi(4v_D)^3}\Big[-(-1)^{\sigma_1+\sigma_2}(|\omega_n|+|\omega_m|)+ \frac{6\omega_n\omega_m+(-1)^{\sigma_1+\sigma_2}(q^2v_D^2+3(\omega_n^2+\omega_m^2))}{6qv_D}\varphi(q,\omega_n,\omega_m)\Big]\ . \tag{4.63b}$$

The following property then gives us the appropriate form of $\operatorname{Re}\mathcal{D}^{\sigma_1\sigma_2}_{nm}(0)$ in Table 3:

$$\lim_{q\to0}\frac{i}{q}\log\left(\frac{-iqv_D-X}{iqv_D-X}\right) \;=\; -\frac{2}{X} \tag{4.64}$$

System	Dirac Semimetal
$(\boldsymbol{\sigma_1}, \boldsymbol{\sigma_2}, \mathbf{sgn}(\boldsymbol{\omega_m}\boldsymbol{\omega_n}))$	
$(0,0,+)$	$\frac{\omega_1^2/v_D^3}{i\omega_1 - i\omega_2}\log\left(\frac{\Lambda v_D + i\omega_1}{i\omega_1}\right) - \frac{\omega_2^2/v_D^3}{i\omega_1 - i\omega_2}\log\left(\frac{\Lambda v_D + i\omega_2}{i\omega_2}\right)$
$(0,0,-)$	$\frac{\omega_1^2/v_D^3}{i\omega_1 - i\omega_2}\log\left(\frac{\Lambda v_D + i\omega_1}{i\omega_1}\right) - \frac{\omega_2^2/v_D^3}{i\omega_1 - i\omega_2}\log\left(\frac{\Lambda v_D + i\omega_2}{i\omega_2}\right)$
$(0,1,+)$	$\frac{\omega_1^2/v_D^3}{i\omega_1 + i\omega_2}\log\left(\frac{\Lambda v_D + i\omega_1}{i\omega_1}\right) - \frac{\omega_2^2/v_D^3}{i\omega_1 + i\omega_2}\log\left(\frac{\Lambda v_D + i\omega_2}{i\omega_2}\right)$
$(0,1,-)$	$\frac{\omega_1^2/v_D^3}{i\omega_1 + i\omega_2}\log\left(\frac{\Lambda v_D + i\omega_1}{i\omega_1}\right) - \frac{\omega_2^2/v_D^3}{i\omega_1 + i\omega_2}\log\left(\frac{\Lambda v_D + i\omega_2}{i\omega_2}\right)$
$(1,1,+)$	$-\frac{\omega_1^2/v_D^3}{i\omega_1 - i\omega_2}\log\left(\frac{\Lambda v_D + i\omega_1}{i\omega_1}\right) + \frac{\omega_2^2/v_D^3}{i\omega_1 - i\omega_2}\log\left(\frac{\Lambda v_D + i\omega_2}{i\omega_2}\right)$
$(1,1,-)$	$-\frac{\omega_1^2/v_D^3}{i\omega_1 - i\omega_2}\log\left(\frac{\Lambda v_D + i\omega_1}{i\omega_1}\right) + \frac{\omega_2^2/v_D^3}{i\omega_1 - i\omega_2}\log\left(\frac{\Lambda v_D + i\omega_2}{i\omega_2}\right)$

TABLE 3. Tabulated results for $\mathcal{D}^{\sigma_1\sigma_2}_{nm}(0)$ in $d = 3$ for Dirac Semimetals. Real valued coefficients of proportionality have been suppressed in each case. Note that $(-1)^{\sigma_i}$ determines the band, such that 0 and 1 are conduction and valence respectively.

Equation (4.63) was obtained using dimensional regularization to suppress UV divergent terms that scale like Λ and $\log\Lambda$, see Reference [66]. This is because the bands are not linear to the Brillouin zone boundary, and crystals have smooth cut-offs [67]. This trick has the unfortunate side effect of also suppressing any terms with a logarithmic non-analyticity, i.e. the terms that scaled as $\Omega\log(\Lambda v_D/\Omega)$ in Table 3, for Λ a UV cut-off, are not present in the result. By dimensional analysis we could conclude that for $q \neq 0$, we are suppressing terms of the order $\frac{q^2v_D^2+\Omega^2}{v_Dq+\Omega}\log(\Lambda^2/(q^2+\Omega^2/v_D^2))$, which is correct. In the development of the effective field theory in Section 4.7, such terms will be treated exactly.

System	Fermi Liquid	Semiconductor
$(\boldsymbol{\sigma_1}, \boldsymbol{\sigma_2}, \mathbf{sgn})$		
$(0,0,+)$	$\dfrac{im_e^{3/2}\operatorname{sgn}\omega_n}{\sqrt{\mu_{eff}+i\omega_n}+\sqrt{\mu_{eff}+i\omega_m}}$	$\dfrac{m_e^{3/2}}{\sqrt{\mu_{eff}-i\omega_n}+\sqrt{\mu_{eff}-i\omega_m}}$
$(0,0,-)$	$\dfrac{im_e^{3/2}\operatorname{sgn}\omega_n}{\sqrt{\mu_{eff}+i\omega_n}-\sqrt{\mu_{eff}+i\omega_m}}$	$\dfrac{m_e^{3/2}}{\sqrt{\mu_{eff}-i\omega_n}+\sqrt{\mu_{eff}-i\omega_m}}$
$(0,1,+)$	$\dfrac{im_e^{3/2}}{\operatorname{sgn}\omega_n\sqrt{\mu_{eff}+i\omega_n}+i\sqrt{\mu+i\omega_m}}$	$\dfrac{m_e^{3/2}}{\sqrt{\mu_{eff}-i\omega_n}+\sqrt{\mu+i\omega_m}}$
$(0,1,-)$	$\dfrac{im_e^{3/2}}{\operatorname{sgn}\omega_n\sqrt{\mu_{eff}+i\omega_n}+i\sqrt{\mu+i\omega_m}}$	$\dfrac{m_e^{3/2}}{\sqrt{\mu_{eff}-i\omega_n}+\sqrt{\mu+i\omega_m}}$
$(1,1,+)$	$\dfrac{m_e^{3/2}}{\sqrt{\mu-i\omega_n}+\sqrt{\mu+i\omega_m}}$	$\dfrac{m_e^{3/2}}{\sqrt{\mu-i\omega_n}+\sqrt{\mu+i\omega_m}}$
$(1,1,-)$	$\dfrac{m_e^{3/2}}{\sqrt{\mu-i\omega_n}+\sqrt{\mu+i\omega_m}}$	$\dfrac{m_e^{3/2}}{\sqrt{\mu-i\omega_n}+\sqrt{\mu+i\omega_m}}$

TABLE 4. Tabulated results for $\mathcal{D}_{nm}^{\sigma_1\sigma_2}(0)$ in $d = 3$ for Fermi liquids and semiconductor. In the case of semiconductors, due to the bandgap none of the modes have real valued ω poles, i.e. they are all massive. Note that real valued coefficients of proportionality have been suppressed in each case. Note that sgn$\equiv$ sgn $(\omega_n\omega_m)$, and $(-1)^{\sigma_i}$ determines the band, such that 0 and 1 are conduction and valence respectively.

For $\sigma_1 = \sigma_2 = 0$ in Fermi liquids in $d = 3$ we must compute:

$$\begin{aligned}
&\mathcal{D}^{00}_{nm}(\boldsymbol{q}) = \\
&= \frac{1}{8}\frac{1}{(2\pi)^2}\int_{-1}^{1} d\eta \int_0^\infty dp \; \frac{p^2}{i\omega_n + \mu - (p^2 + q^2/4 + pq\eta)/2m_e} \times \\
&\quad \times \frac{1}{i\omega_m + \mu - (p^2 + q^2/4 - pq\eta)/2m_e} \\
&= \frac{1}{16}\frac{1}{(2\pi)^2}\int_{-1}^{1} d\eta \int_0^\infty dp \; \frac{p^2}{i\omega_n + \mu - (p^2 + q^2/4 + pq\eta)/2m_e} \times \\
&\quad \times \frac{1}{i\omega_m + \mu - (p^2 + q^2/4 - pq\eta)/2m_e} + (\eta \to -\eta) \; ,
\end{aligned} \tag{4.65}$$

where in the second line we have used the fact that substituting $\eta \to -\eta$ turns the second integral into the first. The p-integral is now even and can be evaluated by contour integration. We then approximate assuming $qv_{\rm F}, \omega_i \ll \mu$. For $\omega_n\omega_m > 0$ we get:

$$\begin{aligned}
\mathcal{D}^{00}_{nm}(\boldsymbol{q}) \;\; &\approx \;\; \frac{1}{8\pi}\mathrm{sgn}\,(\omega_n + \omega_m)\frac{N_F + \mathcal{O}(m_{\rm e}q, \frac{m_e\omega_i}{v_{\rm F}}) + \mathcal{O}(\frac{q^2}{v_{\rm F}}, \frac{\omega_i^2}{v_{\rm F}^3})}{\sqrt{2 + i(\omega_n + \omega_m)/8\mu - q^2/4k_F^2}}\frac{i}{v_{\rm F}q} \times \\
&\quad \times \log\left[\frac{2k_F\sqrt{2 + i(\omega_n + \omega_m)/2\mu - q^2/4k_F^2} - iq}{2k_F\sqrt{2 + i(\omega_n + \omega_m)/2\mu - q^2/4k_F^2} + iq}\right],
\end{aligned} \tag{4.66a}$$

where the numerical coefficients of the sub-leading terms have been suppressed. Note for $q < k_F$ the log term is always imaginary, so $\mathcal{D}^{00}_{nm}(\boldsymbol{q})$ is overall real. For $q \to 0$ we recover the same behavior as shown in Table 4, provided one assumes $\omega_i \ll \mu$. For

$\omega_n\omega_m < 0$ we obtain the limiting behavior

$$\mathcal{D}^{00}_{nm}(\boldsymbol{q}) \approx \frac{1}{16\pi}\mathrm{sgn}\,(\Omega_{n-m})\frac{N_F + \mathcal{O}(m_e q, \frac{m_e\omega_i}{v_\mathrm{F}}) + \mathcal{O}(\frac{q^2}{v_\mathrm{F}}, \frac{\omega_i^2}{v_\mathrm{F}^3})}{v_\mathrm{F} q} i \log\left[\frac{\Omega_{n-m} - iqv_\mathrm{F}}{\Omega_{n-m} + iqv_\mathrm{F}}\right] + \mathcal{O}\left(\frac{\omega_i}{v_\mathrm{F}^3}\right) . \tag{4.66b}$$

One can readily see from the soft Fermi liquid correlation function, Equation (4.66b), that the semimetal correlation function has the same behavior as a Fermi liquid with $m_e, N_F \to 0$, but $v_\mathrm{F} \to v_D$ fixed. Dimensionality of the coefficients of the q's and ω_i's in the numerator ensure that the only leading order terms that can survive are q^2/v_F and $\omega_i^2/v_\mathrm{F}^3$. If a term like $q\Omega/v_\mathrm{F}^2$ existed in the numerator it would remain in the semimetal limit, but no such term is there. A similar limiting behavior could be derived for the Fermi liquid massive modes in Equation (4.66a), if one had not made the assumption $\mu \gg q, \omega_i$ in obtaining it. However, while an explicit analysis of the massive modes is technically feasible, the resulting expressions are rather cumbersome, and thus will not be discussed in this text.

Before moving on, let us briefly return to the suppressed logarithmic non-analyticities in the DSM case. If one were to impose a UV cut-off in the Fermi liquid theory and perform the above integrals exactly, and never make expansions assuming $\mu \gg q, \omega_i$, one would obtain logarithmic factors that schematically scale as $\log(\Lambda/(k_F + q + \Omega))$. In the limit $k_F \to 0$, this function scales as the logarithmic terms we suppressed in the DSM case through dimensional regularization. This observation ultimately reflects the equivalence of different regularization schemes, i.e. instead of using dimensional regularization, we could have introduced an artificial mass term to the DSM propagators, and then taken the limit of vanishing mass to reproduce our results [66].

One could repeat the entire soft mode exercise in $d = 2$ to compare Fermi liquids with graphene, and see the same story play out. Since the expressions involved are significantly more cumbersome, we can make our argument purely with dimensional analysis. The correlation function, $\mathcal{D}^{00}_{nm}(\boldsymbol{q})$, for Fermi liquids in $d = 2$ will once again scale like density of states N over energy E:

$$\mathcal{D}^{00}_{nm}(\boldsymbol{q}) \propto \frac{N(q, \omega_i)}{E(q, \omega_i)}, \tag{4.67}$$

where $E(0, 0) \propto \mu$ for massive modes ($\omega_n \omega_m > 0$) and $E(0, 0) = 0$ for soft modes ($\omega_n \omega_m < 0$). The density of states term for $d = 2$ expands like:

$$N(q, \omega_i) = N_F + \mathcal{O}(\frac{q}{v_{\mathrm{F}}}, \frac{\omega}{v_{\mathrm{F}}^2}) + \dots \tag{4.68}$$

Therefore, in the semimetal limit $m_e, N_F, \mu \to 0$, $v_{\mathrm{F}} \to v_D$ fixed, the linear terms in q and ω_i are retained, which scale as the bare DOS in graphene, $N(\omega) \propto \omega / v_D^2$.

The Ward identity only identifies which elements of ${}_0Q_{nm}$ are soft. One can show that the same statement holds for arbitrary values of r, the reason being that the $\langle {}_0Q_0 Q \rangle$ and $\langle {}_rQ_r Q \rangle$ correlation functions are related by means of an unbroken symmetry. Consider the following transformation:

$$T_{nm} = \delta_{nm}(\delta_{nn_2} x_r \tau_r + (1 - \delta_{nn_2})\tau_0) \ , \tag{4.69}$$

where $x_r = (1, i, 1, i)_r$. T_{nm} is clearly orthogonal, and $S_0[Q]$ is clearly invariant under $Q \to \tilde{Q} = TQT^T$. Under such a transformation ${}_0Q_{nn_2} = \operatorname{tr} Q_{nn_2} \to \operatorname{tr} \tilde{Q}_{nn_2} \propto {}_rQ_{nn_2}$

for $n \neq n_2$, thus:

$$\begin{aligned}\langle {}_0Q_0Q\rangle &= \int D[Q,\Lambda]e^{S_0[Q,\Lambda]}\mathrm{tr}\,Q\mathrm{tr}\,Q \\ &\to \int D[\tilde{Q},\tilde{\Lambda}]e^{S_0[\tilde{Q},\tilde{Q}]}\mathrm{tr}\,\tilde{Q}\mathrm{tr}\,\tilde{Q} \\ &= \int D[Q,\Lambda]e^{S_0[Q,\Lambda]}\mathrm{tr}\,\tilde{Q}\mathrm{tr}\,\tilde{Q} \propto \langle {}_rQ_rQ\rangle \ .\end{aligned}$$

Hence ${}_rQ$ correlations have the same behavior as those for ${}_0Q$.

The Case of Interacting Electrons

At this point in our analysis, we would be motivated to construct an effective field theory for semimetals that explicitly includes all Q_{nm} modes, unlike the Fermi liquid case in which modes with $\omega_n\omega_m > 0$ are integrated out due to their massiveness. The question that remains is what is the impact of interactions on the conclusions of the non-interacting Ward identity? Formulated more succinctly, can interactions cause certain modes in semimetals to become truly soft or truly massive?

Let us first return to the original action of the problem, and rewrite the interaction between fermionic fields $\psi(x)$ as one mediated by a potential $\varphi(x)$ coupled to the fermions by some constant γ:

$$S = S_0[\psi] + S_0[\varphi] + \gamma\int d\boldsymbol{x}\sum_{n,m,\sigma}\varphi_m(\boldsymbol{x})\bar{\psi}^\sigma_{n+m}(\boldsymbol{x})\psi^\sigma_n(\boldsymbol{x}) \ . \tag{4.70}$$

In this form it is easy to see that the interacting part of the action is also invariant under rotations between retarded and advanced degrees of freedom for vanishing external frequency ($\Omega_m \to 0$). Hence, any saddle point solution of the action that

breaks this rotational symmetry is a Goldstone mode, even when interactions are included. We can now repeat the Ward identity derivation, including interaction terms. To this end, within the $\mathcal{A}_0$ term, we can shift $\tilde{\Lambda} \to \langle\tilde{\Lambda}\rangle + i\delta\tilde{\Lambda}$, where $\langle\tilde{\Lambda}\rangle$ is determined by the saddle point. Then we arrive at a new Ward identity:

$$\begin{aligned} & -4i[i\Omega_{1-2} + (\xi^*)^{\sigma_2}(\boldsymbol{k}-\boldsymbol{q}/2) - (\xi^*)^{\sigma_1}(\boldsymbol{k}+\boldsymbol{q}/2)]D^{\sigma_1\sigma_2\sigma_3\sigma_4}_{n_1n_2n_3n_4}(\boldsymbol{k},\boldsymbol{p};\boldsymbol{q}) \\ = \;& J_{1234}\delta_{\boldsymbol{k},\boldsymbol{p}}N_{n_1n_2}(\boldsymbol{k},\boldsymbol{q}) - W^{\sigma_1\sigma_2\sigma_3\sigma_4}_{n_1n_2n_3n_4}(\boldsymbol{k},\boldsymbol{p};\boldsymbol{q}) \; . \end{aligned} \tag{4.71}$$

Here ξ^* is the Hartree-Fock corrected dispersion relation, and

$$N_{n_1n_2}(\boldsymbol{k},\boldsymbol{q}) = \langle {}_0Q^{\sigma_1\sigma_1}_{n_1n_1}(\boldsymbol{k}+\boldsymbol{q}/2)\rangle - \langle {}_0Q^{\sigma_2\sigma_2}_{n_2n_2}(\boldsymbol{k}-\boldsymbol{q}/2)\rangle \; , \tag{4.72a}$$

$$W^{\sigma_1\sigma_2\sigma_3\sigma_4}_{n_1n_2n_3n_4}(\boldsymbol{k},\boldsymbol{p};\boldsymbol{q}) = \langle {}_0Q^{\sigma_3\sigma_4}_{n_3n_4}(\boldsymbol{p};-\boldsymbol{q})\delta\mathcal{A}_{\text{int}}\rangle \; , \tag{4.72b}$$

$$\begin{aligned} \delta\mathcal{A}_{\text{int}} = \; & \frac{4T}{V}\sum_{\substack{\boldsymbol{p}_1,\boldsymbol{q}_1\\ n_1'n_2'}}\sum_{\substack{\alpha\\ r=0,3}} \Gamma^d(\boldsymbol{q}_1)\,{}_0Q^{\alpha\alpha}_{n_1'n_2'}(\boldsymbol{p}_1,\boldsymbol{p}_1-\boldsymbol{q}_1) \\ & \times\Big[{}_rQ^{\sigma_2\sigma_1}_{n_2,n_2'-n_1'+n_1}(\boldsymbol{k}+\boldsymbol{q}/2,\boldsymbol{k}-\boldsymbol{q}/2+\boldsymbol{q}_1) - \\ & -(-1)^r\,{}_rQ^{\sigma_2\sigma_1}_{n_2'-n_1'+n_2,n_1}(\boldsymbol{k}+\boldsymbol{q}/2-\boldsymbol{q}_1,\boldsymbol{k}-\boldsymbol{q}/2)\Big] \; . \end{aligned} \tag{4.72c}$$

Provided that the saddle point selected is still that of a semimetal, the one point part of the correlation function, $N_{n_1n_2}(\boldsymbol{k},\boldsymbol{q})$ will behave as in the noninteracting case, only with renormalized Dirac velocities, and quasi particle residues. If we were to select a saddle point that broke the semimetal phase, either opening a gap or creating a Fermi surface, the conclusions of the previous section would not apply, and we would be in a semiconductor or Fermi liquid like regime respectively. The semimetal saddle point,

however, is perturbatively stable [70–75]. Therefore, the only significant alteration in the Ward identity's behavior would have to originate from the 3-point correlation function $W^{\sigma_1\sigma_2\sigma_3\sigma_4}_{n_1n_2n_3n_4}(\boldsymbol{k},\boldsymbol{p};\boldsymbol{q})$. As W is a function that vanishes with the interaction coupling γ, while $N_{n_1n_2}$ does not, we know W cannot cancel the numerator behavior seen for non-interacting semimetals except for special values of γ. Let us now assume the worst happens, the interaction term somehow adds a constant numerator to one channel of $\operatorname{sgn}\omega_m\omega_n$, but not the other. Then, we would still be compelled to explicitly account for all the bosonic modes in an effective field theory, because the vanishing denominator of a soft-mode with or without zero weight can impact the non-analytic dependence of observables on frequency and wave number. In the following section, we will therefore derive an effective field theory that accounts for all Q degrees of freedom.

Expansion About The Dirac Semimetal Saddle Point

Action

Having determined the saddle point solutions in Equation (4.30), we can now expand the action about them. Letting $\tilde{\Lambda} \to \tilde{\Lambda}^{\rm sp} + \Lambda$ and $\tilde{Q} \to Q^{\rm sp} + Q$ in $\mathcal{A}[\tilde{Q}, \tilde{\Lambda}]$ as defined in (4.19) yields:

$$\mathcal{A} \;=\; \mathcal{A}^{\rm sp} + \mathcal{A}^{(2)} + \Delta\mathcal{A} \;, \tag{4.73}$$

where $\mathcal{A}^{\rm sp}$ is the saddle point action, $\mathcal{A}^{(2)}$ denotes the Gaussian fluctions and $\Delta\mathcal{A}$ contains fluctuations of cubic or higher order resulting from the expansion. Note that as these come from the Tr $\log G$ term, they only contain powers of $\bar{\Lambda}$:

$$\Delta\mathcal{A} \;=\; -\sum_{n=3}^{\infty} \frac{i^n}{2n} {\rm Tr}\,(G^{\rm sp}\Lambda)^n \;. \tag{4.74}$$

The Gaussian part is found to be:

$$\mathcal{A}^{(2)} \;=\; \frac{1}{4}{\rm Tr}\;(G_{\rm sp}\Lambda G_{\rm sp}\Lambda) + {\rm Tr}\,(\Lambda^T Q) + \mathcal{A}_{\rm int}[Q] \;, \tag{4.75}$$

with $G_{\rm sp}$ defined by:

$$\begin{aligned} (G_{\rm sp})^{\alpha\beta}_{mn}(\boldsymbol{k},\boldsymbol{p}) \;&=\; -\delta_{\boldsymbol{k},\boldsymbol{p}}\delta^{\alpha\beta}_{mn}\frac{\tau_0 \otimes (i\omega_n s_0 + \alpha v_D \boldsymbol{p}\cdot\boldsymbol{\sigma})}{\omega_n^2 + p^2 v_D^2} \\ &\equiv\; \delta_{\boldsymbol{k},\boldsymbol{p}}\delta^{\alpha\beta}_{mn} G^{\alpha}_{n}(\boldsymbol{p}) \;, \end{aligned} \tag{4.76}$$

where we have let $v_D^* \equiv v_D$, and $\boldsymbol{\sigma}$ is acting in s-space. It is now convenient to define the inverse Green function $G_{\rm sp}^{-1} \equiv L$:

$$\begin{aligned} L_{mn}^{\alpha\beta}(\boldsymbol{k},\boldsymbol{p}) &= \delta_{\boldsymbol{k},\boldsymbol{p}}\delta_{mn}^{\alpha\beta}\tau_0 \otimes (i\omega_n s_0 - \alpha v_D \boldsymbol{p}\cdot\boldsymbol{\sigma}) \\ &\equiv \delta_{\boldsymbol{k},\boldsymbol{p}}\delta_{mn}^{\alpha\beta} L_n^\alpha(\boldsymbol{p}) \ . \end{aligned} \tag{4.77}$$

Next, for any function of m 4-momenta f, define $f_{1\ldots m} \equiv f_{n_1\ldots n_m}(\boldsymbol{p_1}\ldots\boldsymbol{p_m})$, and the operation $f^\ddagger_{1\ldots m} = f_{n_1\ldots n_m}(\boldsymbol{p}_1\ldots\boldsymbol{p}_m)^\ddagger = f_{n_1\ldots n_m}(-\boldsymbol{p}_1\ldots-\boldsymbol{p}_m)$. Finally, we will now impose on all objects with two 4-momenta f_{12} that $n_1 \geq n_2$. Observe that ${}^i_r L_1^\alpha = (1/4)\mathrm{tr}\,((\tau_r\otimes s_i)^\dagger L_1^\alpha) \propto \delta_{r0}$. Letting ${}^i_0 L_1^\alpha \equiv {}^i L_1^\alpha$ and ${}^i G_1^\alpha = \frac{1}{4}\mathrm{tr}\,(\tau_0\otimes s_i^\dagger)G_1^\alpha$, we can define the matrix $M_{12}^{\alpha\beta}$ and its inverse:

$${}^{ij}M_{12}^{\alpha\beta} \equiv \sum_{i_1,i_2} {}^{i_1}L_1^{\alpha i_2} L_2^\beta \mathrm{tr}\,(s_{i_1}s_i s_{i_2}s_j) \ , \tag{4.78a}$$

$${}^{ij}M^{-1}{}_{12}^{\alpha\beta} = \frac{1}{4}\sum_{i_1,i_2} {}^{i_1}G_1^{\alpha i_2} G_2^\beta \mathrm{tr}\,(s_i^\dagger s_{i_2} s_j^\dagger s_{i_1}) \tag{4.78b}$$

The inverse is only defined on the i,j indices. Some useful properties of the M-matrix are:

$${}^{ij}M_{12}^{\alpha\beta} = {}^{ji}M_{21}^{\beta\alpha} \ , \tag{4.78c}$$

$${}^{ij}M^{-1}{}_{12}^{\alpha\beta} = {}^{ji}M^{-1}{}_{21}^{\beta\alpha} \ , \tag{4.78d}$$

We are now in a position to decouple Λ and Q at the Gaussian level. This comes at the cost of coupling them at all orders higher than the quadratic part of the action, but it provides us a Gaussian theory for which we can define individual Q and Λ

propagators. Let

$$\Lambda_{12}^{\alpha\beta} \to 2L_1^{\alpha}\bar{\Lambda}_{12}^{\alpha\beta}L_2^{\beta} - 2L_1^{\alpha}\bar{Q}_{12}^{\alpha\beta}L_2^{\beta}, \tag{4.79}$$

where we have defined

$${}_{r}^{i}\bar{Q}_{12}^{\alpha\beta} = \begin{pmatrix} + \\ + \\ - \\ - \end{pmatrix}_r \begin{pmatrix} + \\ - \\ + \\ - \end{pmatrix}_i {}_{r}^{i}Q_{12}^{\alpha\beta} \ . \tag{4.80}$$

A transformation like Equation (4.79), where $M \to A\bar{M}A$ for $n\times n$ matrices M and A, imparts a Jacobian to the measure of the partition function $\mathcal{D}[M] \to det(A)^{2n}\mathcal{D}[M]$. We generally don't have to worry about this term as it will factor out in the partition function's normalization term whenever we compute expectation values. Under the transformation the Gaussian action minus the interaction contribution $\mathcal{A}_0^{(2)} = \mathcal{A}^{(2)} - \mathcal{A}_{\text{int}}$, becomes:

$$A_0^{(2)} = 4\sum_{\substack{i,j,r \\ \cdots}} \begin{pmatrix} + \\ - \\ - \\ + \end{pmatrix}_r \begin{pmatrix} + \\ - \\ - \\ - \end{pmatrix}_j {}^{ij}M_{12}^{\alpha\beta} I_{12} \left({}_{r}^{i}\bar{\Lambda}_{12}^{\alpha\beta\, j}{}_{r}\bar{\Lambda}_{12}^{\alpha\beta\ddagger} - {}_{r}^{i}\bar{Q}_{12}^{\alpha\beta\, j}{}_{r}\bar{Q}_{12}^{\alpha\beta\ddagger} \right) \ , \tag{4.81}$$

where $I_{12} = 1 - \delta_{n_1 n_2}/2$ prevents over counting of the $n_1 = n_2$ terms. Then the higher order terms become:

$$\Delta\mathcal{A} = -\sum_{n=3}^{\infty} \frac{(-2i)^n}{2n} \text{Tr}\left(((\bar{Q} - \bar{\Lambda})L)^n\right) \ . \tag{4.82}$$

We will need the third order term later on, so let us explicitly write it here. To that end, let us define

$$\begin{aligned} \not{Q}_{12}^{\alpha\beta} &= [\bar{Q} - \bar{\Lambda}]_{12}^{\alpha\beta}\Theta(n_1 \geq n_2) \ , \\ ({}^{i}_{r}Q_{12}^{\alpha\beta})^{+} &\equiv \begin{pmatrix} + \\ + \\ + \\ - \end{pmatrix}_r \begin{pmatrix} + \\ - \\ - \\ - \end{pmatrix}_i {}^{i}_{r}\not{Q}_{21}^{\beta\alpha\ddagger}\Theta(n_2 > n_1) \ , \end{aligned} \tag{4.83}$$

We emphasize that the object $\not{Q}^{+}$ is the only one to violate the $n_1 \geq n_2$ rule. With this definition we have:

$$\Delta\mathcal{A}^{(3)} = -\frac{4i}{3}\mathrm{Tr}\left((\not{Q} + \not{Q}^{+})L\right)^3 \ , \tag{4.84a}$$

$$\Delta\mathcal{A}^{(4)} = -2\mathrm{Tr}\left((\not{Q} + \not{Q}^{+})L\right)^4 \ , \tag{4.84b}$$

We can now write down the Gaussian action. For simplicity we will focus on the impact of the direct channel, although an extension to include all interaction channels can be done. The Gaussian level action reads:

$$\begin{aligned} \mathcal{A} = \ & 4\sum_{\substack{i,j,r \\ \dots}} \begin{pmatrix} + \\ - \\ - \\ + \end{pmatrix}_r \begin{pmatrix} + \\ - \\ - \\ - \end{pmatrix}_j {}^{ij}M_{12}^{\alpha\beta} I_{12}{}^{i}_{r}\bar{\Lambda}_{12}{}^{\alpha\beta j}_{\ r}\bar{\Lambda}_{12}^{\alpha\beta\ddagger} \\ & -4\sum_{\substack{i,j,r \\ \dots}} \begin{pmatrix} 0 \\ - \\ - \\ 0 \end{pmatrix}_r \begin{pmatrix} + \\ - \\ - \\ - \end{pmatrix}_j {}^{ij}M_{12}^{\alpha\beta} I_{12}{}^{i}_{r}\bar{Q}_{12}{}^{\alpha\beta j}_{\ r}\bar{Q}_{12}^{\alpha\beta\ddagger} \\ & -\sum_{\substack{r=0,3 \\ \dots}} \begin{pmatrix} + \\ - \\ + \\ - \end{pmatrix}_i \begin{pmatrix} + \\ + \\ - \\ + \end{pmatrix}_j \begin{pmatrix} + \\ 0 \\ 0 \\ + \end{pmatrix}_r {}^{ij}W_{12,34}^{\alpha\beta,\mu\nu} I_{12}{}^{i}_{r}Q_{12}{}^{\alpha\beta j}_{\ r}Q_{34}^{\alpha\beta\ddagger} \ , \end{aligned} \tag{4.85}$$

where we have specified that Q is purely non-interacting in the $r = 1, 2$ channels in the second line. The W tensor is the Gaussian level coupling in the $r = 0, 3$ channels

including both non-interacting and interacting contributions:

$$\frac{1}{4}{}^{ij}W^{\alpha\beta,\mu\nu}_{12,34} = {}^{ij}M^{\alpha\beta}_{12}\delta^{\alpha\mu}_{13}\delta^{\beta\nu}_{24} - \delta^{ij}\frac{4T}{V}\delta_{1-2,3-4}\Gamma^{d}_{1-2}\delta^{\alpha\beta}\delta^{\mu\nu}\delta^{i0} \ . \tag{4.86}$$

We will need the inverse of the tensor W^{-1}, which is defined by satisfying the following identity (repeated indices are summed over):

$${}^{ij}\mathbb{1}^{\alpha\beta,\mu\nu}_{12,34} = \delta^{ij}\delta^{\alpha\mu}_{13}\delta^{\beta\nu}_{24} = {}^{ik}W^{\alpha\beta,\bar{\mu}\bar{\nu}}_{12,\bar{3}\bar{4}}{}^{kj}W^{-1}{}^{\bar{\mu}\bar{\nu},\mu\nu}_{\bar{3}\bar{4},34} = {}^{ik}W^{-1}{}^{\alpha\beta,\bar{\mu}\bar{\nu}}_{12,\bar{3}\bar{4}}{}^{kj}W^{\bar{\mu}\bar{\nu},\mu\nu}_{\bar{3}\bar{4},34} \ . \tag{4.87}$$

While W^{-1} can be constructed for all interactions, we will initially want to focus on what the direct channel does to the semimetal system. This corresponds to setting $\Gamma^{e}_{i} \equiv 0$ in Equation (4.86). In such a case the inverse is given by:

$${}^{ij}W^{-1}{}^{\alpha\beta,\mu\nu}_{12,34} = \frac{1}{4}{}^{ij}M^{-1}{}^{\alpha\beta}_{12}\delta^{\alpha\mu}_{13}\delta^{\beta\nu}_{24} + \frac{T}{V}\tilde{\Gamma}^{d}_{1-2}{}^{i0}M^{-1}{}^{\alpha\beta}_{12}{}^{0j}M^{-1}{}^{\mu\nu}_{34}\delta^{\alpha\beta}\delta^{\mu\nu}\delta_{1-2,3-4} \tag{4.88}$$

See Appendix G for properties of this tensor. The function $\tilde{\Gamma}^{d}_{1-2}$ is the random phase approximation (RPA) corrected interaction

$$\tilde{\Gamma}^{d}_{1-2} \equiv \frac{\Gamma^{d}_{1-2}}{1-\Gamma^{d}_{1-2}\chi_{1-2}} \ , \tag{4.89}$$

with the susceptibility χ_{1-2} given by:

$$\begin{aligned}\chi_{1-2} &\equiv \chi_m(\boldsymbol{q}) = 4\frac{T}{V}\sum_{\sigma 34}\delta_{1-2,3-4}{}^{00}M^{-1}{}^{\sigma\sigma}_{3,4} \\ &= 4\frac{T}{V}\sum_{\sigma \boldsymbol{p} n}{}^{00}M^{-1}{}^{\sigma\sigma}_{n+m,n}(\boldsymbol{p}+\boldsymbol{q},\boldsymbol{p}) \\ &\equiv \frac{T}{V}\sum_{3,4}\delta_{1-2,3-4}\mathrm{Tr}\left(G_3 G_4\right) \ . \end{aligned} \tag{4.90}$$

The first line is specific to DSMs but the second line is general to all semimetals. By virtue of the fact that the effective field theory naturally builds in RPA corrections at the Gaussian level, it is no longer perturbative in the interaction coefficient Γ. This is because Γ is now included to all orders in the Gaussian theory. The higher order terms, $\Delta\mathcal{A}$, in the EFT are still *perturbative*, but now in a loop sense.

For any calculation, we will need the exact form of $\tilde{\Gamma}$ in equation (4.89), which in turn requires χ_{1-2}. This can be directly computing using the Feynman trick at $T = 0$ [66]. Setting $(\Omega_m, \boldsymbol{q}) = (\omega_{n_1} - \omega_{n_2}, \boldsymbol{k}_1 - \boldsymbol{k}_2)$, we obtain

$$(d=3)\ \chi_m(\boldsymbol{q}) = -\frac{q^2}{\pi^2 v_D}\left(\frac{11}{36} + \frac{1}{6}\log(\frac{\Lambda^2}{q^2+\Omega_m^2/v_D^2})\right) \ , \tag{4.91a}$$

$$(d=2)\ \chi_m(\boldsymbol{q}) = -\frac{1}{4}\frac{q}{v_D}\frac{1}{\sqrt{1+z_m^2}} \ , \tag{4.91b}$$

with $z_m = \Omega_m/v_D q$. For long ranged interactions, the purely quadratic term in $\chi_m(\boldsymbol{q})$ for $d = 3$ can be treated as a correction to the electric permittivity ϵ_r. The polarization bubble in Eq. (4.91a) agrees modulo a factor of 2 with Abrikosov and Beneslavskii[6] and Eq. (4.91b) agrees exactly with Kotov et al. for graphene [4, 76]. It is also worth observing that $\chi_m(\boldsymbol{q})$ scales with one additional power of q (or equivalently

[6]They didn't allow for two cones of different chirality in their original calculation. The polarization bubble scales linearly with the number of Dirac cones

Ω_m) than the soft modes discussed in Section 4.6, which reflects the fact that $\chi_m(\boldsymbol{q})$ is proportional to a frequency integral of the soft modes.

The field theory we have derived is a local one in the sense that all vertices are finite or vanish in the limit of small momentum and or wavelength, similar to the case of the field theory derived for Fermi liquids in Reference [14]. This conclusion can be reached by means of a scaling argument. For a semimetal, frequency and momentum scale the same $\Omega \sim k$, we can therefore perform the scaling analysis purely in terms of Ω. Now consider the higher order terms in the action $\Delta\mathcal{A}^{(n)} \propto L^n \mathcal{Q}^n$. As the inverse Green function scales as $L \sim \Omega$, then $\Delta\mathcal{A}^{(n)} \sim \Omega^n \mathcal{Q}^n$, which vanishes in the small Ω limit. The Gaussian term is subtler, as the term due to the electron-electron interaction scales as $(T/V)\Gamma^d$. While $\Gamma^d \sim \Omega^{-2}(\log|\Omega|)^{-1}$, the T/V term protects this from being divergence in the zero frequency limit because $T \sim \Omega$ and $V \sim k^{-d}$, and thus $(T/V)\Gamma^d \sim \Omega^2(\log|\Omega|)^{-1}$.

Propagators

Any calculation for which we will employ the effective field theory will involve $\langle QQ \rangle$ correlation functions. In the $r = 0, 3$ channels they are given by

$$\begin{aligned}
&\langle {}^{i}_{r}Q^{\alpha\beta\, j}_{12\ s}Q^{\mu\nu\ddagger}_{34}\rangle = \frac{1}{4I_{12}}\delta_{rs}T^4_r T^1_i T^1_j \left[T^{2\,ji}_j W^{-1\,\mu\nu,\alpha\beta}_{34,12} + T^{2\,ij}_i W^{-1\,\alpha\beta,\mu\nu\ddagger}_{12,34}\right] \\
&= \frac{1}{4I_{12}}\delta_{r,s}T^4_r T^1_i T^1_j \times \\
&\times \left[T^2_j \left(\frac{1}{4}{}^{ji}M^{-1\,\mu\nu}_{34}\delta^{\alpha\mu}_{13}\delta^{\beta\nu}_{24} + T^3_r \frac{T}{V}\tilde{\Gamma}^{d}_{3-4}{}^{j0}M^{-1\,\mu\nu 0i}_{34}M^{-1\,\alpha\beta}_{12}\delta^{\alpha\beta}\delta^{\mu\nu}\delta_{1-2,3-4}\right)\right. \\
&\left. + T^2_i \left(\frac{1}{4}{}^{ij}M^{-1\,\alpha\beta\ddagger}_{12}\delta^{\alpha\mu}_{13}\delta^{\beta\nu}_{24} + T^3_r \frac{T}{V}\tilde{\Gamma}^{d}_{3-4}{}^{i0}M^{-1\,\alpha\beta\ddagger 0j}_{12}M^{-1\,\mu\nu\ddagger}_{34}\delta^{\alpha\beta}\delta^{\mu\nu}\delta_{1-2,3-4}\right)\right] ,
\end{aligned} \tag{4.92}$$

where we have defined:

$$T_i^1 = \begin{pmatrix} + \\ - \\ + \\ - \end{pmatrix}_i \quad T_i^2 = \begin{pmatrix} + \\ - \\ - \\ - \end{pmatrix}_j \tag{4.93a}$$

$$T_r^3 = \begin{pmatrix} + \\ 0 \\ 0 \\ + \end{pmatrix}_r \quad T_r^4 = \begin{pmatrix} + \\ - \\ - \\ + \end{pmatrix}_r , \tag{4.93b}$$

That the propagator is a sum of two terms differing under the operation $i \leftrightarrow j$ composed with ‡ can be understood as follows. The Q-fields are classical (bosonic) fields, so the propagator reflect the fact that they commute, which amounts to being symmetric under the operation we just defined. However, the Gaussian level coupling is not invariant under this operation, so the propagator must take the form of a sum over terms reflecting the operation. This can be seen explicitly by computing the propagator from first principles using source functions. It is worth noting that the Gaussian propagator for semimetals has presented complications that did not arise in the case of a clean or disordered Fermi liquid. Technically speaking, this is because spin orbit coupling was not present in those cases, and thus the tensor M was diagonal in the i, j indices, and depended only on the modulus of momentum $|\boldsymbol{p}|$, for which a change in sign of $\boldsymbol{p}$ doesn't affect the propagator.[7]

It is important to note that all higher order corrections to $\mathcal{A}$ depend on $\mathcal{Q}$, that is $\Delta\mathcal{A}[\bar{Q}, \bar{\Lambda}] \equiv \Delta\mathcal{A}[\mathcal{Q}]$. This means that if one were to analyze the derived EFT by means of a loop expansion, one will never simply compute $\langle\Lambda\Lambda\rangle$ correlation functions, but instead compute either $\langle Q\mathcal{Q}\rangle$ or $\langle\mathcal{Q}\mathcal{Q}\rangle$ correlation functions. This is sensible as

[7]Since developing this theory, a slicker mechanism of expressing the chirality quantum number in the basis of SU(2) was devised, which greatly simplifies many of the expressions here. This will be featured in future works.

Λ is not actually a physical field that fluctuates in the system, it is a mathematical construct (Lagrange Multiplier field) which exists to constrain Q to bi-fermion fields. If we were to treat Λ perturbatively, it would relax the constraint on Q, effectively introducing spurious soft modes to the partition function. Notice at the Gaussian level, the $\bar{Q}$ and Λ terms have the same coefficients, but with opposite signs. This means that the $\langle \mathcal{Q}\mathcal{Q} \rangle$ correlation function contains only the interaction dependent part of the of the $\langle \bar{Q}\bar{Q} \rangle$ correlation function. We are therefore able to treat Λ exactly by formerly eliminating it from the field theory in favor of Feynman rules for $\bar{Q}$ and $\mathcal{Q}$. The $\langle \mathcal{Q}\mathcal{Q} \rangle$ correlation function is given by

$$\begin{aligned}
&\langle {}^{i}_{r}\mathcal{Q}^{\alpha\beta\, j}_{12\ s}\mathcal{Q}^{\mu\nu\,\ddagger}_{34}\rangle = \\
&\frac{1}{4I_{12}}\delta_{rs}T^{3}_{r}\tilde{\Gamma}^{d}_{1-2}\delta_{1-2,3-4}\delta^{\alpha\beta}\delta^{\mu\nu}\frac{T}{V}\left[T^{2}_{j}\ {}^{j0}M^{-1\,\mu\nu 0i}_{\ \ 34}M^{-1\,\alpha\beta}_{\ \ 12}+T^{2}_{i}\ {}^{i0}M^{-1\,\alpha\beta\,\ddagger 0j}_{\ \ 12}M^{-1\,\mu\nu\,\ddagger}_{\ \ 34}\right] .
\end{aligned} \tag{4.94}$$

We are now equipped with an effective field theory and Feynman rules that can be used to determine the impact of strong interactions on various physical observables. The usefulness of the field theory will then become evident, as it will enable us to identify the leading non-analytic corrections to physical observables, which as we discussed in the Chapter III, correspond to long time-tail and long-ranged correlation behavior. The ultimate power of the field theory is that due to the highly local nature of the higher order $\mathcal{A}^{(n)}$ terms in the theory, one can perform a renormalization group analysis of the scaling of observables following the techniques of Reference [77], which will be the focus of future works.

CHAPTER V

A SIMPLE APPLICATION: A LOOP EXPANSION FOR THE DENSITY OF STATES

The Loop Expansion for the Density of States

A simple yet interesting application of the EFT is determining the non-analytic corrections to the density of states (DOS). Recall from Equation (4.27), the DOS is given by:

$$\begin{aligned} N(\omega) &= \frac{4}{\pi}\mathrm{Re}\sum_{\alpha}\langle {}^0_0\tilde{Q}^{\alpha\alpha}_{nn}(\boldsymbol{x},\boldsymbol{x})\rangle|_{i\omega_n\to\omega+i0^+} \\ &\equiv \mathrm{Re}\, N(i\omega_n)|_{i\omega_n\to\omega+i0^+}\ , \end{aligned} \tag{5.1}$$

where $\tilde{Q}$ is the field in $\mathcal{A}[\tilde{Q},\tilde{\Lambda}]$. Then we have:

$$\begin{aligned} N(i\omega_n) &= \frac{4}{\pi}\sum_{\alpha}{}^0_0Q^{\mathrm{sp}\alpha\alpha}_{nn}(\boldsymbol{x},\boldsymbol{x}) + \frac{4}{\pi}\mathrm{Re}\sum_{\alpha}\langle {}^0_0\tilde{Q}^{\alpha\alpha}_{nn}(\boldsymbol{x},\boldsymbol{x})\rangle \\ &\equiv N^{\mathrm{sp}}(i\omega_n) + \delta N(i\omega_n)\ , \end{aligned} \tag{5.2}$$

where $N^{\mathrm{sp}}(i\omega_n) \propto -\omega_n^{(d-1)}/v_D^d$ for semimetals, and v_D is the Hartree-Fock corrected Dirac velocity. Computing $\langle {}^0_0Q^{\alpha\alpha}_{nn}(\boldsymbol{x},\boldsymbol{x})\rangle$ requires us to use a loop expansion in the

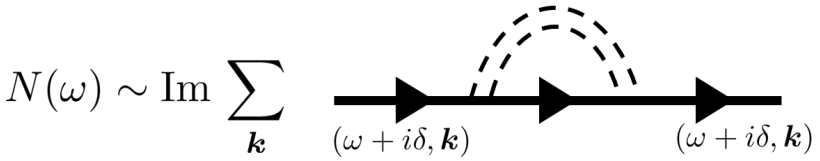

FIGURE 4. Diagrammatic representation of DOS calculation.

EFT. To one loop order $\Delta\mathcal{A}^{(3)}$ contributes:

$$\begin{aligned}
\delta N(i\omega_n) &= -\frac{4}{\pi}\mathrm{Re}\,\frac{1}{V}\sum_{\alpha,\boldsymbol{k}}\langle {}^0_0 Q^{\alpha\alpha}_{nn}(\boldsymbol{x},\boldsymbol{x})\Delta\mathcal{A}^{(3)}\rangle_{\mathcal{A}^{(2)}} \\
&= -\frac{16i}{3\pi}\mathrm{Re}\,\frac{1}{V}\sum_{\alpha,\boldsymbol{k}}\langle {}^0_0 Q^{\alpha\alpha}_{nn}(\boldsymbol{k},\boldsymbol{k})\mathrm{Tr}\,\left[((\not{Q}+\not{Q}^{+})L)^3\right]\rangle_{\mathcal{A}^{(2)}} \\
&\equiv -\frac{16i}{3\pi}\mathrm{Re}\,F(i\omega_n)\ , \\
\Rightarrow \delta N(\omega) &= \frac{16}{3\pi}\mathrm{Im}\,F(i\omega_n)|_{i\omega_n\to\omega+i0+}\ . \qquad (5.3)
\end{aligned}$$

Our goal is now to show that $F(i\omega_n)$ is equivalent to the diagram in Figure 4, as was found by Belitz and Kirkpatrick in Reference [14]. Following Appendix H:

$$\begin{aligned}
F(i\omega_n) &= \frac{T}{V^2}6\sum_{\substack{\boldsymbol{p},\boldsymbol{k}\\ m\neq n}} -\frac{(i\omega_n)^2 i\omega_m - i\omega_m v_D^2 k^2 - 2i\omega_n v_D \boldsymbol{k}\cdot\boldsymbol{p}}{(\omega_n^2+v_D^2k^2)^2(\omega_m^2+v_D^2p^2)}\tilde{\Gamma}^d_{n-m}(\boldsymbol{k}-\boldsymbol{p}) \\
&= \frac{6}{4}\frac{T}{V^2}\sum_{\substack{\boldsymbol{p},\boldsymbol{k}\\ m\neq n}} \mathrm{tr}\ \left(G(i\omega_n,\boldsymbol{k})G(i\omega_m,\boldsymbol{p})G(i\omega_n,\boldsymbol{k})\right)\tilde{\Gamma}^d_{n-m}(\boldsymbol{k}-\boldsymbol{p})\ , \qquad (5.4)
\end{aligned}$$

where the first line is specific to $d = 3$ dimensions (DSMs) and the second is general to all dimensions. The difference between the first line for a DSM and graphene is simply a factor of 2. The goal is to now asymptotically analyze this integral in $d = 2$ and $d = 3$ dimensions, for both long ranged and short ranged interactions. We will explicitly present the long ranged interaction correction to the DOS for DSMs in this chapter, and refer to Appendices H, I and J for the other cases.

Non-Analytic Corrections to the Density of States in Dirac Semimetals

We want to determine the leading non-analytic behavior of the following term as $T \to 0$:

$$F(i\omega_n) = \frac{6}{4}\frac{T}{V^2}\sum_{\substack{\boldsymbol{q},\boldsymbol{k}\\ m\neq 0}} \text{tr}\ \left(G(i\omega_n,\boldsymbol{k})G(i\omega_n+i\Omega_m,\boldsymbol{q}+\boldsymbol{k})G(i\omega_n,\boldsymbol{k})\right)\tilde{\Gamma}^d_m(\boldsymbol{q})\ , \quad (5.5)$$

for the screened interactions derived in Equation (4.91a):

$$\begin{aligned} \text{Long Ranged}\quad \tilde{\Gamma}^d_l(\Omega_m,q) &= \frac{\Gamma}{q^2+nq^2\log(\Lambda^2 v_D^2/(q^2v_D^2+\Omega_m^2))} \\ &\equiv \frac{\Gamma}{q^2}V_l(q^2v_D^2+\Omega^2)\ , \quad (5.6a) \\ \text{Short Ranged}\quad \tilde{\Gamma}^d_s(\Omega_m,q) &= \frac{\Gamma}{1+\epsilon_r q^2+nq^2\log(\Lambda^2 v_D^2/(q^2v_D^2+\Omega_m^2)} \\ &\equiv \frac{\Gamma}{q^2}V_s(q^2,q^2v_D^2+\Omega^2)\ . \quad (5.6b) \end{aligned}$$

In the long ranged case we have absorbed the dimensionless renormalized permittivity ϵ_0 into Γ, and defined the screening parameter n. If we use the bare Coulomb interaction as the basis for the long ranged interaction, then n is a dimensionless constant proportional to the fine structure constant $\alpha_f = 1/137$:

$$n = \frac{\Gamma}{6\pi^2 v_D} \sim \frac{\alpha_f}{6\pi^2}\frac{c}{v_D}\ . \quad (5.7)$$

As discussed in Chapter II the Dirac velocity in a DSM is on the order of $v_D \sim \mathcal{O}(10^{-5}c)$ to $1\mathcal{O}(10^{-4}c)$, resulting in n being a number on the order of $\mathcal{O}(1)$ to $\mathcal{O}(10)$. This is reminiscent of the density parameter $r_s \sim \mathcal{O}(1)$ to $\mathcal{O}(10)$ in metals, and the reason we refer to the Dirac Semimetal as *strongly interacting*. We will refer to the limit $n \to 0$ as the *unscreened limit*, which will be useful in assessing the validity of

our results. In the unscreened limit:

$$\lim_{n\to 0} V_l(q^2 v_D^2 + \Omega^2) = 1 \tag{5.8a}$$

$$\lim_{n\to 0} V_s(q^2, q^2 v_D^2 + \Omega^2) = \frac{1}{1/q^2 + \epsilon_r} \tag{5.8b}$$

The case of a long ranged interaction

We will begin by calculating the right hand side of Equation (5.5) for the screened long ranged interaction in Equation (5.6a), the short ranged case is found in Appendix I . We can isolate the RPA corrected self energy term in $F(i\omega_n)$ as:

$$\begin{aligned}
\Sigma_l^\alpha(i\omega_n, \boldsymbol{k}) &= \frac{T}{V}\sum_{\boldsymbol{q}, m\neq 0} G^\alpha(i\omega_n + i\Omega_m, \boldsymbol{q}+\boldsymbol{k})\tilde{\Gamma}_m^d(\boldsymbol{q}) \\
&\to \frac{1}{(2\pi)^4}\int d\Omega \int d\boldsymbol{q}\; G^\alpha(i\omega_n + i\Omega, \boldsymbol{q}+\boldsymbol{k})\tilde{\Gamma}_l^d(\Omega, p) \\
&= -\frac{\Gamma}{(2\pi)^4}\int d\Omega \int d\boldsymbol{q}\; \frac{1}{q^2}\frac{i(\omega_n+\Omega)\sigma_0 + \alpha v_D(\boldsymbol{q}+\boldsymbol{k})\cdot\boldsymbol{\sigma}}{(\omega_n+\Omega)^2 + v_D^2(\boldsymbol{q}+\boldsymbol{k})^2} V_l(q^2 v_D^2 + \Omega^2)\ .
\end{aligned} \tag{5.9}$$

In the second line we have used that fact that for $T \to 0$, the point $m = 0$ is measure zero on the real line and its inclusion/exclusion does not affect the integral. Let us first compute the leading non-analytic behavior[1] of the unscreened self energy term

[1]The self energy integral can be done exactly for the unscreened interaction, but not for the screened one. We restrict ourselves to using asymptotics to determine the leading behavior in the unscreened case to have a basis of comparison for the harder problem.

$\tilde{\Sigma}_l^\alpha(i\omega_n, \boldsymbol{k})$. Shifting $\Omega \to \Omega - \omega_n$ and scaling out v_D we have:

$$\begin{aligned}
\tilde{\Sigma}_l^\alpha(i\omega_n, \boldsymbol{k}) &= -\alpha \frac{\Gamma}{8\pi^2} \int_0^{\Lambda v_D} dq \int_{-1}^{1} d\eta \; \frac{\boldsymbol{v_D k} + \boldsymbol{q}}{|\boldsymbol{v_D k} + \boldsymbol{q}|} \cdot \boldsymbol{\sigma} \\
&= -\alpha \frac{\Gamma}{8\pi^2} \boldsymbol{k} \cdot \boldsymbol{\sigma} \int_0^{\Lambda/k} dq \int_{-1}^{1} d\eta \; \frac{1 + q\eta}{|\hat{k} + \boldsymbol{q}|} \\
&\approx -\alpha \frac{\Gamma}{8\pi^2} \boldsymbol{k} \cdot \boldsymbol{\sigma} \int_1^{\Lambda/k} dq \int_{-1}^{1} d\eta \; \left(\eta + \frac{(1-\eta^2)}{q} \right) \\
&= -\alpha \frac{2\Gamma}{3(2\pi)^2} \boldsymbol{k} \cdot \boldsymbol{\sigma} \; \log\left(\frac{\Lambda}{q}\right). \qquad (5.10)
\end{aligned}$$

Observe that the denominator in the first line has an infrared (IR) divergence when $k = 0$. By scaling q with k in the second line, we have transfered the $k = 0$ divergence to the ultraviolet (UV) upper limit of the integral. In the third line we Taylor expanded the integrand around $q = \infty$, and cut off the integral at 1 to protect ourselves from introducing fictitious IR blow-ups. The $\log(\Lambda/q)$ term of our unscreened self energy term agrees with the results of Abrikosov and Beneslavskii, and Throckmorton *et al.* [4, 78]. Note that Γ and v_D have the same units for the long ranged interaction, and as a consequence $\tilde{\Sigma}_l^\alpha(i\omega_n, \boldsymbol{k})$ does not scale with v_D.

We will next determine the self energy $\Sigma_l^\alpha(i\omega_n, \boldsymbol{k})$, for the screened interaction, but first we will prove that any non-analytic behavior will originate from $k \to 0$. To do this, let us set $\boldsymbol{k} = 0$ in the denominator of the integrand in Equation (5.9):

$$\begin{aligned}
& -\frac{\Gamma}{(2\pi)^4} \int d\Omega \int d\boldsymbol{q} \; \frac{1}{q^2} \frac{i(\omega_n + \Omega)\sigma_0 + \alpha v_D (\boldsymbol{q} + \boldsymbol{k}) \cdot \boldsymbol{\sigma}}{(\omega_n + \Omega)^2 + v_D^2 q^2} V_l(q^2 v_D^2 + \Omega^2) \\
= \; & -\frac{2\Gamma}{(2\pi)^3} \int d\Omega \int dq \; \frac{i(\omega_n + \Omega)\sigma_0 + \alpha v_D \boldsymbol{k} \cdot \boldsymbol{\sigma}}{(\omega_n + \Omega)^2 + v_D^2 q^2} V_l(q^2 v_D^2 + \Omega^2) \\
= \; & -\frac{2\Gamma}{(2\pi)^3 v_D} \int_0^{\Lambda v_D} d\bar{q} \int_0^{2\pi} d\theta \; \bar{q} \frac{i(\omega_n + \bar{q}\cos\theta)\sigma_0 + \alpha v_D \boldsymbol{k} \cdot \boldsymbol{\sigma}}{\bar{q}^2 + \omega_n^2 + 2\omega_n \bar{q}\cos\theta} V_l(\bar{q}^2) \\
= \; & -\frac{\Gamma}{(2\pi)^2 v_D} \int_0^{\Lambda v_D} d\bar{q} \; \bar{q} V_l(\bar{q}^2) \left(\frac{\alpha v_D \boldsymbol{k} \cdot \boldsymbol{\sigma}}{|\bar{q}^2 - \omega_n^2|} + i\sigma_0 \frac{\mathrm{sgn}\,(\bar{q} - |\omega_n|) - 1}{2\omega_n} \right) \qquad (5.11)
\end{aligned}$$

In the third line we switched to the coordinate system $\bar{q}(\cos\theta, \sin\theta) = (\Omega, q)$. In the first term in the final line the integral is infinite for any real value of ω_n, and it is some kind of logarithmic infinity due the first order pole for $\bar{q}$ at $|\omega_n|$ (the presence of V_l will lightly modify this statement as we will see). Thus, the linear $\boldsymbol{k}$ term will kill the infinity as $k \to 0$, meaning $\Sigma_l^\alpha(i\omega_n, \boldsymbol{k})$ does not have an IR divergence, but its derivative with respect to $\boldsymbol{k}$ does. The second term is integrable, and can be done exactly:

$$\begin{aligned} -\frac{1}{2\omega_n}\int_0^{\omega_n} d\bar{q} V_l(\bar{q}^2)\frac{\bar{q}}{2\omega_n} &= \Lambda^2 v_D^2 \frac{e^{1/n}}{2n}\mathrm{Ei}\left(\frac{-1}{n} + 2\log\left(\frac{|\omega_n|}{\Lambda v_D}\right)\right) \\ &\approx \frac{\omega_n}{4(1 - 2n\log(|\omega_n|/\Lambda v_D))} \end{aligned} \tag{5.12}$$

The last line is true for $\omega_n \to 0$, and we have used the asymptotic form of the exponential integral $\mathrm{Ei}(x) \approx e^x/x$ for $x \to \infty$. While the function in Equation (5.12) is non-analytic in ω_n, it goes to 0 faster than linearly as $\omega_n \to 0$. The conclusion of this discussion is that the leading non-analytic behavior in $\Sigma_l^\alpha(i\omega_n, \boldsymbol{k})$ comes from $\Sigma_l^\alpha(0, \boldsymbol{k})$ which we will now calculate asymptotically.

$$\begin{aligned} \Sigma_l^\alpha(0,\boldsymbol{k}) &= -\alpha\frac{\Gamma}{(2\pi)^3}\boldsymbol{k}\cdot\boldsymbol{\sigma}\int_0^{\Lambda v_D} dq \int d\Omega \int_{-1}^{1} d\eta\ V_l(q^2+\Omega^2)\frac{\frac{q}{v_D k}\eta + 1}{\Omega^2 + q^2 + 2qk\eta v_D + v_D^2 k^2} \\ &= -\alpha\frac{\Gamma}{(2\pi)^3}\boldsymbol{k}\cdot\boldsymbol{\sigma}\int_0^{\Lambda v_D} d\bar{q} \int_0^{\pi} d\theta \int_{-1}^{1} d\eta\ \bar{q} V_l(\bar{q}^2)\frac{\frac{\bar{q}}{v_D k}\eta\sin\theta + 1}{\bar{q}^2 + 2\bar{q}\eta\sin\theta v_D k + v_D^2 k^2} \\ &= -\alpha\frac{\Gamma}{(2\pi)^3}\boldsymbol{k}\cdot\boldsymbol{\sigma}\int_0^{\Lambda/k} d\bar{q} \int_0^{\pi} d\theta \int_{-1}^{1} d\eta\ V_l(\bar{q}^2 k^2 v_D^2)\frac{\bar{q}^2\eta\sin\theta + \bar{q}}{\bar{q}^2 + 2\bar{q}\eta\sin\theta + 1} \\ &\approx -\alpha\frac{\Gamma}{(2\pi)^3}\boldsymbol{k}\cdot\boldsymbol{\sigma}\int_1^{\Lambda/k} d\bar{q} \int_0^{\pi} d\theta \int_{-1}^{1} d\eta\ V_l(\bar{q}^2 k^2 v_D^2)\bar{q}\left(\frac{\eta\sin\theta}{\bar{q}} + \frac{1 - 2\eta^2\sin^2\theta}{\bar{q}^2}\right) \\ &= -\alpha\frac{\Gamma 4\pi}{3(2\pi)^3}\boldsymbol{k}\cdot\boldsymbol{\sigma}\int_{k/\Lambda}^{1} d\bar{q}\ \frac{1}{\bar{q}} V_l(\bar{q}^2/\Lambda^2) \\ &= -\alpha\frac{2}{3}\frac{\Gamma}{(2\pi)^2}\boldsymbol{k}\cdot\boldsymbol{\sigma}\frac{\log\left(1 + 2n\log\left(\frac{\Lambda}{k}\right)\right)}{2n}\ . \end{aligned} \tag{5.13}$$

In the the third line we scaled $\bar{q}$ with k to move the IR divergence to the UV, and then asymptotically expanded the last term in the integrand, cutting it off in the fourth line to prevent artificial infinities. The $k\log(1+2n\log 1/k)$ non-analyticity is very interesting. Using Fermionic many body theory, Throckmorton *et al.* calculated $\Sigma_l^\alpha(0,\boldsymbol{k})$ to two loop order, and found a leading $k\log^2(\Lambda/k)$ non-analyticity. However, if you consider their entire expression for $\Sigma_l^\alpha(0,\boldsymbol{k})$, it contains the first two terms of $(1/2n)\ \log(1+2n\log(\Lambda/k))$ when Taylor expanded with respect to $2n$. By using

$$\lim_{n\to 0}\frac{\log 1+2n\log x}{2n}=\log x\ , \tag{5.14}$$

we recover Equation (5.9) in the unscreened limit. We can now plug $\Sigma_l^\alpha(0,\boldsymbol{k})$ into Equation (5.5):

$$\begin{aligned} F(i\omega_n) &\approx -\frac{6}{4}\frac{2}{3}\frac{\Gamma}{(2\pi)^2}\frac{1}{(2\pi)^3}\int d\boldsymbol{k}\ \mathrm{tr}\left(G(i\omega_n,\boldsymbol{k})^2(\tau_z\otimes\boldsymbol{k}\cdot\boldsymbol{\sigma})\right)\frac{\log\left(1+2n\log\left(\frac{\Lambda}{k}\right)\right)}{2n} \\ &= -i\Gamma\frac{16}{(2\pi)^4}\frac{\Lambda}{v_D^3}\omega_n\int_0^1 dk\ \frac{k^4}{(k^2+(\omega_n/\Lambda v_D)^2)^2}\frac{\log\left(1+2n\log\left(\frac{1}{k}\right)\right)}{2n}\ . \end{aligned} \tag{5.15}$$

In the second line we have scaled k with Λ. This is a tricky integral to apply the usual asymptotic analysis to, because the integrand goes to 0 at both ends of the integral, so any non-analyticity is coming from the middle. If we set $\omega_n=0$ in the integrand we obtain a finite, non-zero number. One could try Taylor expanding the first term for small ω_n, but you would see that all terms of the Taylor expansion contribute to the same order of the leading ω_n non-analyticity. We will have to use a different technique, so let us first demonstrate it works for the leading non-analyticity in the unscreened case, and then apply it to the screened case. Defining $\bar{\omega}_n=\omega_n/\Lambda v_D$, for

unscreened interactions we have:

$$\tilde{F}(i\omega_n) = i\Gamma \frac{16}{(2\pi)^4} \frac{\Lambda}{v_D^3} \omega_n \int_0^1 dk \ \frac{k^4}{(k^2+\bar{\omega}_n^2)^2} \log(k) \ . \tag{5.16}$$

We will do this integral using complex analysis. We could use the branch cut along the

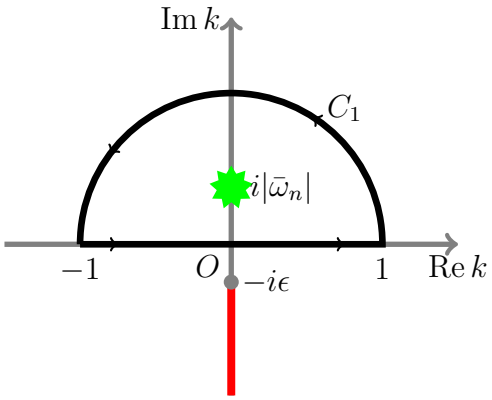

FIGURE 5. Integration path used to evaluate Equation (5.16).

negative axis, or we could equivalently introduce a small parameter $i\epsilon$ to $\log(k+i\epsilon)$, so that the branch cut is now along the imaginary axis interval $(-i\epsilon, -i\infty)$. Evaluating the contour integral in Figure 5 we obtain:

$$\begin{aligned} f(\bar{\omega}_n) &= \int_C dk \ \frac{k^4}{(k^2+\bar{\omega}_n^2)^2} \log(k+i\epsilon) \\ &= \int_0^1 dk \ \frac{k^4}{(k^2+\bar{\omega}_n^2)^2} \left(\log(k+i\epsilon) + \log(-k+i\epsilon)\right) + \int_{C_1} dk \ \frac{k^4}{(k^2+\bar{\omega}_n^2)^2} \log(k+i\epsilon) \\ &\xrightarrow[\epsilon\to 0]{} \int_0^1 dk \ \frac{k^4}{(k^2+\bar{\omega}_n^2)^2} \left(2\log(|k|) + i\pi\right) + \int_0^\pi d\theta \ i \frac{e^{i5\theta}}{(e^{i2\theta}+\bar{\omega}_n^2)^2} i\theta \ . \end{aligned} \tag{5.17}$$

We then have

$$\tilde{F}(i\omega_n) = i\Gamma \frac{16}{(2\pi)^4} \frac{\Lambda}{v_D^3} \omega_n \frac{1}{2} \mathrm{Re} \lim_{\epsilon\to 0} \left(f(\bar{\omega}_n) - \int_0^\pi d\theta \ i \frac{e^{i5\theta}}{(e^{i2\theta}+\bar{\omega}_n^2)^2} i\theta \right) \ . \tag{5.18}$$

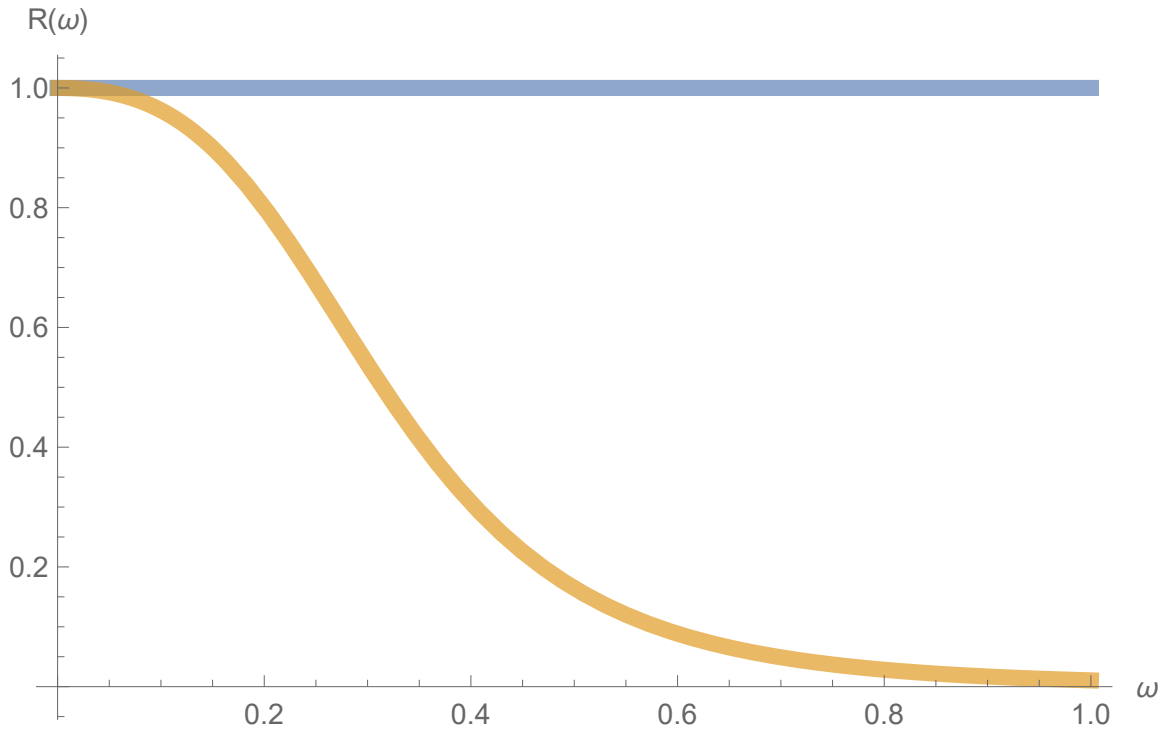

FIGURE 6. We have plotted $R(\omega) = g_2(\omega)/g_1(\omega)$ versus ω to show that the asymptotic analysis is correct.

The last term in Equation (5.18) only contains analytic $\bar{\omega}_n$ contributions to $f(\bar{\omega}_n)$, and since $\bar{\omega}_n \ll 1$, to leading order we can set it equal to zero in the denominator. Then we can compute

$$\int_0^\pi d\theta \; ie^{i\theta} i\theta = 2 - i\pi \; . \tag{5.19}$$

Next, we use the residue theorem to determine:

$$f(\bar{\omega}_n) \;\; = \;\; -\text{sgn}\,(\bar{\omega}_n)\frac{\pi}{2}((1 + i3\pi/2)\bar{\omega}_n + 3\bar{\omega}_n \log|\bar{\omega}_n|) \; . \tag{5.20}$$

In Figure 6 we numerically demonstrate that the asymptotic behavior

$$g_1(\bar{\omega}_n) \equiv \frac{1}{2}\mathrm{Re}\left(f(\bar{\omega}_n) - 2\right) \approx \int_0^1 dk \ \frac{k^4}{(k^2+\bar{\omega}_n^2)^2} \log k \equiv g_2(\bar{\omega}_n) \ , \tag{5.21}$$

is correct. Putting everything together, the leading non-analytic behavior of $\tilde{F}(i\omega_n)$ is:

$$\tilde{F}(i\omega_n) \approx -i \ \mathrm{sgn}\left(\omega_n\right) \frac{6}{(2\pi)^3} \frac{\Gamma}{v_D} \frac{\omega_n^2}{v_D^3} \log\left(\frac{|\omega_n|}{\Lambda v_D}\right) \ . \tag{5.22}$$

We could anticipate from the earlier discussion of Σ and $\tilde{\Sigma}$, that the leading non-analytic behavior in F comes from sending $-\log|\omega_n|$ to $\log(1-n\log|\omega_n|)/2n$. This will prove correct, but getting there takes a couple steps. First, the integrand with $\omega_n = 0$ in the denominator of Equation (5.15) can be done exactly:

$$\begin{aligned} F_0(i\omega_n) &\equiv \lim_{\omega_n \to 0} \left(\frac{F(i\omega_n)}{\omega_n}\right) = -i\Gamma \frac{16}{(2\pi)^4} \frac{\Lambda}{v_D^3} \omega_n \int_0^1 dk \ \frac{\log\left(1 + 2n \log\left(\frac{1}{k}\right)\right)}{2n} \\ &= i\Gamma \frac{16}{(2\pi)^4} \frac{\Lambda}{v_D^3} \omega_n e^{1/2n} \frac{\mathrm{Ei}(-\frac{1}{2n})}{2n} \end{aligned} \tag{5.23}$$

Next, let us Taylor expand the $\log(1+2n\log 1/k)$ term in Equation (5.15) in $2n$ using $\log 1/k > 0$ for $k \in (0,1)$:

$$F(i\omega_n) = -i\Gamma \frac{16}{(2\pi)^4} \frac{\Lambda}{v_D^3} \omega_n \int_0^1 dk \ \frac{k^4}{(k^2 + (\omega_n/\Lambda v_D)^2)^2} \frac{1}{2n} \sum_{j=1}^{\infty} -\frac{(2n)^j \log^j(k)}{j} \ . \tag{5.24}$$

We are interested in obtaining the following integral:

$$i_j(\bar{\omega}_n) = \lim_{\epsilon \searrow 0} \int_0^1 dx \ \log^j(x + i\epsilon) \frac{x^4}{(x^2+\bar{\omega}_n^2)^2} \ . \tag{5.25}$$

Let us again use the contour C defined in Figure 5, and define the integral:

$$\begin{aligned} k_j(\bar{\omega}_n) &= \lim_{\epsilon \searrow 0} \int_C dx \ \log^j(x + i\epsilon) \frac{x^4}{(x^2 + \bar{\omega}_n^2)^2} \\ &= \underbrace{\int_0^\pi d\theta \frac{i e^{i5\theta} (i\theta)^j}{(e^{2i\theta} + \bar{\omega}_n^2)^2}}_{\equiv K_j(\bar{\omega}_n)} + \int_0^1 dx \ \left(\log^j(x) + (\log(x) + i\pi)^j \right) \frac{x^4}{(x^2 + \bar{\omega}_n^2)^2} \ . \end{aligned} \tag{5.26}$$

One can then derive the following recursion relationship between $i_j(\bar{\omega}_n)$ and $k_j(\bar{\omega}_n)$:

$$i_j(\bar{\omega}_n) = \frac{1}{2} \left(\text{Re} \left[k_j(\bar{\omega}_n) - K_j(\bar{\omega}_n) \right] - \sum_{\substack{a=1 \\ a\text{-even}}}^{j} \binom{j}{a} i_{j-a}(\bar{\omega}_n) (i\pi)^a \right) \tag{5.27}$$

Since we are interested in the *leading* non-analytic behavior of $F(i\omega_n)$, we ultimately only need to determine the leading non-analytic behavior of each $i_j(\bar{\omega}_n)$, which we will see scale as $|\bar{\omega}_n| \log^j(|\bar{\omega}_n|)$. The lesser terms will scale as $|\bar{\omega}_n| \log^{a<j}(|\bar{\omega}_n|)$ for $a \in \mathbb{N}$, and upon inserting these terms back into the sum in Equation (5.24), they ultimately yield terms in $F(i\omega_n)$ like $\omega_n^2 \log(1 - 2n \log|\bar{\omega}_n|) / \log^s|\bar{\omega}_n|$ for $s \in \mathbb{N}$. These terms are non-analytic, but vanish faster than the leading term, so can be safely ignored. Additionally, all of the terms $K_j(\bar{\omega}_n)$ are analytic in $\bar{\omega}_n$, and when resummed will not contribute any non-analyticities in $\bar{\omega}_n$, moreover their constant terms $K_j(0)$ only provide another way to calculate $F_0(i\omega_n)$. To leading order in $\bar{\omega}_n \ll 1$, we obtain by the residue theorem:

$$k_j(\bar{\omega}_n) \approx -\text{sgn}(\bar{\omega}_n) \bar{\omega}_n \frac{3\pi}{2} \log^j(|\bar{\omega}_n|) \ . \tag{5.28}$$

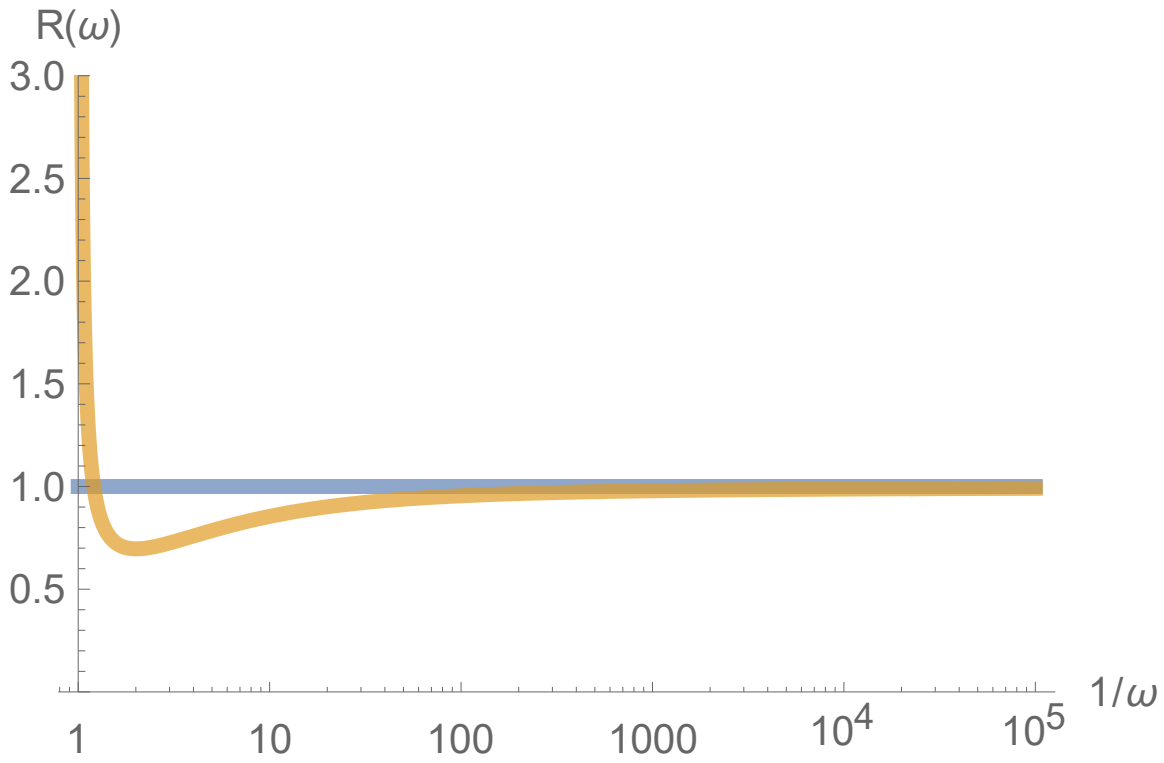

FIGURE 7. We have plotted $R(\omega) = g_4(i\omega)/g_3(i\omega)$ versus $1/\omega$ to show that the asymptotic analysis is correct.

By Equation (5.27) and the discussion above we have:

$$\begin{aligned} F(i\omega_n) - F_0(i\omega_n) &\approx -i\Gamma \frac{16}{(2\pi)^4} \frac{\Lambda}{v_D^3} \omega_n \frac{1}{2n} \sum_{j=1}^{\infty} -\frac{(2n)^j k_j(\bar{\omega}_n)}{j} \\ &\approx -i \ \mathrm{sgn}\,(\omega_n) \frac{6}{(2\pi)^3} \frac{\Gamma}{v_D} \frac{\omega_n^2}{v_D^3} \frac{\log\left(1 - 2n \log\left(\frac{|\omega_n|}{\Lambda v_D}\right)\right)}{2n} \ . \end{aligned} \tag{5.29}$$

This result again yields the unscreened calculation for $n \to 0$. Numerical comparison of the asymptotic result can be found in Figure 7, using the following functions:

$$g_3(\omega) \equiv -\frac{3\pi}{4} \frac{|\omega|}{2n} \log(1 - 2n \log(|\omega|)) \ , \tag{5.30a}$$

$$g_4(\omega) \equiv \int_0^1 dk \ \left(\frac{k^4}{(k^2 + \omega^2)^2} - 1 \right) \frac{\log(1 - 2n \log(x))}{2n} \ . \tag{5.30b}$$

The DOS non-analyticity can finally be determined by analytic continuation of $i\omega_n \to \omega + i0^+$, and then computing the imaginary part of F:

$$\begin{aligned} \operatorname{Im} F(\omega + i0^+) &\approx \frac{3}{(2\pi)^3}\frac{\Gamma}{v_D}\frac{\omega^2}{v_D^3}\frac{1}{2n}\log\left(\left[1 - 2n\log\left(\frac{|\omega|}{\Lambda v_D}\right)\right]^2 + n^2\pi^2\right) \\ &\xrightarrow[\omega\to 0]{} \frac{6}{(2\pi)^3}\frac{\Gamma}{v_D}\frac{\omega^2}{v_D^3}\frac{1}{2n}\log\left(2n\left|\log\left(\frac{|\omega|}{\Lambda v_D}\right)\right|\right) \ . \end{aligned} \tag{5.31}$$

We will discuss this result in the following section.

Density of States Non-Analyticities for All Cases

Using Equation (5.3) and the results of the previous section and Appendices H and I, we find for DSMs the DOS correction goes for screened long range, unscreened long range and screened short ranged interactions respectively as:

$$\delta N_{LRS}^{DSM}(\omega) = \mathcal{O}(\omega^2) + \frac{4\Gamma}{\pi^4 v_D}\frac{\omega^2}{v_D^3}\frac{1}{2n}\log\left(2n\left|\log\left(\frac{|\omega|}{\Lambda v_D}\right)\right|\right) \ , \tag{5.32a}$$

$$\delta N_{LRUS}^{DSM}(\omega) = \mathcal{O}(\omega^2) + \frac{4\Gamma}{\pi^4 v_D}\frac{\omega^2}{v_D^3}\log\left(\frac{|\omega|}{\Lambda v_D}\right) \ , \tag{5.32b}$$

$$\delta N_{SR}^{DSM}(\omega) = \mathcal{O}(\omega^2, \omega^4) + \frac{4\Gamma}{\pi^5 v_D^6}\omega^4\log\left(\frac{|\omega|}{\Lambda v_D}\right) \ . \tag{5.32c}$$

Note that for units of energy E, and length L, the units of Γ are $[\Gamma] = E \times L$ for long ranged interactions and $[\Gamma] = E \times L^d$ for short. By the results of Appendices H and J we find the DOS corrections for graphene go as:

$$\delta N_{LR}^{gra}(\omega) = \mathcal{O}(\omega) + c_1(n_g)|\omega|\frac{\Gamma}{v_D^3} - c_2(n_g)|\omega|\frac{\Gamma}{v_D^3}\log\left(\frac{|\omega|}{\Lambda v_D}\right) \ , \tag{5.33a}$$

$$\delta N_{SR}^{gra}(\omega) = \mathcal{O}(|\omega|, \omega^2) + \frac{2}{5\pi^2}\frac{\Gamma n_g}{v_D^5}\omega^2|\omega|\log\left(\frac{|\omega|}{\Lambda v_D}\right) \ , \tag{5.33b}$$

where the $c_i(n_g)$'s are real and positive dimensionless constants, and defined in Appendix J. In this case, the units $[\Gamma] = E \times L$ and $[n_g] = 1$ for long ranged interactions, and $[\Gamma] = E \times L^2$ and $[n_g] = L$ for short. Note that in Fermi liquids we would already identify $|\omega|$ as a non-analytic correction to the DOS in 2D, but since graphene's bare DOS is proportional to $|\omega|$, it is merely a correction to the $\frac{1}{v_D^2}$ coefficient in the bare DOS.

Now that we have our non-analytic corrections to the DOS, let us understand them in detail. First recall that $\Gamma > 0$ for repulsive electron-electron interactions, and $\Gamma < 0$ for attractive. We will assume that $\Gamma > 0$ for the long ranged interactions, which is the case if it is Coulomb. Notice that the $\omega^2 \log|\log|\omega||$ non-analytic term is unique to $\delta N_{LRS}^{DSM}(\omega)$, and dominates $N(\omega)$ at small frequencies. $\Gamma > 0$ implies $\delta N_{LRS}^{DSM}(\omega)$ yields a net *increase* to the DOS near the Fermi energy/Dirac point. Compare this to the unscreened case $\delta N_{LRUS}^{DSM}(\omega)$, in which the $\omega^2 \log|\omega|$ is negative as ω approaches zero. As $N(\omega = 0) = 0$ in a DSM, this would actually drive the DOS to be negative for a range of ω near the Fermi energy, a clearly unphysical result. The RPA built into the effective field theory we derived is therefore *essential* to obtaining physically sensible results for long ranged interactions in DSMs. For $\Gamma > 0$, the leading non-analyticity of $\delta N_{SR}^{DSM}(\omega)$ is also negative for small frequencies, but as it comes with a ω^4 prefactor, it cannot drive the DOS to be negative due to the bare DOS scaling as ω^2. Moreover, as phonons are always present in the crystal systems, we can assume attractive short ranged interactions exist, in which case $\Gamma < 0$, and $\delta N_{SR}^{DSM}(\omega)$ yields an increase to the DOS about the Fermi energy.

For graphene, we find the well known result in $\delta N_{LR}^{gra}(\omega)$ that the long ranged interactions provide a net increase to the DOS about the Fermi energy, and in the unscreened limit of $n_g \to 0$, this is still true as $c(0) > 0$ [76, 79]. When $\Gamma > 0$, the

short ranged DOS non-analyticity provides a negative correction to the DOS, but cannot drive the system to an unphysical state because both the bare $|\omega|$ and long range corrected $-|\omega|\log|\omega|$ scaling of the DOS protect the positivity of $N(\omega)$ for $\omega \to 0$.

It is additionally interesting to compare our results to those for Fermi liquids as enumerated in References [14] and [77], as there are some striking resemblances between the two cases which give insight to the physics behind the results. The leading non-analytic corrections to the DOS in a Fermi liquid scale as[2] :

$$\begin{aligned} (1 < d < 3) \qquad & \delta N(\omega) \propto |\omega|^{d-1} \ , \\ (d = 3) \qquad & \delta N(\omega) \propto |\omega|^2 \log|\omega| \ . \end{aligned}$$

The scaling here assumes short ranged interactions for $d < 3$, and in $d = 3$ is independent of short or long ranged interactions because the polarization bubble for Fermi liquids ultimately screens the interaction to make it short ranged. Also, because a Fermi liquid possess a Fermi surface with non-zero density of states $N(\omega = 0) = N_F$, it isn't unphysical if $\delta N(\omega) < 0$ for $\omega \to 0$. The most interesting observation to make is that for semimetals with long ranged interactions and Fermi liquids, modulo logarithmic factors, the leading non-analyticities occur at the same power of ω. This can be understood as a competition between the scaling DOS and the interaction with momentum. In Fermi liquids, a ratio of the density of states at the Fermi level N_F to a short ranged interaction appears in the relevant integrals, to lowest order in momentum the ratio scales as $N_F/(1/\Gamma)$. In semimetals it is a ratio of a vanishing

[2]Technically, the $|\omega|$ non-analyticity in $d = 2$ for Fermi liquids comes with a prefactor of zero. This is a consequence of the prefactor changing signs as d crosses through 2. However, it is how $\delta N(\omega)$ would scale if the integral did not accidentally vanish.

DOS to a long range interaction that appears, which for small momentum goes like $p^{d-1}/(p^{d-1}/\Gamma)f(p)$, for $f(p)$ a function that takes into account screening and doesn't impact the algebraic scaling of p. The other terms are analogous in the integrals for Fermi liquids and semimetals, so the vanishing DOS effectively eats the long range interaction, modulo the screening factor, and hence we end up with the same power of ω in the non-analyticities. We have already noted that the effect of screening in $d = 3$ for DSMs modifies the $\log|\omega|$ portion of the non-analyticity to $\log|\log|\omega||$. This is special to $d = 3$ because it requires the $\log|\Lambda/(q^2 + \omega^2/v_D^2)|$ screening induced by the RPA in Equation (4.91a), which can only happen in $d = 3$ due to the momentum integration measure.

In the short ranged interaction case for semimetals, the competition of DOS and interaction weakens to scale as $p^{d-1}/(1/\Gamma)$ for small momentum, making the non-analyticities appears with at least an additional power of ω^{d-1} over the long ranged case, which comes from the bare DOS in the semimetal. In $d = 3$, the $\omega^4 \log|\omega|$ non-analyticity reflects this for DSMs. In $d = 2$, the term that would yield $\omega^2 \log|\omega|$ dependence in $\delta N_{SR}^{gra}(\omega)$ vanishes due to an angular integration. [3] However, a ω^2 correction is present in $\delta N_{SR}^{gra}(\omega)$, and represents the analytic contribution that follows our scaling argument.

By the physical intuition outlined above, equipped with the knowledge of the Fermi liquid case, we could have anticipated the algebraic form of the DOS non-analyticities in semimetals. Determining the subtle differences that arose due to the complicated nature of screening in semimetals required the machinery of the effective field theory.

[3] This is actually reminiscent of the vanishing prefactor in $d = 2$ for Fermi liquids, detailed in the previous footnote.

Outlook to Other Observables

The insight we just developed to understand the subtle differences in the DOS corrections for Fermi liquids and semimetals extends to other observables. For example, while not explicitly derived in this work, we have found the leading non-analytic correction to the static spin susceptibility in DSMs to scale as $\chi_s(q) \sim q^2 \log|q| \times \log|\log|q||$, while in Fermi liquids the non-analyticity is of the form $\chi_s(q) \sim N_F q^2 \log|q|$. Through careful analysis of the diagrammatic expressions and relevant integrals for $\chi_s(q)$, one can conclude that the $q^2 \log|q|$ scaling in the Fermi liquid calculation becomes $q^2 \log|\log|q||$ if one replaces the internal DOS-interaction term $N_F\Gamma$, with $\Gamma/\log|p|$ to schematically represent the IR scaling of the same term for a Dirac semimetal. Finally, the additional $\log|q|$ in $\chi_s(q)$ for the DSM case can be understood as resulting from the factor of $\log[\Lambda/(p^2+\Omega^2/v_D^2)]$ present in the DSM soft modes, as discussed in Section 4.6, and showcased in Equation (4.91a).

CHAPTER VI

SUMMARY AND CONCLUSION

We must ultimately assess the merit of the endeavor undertaken in this dissertation, and discuss the usefulness of the field theory we derived. This is most easily done by comparing our effective field theory with those of other systems. Disordered Fermi liquids were the original material system for which the Q-matrix formalism was developed [65]. In that case the bosonization program is far simpler due to a following facts: first, the Green function in a disordered Fermi liquid is massive as disorder gives electrons an inelastic scattering time, which in turn makes the saddle point for $Q_{nn}(x)$ homogeneous, that is $Q^{\mathrm{sp}}_{nn}(\boldsymbol{x}) \propto \mathrm{sgn}\,(\omega_n)$. This enables one to construct a non-linear sigma model for the soft and massive Q correlations using the non-linear constraint $Q^2 = \mathbb{1}$. Additionally, there is only one set of soft modes, the diffusons, so many massive modes are integrated out and one arrives at a very simple field theory [64]. In a clean Fermi liquid the situation is already more complicated, the saddle point is not homogeneous, and the ground state possesses an infinite number of Goldstone modes as discussed in Section 4.6. Still, one is able to integrate out the massive modes from this theory, and arrive at an effective field theory, though one more complicated than the disordered case. The additional complications of cleanliness principally impact calculations, making them require some more effort to arrive at a conclusion.

In the semimetal case, we have proven in Section 4.6 that all bosonic modes are soft , thus there are no massive modes to integrate out in the effective field theory. The structure of the Dirac semimetal Hamiltonian also makes the Gaussian level operator for the bosonic theory complicated and nearly unwieldy. We saw in Chapter

V that massaging loop expansions into a tractable form requires a non-trivial amount of effort. Since the original development of the field theory, a slicker form with a simpler Gaussian action has been obtained. This was achieved by expressing the chiral quantum number matrices in the basis of $SU(2)$, as was done for spin and particle-hole quantum numbers in Section 4.1. It is still non-trivial to calculate observables in this field theory. Thus, if one is merely interested in obtaining low order diagrammatic corrections to observables, it would be advisable to use the fermionic formulation of the action, and employ regular many body theory. The reason the effective field theory is a powerful tool is because it allows us to conclude that any non-analyticities we determine from it are asymptotically exact. This follows from the discussion of the scaling of the vertices of cubic and higher order terms in the field theory in Section 4.7. In the Fermi liquid case, the irrelevance of the higher order terms was formally proven using renormalization group arguments in Reference [77], which future works will be extended to the Dirac semimetal case. The appeal of precisely determined non-analyticities is undeniable. Exact expressions are unattainable in regular many body perturbation theory, due the simple reason one cannot be sure if the next order Feynman diagram in a calculation will introduce a stronger non-analyticity, or cancel an old one. The perils of many body perturbation theory is exemplified by the density of states discussion in Chapter V. Calculating corrections to one loop order using the *fermionic* theory yields an unphysical result, and one needs to perform a sum over all loop orders to obtain the correct result. By incorporating the random phase approximation at the Gaussian level of the action, no such effort is needed when using the *bosonic* field theory.

APPENDIX A

INTERACTION DECOMPOSITION

Consider the full interaction (ignoring cone indices):

$$\begin{aligned} S_{\rm int} &= -\frac{T}{2V}\sum_{\substack{k,p,q\\ \sigma_1,\sigma_2}} v(\boldsymbol{q})\bar{\psi}_{\sigma_1}(k)\bar{\psi}_{\sigma_2}(p+q)\psi_{\sigma_2}(p)\psi_{\sigma 1}(k+q)\times \\ &\quad\times\left[\Theta(|\boldsymbol{q}|<\lambda)+\Theta(|\boldsymbol{q}|>\lambda)\left\{\Theta(|\boldsymbol{p}-\boldsymbol{k}|<\lambda)+\Theta(|\boldsymbol{p}+\boldsymbol{k}+\boldsymbol{q}|<\lambda)\right\}\right. \\ &\quad\left.+\Theta(|\boldsymbol{q}|>\lambda)\left\{\Theta(|\boldsymbol{p}-\boldsymbol{k}|>\lambda)+\Theta(|\boldsymbol{p}+\boldsymbol{k}+\boldsymbol{q}|>\lambda)\right\}\right]\ . \end{aligned} \tag{A.1}$$

We have split the momentum sums in a seemingly odd way, and introduced some large momentum cutoff λ. Using the innerproduct notation $(a,b)=a^\dagger b$, the first 3 terms, upon massaging, respectively become:

$$S^{\rm d}_{\rm int} = -\frac{T}{2V}\sum_\alpha\sum_{k,p}{\sum_q}' v(\boldsymbol{q})(\psi^\alpha(k),s_0\psi^\alpha(k+q))(\psi^\beta(p+q),s_0\psi^\beta(p))\ , \tag{A.2a}$$

$$\begin{aligned} S^{\rm e}_{\rm int} &= -\frac{T}{2V}\sum_\alpha\sum_{i=0}^{3}\begin{pmatrix}-\\+\\+\\+\end{pmatrix}_i\sum_{k,p}{\sum_q}'\frac{1}{2}v(\boldsymbol{p}-\boldsymbol{k})\Theta(|\boldsymbol{p}-\boldsymbol{k}|>\lambda)\times \\ &\quad\times(\psi(k)^\alpha,s_i\psi(k+q))(\psi^\beta(p+q),s_i\psi(p))\ , \end{aligned} \tag{A.2b}$$

$$S^{\rm c}_{\rm int} = -\frac{T}{2V}\sum_{\substack{\alpha,\beta\\ \sigma,\sigma'}}\sum_{k,p}{\sum_q}' v(\boldsymbol{p}+\boldsymbol{k})\Theta(|\boldsymbol{p}+\boldsymbol{k}|>\lambda)\bar{\psi}^\alpha_\sigma(k)\bar{\psi}^\beta_{\sigma'}(-k+q)\psi^\beta_{\sigma'}(p+q)\psi^\alpha_\sigma(-p)\ . \tag{A.2c}$$

Due to the momentum restrictions, the remaining terms will not impart hydrodynamic content. Note that even though there is no Fermi Surface, the nature of the decomposition screens the exchange channel. We will need to further decompose

the Cooper channel into its singlet and triplet components:

$$S_{\rm int}^{{\rm c}(s)} = -\frac{T}{2V}\sum_{\sigma\neq\sigma'}\sum_{\substack{\alpha,\beta\\ \gamma,\delta}}\sum_{k,p}\sum_{q}\Gamma^{c(s)\,\alpha\beta\gamma\delta}_{\boldsymbol{k},\boldsymbol{p}}(\boldsymbol{q})\bar{\psi}^{\alpha}_{\sigma}(k)\bar{\psi}^{\beta}_{\sigma'}(-k+q)\psi^{\delta}_{\sigma'}(p+q)\psi^{\gamma}_{\sigma}(-p)\ , \tag{A.3a}$$

$$S_{\rm int}^{{\rm c}(t)} = -\frac{T}{2V}\sum_{\sigma,\sigma'}\sum_{\substack{\alpha,\beta\\ \gamma,\delta}}\sum_{k,p}\sum_{q}\Gamma^{c(t)\,\alpha\beta\gamma\delta}_{\boldsymbol{k},\boldsymbol{p}}(\boldsymbol{q})\bar{\psi}^{\alpha}_{\sigma}(k)\bar{\psi}^{\beta}_{\sigma'}(-k+q)\psi^{\delta}_{\sigma'}(p+q)\psi^{\gamma}_{\sigma}(-p)\ . \tag{A.3b}$$

Defining $\tilde{v}(\boldsymbol{p})\equiv v(\boldsymbol{p})\Theta(|\boldsymbol{p}|>\lambda)$, we can write the Cooper singlet and triplet specific interactions as:

$$\Gamma^{c(s/t)\,\alpha\beta\gamma\delta}_{\boldsymbol{k},\boldsymbol{p}}(\boldsymbol{q}) = \frac{1}{2}\left(\tilde{v}(\boldsymbol{p}+\boldsymbol{k})\delta^{\alpha\gamma}\delta^{\beta\delta}\pm\tilde{v}(\boldsymbol{p}-\boldsymbol{k}+\boldsymbol{q})\delta^{\alpha\delta}\delta^{\beta\gamma}\right)\ . \tag{A.4}$$

Some useful properties are:

$$\begin{aligned}\Gamma^{c(s/t)\,\alpha\beta\gamma\delta}_{\boldsymbol{k},\boldsymbol{p}}(\boldsymbol{q}) &= \Gamma^{c(s/t)\,\alpha\beta\gamma\delta}_{-\boldsymbol{p},-\boldsymbol{k}}(\boldsymbol{q}) = \Gamma^{c(s/t)\,\alpha\beta\gamma\delta}_{-\boldsymbol{k},-\boldsymbol{p}}(-\boldsymbol{q})\\ &= \Gamma^{c(s/t)\,\alpha\beta\gamma\delta}_{\boldsymbol{p}+\boldsymbol{q},\boldsymbol{k}-\boldsymbol{q}}(\boldsymbol{q}) = \pm\Gamma^{c(s/t)\,\alpha\beta\delta\gamma}_{\boldsymbol{k},-\boldsymbol{p}-\boldsymbol{q}}(\boldsymbol{q}) = \pm\Gamma^{c(s/t)\,\alpha\beta\delta\gamma}_{\boldsymbol{p}+\boldsymbol{q},-\boldsymbol{k}}(\boldsymbol{q})\ .\end{aligned} \tag{A.5}$$

APPENDIX B

BISPINOR ADDENDUM

Checking that (4.11) holds. First let's make the observation, that since ψ are Grassmannian fields, it is easy to show that:

$$\sum_{\boldsymbol{k}} \eta_n^+(\boldsymbol{k}) \cdot \eta_m(\boldsymbol{k}) = \frac{i}{2} \sum_{\sigma,\boldsymbol{k}} (\bar{\psi}_{m,\sigma}(\boldsymbol{k})\psi_{n,\sigma}(\boldsymbol{k}) + \bar{\psi}_{n,\sigma}(\boldsymbol{k})\psi_{m,\sigma}(\boldsymbol{k})) \ .$$

Next observe for $G_0^{-1}(k) = \mathrm{diag}(A_1(k), A_2(k))$ where $A_{1,2}$ are 2x2 matrices:

$$\begin{aligned} \eta^+ A \eta &= \frac{i}{2} \begin{pmatrix} s_2^2\psi \\ s_2\bar{\psi} \end{pmatrix} A \begin{pmatrix} \bar{\psi} \\ s_2\psi \end{pmatrix} \\ &= \frac{i}{2} \left(-\psi^T A_1 \bar{\psi} + \psi^\dagger s_2^T A_2 s_2 \psi\right) = \frac{i}{2} \left(\psi^\dagger A_1^T \psi + \psi^\dagger s_2^T A_2 s_2 \psi\right) \ , \end{aligned} \tag{B.1}$$

due to anticommuting ψ's. The structure of G_0^{-1} is essential, as ψ-ψ couplings do not occur in the original Hamiltonian. Then we require that the spin-quaternion Hamiltonian returns the original action:

$$i \sum_k \psi^\dagger(k) H(k) \psi(k) = \sum_k \eta_n^+(\boldsymbol{k}) G_0^{-1}(k) \eta_n(\boldsymbol{k}) \ ,$$

and is charge conjugate symmetric:

$$(\tau_1 \otimes \mathbb{1}) H (\tau_1^* \otimes \mathbb{1}) = H \ . \tag{B.2}$$

This implies

$$\sum_k \psi^\dagger(i\omega_n, -\boldsymbol{k}) A_1(k)^T \psi(i\omega_n, -\boldsymbol{k}) = \sum_k \psi^\dagger(k) H(k) \psi(k) \Rightarrow A_1^T(i\omega_n, -\boldsymbol{k}) = H(k) \ ,$$
$$\sum_k \psi^\dagger(i\omega_n, \boldsymbol{k}) s_2^T A_2(k) s_2 \psi(i\omega_n, \boldsymbol{k}) = \sum_k \psi^\dagger(k) H(k) \psi(k) \Rightarrow -s_2 A_2(i\omega_n, \boldsymbol{k}) s_2 = H(k) \ .$$

We have used $s_2 = -s_2^T$. Then we immediately obtain the Lagrangian in the main text (4.12).

APPENDIX C

SADDLE POINT ADDENDUM

In this Appendix we explain the derivation of the $\tilde{\Lambda}$ part of the saddle point Equation (4.31). The aim is to obtain a form of the saddle point equation for $\tilde{\Lambda}$ in the spin-quaternion basis, using standard matrix notation we have:

$$ {}^{i}_{r}\tilde{\Lambda} = \frac{1}{4}\sum_{k,j}(\tau_r \otimes s_j)_{kj}\tilde{\Lambda}_{jk} \ . \tag{C.1} $$

Then Equation (4.30) for $\tilde{\Lambda}$ tells us:

$$ \tilde{\Lambda}_{ij} = -\frac{\delta}{\delta Q_{ij}}A_{\rm int} \ . \tag{C.2} $$

Schematically, the RHS of the above equation goes like:

$$ \begin{aligned} \frac{\delta}{\delta Q_{kj}}A_{\rm int} &= \frac{\delta}{\delta Q_{ij}}\frac{T}{2V}\sum \ldots \mathrm{tr}\left((\tau_r \otimes s_i)^\dagger Q\right)\mathrm{tr}\left((\tau_r \otimes s_i)^\dagger Q\right) \\ &= \frac{T}{2V}\sum \ldots 2 \times (\tau_r \otimes s_i)^\dagger_{jk}\mathrm{tr}\left((\tau_r \otimes s_i)^\dagger Q\right) \ . \end{aligned} \tag{C.3} $$

The 2 in the second line is because our Ansatz combines the two derivative terms. Note that there is currently a sum over i and r, but if we want to pick out a specific ${}^{i}_{r}\tilde{\Lambda}$ component, this requires tracing with respect to the appropriate $\tau_r \otimes s_i$ tensor:

$$ {}^{i}_{r}\left(\frac{\delta}{\delta Q}A_{\rm int}\right) = \frac{T}{4}\sum \ldots (\tau_r \otimes s_i)_{kj}(\tau_r \otimes s_i)^\dagger_{jk}\mathrm{tr}\left((\tau_r \otimes s_i)^\dagger Q\right) = 4\sum \ldots {}^{i}_{r}Q \ . \tag{C.4} $$

APPENDIX D

SELF-ENERGY ω-TERM

Given $\tilde{\Lambda}^0_n(\boldsymbol{p}) = 0$, we have:

$$\begin{aligned}\tilde{\Lambda}^0_n(\boldsymbol{p}) &= = -4\frac{T}{V}\sum_{m,\boldsymbol{k}}\Gamma^s(\boldsymbol{p},\boldsymbol{k},0)Q^0_m(\boldsymbol{k}) \\ &= 2i\frac{T}{V}\sum_{m,\boldsymbol{k}}(v(0) - \frac{1}{2}v(\boldsymbol{p}-\boldsymbol{k})\frac{I\omega_n - I\tilde{\Lambda}^0_m(\boldsymbol{p})}{(I\omega_m - I\tilde{\Lambda}^0_m(\boldsymbol{p}))^2 + v_D^2(\boldsymbol{k}+\tilde{\boldsymbol{\Lambda}}_m(\boldsymbol{k}))^2} \quad \text{(D.1)}\end{aligned}$$

By charge neutrality $v(0) = 0$. Next, if we assume $\tilde{\boldsymbol{\Lambda}}_m(\boldsymbol{k})$ is m independent, then we see at least at $T = 0$, this integral is odd in ω when we assume $\tilde{\Lambda}^0_n(\boldsymbol{p}) = const.$ So iteratively we would always get 0 (i.e. 0 is a solution).

APPENDIX E

INVARIANCE OF τ_2 UNDER $T^{(\pm)}$

We demonstrate the invariance of τ_2 under the action of $T^{(\pm)}$:

$$
\begin{aligned}
&\sum_{l,\gamma} \int d\boldsymbol{y}\; \hat{T}^{(\pm)}(\boldsymbol{x},\boldsymbol{y}) \left(\hat{T}^{(\pm)}(\boldsymbol{y},\boldsymbol{z})\right)^T \\
= \;& \mathbb{1} + \sum_{l,\gamma} \int d\boldsymbol{y}\; \left(\delta_{n1}\delta_{l2}\delta^{\alpha\sigma}\delta^{\gamma\tau} \mp \delta_{n2}\delta_{l1}\delta^{\alpha\tau}\delta^{\gamma\sigma}\right) \varphi^{(\pm)}(\boldsymbol{x},\boldsymbol{y})\delta(\boldsymbol{y}-\boldsymbol{z})\delta^{\beta\gamma}\delta_{ml} + \\
&+ \sum_{l,\gamma} \int d\boldsymbol{y}\; \left(\delta_{m1}\delta_{l2}\delta^{\beta\sigma}\delta^{\gamma\tau} \mp \delta_{m2}\delta_{l1}\delta^{\beta\tau}\delta^{\gamma\sigma}\right) \varphi^{(\pm)}(\boldsymbol{z},\boldsymbol{y})\delta(\boldsymbol{x}-\boldsymbol{y})\delta^{\alpha\beta}\delta_{nl} \\
= \;& \mathbb{1} + \left(\delta_{n1}\delta_{m2}\delta^{\alpha\sigma}\delta^{\beta\tau} \mp \delta_{n2}\delta_{m1}\delta^{\alpha\tau}\delta^{\beta\sigma}\right) \varphi^{(\pm)}(\boldsymbol{x},\boldsymbol{z}) \\
&+ \left(\delta_{m1}\delta_{n2}\delta^{\beta\sigma}\delta^{\alpha\tau} \mp \delta_{m2}\delta_{n1}\delta^{\beta\tau}\delta^{\alpha\sigma}\right) \varphi^{(\pm)}(\boldsymbol{z},\boldsymbol{x}) \\
= \;& \mathbb{1}\,. \qquad \text{(E.1)}
\end{aligned}
$$

APPENDIX F

DETERMINING $\delta\mathcal{A}_0$

Under the transformation of $\tilde{\Lambda}$, the free part of the action goes as:

$$\begin{aligned}\frac{1}{2}\mathrm{Tr}\,\ln(G^{-1}-i\delta\tilde{\Lambda}) &= \frac{1}{2}\mathrm{Tr}\,\ln(G^{-1}(\mathbb{1}-iG\delta\tilde{\Lambda})) = \frac{1}{2}\mathrm{Tr}\,(\ln G^{-1}\ln(\mathbb{1}-iG\delta\tilde{\Lambda})) \\ &\approx \frac{1}{2}\mathrm{Tr}\,(\ln G^{-1}-iG\delta\tilde{\Lambda}) \equiv \mathcal{A}_0+\delta\mathcal{A}_0\ . \end{aligned} \tag{F.1}$$

Recall the definition:

$$\begin{aligned}\delta\tilde{\Lambda}^{\beta\alpha}_{mn}(\boldsymbol{y},\boldsymbol{x}) &= \int d\boldsymbol{z}\left[\varphi^{(\pm)}(\boldsymbol{x},\boldsymbol{z})\left(\delta^{\alpha\sigma}_{n1}\tilde{\Lambda}^{\beta\tau}_{m2}(\boldsymbol{y},\boldsymbol{z})\mp\delta^{\alpha\tau}_{n2}\tilde{\Lambda}^{\beta\sigma}_{m1}(\boldsymbol{y},\boldsymbol{z})\right)+\right. \\ &\quad \left.\varphi^{(\pm)}(\boldsymbol{y},\boldsymbol{z})\left(\delta^{\beta\sigma}_{m1}\tilde{\Lambda}^{\tau\alpha}_{2n}(\boldsymbol{z},\boldsymbol{x})\mp\delta^{\beta\tau}_{m2}\tilde{\Lambda}^{\sigma\alpha}_{1n}(\boldsymbol{z},\boldsymbol{x})\right)\right]\ . \end{aligned} \tag{F.2}$$

We must calculate the resulting change in the action:

$$\begin{aligned}\delta\mathcal{A}_0 &= -\frac{i}{2}\mathrm{Tr}\,G\delta\tilde{\Lambda} \\ &= -\frac{i}{2}\sum_{n,m,\alpha,\beta}\int d\boldsymbol{x}d\boldsymbol{y}\ \mathrm{tr}\,G^{\alpha\beta}_{nm}(\boldsymbol{x},\boldsymbol{y})\delta\tilde{\Lambda}^{\beta\alpha}_{mn}(\boldsymbol{y},\boldsymbol{x}) \\ &= -\frac{i}{2}\sum_{m,\alpha}\int d\boldsymbol{x}d\boldsymbol{y}d\boldsymbol{z}\ \mathrm{tr}\,\Big[G^{\sigma\alpha}_{1m}(\boldsymbol{x},\boldsymbol{y})\tilde{\Lambda}^{\alpha\tau}_{m2}(\boldsymbol{y},\boldsymbol{z})\varphi^{(\pm)}(\boldsymbol{x},\boldsymbol{z}) \\ &\quad \mp G^{\tau\alpha}_{2m}(\boldsymbol{x},\boldsymbol{y})\tilde{\Lambda}^{\alpha\sigma}_{m1}(\boldsymbol{y},\boldsymbol{z})\varphi^{(\pm)}(\boldsymbol{x},\boldsymbol{z}) \\ &\quad +G^{\alpha\sigma}_{m1}(\boldsymbol{x},\boldsymbol{y})\tilde{\Lambda}^{\tau\alpha}_{2m}(\boldsymbol{z},\boldsymbol{x})\varphi^{(\pm)}(\boldsymbol{y},\boldsymbol{z})\mp G^{\alpha\tau}_{m2}(\boldsymbol{x},\boldsymbol{y})\tilde{\Lambda}^{\sigma\alpha}_{1m}(\boldsymbol{z},\boldsymbol{x})\varphi^{(\pm)}(\boldsymbol{y},\boldsymbol{z})\Big]\ . \end{aligned}$$

Now we use the property that $-iG\tilde{\Lambda}\equiv G(-i\tilde{\Lambda}+G_0^{-1}-G_0^{-1})\equiv\mathbb{1}-GG_0^{-1}$. The matrix contractions have suppressed sums and integrals, we will explicitly include $\delta(x)$ like objects in the following derivation to preserve the triple integral and sum structure.

It then follows that:

$$\begin{aligned}\delta\mathcal{A}_0 \;=\; & \frac{1}{2}\sum_{m,\alpha}\int d\boldsymbol{x}d\boldsymbol{y}d\boldsymbol{z}\ \mathrm{tr}\ \big[\big(\delta^{\sigma\tau}_{12}\delta(\boldsymbol{x}-\boldsymbol{z})\delta(\boldsymbol{y})\delta^{\alpha} - G^{\sigma\alpha}_{1m}(\boldsymbol{x},\boldsymbol{y})(G_0^{-1})^{\alpha\tau}_{m2}(\boldsymbol{y},\boldsymbol{z})\big)\,\varphi^{(\pm)}(\boldsymbol{x},\boldsymbol{z}) \\ & \mp\big(\delta^{\sigma\tau}_{12}\delta(\boldsymbol{x}-\boldsymbol{z})\delta(\boldsymbol{y})\delta^{\alpha} - G^{\tau\alpha}_{2m}(\boldsymbol{x},\boldsymbol{y})(G_0^{-1})^{\alpha\sigma}_{m1}(\boldsymbol{y},\boldsymbol{z})\big)\,\varphi^{(\pm)}(\boldsymbol{x},\boldsymbol{z}) \\ & +\big(\delta^{\sigma\tau}_{12}\delta(\boldsymbol{z}-\boldsymbol{y})\delta(\boldsymbol{x})\delta^{\alpha} - (G_0^{-1})^{\tau\alpha}_{2m}(\boldsymbol{z},\boldsymbol{x})G^{\alpha\sigma}_{m1}(\boldsymbol{x},\boldsymbol{y})\big)\,\varphi^{(\pm)}(\boldsymbol{y},\boldsymbol{z}) \\ & \mp\big(\delta^{\sigma\tau}_{12}\delta(\boldsymbol{z}-\boldsymbol{y})\delta(\boldsymbol{x})\delta^{\alpha} - (G_0^{-1})^{\sigma\alpha}_{1m}(\boldsymbol{z},\boldsymbol{x})G^{\alpha\tau}_{m2}(\boldsymbol{x},\boldsymbol{y})\big)\,\varphi^{(\pm)}(\boldsymbol{y},\boldsymbol{z})\big]\ . \end{aligned} \tag{F.3}$$

It is trivial to see $\varphi^{(-)}(\boldsymbol{x},\boldsymbol{x}) = 0$, and the $\varphi^{(+)}(\boldsymbol{x},\boldsymbol{x})$ terms kill each other because they come as $\pm$. We are now in the position to apply the triple convolution theorem:

$$\int d\boldsymbol{x}d\boldsymbol{y}d\boldsymbol{z}\ (G_0^{-1})_{nm}(\boldsymbol{z},\boldsymbol{x})G_{rs}(\boldsymbol{x},\boldsymbol{y})\varphi(\boldsymbol{y},\boldsymbol{z}) = \sum_{\boldsymbol{q},\boldsymbol{p},\boldsymbol{k}}(G_0^{-1})_{nm}(\boldsymbol{k},\boldsymbol{p})G_{rs}(\boldsymbol{p},\boldsymbol{q})\varphi(\boldsymbol{q},\boldsymbol{k})\ . \tag{F.4}$$

Recalling

$$(G_0^{-1})^{\beta\alpha}_{nm}(\boldsymbol{p},\boldsymbol{q}) = \delta_{nm}\delta_{\boldsymbol{p},\boldsymbol{q}}\tau_0\ \mathrm{diag}(i\omega_n - \xi^{-}(\boldsymbol{p}), i\omega_n - \xi^{+}(\boldsymbol{p}))\ , \tag{F.5}$$

we have:

$$\begin{aligned}
\delta\mathcal{A}_0 &= -\frac{1}{2}\sum_{m,\alpha}\sum_{\boldsymbol{p},\boldsymbol{q},\boldsymbol{k}} \mathrm{tr}\,\big[\pm G^{\sigma\alpha}_{1m}(\boldsymbol{p},\boldsymbol{q})(G_0^{-1})^{\alpha\tau}_{m2}(\boldsymbol{q},\boldsymbol{k})\varphi^{(\pm)}(\boldsymbol{k},\boldsymbol{p}) - ((\tau,1)\leftrightarrow(\sigma,2)) \\
&\quad +(G_0^{-1})^{\tau\alpha}_{2m}(\boldsymbol{p},\boldsymbol{q})G^{\alpha\sigma}_{m1}(\boldsymbol{q},\boldsymbol{k})\varphi^{(\pm)}(\boldsymbol{k},\boldsymbol{p}) \mp ((\tau,1)\leftrightarrow(\sigma,2))\big] \\
&= \frac{1}{2}\sum_{\boldsymbol{p},\boldsymbol{k}} \mathrm{tr}\,\big[i\Omega_{1-2}\times\varphi^{(\pm)}(\boldsymbol{k},\boldsymbol{p})\left(G^{\tau\sigma}_{21}(\boldsymbol{p},\boldsymbol{k}) \pm G^{\sigma\tau}_{12}(\boldsymbol{p},\boldsymbol{k})\right) + \\
&\quad +(-\xi^{\sigma}(\boldsymbol{k})+\xi^{\tau}(\boldsymbol{p}))\times\varphi^{(\pm)}(\boldsymbol{k},\boldsymbol{p})G^{\tau\sigma}_{21}(\boldsymbol{p},\boldsymbol{k}) \\
&\quad \pm(-\xi^{\sigma}(\boldsymbol{p})+\xi^{\tau}(\boldsymbol{k}))\times\varphi^{(\pm)}(\boldsymbol{k},\boldsymbol{p})G^{\sigma\tau}_{12}(\boldsymbol{p},\boldsymbol{k})\big] \\
&= \frac{1}{2}\sum_{\boldsymbol{p},\boldsymbol{k}}\varphi^{(\pm)}(\boldsymbol{k},\boldsymbol{p})\mathrm{tr}\,\left[i\Omega_{1-2}+(-\xi^{\sigma}(\boldsymbol{k})+\xi^{\tau}(\boldsymbol{p}))\right]\times\left(G^{\tau\sigma}_{21}(\boldsymbol{p},\boldsymbol{k})+G^{\sigma\tau}_{12}(-\boldsymbol{k},-\boldsymbol{p})\right)\ .
\end{aligned} \tag{F.6}$$

In the last line we use the property:

$$\varphi^{(\pm)}(\boldsymbol{p},\boldsymbol{q}) = \pm\varphi^{(\pm)}(-\boldsymbol{q},-\boldsymbol{p})\ . \tag{F.7}$$

Finally, we have the following expressions for δQ in momentum space:

$$\begin{aligned}
\delta Q^{\alpha\beta}_{nm}(\boldsymbol{k},\boldsymbol{p}) &= \sum_{\boldsymbol{q}}\varphi^{(\pm)}(-\boldsymbol{p},\boldsymbol{q})\left(\delta^{\beta\sigma}_{m1}Q^{\alpha\tau}_{n2}(\boldsymbol{k},-\boldsymbol{q}) \mp \delta^{\beta\tau}_{m2}Q^{\alpha\sigma}_{n1}(\boldsymbol{k},-\boldsymbol{q})\right) + \\
&\quad +\varphi^{(\pm)}(\boldsymbol{k},\boldsymbol{q})\left(\delta^{\alpha\sigma}_{n1}Q^{\tau\beta}_{2m}(\boldsymbol{q},\boldsymbol{p}) \mp \delta^{\alpha\tau}_{n2}Q^{\sigma\beta}_{1m}(\boldsymbol{q},\boldsymbol{p})\right) && \text{(F.8a)} \\
\delta Q^{\sigma\tau}_{12}(\boldsymbol{k},\boldsymbol{p}) &= \sum_{\boldsymbol{q}}\varphi^{(\pm)}(\boldsymbol{k},\boldsymbol{q})Q^{\tau\tau}_{22}(\boldsymbol{q},\boldsymbol{p}) - Q^{\sigma\sigma}_{11}(\boldsymbol{k},\boldsymbol{q})\varphi^{(\pm)}(\boldsymbol{q},\boldsymbol{p}) && \text{(F.8b)} \\
\langle\delta Q^{\alpha\beta}_{nm}(\boldsymbol{k},\boldsymbol{p})\rangle &= \varphi^{(\pm)}(\boldsymbol{k},\boldsymbol{p})\Big[\pm\delta^{\beta\sigma}_{m1}\delta^{\alpha\tau}_{n2}\left(\langle Q^{\tau\tau}_{22}(\boldsymbol{k},\boldsymbol{k})\rangle - \langle Q^{\sigma\sigma}_{11}(\boldsymbol{p},\boldsymbol{p})\rangle\right) + \\
&\quad +\delta^{\alpha\sigma}_{n1}\delta^{\tau\beta}_{2m}\left(\langle Q^{\tau\tau}_{22}(\boldsymbol{p},\boldsymbol{p})\rangle - \langle Q^{\sigma\sigma}_{11}(\boldsymbol{k},\boldsymbol{k})\rangle\right)\Big] && \text{(F.8c)}
\end{aligned}$$

APPENDIX G

W^{-1} TENSOR PROPERTIES

From the exact form of the W^{-1} tensor, Equation (4.88) it is easy to see that:

$$ {}^{ij}W^{-1}{}^{\alpha\beta,\mu\nu}_{12,34} = {}^{ji}W^{-1}{}^{\mu\nu,\alpha\beta}_{34,12} , \tag{G.1} $$

$$ {}^{ij}W^{-1}{}^{\alpha\beta,\mu\nu\ddagger}_{12,34} = {}^{ij}W^{-1}{}^{\bar{\alpha}\bar{\beta},\bar{\mu}\bar{\nu}}_{12,34} , \tag{G.2} $$

where $(\bar{\pm}) = \mp$.

APPENDIX H

DOS CORRECTION IN DETAIL

In Chapter V we defined:

$$F(i\omega_n) = \frac{6}{4}\frac{T}{V}{\sum_{\substack{m\neq 0\\ \boldsymbol{q}}}}' I(i\omega_n, i\omega_n + i\Omega_m, v_D\boldsymbol{q})\tilde{\Gamma}^d_m(\boldsymbol{q}) \ , \tag{H.1a}$$

$$I(i\omega_n, i\omega_m, v_D\boldsymbol{q}) = \frac{1}{V}\sum_{\boldsymbol{k}} \mathrm{tr}\left(G(i\omega_n, \boldsymbol{k})G(i\omega_m, \boldsymbol{k}+\boldsymbol{q})G(i\omega_n, \boldsymbol{k})\right) . \tag{H.1b}$$

We have brought back the notation of $\sum_q{}'$ to mean $|\boldsymbol{q}| < \Lambda$, an arbitrary cutoff we chose in Appendix A to split up the interaction channels. At the end of this Appendix we will prove $F(i\omega_n) = -F(-i\omega_n)$, so that we only need to evaluate $F(i\omega_n)$ for $\omega_n > 0$. Let $\Sigma \equiv \not{Q} + \not{Q}^+$, then by Wick's theorem, and cyclic symmetry of the summation, $F(i\omega_n)$ in (5.3) becomes[1]:

$$F(i\omega_{n_7}) = \frac{3}{V}\sum_{\substack{\cdots\\ \neg n_7}} \tilde{t}^{i_1 i_2 i_3 i_4 i_5 i_6}_{r_1 r_3 r_5} \times {}^{i_2}L_2^{\sigma_2 i_4} L_3^{\sigma_3 i_6} L_1^{\sigma_1} \langle {}^0_0 Q^{\alpha\alpha i_1}_{77\ r_1} \Sigma^{\sigma_1\sigma_2}_{12}\rangle \langle {}^{i_3}_{r_3} \Sigma^{\sigma_2\sigma_3 i_5}_{23\ \ r_5} \Sigma^{\sigma_3\sigma_1}_{31}\rangle \ , \tag{H.2}$$

for $\tilde{t}^{i_1 i_2 i_3 i_4 i_5 i_6}_{r_1 r_3 r_5} = \mathrm{tr}\,(s_{i_1}\cdots s_{i_6})\mathrm{tr}\,(\tau_{r_1}\tau_{r_3}\tau_{r_5})$. Due to the fact that

$$\langle {}^0_0 Q^{\alpha\alpha i_1}_{77\ r_1} \not{Q}^{\sigma_1\sigma_2(+)}_{12}\rangle \propto \delta_{r_1,0}(\delta_{71}\delta_{72}(\ldots) + \delta_{0,1-2}(\ldots)) \ , \tag{H.3}$$

[1]Note that $\ldots, \neg n_7$ means sum over everything but n_7

we have the constraints $n_1 = n_2$ and $r_1 = 0$. It then follows that:

$$\tilde{t}^{i_1 i_2 i_3 i_4 i_5 i_6}_{0 r_3 r_5} = 2 \begin{pmatrix} + \\ - \\ - \\ - \end{pmatrix}_{r_3} \delta_{r_3 r_5} \mathrm{tr}\left(s_{i_1} \cdots s_{i_6}\right) \equiv 2 \begin{pmatrix} + \\ - \\ - \\ - \end{pmatrix}_{r_3} \delta_{r_3 r_5} t^{i_1 \ldots i_6} \ . \tag{H.4}$$

This property kills the $\not{Q}^+$ contraction because $\not{Q}^+_{12}$ has a built in $n_2 > n_1$ (see Equation (4.83)). Finally, as discussed above $\langle Q\not{Q}\rangle = \langle QQ\rangle$ as Q and $\bar{\Lambda}$ don't couple, thus:

$$F(i\omega_{n_7}) \ = \ \frac{6}{V} \sum_{\substack{\cdots \\ \neg n_7}} \begin{pmatrix} + \\ - \\ - \\ - \end{pmatrix}_{r} t^{i_1 i_2 i_3 i_4 i_5 i_6} \times {}^{i_2}L_2^{\sigma_2 i_4} L_3^{\sigma_3 i_6} L_1^{\sigma_1} \langle {}^0_0 Q^{\alpha\alpha i_1}_{77\ 0} \bar{Q}^{\sigma_1\sigma_2}_{12} \rangle \langle {}^{i_3}_{r} \Sigma^{\sigma_2\sigma_3 i_5}_{23\ r} \Sigma^{\sigma_3\sigma_1}_{31} \rangle \ . \tag{H.5}$$

Let us evaluate the relevant correlation functions:

$$\begin{aligned}
& \langle {}^0_0 Q^{\alpha\alpha i_1}_{77\ 0} \bar{Q}^{\sigma_1\sigma_2}_{12} \rangle \\
\equiv \ & \begin{pmatrix} + \\ - \\ + \\ - \end{pmatrix}_{i_3} \langle {}^0_0 Q^{\alpha\alpha i_1}_{77\ 0} Q^{\sigma_1\sigma_2\ddagger}_{\bar{1}\bar{2}} \rangle \Theta(n_1 \geq n_2) \\
= \ & \frac{1}{4 I_{77}} \Bigg[\begin{pmatrix} + \\ - \\ - \\ - \end{pmatrix}_{i_1} \left(\frac{1}{4} {}^{i_1 0} M^{-1\,\sigma_1\sigma_2}_{\bar{1}\bar{2}} \delta^{\alpha\sigma_1}_{7\bar{1}} \delta^{\alpha\sigma_2}_{7\bar{2}} + \frac{T}{V} \tilde{\Gamma}^d_{1-2} \delta_{\bar{1},\bar{2}}{}^{i_1 0} M^{-1\,\sigma_1\sigma_2 00}_{\quad 12} M^{-1\,\alpha\alpha}_{\quad 77} \right) \\
+ \ & \begin{pmatrix} + \\ - \\ - \\ - \end{pmatrix}_{0} \left(\frac{1}{4} {}^{0 i_1} M^{-1\,\alpha\alpha\ddagger}_{\quad 77} \delta^{\alpha\sigma_1}_{7\bar{1}} \delta^{\alpha\sigma_2}_{7\bar{2}} + \frac{T}{V} \tilde{\Gamma}^d_{1-2} \delta_{\bar{1},\bar{2}}{}^{00} M^{-1\,\alpha\alpha\ddagger 0 i_1}_{\quad 77} M^{-1\,\sigma_1\sigma_2\ddagger}_{\quad 12} \right) \Bigg] \Theta(n_1 \geq n_2) \\
= \ & \frac{1}{8} \delta^{\alpha\sigma_1}_{7\bar{1}} \delta^{\alpha\sigma_2}_{7\bar{2}} \left[\begin{pmatrix} + \\ - \\ - \\ - \end{pmatrix}_{i_1} {}^{i_1 0} M^{-1\,\sigma_1\sigma_2}_{\bar{1}\bar{2}} + {}^{0 i_1} M^{-1\,\alpha\alpha\ddagger}_{\quad 77} \right] \Theta(n_1 \geq n_2) \ .
\end{aligned} \tag{H.6}$$

Where we have used $\tilde{\Gamma}_0^d \equiv 0$ via the Jellium model argument again, and $I_{77} = 1/2$. For $F(i\omega_{n_7})$ we get:

$$\begin{aligned} F(i\omega_{n_7}) &= \frac{3}{4V} \sum_{\substack{\cdots \\ \neg n_7}} \begin{pmatrix} + \\ - \\ - \\ - \end{pmatrix}_r t^{i_1 i_2 i_3 i_4 i_5 i_6} \times {}^{i_2}L_{\bar{7}}^{\alpha i_4} L_3^{\sigma_3 i_6} L_{\bar{7}}^{\alpha} \times \\ &\quad \times \left[\begin{pmatrix} + \\ - \\ - \\ - \end{pmatrix}_{i_1} {}^{i_1 0}M^{-1}{}_{\bar{1}\bar{2}}^{\alpha\alpha} + {}^{0 i_1}M^{-1}{}_{77}^{\alpha\alpha\ddagger} \right] \Theta(n_1 \geq n_2) \langle {}_r^{i_3}\Sigma_{\bar{7}3}^{\alpha\sigma_3}{}_r^{i_5}\Sigma_{3\bar{7}}^{\sigma_3\alpha} \rangle \ . \end{aligned} \tag{H.7}$$

From Section 4.7 we have have that:

$$\begin{aligned} \langle {}_r^{i_3}\not{Q}_{\bar{7}3}^{\alpha\sigma_3}{}_r^{i_5}\not{Q}_{3\bar{7}}^{\sigma_3\alpha\dagger} \rangle &\propto \begin{pmatrix} + \\ 0 \\ 0 \\ + \end{pmatrix}_r \Theta(n_3 \geq n_7)\Theta(n_7 \geq n_3) \\ \langle {}_r^{i_3}\not{Q}_{\bar{7}3}^{\alpha\sigma_3 +}{}_r^{i_5}\not{Q}_{3\bar{7}}^{\sigma_3\alpha +} \rangle &\propto \begin{pmatrix} + \\ 0 \\ 0 \\ + \end{pmatrix}_r \Theta(n_3 > n_7)\Theta(n_7 > n_3) \end{aligned} \tag{H.8}$$

Both of these terms will drop about because of the $\begin{pmatrix} + \\ - \\ - \\ - \end{pmatrix}_r$ in $F(i\omega_{n_7})$ (the second one would drop out from frequency restriction too). The terms that survive are:

$$\langle {}^{i_3}_{r}\mathcal{Q}^{\alpha\sigma_3}_{\bar{7}3}\,{}^{i_5}_{r}\mathcal{Q}^{\sigma_3\alpha+}_{3\bar{7}}\rangle = \begin{pmatrix} + \\ 0 \\ 0 \\ - \end{pmatrix}_r \begin{pmatrix} + \\ - \\ - \\ - \end{pmatrix}_{i_5} \frac{T}{4V}\tilde{\Gamma}^d_{\bar{7}-3}\delta^{\alpha\sigma_3}\Theta(n_7 > n_3)\times$$
$$\times \left[\begin{pmatrix} + \\ - \\ - \\ - \end{pmatrix}_{i_5} {}^{i_5 0}M^{-1}{}^{\alpha\sigma_3 0 i_3}_{\bar{7}3} M^{-1}{}^{\alpha\sigma_3}_{\bar{7}3} + \begin{pmatrix} + \\ - \\ - \\ - \end{pmatrix}_{i_3} {}^{i_3 0}M^{-1}{}^{\alpha\sigma_3\ddagger 0 i_5}_{\bar{7}3} M^{-1}{}^{\alpha\sigma_3\ddagger}_{\bar{7}3} \right] ,$$
$$\langle {}^{i_3}_{r}\mathcal{Q}^{\alpha\sigma_3+}_{\bar{7}3}\,{}^{i_5}_{r}\mathcal{Q}^{\sigma_3\alpha}_{3\bar{7}}\rangle = \begin{pmatrix} + \\ 0 \\ 0 \\ - \end{pmatrix}_r \begin{pmatrix} + \\ - \\ - \\ - \end{pmatrix}_{i_3} \frac{T}{4V}\tilde{\Gamma}^d_{\bar{7}-3}\delta^{\alpha\sigma_3}\Theta(n_3 > n_7)\times$$
$$\times \left[\begin{pmatrix} + \\ - \\ - \\ - \end{pmatrix}_{i_3} {}^{i_5 0}M^{-1}{}^{\alpha\sigma_3 0 i_3}_{\bar{7}3} M^{-1}{}^{\alpha\sigma_3}_{\bar{7}3} + \begin{pmatrix} + \\ - \\ - \\ - \end{pmatrix}_{i_5} {}^{i_3 0}M^{-1}{}^{\alpha\sigma_3\ddagger 0 i_5}_{\bar{7}3} M^{-1}{}^{\alpha\sigma_3\ddagger}_{\bar{7}3} \right] .$$

Plugging all this into $F(i\omega_{n_7})$, shifting $\bar{7} \to 7 \to (n, \boldsymbol{k})$ and $3 \to (m, \boldsymbol{p})$, and now writing all the momenta/frequencies explicitly, we obtain:

$$\begin{aligned}
&F(i\omega_n) = \\
&\frac{6T}{16V^2}\sum_{\substack{m\alpha \\ \boldsymbol{p},\boldsymbol{k}}} t^{i_1\dots i_6}(1-\delta_{nm})\tilde{\Gamma}^d_{n-m}(\boldsymbol{k}-\boldsymbol{p})^i_2 L^\alpha_n(\boldsymbol{k})^i_4 L^\alpha_m(\boldsymbol{p})^i_6 L^\alpha_n(\boldsymbol{k})\times \\
&\times \left[\begin{pmatrix} + \\ - \\ - \\ - \end{pmatrix}_{i_1} {}^{i_1 0}M^{-1}{}^{\bar{\alpha}\bar{\alpha}}_{nn}(\boldsymbol{k},\boldsymbol{k}) + {}^{0 i_1}M^{-1}{}^{\alpha\alpha}_{nn}(\boldsymbol{k},\boldsymbol{k}) \right] \times \\
&\times \Big[{}^{i_5 0}M^{-1}{}^{\alpha\alpha}_{nm}(\boldsymbol{k},\boldsymbol{p}) {}^{0 i_3}M^{-1}{}^{\alpha\alpha}_{nm}(\boldsymbol{k},\boldsymbol{p}) + \\
&+ \begin{pmatrix} + \\ - \\ - \\ - \end{pmatrix}_{i_3} \begin{pmatrix} + \\ - \\ - \\ - \end{pmatrix}_{i_5} {}^{i_3 0}M^{-1}{}^{\bar{\alpha}\bar{\alpha}}_{nm}(\boldsymbol{k},\boldsymbol{p}) {}^{0 i_5}M^{-1}{}^{\bar{\alpha}\bar{\alpha}}_{nm}(\boldsymbol{k},\boldsymbol{p}) \Big] ,
\end{aligned} \tag{H.9}$$

where we have employed ${}^{ij}M^{-1}{}^{\alpha\beta^\ddagger}_{12} = {}^{ij}M^{-1}{}^{\alpha\beta}_{n_1n_2}(-\boldsymbol{p_1}, -\boldsymbol{p_2}) = {}^{ij}M^{-1}{}^{\bar{\alpha}\bar{\beta}}_{n_1n_2}(\boldsymbol{p_1}, \boldsymbol{p_2})$ for $(\bar{\pm}) = \mp$. Massaging this by hand is somewhat of a fool's errand because of how ugly t and M are when expressed in terms of well known tensors. However, plugging the objects into Mathematica allows it to evaluate the sum simply, and we get:

$$\begin{aligned} F(i\omega_n) &= \frac{T}{V^2} 6 \sum_{\substack{\boldsymbol{p},\boldsymbol{k} \\ m}} \frac{-(i\omega_n)^2 i\omega_m - i\omega_m v_D^2 k^2 - 2i\omega_n v_D \boldsymbol{k}\cdot\boldsymbol{p}}{(\omega_n^2 + v_D^2 k^2)^2(\omega_m^2 + v_D^2 p^2)} (1-\delta_{nm}) \tilde{\Gamma}^d_{n-m}(\boldsymbol{k}-\boldsymbol{p}) \\ &= \frac{6}{4}\frac{T}{V^2} \sum_{\substack{\boldsymbol{p},\boldsymbol{k} \\ m}} \mathrm{tr}\ \left(G(i\omega_n, \boldsymbol{k}) G(i\omega_m, \boldsymbol{p}) G(i\omega_n, \boldsymbol{k})\right) (1-\delta_{nm}) \tilde{\Gamma}^d_{n-m}(\boldsymbol{k}-\boldsymbol{p})\ , \end{aligned} \tag{H.10}$$

where the first line is specific to DSMs in $d = 3$, and the second line is for general dimensions (The difference between the first line for 3D DSMs and graphene is simply a factor of 2.) Finally let us demonstrate $F(i\omega_n) = -F(-i\omega_n)$:

$$\begin{aligned} F(-i\omega_n) &= \text{const} \times \sum_{\substack{\boldsymbol{q},\boldsymbol{p} \\ m\neq 0}} \mathrm{tr}\left(G^2(-i\omega_n, \boldsymbol{p}) G(-i\omega_n + i\Omega_m, \boldsymbol{p}+\boldsymbol{q})\right) \tilde{\Gamma}^d_m(\boldsymbol{q}) \\ &= \text{const} \times \sum_{\substack{\boldsymbol{q},\boldsymbol{p} \\ m\neq 0}} (-1) \mathrm{tr}\left(G^2(i\omega_n, \boldsymbol{p}) G(i\omega_n - i\Omega_m, \boldsymbol{p}+\boldsymbol{q})\right) \tilde{\Gamma}^d_m(\boldsymbol{q}) \\ &= -\text{const} \times \sum_{\substack{\boldsymbol{q},\boldsymbol{p} \\ m\neq 0}} \mathrm{tr}\left(G^2(i\omega_n, \boldsymbol{p}) G(i\omega_n + i\Omega_m, \boldsymbol{p}+\boldsymbol{q})\right) \tilde{\Gamma}^d_m(\boldsymbol{q}) \\ &= -F(i\omega_n)\ , \end{aligned} \tag{H.11}$$

where the second line result follows immediately form the explicit form of F, and the third line follows from $\tilde{\Gamma}^d_m = \tilde{\Gamma}^d_{-m}$. In the following Appendix I, we will need the

explicit form of $I(i\omega_n, i\omega_m, v_D\boldsymbol{q})$ as defined in Eq. (H.1):

$$
\begin{aligned}
&I_{\mathrm{DSM}}(i\omega_n, i\omega_m, v_D q) = \\
&\frac{-1}{2\pi v_D^3}\left[\frac{i\omega_n|\omega_m| - i\omega_m|\omega_n|}{v_D^2 q^2 + \Omega_{n-m}^2} + \right. \\
&\left. + \frac{\omega_n + \omega_m}{2v_D q}\log\left(\frac{2iv_D q|\omega_m| - v_D^2 q^2 - (\omega_n - \omega_m)(\omega_n + \omega_m)}{2iv_D q|\omega_n| + v_D^2 q^2 - (\omega_n - \omega_m)(\omega_n + \omega_m)}\right)\right] , \qquad \text{(H.12a)} \\
&I_{\mathrm{gra}}(i\omega_n, i\omega_m, v_D q) = \\
&-\frac{2i\Omega_{n-m}}{\pi v_D^2(v_D^2 q^2 + \Omega_{n-m}^2)} + \frac{i\omega_m(v_D^2 q^2 - (\omega_n - \omega_m)(\omega_n + \omega_m))}{v_D^2\pi(v_D^2 q^2 + \Omega_{n-m}^2)^{3/2}(v_D^2 q^2 + (\omega_n + \omega_m)^2)^{1/2}} \times \\
&\times \log\left(\frac{v_D^2 q^2 + \omega_n^2 + \omega_m^2 - \sqrt{(\omega_n + \omega_m)^2(v_D^2 q^2 + \Omega_{n-m}^2)}}{v_D^2 q^2 + \omega_n^2 + \omega_m^2 + \sqrt{(\omega_n + \omega_m)^2(v_D^2 q^2 + \Omega_{n-m}^2)}}\right) . \qquad \text{(H.12b)}
\end{aligned}
$$

These were computed directly using the Feynman parameter trick. One could have chosen to make use of the identity

$$
I(i\omega_n, i\omega_m, v_D\boldsymbol{q}) \;=\; \partial_{i\Omega_{n-m}}\frac{1}{V}\sum_{\boldsymbol{k}}\mathrm{tr}\left(G(i\omega_n, \boldsymbol{k})G(i\omega_m, \boldsymbol{k}+\boldsymbol{q})\right) , \qquad \text{(H.13)}
$$

but the sum in this case is UV divergent, whilst I is not, complicating matters.

APPENDIX I

EXPLICIT EVALUATION OF NON-ANALYTIC CORRECTIONS TO THE DENSITY OF STATES IN DIRAC SEMIMETALS

First consider the DSM case, at $T = 0$ one can massage F to following expression:

$$F(i\omega_n) = F_1(i\omega_n) + F_2(i\omega_n) + F_3(i\omega_n) \tag{I.1a}$$

$$F_1(i\omega_n) = -\frac{3}{\pi v_D^3}\frac{T}{V}\sum_{\boldsymbol{q}}\sum_{m>n}\tilde{\Gamma}^d_m(\boldsymbol{q})\frac{2i\omega_n(\Omega_m-\omega_n)}{v_D^2q^2+\Omega_m^2} \tag{I.1b}$$

$$F_2(i\omega_n) = -\frac{3}{\pi v_D^3}\frac{T}{V}\sum_{\boldsymbol{q}}\sum_{\substack{m\geq -n\\ m\neq 0}}\frac{\tilde{\Gamma}^d_m(\boldsymbol{q})}{2v_Dq}(2\omega_n+\Omega_m)\log\left(\frac{-v_Dq+2i\omega_n+i\Omega_m}{v_D+2i\omega_n+i\Omega_m}\right) \tag{I.1c}$$

$$F_3(i\omega_n) = -\frac{3}{\pi v_D^3}\frac{T}{V}\sum_{\boldsymbol{q}}\sum_{m>n}\frac{\tilde{\Gamma}^d_m(\boldsymbol{q})}{2v_Dq}(2\omega_n-\Omega_m)\log\left(\frac{i\Omega_m-v_Dq}{i\Omega_m+v_Dq}\right)\ . \tag{I.1d}$$

The best way to evaluate F_2 and F_3 is to turn them into a single integral, so one isn't doing asymptotic analysis piece wise, and avoiding any branch cut issues with logs. Shifting $m \to m-2n-1$ in F_2, and ignoring the measure zero term at $T=0$ gives us $F_2+F_3=F_4$ as:

$$\begin{aligned}F_4(i\omega_n) = &-\frac{3}{\pi v_D^3}\frac{T}{V}\sum_{\boldsymbol{q}}\sum_{m>n}\log\left(\frac{i\Omega_m-v_Dq}{i\Omega_m+v_Dq}\right)\times\\ &\times\left[\left(\tilde{\Gamma}^d\left(\boldsymbol{q},\frac{\Omega_m-2\omega_n}{v_Dq}\right)-\tilde{\Gamma}^d\left(\boldsymbol{q},\frac{\Omega_m}{v_Dq}\right)\right)\frac{\Omega_m}{2v_Dq}+\frac{\omega_n}{v_Dq}\tilde{\Gamma}^d\left(\boldsymbol{q},\frac{\Omega_m}{v_Dq}\right)\right]\ ,\end{aligned} \tag{I.2}$$

Recall, in d=3 $\tilde{\Gamma}^d_m(q) \equiv \tilde{\Gamma}^d(q, z_m)$ takes the form:

$$\begin{aligned} \text{Long Ranged} \quad \tilde{\Gamma}^d_l(\Omega_m, q) &= \frac{\Gamma}{q^2 + nq^2 \log(\Lambda^2 v_D^2/(q^2 v_D^2 + \Omega_m^2))} \\ &\equiv \frac{\Gamma}{q^2} V_l(q^2 v_D^2 + \Omega^2) \ , \qquad \text{(I.3a)} \\ \text{Short Ranged} \quad \tilde{\Gamma}^d_s(\Omega_m, q) &= \frac{\Gamma}{1 + \epsilon_r q^2 + nq^2 \log(\Lambda^2 v_D^2/(q^2 v_D^2 + \Omega_m^2)} \\ &\equiv \frac{\Gamma}{q^2} V_s(q^2, q^2 v_D^2 + \Omega^2) \ . \qquad \text{(I.3b)} \end{aligned}$$

for $n = \Gamma/12\pi^2 v_D \ll 1$.

The Case of an Unscreened Long Ranged Interaction

We will present an alternative method for computing the unscreened long ranged interaction's contribution to the DOS non-analyticity, and then use that method to evaluate the screened short ranged interaction case. The unscreened interaction is obtained by setting $n = 0$ in $\tilde{\Gamma}^d_l(\Omega_m, q)$ above. Let us set $T = 0$ and consider $\omega_n \to 0$. Next, define $\lambda = \Lambda v_D/\omega_n$, then upon a change of variables, first $\Omega \to zqv_D$ then $q \to \bar{q}\omega_n/v_D$, F_1 becomes:

$$F_1(i\omega_n) = -\frac{6i\omega_n^4}{4\pi^4 v_D^6} \int_0^\lambda d\bar{q} \left(-\int_0^{1/\bar{q}} + \int_0^{\lambda/\bar{q}} \right) dz \tilde{\Gamma}^d(\bar{q}\omega_n/v_D, z) \frac{z\bar{q} - 1}{1 + z^2} \ . \quad \text{(I.4)}$$

Recall that the Λ cutoff present in both integrals is due to interaction cutoff. In the second integral let $\bar{q} \to \tilde{q}/\omega$, we see immediately that it generates a function $c_1 i\omega_n + c_2 i\omega_n^2$, for c_i real, regardless of interaction type. These do not provide any non-analyticities so we can ignore them. Now consider $\omega_n \to 0$ in the first integral,

asymptotic analysis yields, for long range interactions:

$$\begin{aligned} F_1(i\omega_n) &= \frac{6i\Gamma\omega_n^2}{4\pi^4 v_D^4}\int_0^\lambda d\bar{q}\int_0^{1/\bar{q}} dz\frac{z\bar{q}-1}{1+z^2} = id_1\frac{\Gamma}{\Lambda^2 v_D^6}\omega_n^4 \\ \Rightarrow F_1(z) &= -d_1 i z^4 \mathrm{sgn}\,\mathrm{Im}\,z \end{aligned} \tag{I.5}$$

where the last line follows from the oddness of each $F_i(z)$ across the imaginary axis. Here $d_1 = 3/8\pi^4\epsilon_1$ is a real number. Upon analytic continuation $\mathrm{Im}\,(F_1(\omega + i0+)) \approx -d_1\omega_n^4\Gamma/\Lambda^2 v_D^6$ and provides no non-analyticity. Next we evaluate F_4.

$$\begin{aligned} F_4(i\omega_n) &= -\frac{3}{\pi v_D^3}\frac{4\pi}{(2\pi)^4}\int_0^\Lambda dq\; q^2\int_{\omega_n}^{\Lambda v_D} d\Omega\;\frac{\omega_n}{v_D q}\tilde{\Gamma}^d\left(\boldsymbol{q},\frac{\Omega_m}{v_D q}\right)\log\left(\frac{i\Omega - v_D q}{i\Omega + v_D q}\right) \\ &= -\frac{3\Gamma}{4\pi^4 v_D^4}\omega_n^2\int_0^\lambda d\bar{q}\int_{1/\bar{q}}^{\lambda/\bar{q}} dz\;\log\left(\frac{iz-1}{iz+1}\right) \\ &= -\frac{3\Gamma}{4\pi^4 v_D^4}\omega_n^2\int_0^\lambda d\bar{q}\left[-\int_0^{1/\bar{q}} dz f(z) + \int_0^{\lambda/\bar{q}} dz f(z)\right] \end{aligned} \tag{I.6}$$

Let us consider the limiting behavior of $f(z)$, for $z \to 0$, the log to leading order goes likes $\log((iz-1)/(iz+1)) \approx i\pi - 2iz$. Then to leading order in ω_n we obtain:

$$\begin{aligned} F_4(i\omega_n) &\approx -\frac{3\Gamma\omega_n^2}{4\pi^4 v_D^4}\int_0^\lambda d\bar{q}\left[-\int_0^{1/\bar{q}} dz(i\pi - 2iz) + \int_0^{\lambda/\bar{q}} dz f(z)\right] \\ &\approx \frac{3i\Gamma\omega_n^2}{4\pi^3 v_D^4\epsilon_1}\log\left(\frac{\Lambda v_D}{\omega_n}\right) - \frac{3\Gamma\omega_n^2}{4\pi^4 v_D^4}\int_0^\lambda d\bar{q}\int_0^{\lambda/\bar{q}} dz f(z)\;. \end{aligned}$$

Let us now prove that the $\lambda/\bar{q}$ portion of the integral will not provide any non-analyticities. Let $\bar{q} \to \lambda\tilde{q}$, then we obtain:

$$-\frac{3\Gamma\omega_n\Lambda v_D}{4\pi^4 v_D^4}\int_0^1 d\tilde{q}\int_0^{1/\tilde{q}} dz\log\left(\frac{iz-1}{iz+1}\right)\;. \tag{I.7}$$

The integrand and limits now don't contain ω_n, and thus cannot contribute anything to the non-analytic dependence of $F_4(i\omega_n)$ on ω_n. Now consider such a function $iR(\omega_n) = i\omega_n c_1 + i\omega_n^2 c_2 + \ldots$ for c_i real. Upon analytic continuations, we get $iR(\omega + i0) = i(\omega + i0) + \ldots$, clearly the imaginary part will only pick up even contributions, and they will all be analytic. Hence, for unimportant c_1 and c_2, F_4 becomes:

$$F_4(i\omega_n) \approx \frac{3i\Gamma\omega_n^2}{4\pi^3 v_D^4 \epsilon_1} \log\left(\frac{\Lambda v_D}{\omega_n}\right) + ic_1\omega_n + ic_2\omega_n^2 + \ldots \tag{I.8}$$

Using $\log(|\omega_n|) = (1/2)(\log(i\omega_n) + \log(-i\omega_n))$, and the fact that $F_4(i\omega_n)$ is odd in ω_n, we get for the non-analytic part of F_4:

$$\begin{aligned} F_{NA}(z) &= -i\frac{3\Gamma z^2}{8\pi^3 v_D^4}\left[\log\left(\frac{z}{\Lambda v_D}\right) + \log\left(\frac{-z}{\Lambda v_D}\right)\right] \operatorname{sgn}\operatorname{Im}(z) \\ \Rightarrow \operatorname{Im} F_{NA}(\omega + i0^+) &= \frac{3\Gamma}{4\pi^3 v_D}\frac{\omega^2}{v_D^3}\log\left(\frac{|\omega|}{\Lambda v_D}\right) , \end{aligned} \tag{I.9}$$

which is precisely what we found in Equation (5.22). By equation (5.3), we the leading non-analytic correction to the DOS for unscreened, long ranged interactions is:

$$\delta N_{NA}(\omega) = \frac{4\Gamma}{\pi^4 v_D}\frac{\omega^2}{v_D^3}\log\left(\frac{|\omega|}{\Lambda v_D}\right) . \tag{I.10}$$

The Case of a Short Ranged Interaction

We now need to evaluate equations (I.4) and (I.6) for short ranged interactions. For similar reasons to those in the previous section, the $\int_0^{\lambda/\bar{q}} dz$ portions of each integral will not yield non-analyticities[1]. In this case, the non-analyticity comes from

[1] An additional step of Taylor expanding the interaction is necessary to see this.

F_1:

$$\begin{aligned}
F_1^{NA}(i\omega_n) &= \frac{6i\Gamma\omega_n^4}{4\pi^4 v_D^6}\int_0^\lambda d\bar{q}\int_0^{1/\bar{q}} dz \frac{z\bar{q}-1}{1+z^2} \\
&\quad \times \frac{1}{1+\epsilon_2\bar{q}^2\omega_n^2/v_D^2 + n\bar{q}^2\omega_n^2\log(1+z^2)/v_D^2} \\
&\to \frac{6i\Gamma\omega_n^4}{4\pi^4 v_D^6}\int_0^\lambda d\bar{q}\int_0^{1/\bar{q}} dz \frac{z\bar{q}-1}{1+\epsilon_2\bar{q}^2\omega_n^2/v_D^2} \\
&= -\frac{6i\Gamma\omega_n^4}{8\pi^4 v_D^6}\int_0^\lambda d\bar{q}\ \frac{1}{\bar{q}} - \frac{\epsilon_2\bar{q}\omega_n^2}{1+\epsilon_2\bar{q}^2\omega_n^2/v_D^2} \\
&= -\frac{6i\Gamma\omega_n^4}{8\pi^4 v_D^6}\left(\log\left(\frac{\Lambda v_D}{|\omega_n|}\right) - \frac{1}{2}\log(1+\epsilon_2\Lambda^2)\right) \\
\Rightarrow F_1^{NA}(z) &= \frac{6i\Gamma z^4}{16\pi^4 v_D^6}\left[\log\left(\frac{z}{\Lambda v_D}\right) + \log\left(\frac{-z}{\Lambda v_D}\right)\right]\operatorname{sgn}\operatorname{Im}(z) \\
\Rightarrow \operatorname{Im} F_1^{NA}(\omega+i0^+) &= \frac{6\Gamma}{8\pi^4 v_D^6}\omega^4\log\left(\frac{|\omega|}{\Lambda v_D}\right)\ . \qquad \text{(I.11)}
\end{aligned}$$

The limit in the second line comes from $z \to 1/\bar{q} \to 1/\lambda$, and the $n\log(1+z^2)\bar{q}^2\omega_n^2/v_D^2 \approx n\omega_n^2/v_D^2$ term in this limit can be ignored[2] Finally, F_4 does not provide any non-analyticities, and thus Equation (I.11) is the most important contribution to the DOS.[3]

[2]Dimensionality check: in 3d short ranged $[\Gamma] = E \times V$, and $[v_D] = E \times L$.

[3]In the case that $\epsilon_2 = 0$ as is assumed in some cases in the literature [78], $F_1(i\omega_n)$ still provides the $\omega^4\log\omega$ non-analyticity, and F_4 provides a less important $\omega^6\log\omega$ non-analyticity.

APPENDIX J

EXPLICIT EVALUATION OF NON-ANALYTIC CORRECTIONS TO THE DENSITY OF STATES IN GRAPHENE

Recall to compute $\delta N(\omega)$ we need:

$$F(i\omega_n) = \frac{6}{4}\frac{T}{V}\sum_{\substack{m\neq 0\\ \boldsymbol{q}}} I(i\omega_n, i\omega_n + i\Omega_m, v_D\boldsymbol{q})\tilde{\Gamma}^d_m(\boldsymbol{q}) \ , \tag{J.1a}$$

$$I_{\rm gra}(i\omega_n, i\omega_n + i\Omega_m, v_D q) = \frac{2i\Omega_m}{\pi v_D^2(v_D^2q^2+\Omega_m^2)} - \frac{(i\omega_n + i\Omega_m)(\Omega_m(2\omega_n+\Omega_m)+v_D^2q^2)}{v_D^2\pi(v_D^2q^2+\Omega_m^2)^{3/2}(v_D^2q^2+(2\omega_n+\Omega_m)^2)^{1/2}} l(qv_D, \Omega_m, \omega_n) \ , \tag{J.1b}$$

$$l(qv_D, \Omega_m, \omega_n) = \log\left(\frac{v_D^2q^2+\omega_n^2+(\omega_n+\Omega_m)^2 - \sqrt{(v_D^2q^2+(2\omega_n+\Omega_m)^2)(v_D^2q^2+\Omega_m^2)}}{v_D^2q^2+\omega_n^2+(\omega_n+\Omega_m)^2 + \sqrt{(v_D^2q^2+(2\omega_n+\Omega_m)^2)(v_D^2q^2+\Omega_m^2)}}\right) \ . \tag{J.1c}$$

Recall, in d=2 $\tilde{\Gamma}^d_m(q) \equiv \tilde{\Gamma}^d(q, z_m)$ takes the form:

$$\text{Long Ranged} \quad \tilde{\Gamma}^d(q, z_m) = \frac{\Gamma}{q + n_g q(1+z_m^2)^{-1/2}} \ , \tag{J.2a}$$

$$\text{Short Ranged} \quad \tilde{\Gamma}^d(q, z_m) = \frac{\Gamma}{1 + n_g q(1+z_m^2)^{-1/2}} \ . \tag{J.2b}$$

with $n_g = \Gamma/4v_D$. For long ranged interactions n_g is dimensionless, for short ranged it has units $[n_g] = L$ (equivalently for long ranged interactions $[\Gamma] = E \times L$ and short ranged $\Gamma = E \times L^2$). We see that since $\tilde{\Gamma}^d_m(\boldsymbol{q})$ is even in m, the sum over the first term in $I_{\rm gra}$ drops out in both interaction cases. We can write the $T \to 0$ limit of

$F(i\omega_n)$ as:

$$\begin{aligned} F(i\omega_n) &= -\frac{6}{4}\frac{1}{v_D^2\pi(2\pi)^3}\int_0^\infty d\vec{q}\int_0^{\Lambda v_D} d\Omega \frac{\tilde{\Gamma}^d(q,\Omega/qv_D)}{(v_D^2q^2+\Omega^2)^{3/2}} \times \\ &\times \left[\frac{(i\omega_n+i\Omega)(\Omega(2\omega_n+\Omega)+v_D^2q^2)l(qv_D,\Omega,\omega_n)}{(v_D^2q^2+(2\omega_n+\Omega)^2)^{1/2}} + (\Omega\to-\Omega)\right] \\ &= -\frac{i3\omega_n^2}{v_D^4(2\pi)^3}\int_0^\infty d\bar{q}\ \bar{q}\int_0^\lambda d\bar{\Omega}\frac{\tilde{\Gamma}^d(\bar{q}\omega_n/v_D,\bar{\Omega}/\bar{q})}{(\bar{q}^2+\bar{\Omega}^2)^{3/2}} \times \\ &\times \left[\frac{(1+\bar{\Omega})(\bar{\Omega}(2+\bar{\Omega})+\bar{q}^2)l(\bar{q},\bar{\Omega},1)}{(\bar{q}^2+(2+\bar{\Omega})^2)^{1/2}} + (\bar{\Omega}\to-\bar{\Omega})\right] , \end{aligned} \tag{J.3}$$

where a Λv_D cutoff has been imposed on the frequency integral which would otherwise be UV divergent. In the second line we used $\Omega \to \bar{\Omega}\omega_n$ and $q \to \bar{q}\omega_n/v_D$. Once again we define $\lambda = \Lambda v_D/\omega_n$.

The Case of a Long Ranged Interaction

Let us now first consider the case of a long ranged interaction, in which case we can write $\tilde{\Gamma}^d(\bar{q}\omega_n/v_D,\bar{\Omega}/\bar{q}) = \frac{v_D}{\omega_n\bar{q}}v\left(\frac{\bar{\Omega}}{\bar{q}}\right)$. Then consider some number $c \gg 1$, and break up the frequency integral.

$$\begin{aligned} F(i\omega_n) &= -\frac{i3\omega_n}{v_D^3(2\pi)^3}\int_0^\infty d\bar{q}\left(\int_0^c + \int_c^\lambda\right) d\bar{\Omega}\frac{v\left(\frac{\bar{\Omega}}{\bar{q}}\right)}{(\bar{q}^2+\bar{\Omega}^2)^{3/2}} \times \\ &\times \left[\frac{(1+\bar{\Omega})(\bar{\Omega}(2+\bar{\Omega})+\bar{q}^2)l(\bar{q},\bar{\Omega},1)}{(\bar{q}^2+(2+\bar{\Omega})^2)^{1/2}} + (\bar{\Omega}\to-\bar{\Omega})\right] . \end{aligned} \tag{J.4}$$

The first integral is ω_n independent, and just yields a $\mathcal{O}(\omega_n)$ contribution to the DOS correction. Let's focus on the second integral for $\omega_n \to 0$. Since $c \gg 1$, we have $\bar{\Omega} \gg 1$ and:

$$\sqrt{\bar{q}^2+\bar{\Omega}^2+4\bar{\Omega}+4} \approx \sqrt{\bar{q}^2+\bar{\Omega}^2} + \frac{2+2\bar{\Omega}}{\sqrt{\bar{q}^2+\bar{\Omega}^2}} - \frac{2(1+\bar{\Omega})^2}{(\bar{q}^2+\bar{\Omega}^2)^{3/2}} . \tag{J.5}$$

Then for $\bar{\Omega} \to \infty$ we have

$$\begin{aligned} l(\bar{q},\bar{\Omega},1) &\approx \log\left(\frac{\bar{q}^2+1+(1+\bar{\Omega})^2-(\bar{q}^2+\bar{\Omega}^2)-2-2\bar{\Omega}+\frac{2(1+\bar{\Omega})^2}{\bar{q}^2+\bar{\Omega}^2}}{\bar{q}^2+1+(1+\bar{\Omega})^2+(\bar{q}^2+\bar{\Omega}^2)+2+2\bar{\Omega}-\frac{2(1+\bar{\Omega})^2}{\bar{q}^2+\bar{\Omega}^2}}\right) \\ &\approx \log\left(\frac{(1+\bar{\Omega})^2}{(\bar{q}^2+\bar{\Omega}^2)^2}\right) \approx 2\log\left(\frac{|\bar{\Omega}|}{\bar{q}^2+\bar{\Omega}^2}\right) \ . \end{aligned} \tag{J.6}$$

To leading order $l(\bar{q},\bar{\Omega},1)$ is even in $\bar{\Omega}$, expanding the numerator in Eq. (J.4) we get to leading order:

$$\begin{aligned} F(i\omega_n) &\approx \mathcal{O}(i\omega_n) - \frac{i12\omega_n}{v_D^3(2\pi)^3}\int_0^\infty d\bar{q}\int_c^\lambda d\bar{\Omega}\frac{(3\bar{\Omega}^2+\bar{q}^2)}{(\bar{q}^2+\bar{\Omega}^2)^2}v\left(\frac{\bar{\Omega}}{\bar{q}}\right)\log\left(\frac{|\Omega|}{\bar{q}^2+\bar{\Omega}^2}\right) \\ &= \mathcal{O}(i\omega_n) + \frac{i12\omega_n}{v_D^3(2\pi)^3}\int_c^\lambda d\bar{\Omega}\int_0^\infty dx\ \frac{1}{\bar{\Omega}}\frac{3+x^2}{(1+x^2)^2}v\left(\frac{1}{x}\right)\left[\log\left(\bar{\Omega}\right)+\log\left(1+x^2\right)\right] \\ &= \mathcal{O}(i\omega_n) - i\omega_n\frac{\Gamma}{v_D^3}c_1(n_g)\log\left(\frac{|\omega_n|}{\Lambda v_D}\right) + i\omega_n\frac{\Gamma}{v_D^3}c_2(n_g)\log^2\left(\frac{|\omega_n|}{\Lambda v_D}\right) \ , \end{aligned} \tag{J.7}$$

where in the second line we let $\bar{q} \to \bar{\Omega}x$, and $c_1(n_g)$ and $c_2(n_g)$ are real constants that depend on n_g. They can be easily numerically evaluated given a value of n_g. By the techniques of the previous section, upon analytic continuation we yield for the DOS correction:

$$\delta N(\omega) = \frac{16}{3\pi}\mathrm{Im}\, F(\omega+i0^+) \approx +|\omega|\frac{\Gamma}{v_D^3}c_1(n_g) - |\omega|c_2(n_g)\frac{\Gamma}{v_D^3}\log\left(\frac{|\omega|}{\Lambda v_D}\right) \ , \tag{J.8}$$

where we have absorbed multiplicative factors into the c_i's. They are defined as:

$$c_1(n_g) \equiv \frac{8}{\pi^3}\int_0^\infty dx\ \frac{3+x^2}{(1+x^2)^2}\frac{1}{1+n_g(1+1/x^2)^{-1/2}} \ , \tag{J.9a}$$

$$c_2(n_g) \equiv \frac{32}{\pi^3}\int_0^\infty dx\ \frac{3+x^2}{(1+x^2)^2}\frac{\log(1+x^2)}{1+n_g(1+1/x^2)^{-1/2}} \ . \tag{J.9b}$$

The Case of a Short Ranged Interaction

In the short ranged interaction case, Equation (J.3) is UV divergent for $q \to 0$, so it now reads:

$$\begin{aligned} F(i\omega_n) &= -\frac{3i\omega_n^2\Gamma}{v_D^4(2\pi)^3}\int_0^\lambda d\tilde{q}\int_{-\lambda}^\lambda d\bar{\Omega}\ \frac{\tilde{q}}{1+\tilde{q}^2\frac{\omega_n}{v_D}n_g(\tilde{q}^2+\bar{\Omega}^2)^{-1/2}}\frac{1}{(\tilde{q}^2+\bar{\Omega}^2)^{3/2}}\times \\ &\quad\times\frac{(1+\bar{\Omega})(\bar{\Omega}(2+\bar{\Omega})+\tilde{q}^2)}{(\tilde{q}^2+(2+\bar{\Omega})^2)^{1/2}}l(\tilde{q},\bar{\Omega},1)\ . \end{aligned} \tag{J.10}$$

Upon changing coordinates to $\bar{q}(\cos\theta,\sin\theta)=(\bar{\Omega},\tilde{q})$ we get:

$$\begin{aligned} F(i\omega_n) &= -\frac{3i\omega_n^2\Gamma}{v_D^4(2\pi)^3}\int_0^\lambda d\tilde{q}\int_{-1}^1 d\eta\ \frac{1}{\bar{q}+\bar{q}^2\frac{\omega_n}{v_D}n_g(1-\eta^2)}\frac{(1+\bar{q}\eta)(\bar{q}^2+2\bar{q}\eta)}{(\bar{q}^2+4\bar{q}\eta+4)^{1/2}}\times \\ &\quad\times\log\left(\frac{\bar{q}^2+2\bar{q}\eta+2-\bar{q}\sqrt{\bar{q}^2+4\bar{q}\eta+4}}{\bar{q}^2+2\bar{q}\eta+2+\bar{q}\sqrt{\bar{q}^2+4\bar{q}\eta+4}}\right)\ . \end{aligned} \tag{J.11}$$

This integral doesn't blow up in the IR limit because the log term in the last line goes to zero as $-2\bar{q}$ as $\bar{q}\to 0$. We are interested in the leading behavior as $\bar{\omega_n}\to 0$, so we Taylor expand the integrand around $\omega_n = 0$ and for $\bar{q}\to\infty$, yielding:

$$\begin{aligned} F(i\omega_n) &\approx -\frac{3i\omega_n^2\Gamma}{v_D^4(2\pi)^3}\int_1^\lambda d\tilde{q}\int_{-1}^1 d\eta\ \frac{(1+\bar{q}\eta)(\bar{q}^2+2q\eta)}{q}\left(\frac{1}{\bar{q}}-\frac{2\eta}{\bar{q}^2}\right)\times \\ &\quad\times\left(1+\frac{n_g\omega_n\bar{q}(-1+\eta^2)}{v_D}\right)\left[\log(\eta^2)-\log(\bar{q}^2)\right]\ . \end{aligned} \tag{J.12}$$

The leading non-analytic behavior of $F(i\omega_n)$ for $i\omega_n\to 0$ is then:

$$\begin{aligned} F(i\omega_n) &\approx i\frac{2}{5\pi^3}\omega_n^3\frac{\Gamma n_g}{v_D^5}\log^2\left(\frac{|\omega_n|}{\Lambda v_D}\right) \\ &\quad+\mathcal{O}\left(\omega^3\log\left(\frac{|\omega_n|}{\Lambda v_D}\right),\omega\log\left(\frac{|\omega_n|}{\Lambda v_D}\right),\omega_n\right)\ . \end{aligned} \tag{J.13}$$

Upon computing $\mathrm{Im}\, F(\omega + i0^+)$ the terms not explicitly written down vanish, or are less important. Finally, analytic continuation yields the leading non-analytic DOS correction:

$$\delta N(\omega) \;=\; \frac{2}{5\pi^2}\frac{\Gamma n_g}{v_D^5}\omega^2|\omega|\log\left(\frac{|\omega|}{\Lambda v_D}\right)\;. \tag{J.14}$$

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