PERIODIC MARGOLIS SELF MAPS AT $P = 2$

by

LEANNE ELIZABETH MERRILL

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Student: Leanne Elizabeth Merrill

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This dissertation has been accepted and approved in partial fulfillment of the requirements for the Doctor of Philosophy degree in the Department of Mathematics by:

Dr. Hal Sadofsky	Chair
Dr. Boris Botvinnik	Core Member
Dr. Dev Sinha	Core Member
Dr. Christopher Sinclair	Core Member
Dr. Malcom Wilson	Institutional Representative

and

Dr. Sara D. Hodges	Interim Vice Provost and Dean of the Graduate School

Original approval signatures are on file with the University of Oregon Graduate School.

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DISSERTATION ABSTRACT

Leanne Elizabeth Merrill

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The Periodicity theorem [HS98] tells us that any finite spectrum supports a v_n-map for some n. We are interested in finding finite 2-local spectra that both support a v_2-map with a low power of v_2 and have few cells. Following the process outlined in [PS94], we study a related class of self-maps, known as u_2-maps, between stably finite spectra. We construct examples of spectra that might be expected to support u_2^1-maps, and then we use Margolis homology and homological algebra computations to show that they do not support u_2^1-maps. We also show that one example does not support a u_2^2-map. The nonexistence of u_2-maps on these spectra eliminates certain examples from consideration by this technique.

CURRICULUM VITAE

NAME OF AUTHOR: Leanne Elizabeth Merrill

GRADUATE AND UNDERGRADUATE SCHOOLS ATTENDED:

University of Oregon, Eugene, OR
The State University of New York at Potsdam, Potsdam, NY

DEGREES AWARDED:

Doctor of Philosophy, Mathematics, 2017, University of Oregon
Master of Science, Mathematics, 2014, University of Oregon
Master of Arts, Mathematics, 2011, The State University of New York at Potsdam
Bachelor of Arts, Mathematics and Music, 2011, The State University of New York at Potsdam

AREAS OF SPECIAL INTEREST:

Algebraic Topology
Stable Homotopy Theory
Chromatic Homotopy Theory

PROFESSIONAL EXPERIENCE:

Graduate Teaching Fellow, Department of Mathematics, University of Oregon, Eugene, 2011–2017

GRANTS, AWARDS AND HONORS:

Graduate Teaching Excellence Award, University of Oregon, 2016

Anderson Distinguished Graduate Teaching Award, University of Oregon, 2016

Johnson Fellowship for Travel and Research, University of Oregon, 2016

Dan Kimble First Year Teaching Award, University of Oregon, 2016

PUBLICATIONS:

A Tale of Two Puzzles: Towers of Hanoi and Spin-Out 2013, Journal of Information Processing, v. 21, pp. 378-392.

Intrinsically linked signed graphs in projective space, 2012, Discrete Mathematics, v. 312, pp. 2009-2022.

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CHAPTER I

INTRODUCTION

1.1. Motivation

We are interested in the homotopy properties of CW-spectra with finitely many cells. For the definitions of spectra and related terms, see Chapter 8 of [Swi02]. For definitions involving p-locality, see Chapter 1 of [Rav92].

Definition 1.1. A *finite spectrum* is a CW-spectrum with finitely many cells.

Such spectra are suspension spectra of CW-complexes, possibly desuspended. We use cohomology theories to study finite spectra and their self-maps. Fix a prime p. There are extraordinary cohomology theories $K(n)_*$, known as the Morava K-theories, which separate p-local spectra into different types. They satisfy $K(n)_*(S^0) = \mathbb{F}_p[v_n, v_n^{-1}]$ where $|v_n| = 2p^n - 2$ (see Proposition 1.5.2 in [Rav92]).

Definition 1.2 (Definition 1.5.3.i from [Rav92])**.** A finite spectrum X is of *type n* if $K(n)_*X \neq 0$ but $K(m)_*X = 0$ for all $m < n$.

All p-local finite spectra are type n for some n. Given a type n complex X, the Periodicity theorem [HS98] guarantees the existence of a self map of X which is an isomorphism on $K(n)_*$, and is thus non-nilpotent. We quite the following restatement of the Periodicity theorem for later use.

Theorem 1.3 (Theorem 1.5.4.i from [Rav92])**.** *Let X be a p-local finite CW-complex of type n. There is a self-map $f\colon \Sigma^{d+i}X \to \Sigma^i X$ for some $i \geq 0$ such that $K(n)_*(f)$ is an isomorphism and $K(m)_*(f)$ is trivial for $m > n$. Such a map is called an v_n*-map.

The Nilpotence theorem [DHS88] tells us that essentially all non-nilpotent maps have this property. The Periodicity theorem does not specify the minimum value of d or the number of cells in X, though d must be a multiple of the degree of v_n. Since we are concerned about the particular value of d in the theorem above, we make the following definition:

Definition 1.4. A v_n^j-map on a spectrum X is a map $f\colon \Sigma^{j(2p^n-2)+i}X \to \Sigma^i X$ for some $i \geq 0$ which induces multiplication by v_n^j on $K(n)_*$.

We would like to study examples where both the the number of cells in X and the power of v_n are small. Typically, a complex with more cells allows a lower power of v_n, while a higher power of v_n requires fewer cells.

The following examples illustrate this principle. In [DM81], two complexes are described that support v_1-maps. $\mathbb{R}P^2$, which has two cells, supports a v_1^4-map and no lower power of v_1. This example realizes the minimal number of cells and one can prove from this example that no two cell complex can support a v_1^k-map for $k < 4$. On the other hand,

$$Y = \Sigma^{-3}\mathbb{R}P^2 \wedge \mathbb{C}P^2$$

has four cells and supports a v_1^1-map, which is the minimal value of d. Any complex supporting a v_1^1-map must have at least four cells. Similarly, for v_2, [BHHM08] describes a four cell complex that supports at v_2^{32}-map, and [BE16] describes several homotopy classes of 32-cell complexes which support a v_2^1-map.

Our goal is to identify complexes with a small number of cells that have v_2-maps with a low power of v_2. We do not succeed in this goal, but we eliminate certain examples from consideration.

1.2. Strategy

The following strategy to find v_n-maps is described in detail [PS94]. It involves introducing other self maps, known as u_i-maps. u_i-maps interpolate between the v_n-maps, potentially giving an inductive technique to go from a v_n- to a v_{n+1}-map. We need to develop some algebraic background to describe u_i-maps in more detail.

1.2.1. The Steenrod Algebra and u_i maps

Let A represent the usual mod 2 Steenrod algebra. It is a theorem of Milnor [Mil58] that A_*, the dual to A, satisfies

$$A_* = \mathbb{Z}/2\,[\xi_1, \xi_2, \dots]$$

Let $P^s_t \in A$ be the element dual to $\xi_t^{2^s}$. By Lemma 15.4 in [Mar83], $(P^s_t)^2 = 0$ if and only if $s < t$. We can order these elements by their degree: $|P^s_t| = 2^s(2^t - 1)$. For example, the first few elements are given in Table 1.1.

element	degree
P^0_1	1
P^0_2	3
P^1_2	6
P^0_3	7

TABLE 1.1. First four P^s_t elements, ordered by degree

Let x_i be the i^{th} element in this list for $i \geq 0$, so $x_0 = P^0_1$. Let $E = E(x_i)$ be the exterior subalgebra of A generated by x_i for $i \geq 0$. $\mathrm{Ext}_E(\mathbb{F}_2, \mathbb{F}_2) = \mathbb{F}_2[u_i]$ where u_i

is a polynomial generator with bidegree $(1,|x_i|)$. Let M be a module over A and consider the following diagram:

$$\begin{array}{ccc} \operatorname{Ext}_A^{*,*}(M,M) & & \\ \downarrow i^* & & \\ \operatorname{Ext}_E^{*,*}(M,M) & \xleftarrow[-\otimes M]{} & \operatorname{Ext}_E^{*,*}(\mathbb{F}_2,\mathbb{F}_2) \end{array}$$

The downward map exists because a free A-resolution of M is also a free E-resolution of M: A is free over $E\left(\{P_t^r : r < t\}\right)$ ([Rav92], Lemma C.3.1) and $E\left(\{P_t^r : r < t\}\right)$ is free over $E(P_t^s)$ for any particular $s < t$. Therefore, A is free over $E(P_s^t)$ for any $s < t$, and so any free A-module is a free E-module by [MM65].

Definition 1.5. An element $f \in \operatorname{Ext}_A^{k,k|x_i|}(M,M)$ is a u_i-map if $i^*(f) = (u_i)^k \otimes 1_M$ for some k.

The above definition is purely algebraic. We are interested in maps between the cohomology of spectra that survive the Adams spectral sequence to get an element of homotopy. In order to describe such maps, we introduce the topological structure used to build the Adams spectral sequence.

Definition 1.6 (2.2.1 in [Rav86])**.** Given a spectrum X, an *Adams resolution for X* is a diagram:

$$\begin{array}{ccc} \vdots & & \\ \downarrow & & \\ X_2 & \xrightarrow{b_2} & C_2(X) \\ i_2\downarrow & & \\ X_1 & \xrightarrow{b_1} & C_1(X) \\ i_1\downarrow & & \\ X & \xrightarrow{b_0} & C_0(X) \end{array}$$

such that for all $i \geq 0$, the following conditions hold:

(1) Each $C_i(X)$ is a wedge of suspensions of $K(\mathbb{Z}/2)$'s;

(2) Each $b_i^*: H^*(C_i(X)) \to H^*(X_i)$ is surjective (here $X := X_0$).

The spectra X_i and the maps between them form an *Adams tower for* X.

Given any spectrum X, such a diagram always exists (see [Rav86]). Define $P_i(X) = H^*\Sigma^i C_i(X)$, and note that by condition (1) the modules $P_i(X)$ are free. We can knit these together into a long exact sequence to form a projective resolution for H^*X as an A-module:

$$\ldots \xrightarrow{d_2} P_2(X) \xrightarrow{d_2} P_1(X) \xrightarrow{d_1} P_0(X) \xrightarrow{d_0} H^*X \to 0$$

Let $K_iX = \ker(d_i: P_i \to P_{i-1}X)$. We note that $K_iX = H^*\Sigma^{i+1}X_{i+1}$, which we will use extensively later.

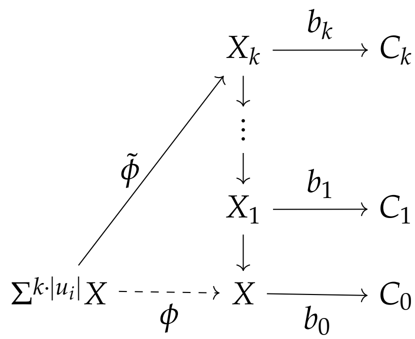
Now we can say what it means for X to have a u_i-map. Let $H^*X := H^*(X; \mathbb{Z}_2)$.

Definition 1.7. Let $f \in \mathrm{Ext}_A^{k,k|x_i|}(H^*X, H^*X)$ be such that $i^*(f) = (u_i)^k \otimes 1_{H^*X}$ for some k. If f is a permanent cycle in the Adams Spectral Sequence, then we say that X has a u_i-map.

Under these hypotheses, the permanent cycle f corresponds to a map

$$\tilde{\phi}: \Sigma^{k\cdot|u_i|}X \to X$$

which lifts to the k^{th} stage of the Adams tower:



The map ϕ is defined to be the composite of $\tilde{\phi}$ with the downward maps.

Lemma 3.1 and Theorem 3.3 in [PS94] give an inductive method to produce a complex supporting a u_{i+1}-map from a complex that supports a u_i-map. Let X be a complex supporting a u_i-map and $\tilde{\phi}$ be as above.

Let

$$Y = \mathrm{cof}(\tilde{\phi})$$

We may assume that $\tilde{\phi}^*\colon H^*X_k \to H^*\Sigma^{k\cdot|u_i|}X$ is onto since we can add copies of A to H^*X_k by choosing $C_{k-1}(X)$ to have more wedge summands of $K(\mathbb{Z}/2)$. Thus we have that

$$H^*Y = \ker\left(\tilde{\phi}^*\colon H^*X_k \to H^*\Sigma^{k\cdot|u_i|}X\right)$$

The proof of Theorem 3.3 in [PS94] shows that Y supports a u_{i+1}^j-map for some j. Inductively taking cofibers in this way allows us to obtain spectra with u_i-maps for all i; however, this does not control the power j.

1.2.2. u_i maps and v_n maps

There is a correspondence between u_i-maps and v_n-maps, described in the proofs of Theorem 3.3 and Corollary 3.5 in [PS94]. They show that for each i, there is cofiber sequence

$$F \to X_i \to X_i'$$

where X_i has a u_i^k-map for some k, X_i' is a finite spectrum, and F has a finite Adams resolution. The proof uses the inductive process described above.

In the case that u_i corresponds to P_{n+1}^0 for the elements P_t^s described above, they show that the u_i-map on X_i induces a v_n-map on X_i'.

The first several cases of this correspondence are given in Table 1.2.

u_i	x_i	v_n	degree
u_0	P_1^0	v_0	1
u_1	P_2^0	v_1	3
u_2	P_2^1	–	6
u_3	P_3^0	v_2	7

TABLE 1.2. Correspondence between u_i- and v_n-maps

1.2.3. Margolis Homology

We see above that u_i-maps can give rise to v_n-maps on associated finite complexes. In order to detect u_i-maps, we use a tool called Margolis homology.

Let $x \in A$ be such that $x^2 = 0$. Let M be a left A-module, and define a vector space map $x\colon M \to M$ by $m \mapsto xm$. Since $x^2 = 0$, $\operatorname{im} x \subseteq \ker x$. So one can define the homology of an A-module M with respect to x to be

$$H(M,x) = \frac{\ker x\colon M \to M}{\operatorname{im} x\colon M \to M}$$

We'll specialize to the cases where $x = x_i = P^s_t$ for $s < t$. These give the Margolis homology groups of a spectrum X, where

$$H(H^*X, x_i) = \frac{\ker x_i\colon H^*X \to H^*X}{\operatorname{im} x_i\colon H^*X \to H^*X}$$

We often use short exact sequences to calculate Margolis homology. When one of the modules in a short exact sequence is free, we will use the following theorem:

Theorem 1.8 (Theorem 19.1 from [Mar83])**.** *If M is a free module, then $H(M, x_i) = 0$ for all i.*

We will also use the following Theorem 2.4 from [Rei17]:

Theorem 1.9. *Let M be a stably finite A-module such that $H(M, x_m) = 0$ for $m < k$, and $H(M, x_k) \neq 0$. Let $\alpha \in \operatorname{Ext}_A^{q,r}(M,M)$ be represented by a function $f\colon P_q(M) \to \Sigma^r M$. Then some power of α is a u_k-map if and only if the induced map $\bar{f} : K_{q-1}(M) \to \Sigma^r(M)$ is an isomorphism on x_k homology.*

This theorem gives us a computationally feasible way of determining whether a given element of $\operatorname{Ext}_A^{q,r}(H^*X, H^*X)$ is a u_k-map for low values of q.

1.3. Results

In this paper, I describe four complexes $Z(\beta_1, \beta_2)$, where $\beta_1, \beta_2 \in \{0,1\}$. These are candidates for complexes with a minimal number of cells supporting a low power of u_2. Using the techniques outlined above, we expect the cofiber of such a map to support a low power of v_2.

We show instead that none of these complexes support

$$u_2^1 \colon \Sigma^6 Z(\beta_1, \beta_2) \to Z(\beta_1, \beta_2)$$

and that $Z(0,0)$ does not support

$$u_2^2 \colon \Sigma^{12} Z(0,0) \to Z(0,0)$$

The main result of the paper is Theorem 5.2.

CHAPTER II

U_1 MAPS AND THE DEFINITION OF $Z(\beta_1, \beta_2)$

Davis and Mahowald show [DM81] that the complex $Y = \Sigma^{-3}\mathbb{R}P^2 \wedge \mathbb{C}P^2$ supports a self-map $v_1^1\colon \Sigma^2 Y \to Y$. Following the notation of [PS94], this corresponds to a map u_1 map $\pi \in \mathrm{Ext}_A^{1,3}(H^*Y, H^*Y)$. Recall that H^*Y has four vector space generators, which we will call $\alpha_0, \alpha_1, \alpha_2, \alpha_3$, where α_i is in degree i. H^*Y is a module over A with module structure given below. All other Sq^{2^i} are zero since they change degree by at least four. The A-module structure is pictured in Figure 2.1.

$$
\begin{aligned}
\mathrm{Sq}^1 \alpha_0 &= \alpha_1 \\
\mathrm{Sq}^2 \alpha_0 &= \alpha_2 \\
\mathrm{Sq}^2 \alpha_1 &= \alpha_3 \\
\mathrm{Sq}^1 \alpha_2 &= \alpha_3
\end{aligned}
$$

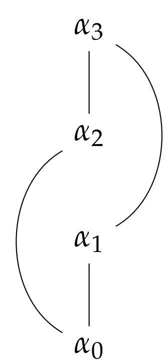


FIGURE 2.1. A-module structure of H^*Y

We will use Y and a v_1-map to build a complex that supports a u_2^i-map.

2.1. Calculation of $K_0(Y)$

We build an Adams resolution for Y as follows. We take $C_0(Y) = K(\mathbb{Z}/2)$, so that $P_0(Y) = A$. Let $Y \to K(\mathbb{Z}/2)$ be the map corresponding to α_0. Then $d_0\colon P_0(Y) \to H^*Y$ is given by $d_0(\mathrm{Sq}^0) = \alpha_0$. Clearly d_0 is surjective. As before, let $K_0 Y = \ker d_0$.

We will need describe the A-module structure of $K_0(Y)$ in a finite range to perform later computations. We note that, above dimension 3, $K_0(Y) \cong A$, since H^*Y is nonzero only in dimensions $0, 1, 2$, and 3. The A-module structure of $K_0(Y)$ through dimension 10 appears in Figure 2.2. We are using shorthand to refer to the Serre-Cartan basis elements of A. For example $(2,1)+(3)$ represents the Serre-Cartan basis element $\mathrm{Sq}^2\mathrm{Sq}^1 + \mathrm{Sq}^3 \in A$. Each arc represents an action of Sq^{2^i}; the value of i can be determined by the difference in dimension. Some Sq^4's and Sq^8's are omitted for clarity.

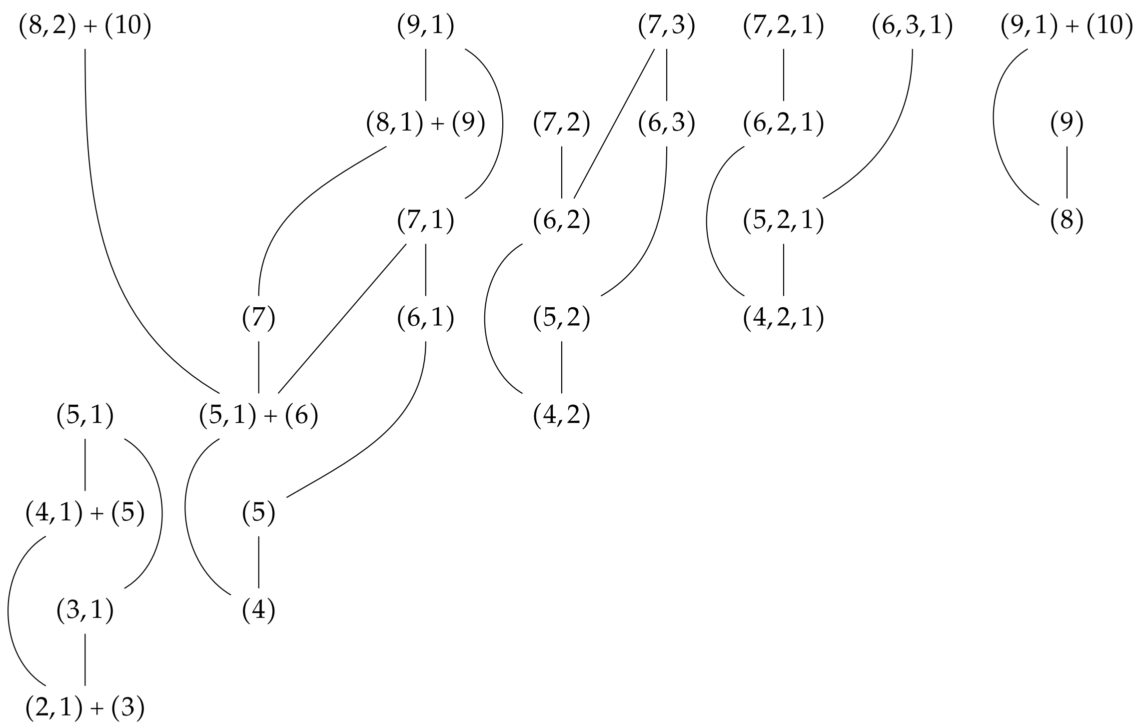


FIGURE 2.2. K_0Y through dimension 10

2.2. Identifying u_1 on Y

In the Adams resolution, we have $K_i(Y) \cong H^*\Sigma^{i+1}Y_{i+1}$. From now on, we may refer to $K_0(Y)$ as $H^*\Sigma Y_1$ (alternatively, $\Sigma^{-1}K_0(Y) = H^*Y_1$).

Now that we have $H^*\Sigma Y_1$, we look for a u_1^1-map on Y. The u_1^1-map corresponds to the $v_1^1\colon \Sigma^2 Y \to Y$ identified in [DM81]. This is represented by an element

$$[\pi] \in \mathrm{Ext}_A^{1,3}(H^*Y, H^*Y)$$

which maps to the polynomial generator $u_1 \otimes 1 \in \mathrm{Ext}^{1,3}_{E(P_2^0)}(H^*Y, H^*Y)$ under the map induced by the inclusion $E(P_2^0) \subset A$. Recall that elements of $\mathrm{Ext}_A^{1,3}(H^*Y, H^*Y)$ are represented by maps $P_1(Y) \to H^*\Sigma^3 Y$ that are cocycles.

We proceed as follows:

1. Calculate $P_1(Y)$, $P_2(Y)$ through dimension 6;
2. Find maps $P_1(Y) \to H^*\Sigma^3 Y$ that are cocycles;
3. Determine if these maps lift u_1^1.

We note that we need only to calculate $P_2(Y), P_1(Y)$ through dimension 6 because $H^*\Sigma^3 Y$ is 0 above dimension 6, so this information will be sufficient to determine whether a map is a cocycle.

Since $P_1(Y)$ must surject onto $H^*\Sigma Y_1 = K_0(Y)$, we choose

$$P_1(Y) = \Sigma^3 A \oplus \Sigma^4 A \oplus \Sigma^6 A \oplus \dots$$

through dimension 6. Denote the generator of $\Sigma^n A$ by ι_n. The map $d_1\colon P_1(Y) \to P_0(Y) = A$ is given by

$$\begin{aligned} d_1(\iota_3) &= \mathrm{Sq}^3 + \mathrm{Sq}^2\,\mathrm{Sq}^1 \\ d_1(\iota_4) &= \mathrm{Sq}^4 \\ d_1(\iota_6) &= \mathrm{Sq}^4\,\mathrm{Sq}^2 \end{aligned}$$

In order to calculate $P_2(Y)$, we must identify $\ker(d_1\colon P_1(Y) \to P_2(Y))$ and choose $P_2(Y)$ so that it maps surjectively onto this kernel. Through dimension 6, this kernel has only one element, which is $(\mathrm{Sq}^3 + \mathrm{Sq}^2\,\mathrm{Sq}^1)\iota_3$. Thus through dimension 6 we have $P_2(Y) = \Sigma^6 A$. Call the generator of this group j_6; then the map $d_2\colon P_2(Y) \to P_1(Y)$ is given by $d_2(j_6) = (\mathrm{Sq}^3 + \mathrm{Sq}^2\,\mathrm{Sq}^1)\iota_3$. This calculation of $P_1(Y)$ and $P_2(Y)$ will be sufficient to find a u_1^1 map.

Next we find maps $P_1(Y) \to H^*\Sigma^3 Y$ that are cocycles. The vector space $\mathrm{Hom}_A(P_1(Y), \Sigma^3 H^* Y)$ has three generators ϕ_3, ϕ_4, ϕ_6, defined as follows:

$$\phi_3(\iota_n) = \begin{cases} \Sigma^3\alpha_0 & \text{if } n = 3 \\ 0 & \text{if } n \neq 3 \end{cases}$$

$$\phi_4(\iota_n) = \begin{cases} \Sigma^3\alpha_1 & \text{if } n = 4 \\ 0 & \text{if } n \neq 4 \end{cases}$$

$$\phi_6(\iota_n) = \begin{cases} \Sigma^3\alpha_3 & \text{if } n = 6 \\ 0 & \text{if } n \neq 6 \end{cases}$$

Now we look at the images of these maps under

$$d_2^*\colon \mathrm{Hom}_A(P_1(Y), H^*\Sigma^3 Y) \to \mathrm{Hom}_A(P_2(Y), H^*\Sigma^3 Y).$$

The group $\mathrm{Hom}_A(P_2(Y), H^*\Sigma^3 Y)$ only has one generator, j_6, in the relevant dimensions. So it is sufficient to evaluate $d_2^*(\phi_i)(j_6)$; if this is zero, then ϕ_i is a cocycle. We have

$$\begin{aligned} d_2^*(\phi_3)(j_6) &= (\phi_3 \circ d_2)(j_6) \\ &= \phi_3((\mathrm{Sq}^3 + \mathrm{Sq}^2\,\mathrm{Sq}^1)\iota_3) \\ &= (\mathrm{Sq}^3 + \mathrm{Sq}^2\,\mathrm{Sq}^1)(\Sigma^3 \alpha_0) \\ &= \Sigma^3(\alpha_3 + \alpha_3) \\ &= 0 \end{aligned}$$

$$\begin{aligned} d_2^*(\phi_4)(j_6) &= (\phi_4 \circ d_2)(j_6) \\ &= \phi_4((\mathrm{Sq}^3 + \mathrm{Sq}^2\,\mathrm{Sq}^1)\iota_3) \\ &= (\mathrm{Sq}^3 + \mathrm{Sq}^2\,\mathrm{Sq}^1)(\Sigma^3 \alpha_1) \\ &= 0 \end{aligned}$$

$$\begin{aligned} d_2^*(\phi_6)(j_6) &= (\phi_6 \circ d_2)(j_6) \\ &= \phi_6((\mathrm{Sq}^3 + \mathrm{Sq}^2\,\mathrm{Sq}^1)\iota_3) \\ &= (\mathrm{Sq}^3 + \mathrm{Sq}^2\,\mathrm{Sq}^1)(\Sigma^3 \alpha_3) \\ &= 0 \end{aligned}$$

Thus, all of ϕ_3, ϕ_4, ϕ_6 are cocycles.

Last, we determine which linear combination of these lift $u_1^1 \otimes 1 \in \mathrm{Ext}^{1,3}_{E(P_2^0)}(H^*Y, H^*Y)$. In order to do so, we make a $E(P_2^0)$-resolution of H^*Y, which we will call $Q_\bullet(Y)$, and using that resolution, we will identify a representative for

u_1^1. Any free A-module is also free over $E(P_2^0)$, so an A-resolution of H^*Y is also an $E(P_2^0)$-resolution. Thus our resolution $P_\bullet(Y)$ is also a $E(P_2^0)$-resolution. Then we will use a map of resolutions $Q_\bullet(Y) \to P_\bullet(Y)$ to see which maps lift our representative u_1^1.

Define $Q_i(Y) = \Sigma^{3i}E(P_2^0) \otimes H^*Y$, so that $Q_\bullet(Y)$ is the minimal resolution of Y as an $E(P_2^0)$-module. More explicitly, this resolution is given by

$$\cdots \to \Sigma^6 E(P_2^0) \otimes H^*Y \xrightarrow{d_2'} \Sigma^3 E(P_2^0) \otimes H^*Y \xrightarrow{d_1'} E(P_2^0) \otimes H^*Y \xrightarrow{d_0'} H^*Y \to 0$$

where the maps are: $d_i'(\Sigma^{3i}1 \otimes \alpha_j) = \Sigma^{3(i-1)}P_2^0 \otimes \alpha_j$ for all $i \geq 1$ and $j = 0,1,2,3$. Observe that $(d')^2 = 0$.

This resolution can be used to compute that

$$\mathrm{Ext}_{E(P_2^0)}(H^*Y, H^*Y) = \mathbb{F}_2[u_1] \otimes \mathrm{Hom}_{\mathbb{F}_2}(H^*Y, H^*Y)$$

u_1 is represented by a map $p \in \mathrm{Hom}_{E(P_2^0)}(Q_1(Y), H^*\Sigma^3 Y)$ so that $p(\Sigma^3 1 \otimes \alpha_j) = \Sigma^3 \alpha_j$.

Now we compare $Q_\bullet$ to the resolution $P_\bullet$ given above. We will create a map between these two resolutions in order to identify the lift of the u_1 representative. Consider the following diagram:

$$\begin{array}{ccccccccc}
\cdots \longrightarrow & P_2(Y) & \xrightarrow{d_2} & P_1(Y) & \xrightarrow{d_1} & P_0(Y) & \xrightarrow{d_0} & H^*Y & \to 0 \\
& \uparrow f_2 & & \uparrow f_1 & & \uparrow f_0 & & \uparrow \mathrm{id} & \\
\cdots \to & \Sigma^6 E(P_2^0) \otimes H^*Y & \xrightarrow{d_2'} & \Sigma^3 E(P_2^0) \otimes H^*Y & \xrightarrow{d_1'} & E(P_2^0) \otimes H^*Y & \xrightarrow{d_0'} & H^*Y & \to 0
\end{array}$$

We now need to define the maps f_i above so that the diagram commutes and is a diagram of $E(P^0_2)$-modules. For f_0 we can choose

$$\begin{aligned} f_0(1 \otimes \alpha_0) &= \mathrm{Sq}^0 \\ f_0(1 \otimes \alpha_1) &= \mathrm{Sq}^1 \\ f_0(1 \otimes \alpha_2) &= \mathrm{Sq}^2 \\ f_0(1 \otimes \alpha_3) &= \mathrm{Sq}^3 \end{aligned}$$

The other images are determined by the $E(P^0_2)$-module structure of $E(P^0_2) \otimes P_0(Y)$, so we have

$$\begin{aligned} f_0(P^0_2 \otimes \alpha_0) &= \mathrm{Sq}^3 + \mathrm{Sq}^2 \mathrm{Sq}^1 \\ f_0(P^0_2 \otimes \alpha_1) &= \mathrm{Sq}^3 \mathrm{Sq}^1 \\ f_0(P^0_2 \otimes \alpha_2) &= \mathrm{Sq}^5 + \mathrm{Sq}^4 \mathrm{Sq}^1 \\ f_0(P^0_2 \otimes \alpha_3) &= \mathrm{Sq}^5 \mathrm{Sq}^1 \end{aligned}$$

Now we define f_1 so that the diagram commutes. We have

$$\begin{aligned} f_1(\Sigma^3 1 \otimes \alpha_0) &= \iota_3 \\ f_1(\Sigma^3 1 \otimes \alpha_1) &= \mathrm{Sq}^1 \iota_3 \\ f_1(\Sigma^3 1 \otimes \alpha_2) &= \mathrm{Sq}^2 \iota_3 \\ f_1(\Sigma^3 1 \otimes \alpha_3) &= \mathrm{Sq}^3 \iota_3 \end{aligned}$$

Again, the other images are determined by the $E(P_2^0)$-module structure of $\Sigma^3 E(P_2^0) \otimes H^*Y$.

This map $f_1 \colon Q_1(Y) \to P_1(Y)$ induces a map

$$f_1^* \colon \mathrm{Hom}_{E(P_2^0)}(P_1(Y), H^*\Sigma^3 Y) \to \mathrm{Hom}_{E(P_2^0)}(Q_1(Y), H^*\Sigma^3 Y).$$

We also have an inclusion

$$i \colon \mathrm{Hom}_A(P_1(Y), H^*\Sigma^3 Y) \subseteq \mathrm{Hom}_{E(P_2^0)}(P_1(Y), H^*\Sigma^3 Y).$$

which compose to give us a map

$$f_1^* \circ i \colon \mathrm{Hom}_A(P_1(Y), H^*\Sigma^3 Y) \to \mathrm{Hom}_{E(P_2^0)}(Q_1(Y), H^*\Sigma^3 Y).$$

An element of $\mathrm{Hom}_A(P_1(Y), H^*\Sigma^3 Y)$ represents a u_1 map if its image under this composition is homologous to the map p described above.

We have already identified three basis elements

$$\phi_3, \phi_4, \phi_6 \in \mathrm{Hom}_A(P_1(Y), H^*\Sigma^3 Y)$$

in dimensions 6 or below, all of which are cocycles and therefore potential u_1 maps. We examine the images of these under $f_1^* \circ i$. In an abuse of notation, we will use ϕ_k to refer to the image $i(\phi_k) \in \mathrm{Hom}_{E(P_2^0)}(P_1(Y), H^*\Sigma^3 Y)$ for $k = 3, 4, 6$. Thus we need to examine the images of ϕ_k under f_1^*. Calculating $f_1^*(\phi_k)$ on $\Sigma^3 1 \otimes \alpha_0$ for $k = 3, 4, 6$, we see that

$$
\begin{aligned}
f_1^*(\phi_3)(\Sigma^3 1 \otimes \alpha_0) &= (\phi_3 \circ f_1)(\Sigma^3 \otimes \alpha_0) \\
&= \phi_3(\mathrm{Sq}^i \iota_3) \\
&= \Sigma^3 \alpha_0
\end{aligned}
$$

Thus, we see that $f_1^*(\phi_3) \in \mathrm{Hom}_{E(P_2^0)}(P_1(Y), \Sigma^3 H^* Y)$ is homologous to p described above. Therefore, ϕ_3 represents a u_1 map on Y.

We also have

$$
\begin{aligned}
f_1^*(\phi_4)(\Sigma^3 1 \otimes \alpha_i) &= (\phi_4 \circ f_1)(\Sigma^3 \otimes \alpha_i) \\
&= \phi_4(\mathrm{Sq}^i \iota_3) \\
&= 0
\end{aligned}
$$

and

$$
\begin{aligned}
f_1^*(\phi_6)(\Sigma^3 1 \otimes \alpha_i) &= (\phi_6 \circ f_1)(\Sigma^3 \otimes \alpha_i) \\
&= \phi_6(\mathrm{Sq}^i \iota_3) \\
&= 0
\end{aligned}
$$

Therefore, $f_1^*(\phi_3 + \beta_1 \phi_4 + \beta_2 \phi_6) = p$ for any $\beta_1, \beta_2 \in \{0, 1\}$.

We note that each possibility for u_1 is a permanent cycle in the Adams spectral sequence: the map

$$
d_2 \colon \mathrm{Ext}_A^{1,3}(H^*Y, H^*Y) \to \mathrm{Ext}_A^{3,2}(H^*Y, H^*Y)
$$

is the zero map because $P_3(Y)$ is zero below dimension 9, and so there are no nonzero maps $P_3Y \to \Sigma^i H^*Y$ for $i < 6$.

2.3. Definition of the Complexes $Z(\beta_1, \beta_2)$

We have described different lifts of u_1^1 described above. Therefore, we have four possible algebraic u_1 maps from

$$(\phi_3 + \beta_1\phi_4 + \beta_2\phi_6)\colon H^*Y_1 \to H^*\Sigma^2 Y$$

where $H^*Y_1 = \Sigma^{-1}K_0Y$. Since these maps are permanent cycles in the Adams spectral sequence, they correspond to topological maps

$$\psi_{\beta_1,\beta_2}\colon \Sigma^2 Y \to Y_1$$

Define four complexes $Z(\beta_1, \beta_2)$ by

$$Z(\beta_1, \beta_2) = \mathrm{cof}\left(\phi_{\beta_1,\beta_2}\right)$$

for $\beta_1, \beta_2 \in \{0, 1\}$.

Then

$$H^*Z(\beta_1, \beta_2) = \ker(\phi_3 + \beta_1\phi_4 + \beta_2\phi_6)$$

since ϕ_{β_1,β_2} are onto in cohomology for each choice of $\beta_1, \beta_2 \in \{0, 1\}$.

This gives us four complexes, all of which must support a u_2^i-map for some i. We show that none of them support a u_2^1-map, and that $Z(0, 0)$ does not support a u_2^2-map either. We will also show that among these four complexes there are at least two different Steenrod algebra structures via a computation in Margolis homology.

We end this chapter with an explicit description of all four $H^*Z(\beta_1,\beta_2)$ in low degrees in Figures 2.3., 2.4., 2.5., and 2.6. The reader will find this helpful in following computations later on. The classes are named according to their images under the inclusions

$$H^*Z(\beta_1,\beta_2) \subseteq H^*Y_1 = \Sigma^{-1}K_0(Y) \subseteq A$$

but the desuspensions and Sq symbols are omitted for brevity. For example, the class labeled “5 1 + 6" in $H^*Z(0,0)$ is shorthand for $\Sigma^{-1}(\mathrm{Sq}^5\mathrm{Sq}^1 + \mathrm{Sq}^6)$.

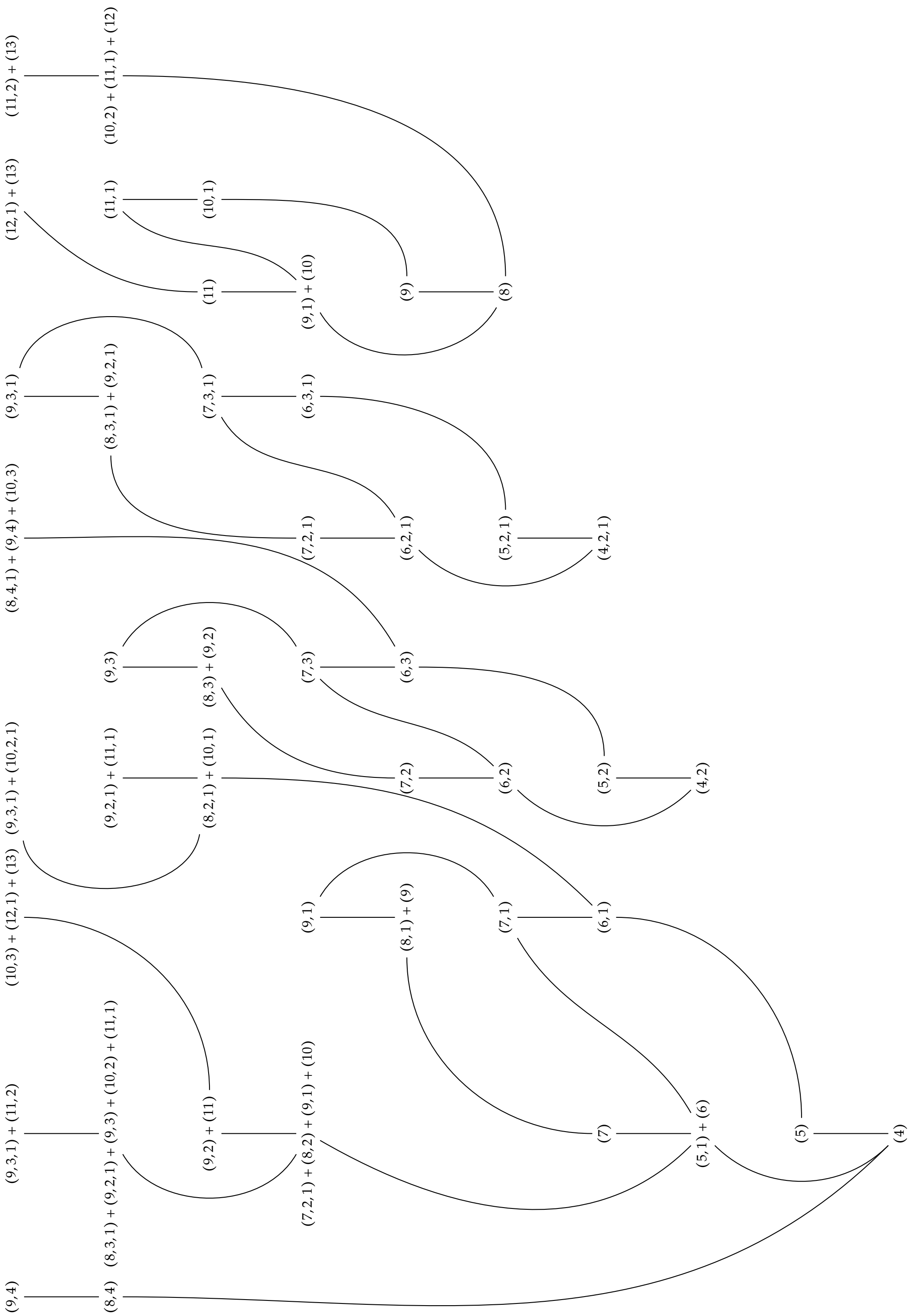


FIGURE 2.3. $H^*Z(0,0)$ through dimension 12

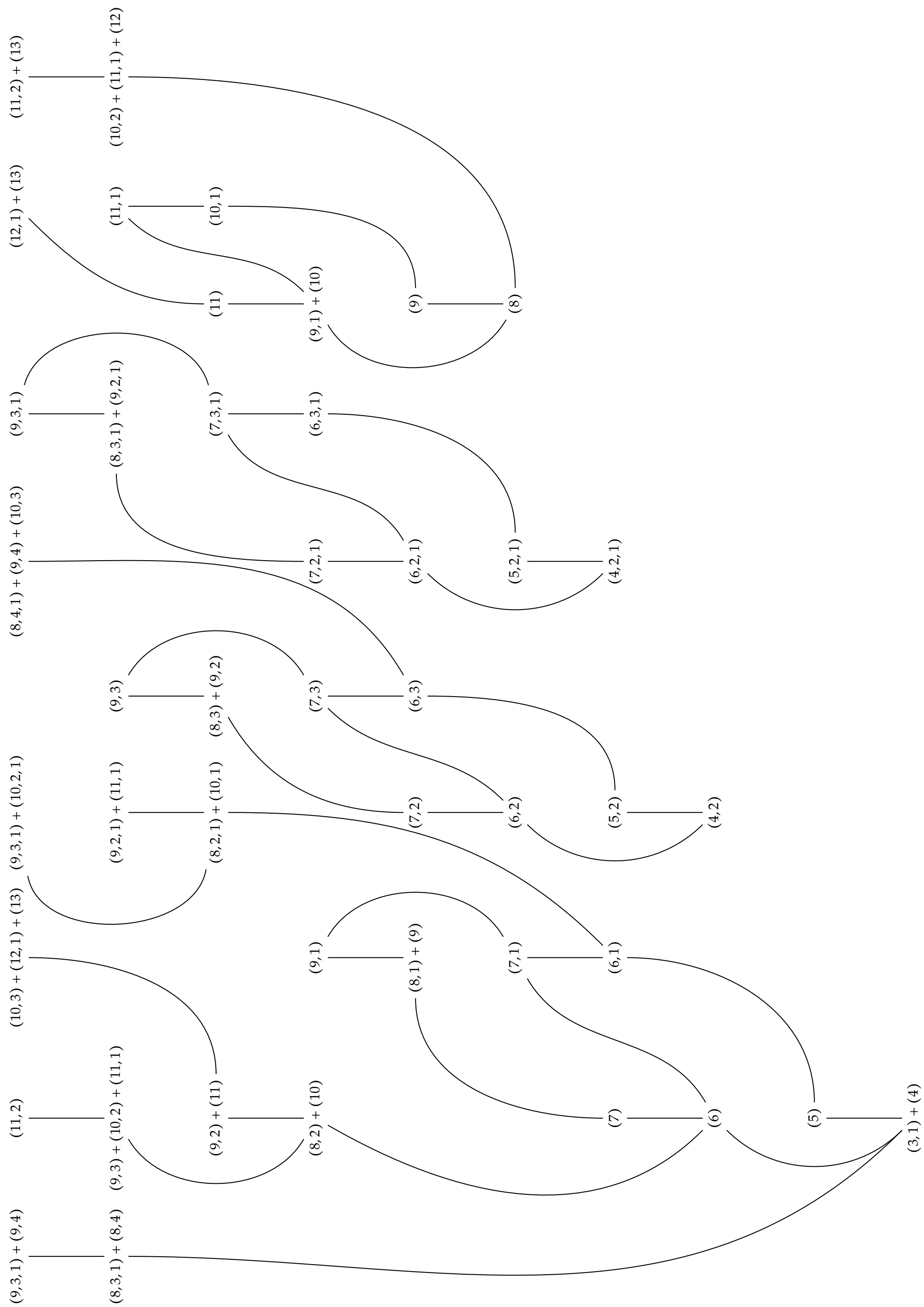


FIGURE 2.4. $H^*Z(1,0)$ through dimension 12

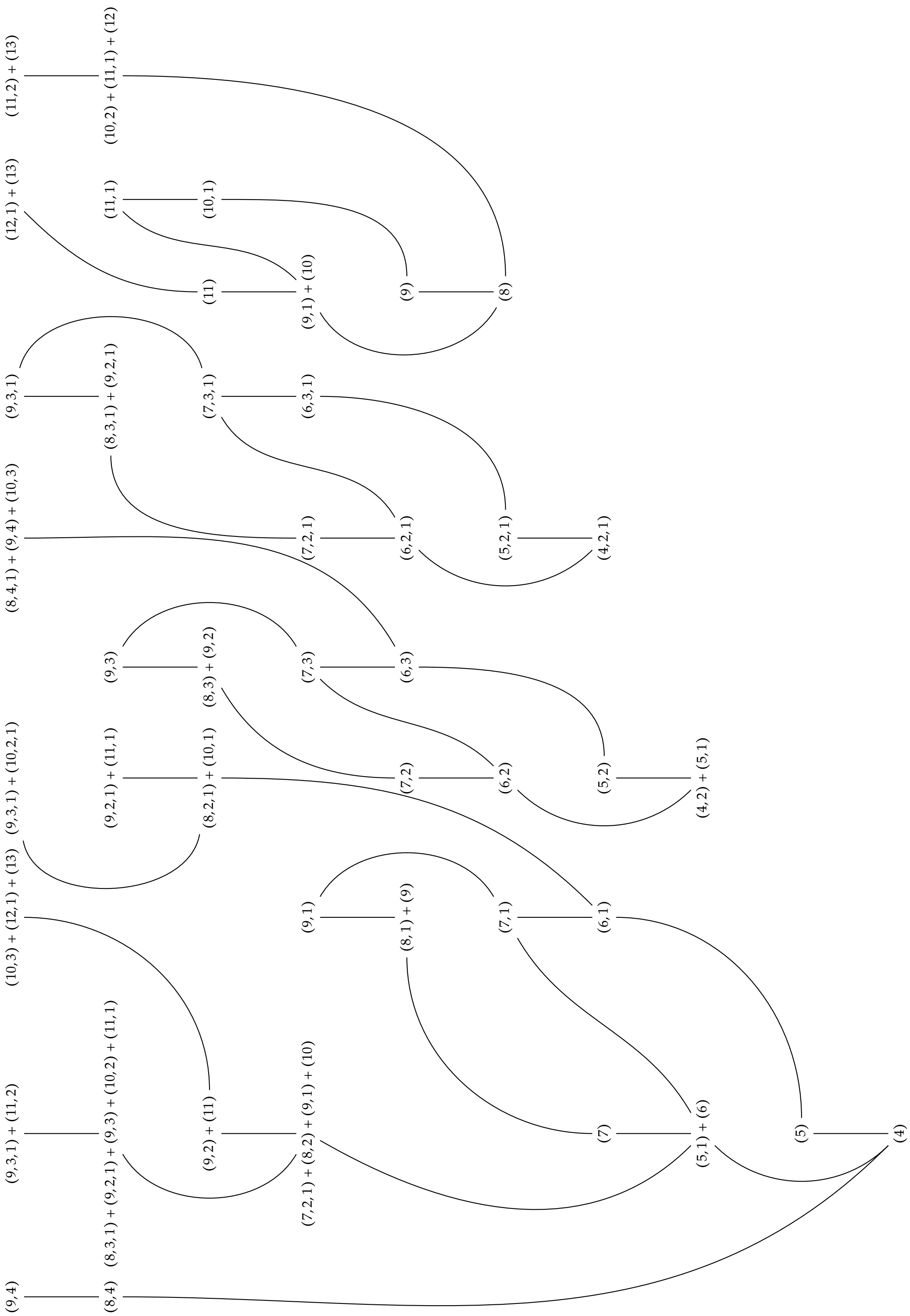


FIGURE 2.5. $H^*Z(0,1)$ through dimension 12

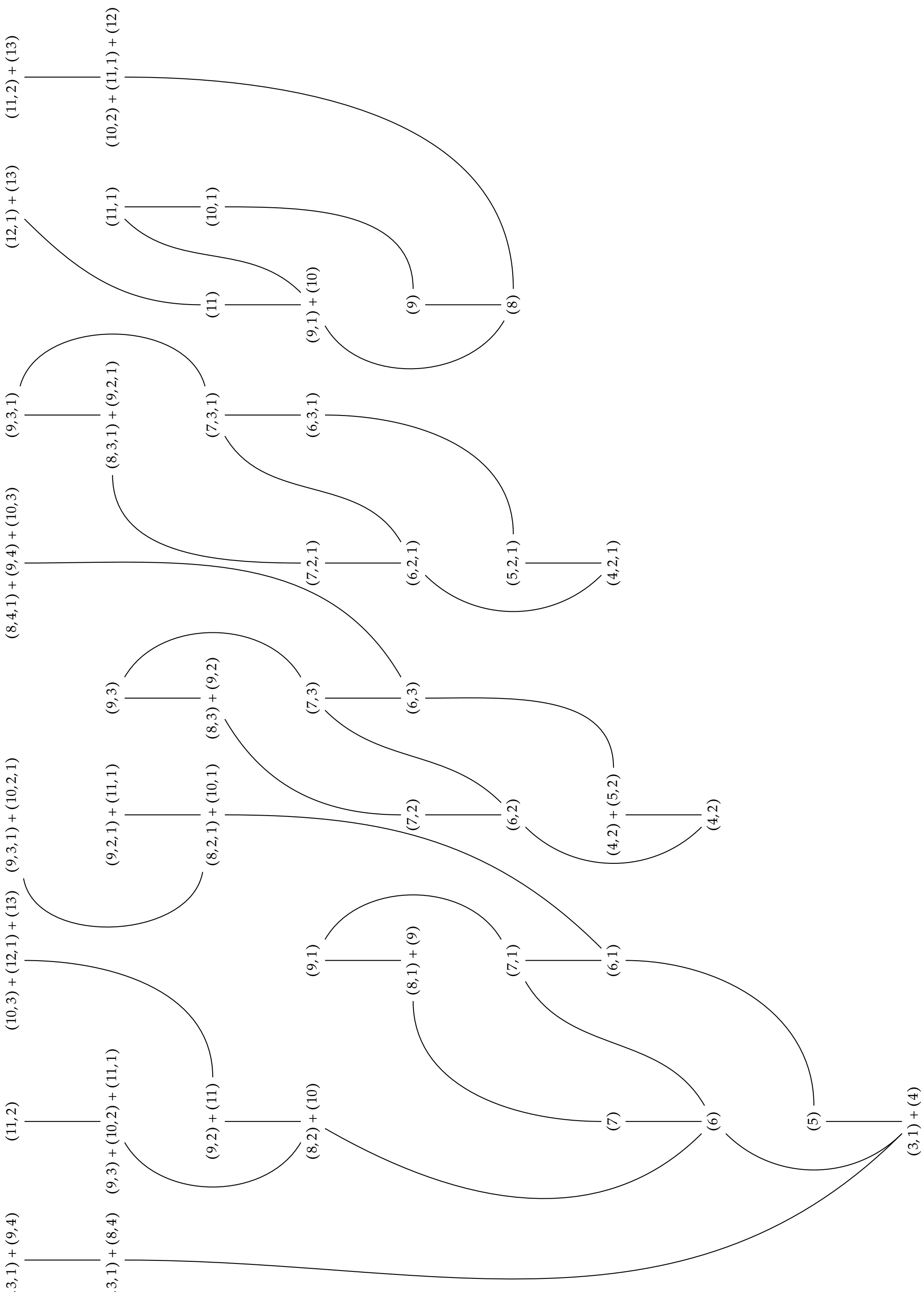


FIGURE 2.6. $H^*Z(1,1)$ through dimension 12

CHAPTER III

MARGOLIS HOMOLOGY COMPUTATIONS

In order to detect a u_2-map on $Z(\beta_1, \beta_2)$, we use the following theorem due to [Rei17]:

Theorem 3.1. *Let M be a stably finite A-module such that $H(M, x_m) = 0$ for $m < k$, and $H(M, x_k) \neq 0$. Let $\alpha \in \mathrm{Ext}_A^{q,r}(M, M)$ be represented by a function $f\colon P_q(M) \to \Sigma^r M$. Then some power of α is a u_k-map if and only if the induced map $\bar{f} : K_{q-1}(M) \to \Sigma^r(M)$ is an isomorphism on x_k homology.*

Specifically, if a map α is a u_2^1-map on $Z(\beta_1, \beta_2)$, it must induce an isomorphism

$$H(K_0 Z(\beta_1, \beta_2), x_2) \xrightarrow{\cong} H(H^* \Sigma^6 Z(\beta_1, \beta_2), x_2)$$

Therefore, we set about calculating the Margolis homology $H(-, x_2)$ of each of these groups. We will prove the following theorem:

Theorem 3.2.
1. *$H(H^* Z(0,0), x_2)$ and $H(H^* Z(1,1), x_2)$ have nonzero classes only in dimensions $5, 6, 7, 8, 9, 10, 11$. $H(K_0 Z(0,0), x_2)$ $H(K_0 Z(1,1), x_2)$ have nonzero classes only in dimensions $11, 12, 13, 14, 15, 16, 17$.*
2. *$H(H^* Z(1,0), x_2)$ and $H(H^* Z(0,1), x_2)$ have nonzero classes only in dimensions $6, 7, 8, 9, 10,$. $H(K_0 Z(1,0), x_2)$ $H(K_0 Z(0,1), x_2)$ have nonzero classes only in dimensions $12, 13, 14, 15, 16$.*

An interesting corollary of this theorem is that although the diagrams representing the Steenrod algebra structure for the four choices of $Z(\beta_1, \beta_2)$ are identical in low degrees, a relatively low Margolis differential, namely x_2, detects a difference in the Steenrod algebra structure between some of the choices for $Z(\beta_1,\beta_2)$: it can tell $Z(0,0)$ from $Z(0,1)$ and $Z(1,0)$, but it cannot distinguish $Z(0,0)$ and $Z(1,1)$, for instance.

In this proof, we will describe identify the ranks of the vector spaces of $H(H^*Z(\beta_1,\beta_2),x_2)$ and $H(K_0Z(\beta_1,\beta_2),x_2)$ for all choices of $\beta_1,\beta_2 \in \{0,1\}$. At the end of the proof we will provide explicit generators for these groups, using the fact that $H^*Z(\beta_1,\beta_2) \subseteq H^*Y_1 \subseteq \Sigma^{-1}A$.

Proof. We make use of two short exact sequences:

$$0 \to K_0(Y) \to A \to H^*Y \to 0$$

and

$$0 \to H^*Z(\beta_1,\beta_2) \to H^*Y_1 \to H^*\Sigma^2 Y \to 0$$

which are related by the fact that $\Sigma^{-1}K_0(Y) = H^*Y_1$. We will apply $H(-,x_2)$ to these sequences and use knowledge of H^*Y to determine elements of $H(H^*Z(\beta_1,\beta_2),x_2)$.

Since $x_2 = \mathrm{Sq}^6 + \mathrm{Sq}^5\mathrm{Sq}^1 + \mathrm{Sq}^4\mathrm{Sq}^2$ has dimension 6, we see that

$$H(H^*Y,x_2) = H^*Y$$

has four classes represented by α_i for $i = 0,1,2,3$. Since A is a free A-module, its Margolis homology is 0. Applying $H(-,x_2)$ to the first short exact sequence, we obtain long exact sequences of the form

$$\cdots \longrightarrow H_n(K_0(Y),x_2) \longrightarrow H_n(A,x_2) \longrightarrow H_n(H^*Y,x_2) \longrightarrow H_{n+6}(K_0(Y),x_2) \longrightarrow H_{n+6}(A,x_2) \longrightarrow H_{n+6}(H^*Y,x_2) \longrightarrow \cdots$$

which give us isomorphisms of the form $H_n(H^*Y, x_2) \cong H_{n+6}(K_0(Y), x_2)$. We can explicitly compute the classes using the connecting homomorphism. We conclude that $K_0(Y)$ has nonzero Margolis homology in dimension 6,7,8,9, and thus $\Sigma^{-1}K_0(Y) = H^*Y_1$ has Margolis homology in dimension 5,6,7,8. We also note that $H(\Sigma^2 H^*Y, x_2) \cong \Sigma^2 H^*Y$. This information is summarized in Table 3.1.

dim	H^*Y	$H^*K_0(Y)$	H^*Y_1	$H^*\Sigma^2Y$
0	α_0	0	0	0
1	α_1	0	0	0
2	α_2	0	0	$\Sigma^2\alpha_0$
3	α_3	0	0	$\Sigma^2\alpha_1$
4	0	0	0	$\Sigma^2\alpha_2$
5	0	0	$\Sigma^{-1}(\mathrm{Sq}^6 + \mathrm{Sq}^5\mathrm{Sq}^1 + \mathrm{Sq}^4\mathrm{Sq}^2)$	$\Sigma^2\alpha_3$
6	0	$\mathrm{Sq}^6 + \mathrm{Sq}^5\mathrm{Sq}^1 + \mathrm{Sq}^4\mathrm{Sq}^2$	$\Sigma^{-1}(\mathrm{Sq}^6\mathrm{Sq}^1 + \mathrm{Sq}^4\mathrm{Sq}^2\mathrm{Sq}^1)$	0
7	0	$\mathrm{Sq}^6\mathrm{Sq}^1 + \mathrm{Sq}^4\mathrm{Sq}^2\mathrm{Sq}^1$	$\Sigma^{-1}(\mathrm{Sq}^6\mathrm{Sq}^2 + \mathrm{Sq}^5\mathrm{Sq}^2\mathrm{Sq}^1)$	0
8	0	$\mathrm{Sq}^6\mathrm{Sq}^2 + \mathrm{Sq}^5\mathrm{Sq}^2\mathrm{Sq}^1$	$\Sigma^{-1}(\mathrm{Sq}^6\mathrm{Sq}^2\mathrm{Sq}^1)$	0
9	0	$\mathrm{Sq}^6\mathrm{Sq}^2\mathrm{Sq}^1$	0	0

TABLE 3.1. Margolis homology of $H^*Y, H^*K_0Y, H^*Y_1, H^*\Sigma^2Y$

Now we will apply $H(-, x_2)$ to the short exact sequence

$$0 \to H^*Z(\beta_1, \beta_2) \to H^*Y_1 \to H^*\Sigma^2Y \to 0$$

which produces long exact sequences of the form

$$\cdots \longrightarrow H_n(H^*Z(\beta_1,\beta_2),x_2) \longrightarrow H_n(H^*Y_1,x_2) \longrightarrow H_n(H^*\Sigma^2 Y,x_2) \longrightarrow$$
$$H_{n+6}(H^*Z(\beta_1,\beta_2),x_2) \longrightarrow H_{n+6}(H^*Y_1,x_2) \longrightarrow H_{n+6}(H^*\Sigma^2 Y,x_2) \longrightarrow \cdots$$

Since $H_n(H^*Y_1,x_2) = 0$ unless $n = 5,6,7,8$, and $H_n(H^*\Sigma^2 Y,x_2) = 0$ unless $n = 2,3,4,5$, we have automatically that $H_{n+6}(H^*Z(\beta_1,\beta_2),x_2) = 0$ unless $n = 5,6,7,8,9,10,11$. Below we show that in the case that $\beta_1 = \beta_2$, all of these dimensions contain nonzero Margolis homology. In the case that $\beta_1 \neq \beta_2$, we have that $H_n(H^*Z(\beta_1,\beta_2),x_2) \neq 0$ in dimensions $6,7,8,9,10$. We prove this in cases:

<u>$H^*Z(0,0)$</u>

<u>$n = 2$:</u> We have $H_2(H^*Y_1,x_2) = H_8(H^*\Sigma^2 Y,x_2) = 0$, so we have

$$0 \to H_2(H^*\Sigma^2 Y,x_2) \xrightarrow{\delta} H_8(H^*Z(0,0),x_2) \to H_8(H^*Y_1,x_2) \to 0$$

Computing the connecting homomorphism δ above shows that $\delta(\Sigma^2\alpha_0) = \Sigma^{-1}(\mathrm{Sq}^6\mathrm{Sq}^3 + \mathrm{Sq}^7\mathrm{Sq}^2 + \mathrm{Sq}^8\mathrm{Sq}^1 + \mathrm{Sq}^9)$. We have, by previous computation, that $H_8(H^*Y_1,x_2)$ is generated by $\Sigma^{-1}(\mathrm{Sq}^6\mathrm{Sq}^2\mathrm{Sq}^1)$. So these are the two generators of $H_8(H^*Z(0,0),x_2)$.

<u>$n = 3$:</u> In this case, we have an isomorphism $H_3(H^*\Sigma^2 Y,x_2) \cong H_9(H^*Z(0,0),x_2)$ since $H_3(H^*Y_1,x_2) = H_9(H^*Y_1,x_2) = 0$.

$\underline{n=4}$: Similar to $n = 3$, we have an isomorphism $H_4(H^*\Sigma^2 Y, x_2) \cong H_{10}(H^*Z(0,0), x_2)$.

$\underline{n=5}$: Since $H_{-1}(H^*\Sigma^2 Y, x_2) = 0$ and $H_{11}(H^*Y_1, x_2) = 0$, we have an exact sequence

$$0 \to H_5(H^*Z(0,0), x_2) \to H_5(H^*Y_1, x_2) \xrightarrow{f_{0,0}} H_5(H^*\Sigma^2 Y, x_2) \to H_{11}(H^*Z(0,0), x_2) \to 0$$

where $f_{0,0} = (\phi_3)_*$.

The generator of $H_5(H^*Y_1, x_2)$ is the class of $\Sigma^{-1}(\mathrm{Sq}^6 + \mathrm{Sq}^5\mathrm{Sq}^1 + \mathrm{Sq}^4\mathrm{Sq}^2)$, which maps to 0 under ϕ_3. Thus we have isomorphisms $H_5(H^*Z(0,0), x_2) \cong H_5(H^*Y_1, x_2)$ and $H_5(H^*\Sigma^2 Y, x_2) \cong H_{11}(H^*Z(0,0), x_2)$. Thus we obtain two nonzero classes in $H(H^*Z(0,0), x_2)$.

$\underline{n=6}$: Since $H_0(H^*\Sigma^2 Y, x_2) = H_6(H^*\Sigma^2 Y, x_2) = 0$, we have an isomorphism $H_6(H^*Z(0,0), x_2) \cong H_6(H^*Y_1, x_2)$.

$\underline{n=7}$: Similar to $n = 6$, we have an isomorphism $H_7(H^*Z(0,0), x_2) \cong H_7(H^*Y_1, x_2)$.

Thus, we have eight nonzero classes in $H(H^*Z(0,0), x_2)$.

$\underline{H^*Z(1,0)}$:

$\underline{n=2}$: We have $H_2(H^*Y_1, x_2) = H_8(H^*\Sigma^2 Y, x_2) = 0$, so we have

$$0 \to H_2(H^*\Sigma^2 Y, x_2) \xrightarrow{\delta} H_8(H^*Z(1,0), x_2) \to H_8(H^*Y_1, x_2) \to 0$$

Computing the connecting homomorphism δ above shows that $\delta(\Sigma^2\alpha_0) = \Sigma^{-1}(\mathrm{Sq}^6\mathrm{Sq}^3 + \mathrm{Sq}^7\mathrm{Sq}^2 + \mathrm{Sq}^8\mathrm{Sq}^1 + \mathrm{Sq}^9)$. We have, by previous computation, that $H_8(H^*Y_1, x_2)$ is generated by $\Sigma^{-1}(\mathrm{Sq}^6\mathrm{Sq}^2\mathrm{Sq}^1)$. So these are the two generators of $H_8(H^*Z(1,0), x_2)$.

$\underline{n=3}$: In this case, we have an isomorphism $H_3(H^*\Sigma^2Y, x_2) \cong H_9(H^*Z(1,0), x_2)$ since $H_3(H^*Y_1, x_2) = H_9(H^*Y_1, x_2) = 0$.

$\underline{n=4}$: Similar to $n = 3$, we have an isomorphism $H_4(H^*\Sigma^2Y, x_2) \cong H_{10}(H^*Z(1,0), x_2)$.

$\underline{n=5}$: Since $H_{-1}(H^*\Sigma^2Y, x_2) = 0$ and $H_{11}(H^*Y_1, x_2) = 0$, we have an exact sequence

$$0 \to H_5(H^*Z(1,0), x_2) \to H_5(H^*Y_1, x_2) \xrightarrow{f_{1,0}} H_5(H^*\Sigma^2Y, x_2) \to H_{11}(H^*Z(1,0), x_2) \to 0$$

where $f_{1,0} = (\phi_3 + \phi_4)_*$.

The generator of $H_5(H^*Y_1, x_2)$ is the class of $\Sigma^{-1}(\mathrm{Sq}^6 + \mathrm{Sq}^5\mathrm{Sq}^1 + \mathrm{Sq}^4\mathrm{Sq}^2)$, which maps to $\Sigma^2\alpha_3$, which generates $H_5(H^*\Sigma^2Y, x_2)$. Thus $(\phi_3 + \phi_4)_*$ is an isomorphism, and so $H_5(H^*Z(1,0), x_2) = H_{11}(H^*Z(1,0), x_2) = 0$.

$\underline{n=6}$: Since $H_0(H^*\Sigma^2Y, x_2) = H_6(H^*\Sigma^2Y, x_2) = 0$, we have an isomorphism $H_6(H^*Z(1,0), x_2) \cong H_6(H^*Y_1, x_2)$.

$\underline{n=7}$: Similar to $n = 6$, we have an isomorphism $H_7(H^*Z(1,0), x_2) \cong H_7(H^*Y_1, x_2)$.

Thus, we have six nonzero classes in $H(H^*Z(1,0), x_2)$.

$\underline{H^*Z(0,1)}$:

$\underline{n=2}$: We have $H_2(H^*Y_1, x_2) = H_8(H^*\Sigma^2 Y, x_2) = 0$, so we have

$$0 \to H_2(H^*\Sigma^2 Y, x_2) \xrightarrow{\delta} H_8(H^*Z(0,1), x_2) \to H_8(H^*Y_1, x_2) \to 0$$

Computing the connecting homomorphism δ above shows that $\delta(\Sigma^2\alpha_0) = \Sigma^{-1}(\mathrm{Sq}^6\mathrm{Sq}^3 + \mathrm{Sq}^7\mathrm{Sq}^2 + \mathrm{Sq}^8\mathrm{Sq}^1 + \mathrm{Sq}^9)$. We have, by previous computation, that $H_8(H^*Y_1, x_2)$ is generated by $\Sigma^{-1}(\mathrm{Sq}^6\mathrm{Sq}^2\mathrm{Sq}^1)$. So these are the two generators of $H_8(H^*Z(0,1), x_2)$.

$\underline{n=3}$: In this case, we have an isomorphism $H_3(H^*\Sigma^2 Y, x_2) \cong H_9(H^*Z(1,0), x_2)$ since $H_3(H^*Y_1, x_2) = H_9(H^*Y_1, x_2) = 0$.

$\underline{n=4}$: Similar to $n = 3$, we have an isomorphism $H_4(H^*\Sigma^2 Y, x_2) \cong H_{10}(H^*Z(0,1), x_2)$.

$\underline{n=5}$: Since $H_{-1}(H^*\Sigma^2 Y, x_2) = 0$ and $H_{11}(H^*Y_1, x_2) = 0$, we have an exact sequence

$$0 \to H_5(H^*Z(0,1), x_2) \to H_5(H^*Y_1, x_2) \xrightarrow{f_{0,1}} H_5(H^*\Sigma^2 Y, x_2) \to H_{11}(H^*Z(0,1), x_2) \to 0$$

where $f_{0,1} = (\phi_3 + \phi_6)_*$.

The generator of $H_5(H^*Y_1, x_2)$ is the class of $\Sigma^{-1}(\mathrm{Sq}^6 + \mathrm{Sq}^5\mathrm{Sq}^1 + \mathrm{Sq}^4\mathrm{Sq}^2)$, which maps to $\Sigma^2\alpha_3$, which generates $H_5(H^*\Sigma^2 Y, x_2)$. Thus $(\phi_3 + \phi_6)_*$ is an isomorphism, and so $H_5(H^*Z(0,1), x_2) = H_{11}(H^*Z(0,1), x_2) = 0$.

$\underline{n=6}$: Since $H_0(H^*\Sigma^2 Y, x_2) = H_6(H^*\Sigma^2 Y, x_2) = 0$, we have an isomorphism $H_6(H^*Z(0,1), x_2) \cong H_6(H^*Y_1, x_2)$.

$\underline{n=7}$: Similar to $n = 6$, we have an isomorphism $H_7(H^*Z(0,1), x_2) \cong H_7(H^*Y_1, x_2)$.

Thus, we have six nonzero classes in $H(H^*Z(1,0), x_2)$.

$\underline{H^*Z(1,1):}$

$\underline{n=2}$: We have $H_2(H^*Y_1, x_2) = H_8(H^*\Sigma^2 Y, x_2) = 0$, so we have

$$0 \to H_2(H^*\Sigma^2 Y, x_2) \xrightarrow{\delta} H_8(H^*Z(1,1), x_2) \to H_8(H^*Y_1, x_2) \to 0$$

Computing the connecting homomorphism δ above shows that $\delta(\Sigma^2\alpha_0) = \Sigma^{-1}(\mathrm{Sq}^6\mathrm{Sq}^3 + \mathrm{Sq}^7\mathrm{Sq}^2 + \mathrm{Sq}^8\mathrm{Sq}^1 + \mathrm{Sq}^9)$. We have, by previous computation, that $H_8(H^*Y_1, x_2)$ is generated by $\Sigma^{-1}(\mathrm{Sq}^6\mathrm{Sq}^2\mathrm{Sq}^1)$. So these are the two generators of $H_8(H^*Z(1,1), x_2)$.

$\underline{n=3}$: In this case, we have an isomorphism $H_3(H^*\Sigma^2 Y, x_2) \cong H_9(H^*Z(1,1), x_2)$ since $H_3(H^*Y_1, x_2) = H_9(H^*Y_1, x_2) = 0$.

$\underline{n=4}$: Similar to $n = 3$, we have an isomorphism $H_4(H^*\Sigma^2 Y, x_2) \cong H_{10}(H^*Z(1,1), x_2)$.

$\underline{n=5}$: Since $H_{-1}(H^*\Sigma^2 Y, x_2) = 0$ and $H_{11}(H^*Y_1, x_2) = 0$, we have an exact sequence

$$0 \to H_5(H^*Z(1,1), x_2) \to H_5(H^*Y_1, x_2) \xrightarrow{f_{1,1}} H_5(H^*\Sigma^2 Y, x_2) \to H_{11}(H^*Z(1,1), x_2) \to 0$$

where $f_{1,1} = (\phi_3 + \phi_4 + \phi_6)_*$.

The generator of $H_5(H^*Y_1, x_2)$ is the class of $\Sigma^{-1}(\mathrm{Sq}^6 + \mathrm{Sq}^5\mathrm{Sq}^1 + \mathrm{Sq}^4\mathrm{Sq}^2)$, which maps to 0 under $\phi_3 + \phi_4 + \phi_6$. Thus we have isomorphisms $H_5(H^*Z(1,1), x_2) \cong H_5(H^*Y_1, x_2)$ and $H_5(H^*\Sigma^2 Y, x_2) \cong H_{11}(H^*Z(1,1), x_2)$. Thus we obtain two nonzero classes in $H(H^*Z(1,1), x_2)$.

<u>$n = 6$:</u> Since $H_0(H^*\Sigma^2 Y, x_2) = H_6(H^*\Sigma^2 Y, x_2) = 0$, we have an isomorphism $H_6(H^*Z(1,1), x_2) \cong H_6(H^*Y_1, x_2)$.

<u>$n = 7$:</u> Similar to $n = 6$, we have an isomorphism $H_7(H^*Z(1,1), x_2) \cong H_7(H^*Y_1, x_2)$.

Thus, we have eight nonzero classes in $H(H^*Z(1,1), x_2)$.

All of this information is summarized in Tables 3.2. and 3.3. below.

dim	$H^*Z(0,0)$	$H^*Z(1,0)$
5	$\Sigma^{-1}(\mathrm{Sq}^6 + \mathrm{Sq}^5\mathrm{Sq}^1 + \mathrm{Sq}^4\mathrm{Sq}^2)$	0
6	$\Sigma^{-1}(\mathrm{Sq}^6\mathrm{Sq}^1 + \mathrm{Sq}^4\mathrm{Sq}^2\mathrm{Sq}^1)$	$\Sigma^{-1}(\mathrm{Sq}^6\mathrm{Sq}^1 + \mathrm{Sq}^4\mathrm{Sq}^2\mathrm{Sq}^1)$
7	$\Sigma^{-1}(\mathrm{Sq}^6\mathrm{Sq}^2 + \mathrm{Sq}^5\mathrm{Sq}^2\mathrm{Sq}^1)$	$\Sigma^{-1}(\mathrm{Sq}^6\mathrm{Sq}^2 + \mathrm{Sq}^5\mathrm{Sq}^2\mathrm{Sq}^1)$
8	$(\mathrm{Sq}^6\mathrm{Sq}^3 + \mathrm{Sq}^7\mathrm{Sq}^2 + \mathrm{Sq}^8\mathrm{Sq}^1 + \mathrm{Sq}^9)$, $\Sigma^{-1}(\mathrm{Sq}^6\mathrm{Sq}^2\mathrm{Sq}^1)$	$(\mathrm{Sq}^6\mathrm{Sq}^3 + \mathrm{Sq}^7\mathrm{Sq}^2 + \mathrm{Sq}^8\mathrm{Sq}^1 + \mathrm{Sq}^9)$, $\Sigma^{-1}(\mathrm{Sq}^6\mathrm{Sq}^2\mathrm{Sq}^1)$
9	$\Sigma^{-1}(\mathrm{Sq}^6\mathrm{Sq}^3\mathrm{Sq}^1 + \mathrm{Sq}^7\mathrm{Sq}^2\mathrm{Sq}^1 + \mathrm{Sq}^9\mathrm{Sq}^1)$	$\Sigma^{-1}(\mathrm{Sq}^6\mathrm{Sq}^3\mathrm{Sq}^1 + \mathrm{Sq}^7\mathrm{Sq}^2\mathrm{Sq}^1 + \mathrm{Sq}^9\mathrm{Sq}^1)$
10	$\Sigma^{-1}(\mathrm{Sq}^7\mathrm{Sq}^3\mathrm{Sq}^1 + \mathrm{Sq}^8\mathrm{Sq}^3 + \mathrm{Sq}^9\mathrm{Sq}^2)$	$\Sigma^{-1}(\mathrm{Sq}^7\mathrm{Sq}^3\mathrm{Sq}^1 + \mathrm{Sq}^8\mathrm{Sq}^3 + \mathrm{Sq}^9\mathrm{Sq}^2)$
11	$\Sigma^{-1}(\mathrm{Sq}^8\mathrm{Sq}^3\mathrm{Sq}^1 + \mathrm{Sq}^9\mathrm{Sq}^2\mathrm{Sq}^1)$	0

TABLE 3.2. Margolis homology of $H^*Z(0,0), H^*Z(1,0)$

dim	$H^*Z(0,1)$	$H^*Z(1,1)$
5	0	$\Sigma^{-1}(\mathrm{Sq}^6 + \mathrm{Sq}^5\mathrm{Sq}^1 + \mathrm{Sq}^4\mathrm{Sq}^2)$
6	$\Sigma^{-1}(\mathrm{Sq}^6\mathrm{Sq}^1 + \mathrm{Sq}^4\mathrm{Sq}^2\mathrm{Sq}^1)$	$\Sigma^{-1}(\mathrm{Sq}^6\mathrm{Sq}^1 + \mathrm{Sq}^4\mathrm{Sq}^2\mathrm{Sq}^1)$
7	$\Sigma^{-1}(\mathrm{Sq}^6\mathrm{Sq}^2 + \mathrm{Sq}^5\mathrm{Sq}^2\mathrm{Sq}^1)$	$\Sigma^{-1}(\mathrm{Sq}^6\mathrm{Sq}^2 + \mathrm{Sq}^5\mathrm{Sq}^2\mathrm{Sq}^1)$
8	$(\mathrm{Sq}^6\mathrm{Sq}^3 + \mathrm{Sq}^7\mathrm{Sq}^2 + \mathrm{Sq}^8\mathrm{Sq}^1 + \mathrm{Sq}^9)$, $\Sigma^{-1}(\mathrm{Sq}^6\mathrm{Sq}^2\mathrm{Sq}^1)$	$(\mathrm{Sq}^6\mathrm{Sq}^3 + \mathrm{Sq}^7\mathrm{Sq}^2 + \mathrm{Sq}^8\mathrm{Sq}^1 + \mathrm{Sq}^9)$, $\Sigma^{-1}(\mathrm{Sq}^6\mathrm{Sq}^2\mathrm{Sq}^1)$
9	$\Sigma^{-1}(\mathrm{Sq}^6\mathrm{Sq}^3\mathrm{Sq}^1 + \mathrm{Sq}^7\mathrm{Sq}^2\mathrm{Sq}^1 + \mathrm{Sq}^9\mathrm{Sq}^1)$	$\Sigma^{-1}(\mathrm{Sq}^6\mathrm{Sq}^3\mathrm{Sq}^1 + \mathrm{Sq}^7\mathrm{Sq}^2\mathrm{Sq}^1 + \mathrm{Sq}^9\mathrm{Sq}^1)$
10	$\Sigma^{-1}(\mathrm{Sq}^7\mathrm{Sq}^3\mathrm{Sq}^1 + \mathrm{Sq}^8\mathrm{Sq}^3 + \mathrm{Sq}^9\mathrm{Sq}^2)$	$\Sigma^{-1}(\mathrm{Sq}^7\mathrm{Sq}^3\mathrm{Sq}^1 + \mathrm{Sq}^8\mathrm{Sq}^3 + \mathrm{Sq}^9\mathrm{Sq}^2)$
11	0	$\Sigma^{-1}(\mathrm{Sq}^8\mathrm{Sq}^3\mathrm{Sq}^1 + \mathrm{Sq}^9\mathrm{Sq}^2\mathrm{Sq}^1)$

TABLE 3.3. Margolis homology of $H^*Z(0,1), H^*Z(1,1)$

Thus we have proven that $H(H^*Z(0,0), x_2)$ and $H(H^*Z(1,1), x_2)$ have nonzero classes only in dimensions $5,6,7,8,9,10,11$, and $H(H^*Z(1,0), x_2)$ and $H(H^*Z(0,1), x_2)$ have nonzero classes only in dimensions $6,7,8,9,10$. This shows that there are at least two different A-module structures amongst the various choices of $Z(\beta_1, \beta_2)$.

To prove the second parts of the statements, we use the fact that

$$K_0Z(\beta_1,\beta_2) = \ker(d_0\colon P_0Z(\beta_1,\beta_2) \to H^*Z(\beta_1,\beta_2))$$

where $P_0Z(\beta_1,\beta_2)$ is the 0^{th} stage of a free resolution of $Z(\beta_1,\beta_2)$. Thus we have a short exact sequence

$$0 \to K_0Z(\beta_1,\beta_2) \to P_0Z(\beta_1,\beta_2) \to H^*Z(\beta_1,\beta_2) \to 0$$

for all choices of β_1, β_2. Since $P_0 Z(\beta_1, \beta_2)$ is free, its Margolis homology is trivial. Thus, by applying $H(-, x_2)$ to the short exact sequence above, we have isomorphisms

$$H_n(H^* Z(\beta_1, \beta_2), x_2) \cong H_{n+6}(K_0 Z(\beta_1, \beta_2), x_2)$$

induced by the connecting homomorphisms in the long exact sequence. This proves the theorem.

□

Now that we know the dimensions in which Margolis homology of $H^* Z(\beta_1, \beta_2)$ and $K_0 Z(\beta_1, \beta_2)$ and nonzero, we will begin to search for elements of

$$\mathrm{Ext}_A^{1,6}(H^* Z(\beta_1, \beta_2), H^* Z(\beta_1, \beta_2))$$

whose representatives

$$f\colon P_1 Z(\beta_1, \beta_2) \to H^* \Sigma^6 Z(\beta_1, \beta_2)$$

induce an isomorphism on Margolis homology. This requires us to calculate the beginning of a projective resolution for each $H^* Z(\beta_1, \beta_2)$. As we will see, the computations are quite similar.

CHAPTER IV

PROJECTIVE RESOLUTIONS FOR $H^*Z(\beta_1,\beta_2)$

In this chapter we explicitly compute the first two stages in a projective resolution for $H^*Z(\beta_1,\beta_2)$ for $\beta_1,\beta_2 \in \{0,1\}$.

For $P_0Z(\beta_1,\beta_2)$, we see from the A-module structure that we can choose

$$P_0Z(\beta_1,\beta_2) = \Sigma^3 A \oplus \Sigma^5 A \oplus \Sigma^6 A \oplus \Sigma^7 A \oplus \bigoplus_{l_k>12} \Sigma^{l_k} A$$

for any $\beta_1,\beta_2 \in \{0,1\}$. Here the last summand stands for the rest of the summands of $P_0Z(\beta_1,\beta_2)$ which have generators in dimensions above 12. Let i_k stand for the generator of $\Sigma^k A$ in $P_0Z(\beta_1,\beta_2)$. We will only need this for $k \leq 12$ where it is unambiguous.

We will compute

$$K_0Z(\beta_1,\beta_2) = \ker(d_0 \colon P_0Z(\beta_1,\beta_2) \to H^*Z(\beta_1,\beta_2)$$

for all choices of β_1 and β_2, and then we will show that

$$P_1Z(\beta_1,\beta_2) = \Sigma^7 A \oplus \Sigma^9 A \oplus \Sigma^{10} A \oplus \Sigma^{12} A \oplus \bigoplus_{l_k>12} \Sigma^{l_k} A$$

for $\beta_1,\beta_2 \in \{0,1\}$. Again, all of the generators beyond those listed will occur above dimension 12.

These partial calculations of $P_\bullet Z(\beta_1,\beta_2)$ will be sufficient to compute

$$\mathrm{Ext}_A^{1,6}(H^*Z(\beta_1,\beta_2), H^*Z(\beta_1,\beta_2)).$$

4.1. $K_0Z(0,0)$ and $P_1Z(0,0)$

Define $d_0\colon P_0Z(0,0) \to H^*Z(0,0)$

$$\begin{aligned} d_0(i_3) &= \Sigma^{-1}\mathrm{Sq}^4 \\ d_0(i_5) &= \Sigma^{-1}\mathrm{Sq}^4\mathrm{Sq}^2 \\ d_0(i_6) &= \Sigma^{-1}\mathrm{Sq}^4\mathrm{Sq}^2\mathrm{Sq}^1 \\ d_0(i_7) &= \Sigma^{-1}\mathrm{Sq}^8 \end{aligned}$$

It is clear that d_0 is surjective through dimension 12 by an examination of Figure 2.3.

We calculate $K_0Z(0,0) = \ker(d_0\colon P_0Z(0,0) \to H^*Z(0,0))$ in Table 4.1. This table describes the inputs and outputs of d_0 by dimension. In the rightmost column, we list possible generators of $K_0Z(0,0)$ in that dimension. The names of these generators are chosen to assist in the calculation of $P_1Z(0,0)$, which follows later. Again, we use shorthand to refer to elements of $P_0Z(0,0)$ and $H^*Z(0,0)$. So for example, $(1)i_3$ means $\mathrm{Sq}^1 i_3$ in $P_0Z(0,0)$, and (5) means $\Sigma^{-1}\mathrm{Sq}^5 \in H^*Z(0,0)$. We use the word "zero" to refer to the zero element of $P_0Z(0,0)$ to avoid confusion with the element Sq^0.

TABLE 4.1. Calculation of $K_0Z(0,0)$ through dimension 12

dimension	input	output	generator of $K_0Z(0,0)$
3	i_3	(4)	
4	$(1)i_3$	(5)	
5	$(2)i_3$	$(5,1)+(6)$	
	i_5	$(4,2)$	

TABLE 4.1. (continued)

dimension	input	output	generator of $K_0Z(0,0)$
6	$(7)i_3$	(7)	
	$(2,1)i_3$	$(6,1)$	
	$(1)i_5$	$(5,2)$	
	i_6	$(4,2,1)$	
7	$(3,1)i_3$	$(7,1)$	$[(3,1)+(4)]i_3+(2)i_5$
	$(4)i_3$	$(6,2)+(7,1)$	
	$(2)i_5$	$(6,2)$	
	$(1)i_6$	$(5,2,1)$	
	i_7	8	
8	$(5)i_3$	$(7,2)$	$(5)i_3+(3)i_5$
	$(4,1)i_3$	$(7,2)+(8,1)+(9)$	
	$(3)i_5$	$(7,2)$	
	$(2,1)i_5$	$(6,3)$	
	$(2)i_6$	$(6,2,1)$	
	$(1)i_7$	(9)	
9	$(6)i_3$	$(7,3)$	$(6)i_3+(3,1)i_5$
	$(5,1)i_3$	$(9,1)$	$[(3,1)+(4)]i_5+(2,1)i_6$
	$(4,2)i_3$	$(7,2,1)+(8,2)+(9,1)+(10)$	
	$(4)i_5$	$(6,3,1)+(7,3)$	
	$(3,1)i_5$	$(7,3)$	
	$(3)i_6$	$(7,2,1)$	
	$(2,1)i_6$	$(6,3,1)$	

TABLE 4.1. (continued)

dimension	input	output	generator of $K_0Z(0,0)$
	$(2)i_7$	$(9,1)+(10)$	
10	$(7)i_3$	zero	$(7)i_3$
	$(6,1)i_3$	$(8,3)+(9,2)$	$(6,1)i_3+[(4,1)+(5)]i_5$
	$(5,2)i_3$	$(9,2)+(11)$	$(5)i_5+(3,1)i_6$
	$(4,2,1)i_3$	$(8,2,1)+(10,1)$	$[(3,1)+(4)]i_6$
	$(5)i_5$	$(7,3,1)$	
	$(4,1)i_5$	$(7,3,1)+(8,3)+(9,2)$	
	$(4)i_6$	$(7,3,1)$	
	$(3,1)i_6$	$(7,3,1)$	
	$(3)i_7$	(11)	
	$(2,1)i_7$	$(10,1)$	
11	$(8)i_3$	$(8,4)$	$(7,1)i_3+(5,1)i_5$
	$(7,1)i_3$	$(9,3)$	$[(5,2,1)+(6,2)+(7,1)]i_3$
	$(6,2)i_3$	$(8,3,1)+(9,2,1)+(9,3)$	$+(4,2)i_5$
		$+(10,2)+(11,1)$	
	$(5,2,1)i_3$	$(9,2,1)+(11,1)$	$(5)i_6$
	$(6)i_5$	zero	$(6)i_5$
	$(5,1)i_5$	$(9,3)$	
	$(4,2)i_5$	$(8,3,1)+(10,2)$	
	$(5)i_6$	zero	
	$(4,1)i_6$	$(8,3,1)+(9,2,1)$	
	$(4)i_7$	$(10,2)+(11,1)+(12)$	

TABLE 4.1. (continued)

dimension	input	output	generator of $K_0Z(0,0)$
	$(3,1)i_7$	$(11,1)$	
12	$(9)i_3$	$(9,4)$	$[(8,1)+(9)]i_3$
	$(8,1)i_3$	$(9,4)$	$(7,2)i_3+(5,2)i_5$
	$(7,2)i_3$	$(9,3,1)+(11,2)$	$(7)i_5$
	$(6,3)i_3$	$(10,3)+(12,1)+(13)$	$(6,1)i_5+(5,1)i_6$
	$(6,2,1)i_3$	$(9,3,1)+(10,2,1)$	$(6)i_6$
	$(7)i_5$	zero	$(6,2,1)i_3+[(4,2)$
	$(6,1)i_5$	$(9,3,1)$	$+(5,1)]i_6$
	$(5,2)i_5$	$(9,3,1)+(11,2)$	
	$(4,2,1)i_5$	$(8,4,1)+(9,4)+(10,3)$	
	$(6)i_6$	zero	
	$(5,1)i_6$	$(9,3,1)$	
	$(4,2)i_6$	$(10,2,1)$	
	$(5)i_7$	$(11,2)+(13)$	
	$(4,1)i_7$	$(11,2)+(12,1)$	

The A-module structure of $K_0Z(0,0)$ through dimension 12 is pictured in Figure 4.1. The only Sq^{2^n} missing in this range is $\mathrm{Sq}^4(\mathrm{Sq}^5\, i_3+\mathrm{Sq}^3\, i_5) = (\mathrm{Sq}^7\,\mathrm{Sq}^2+\mathrm{Sq}^8\,\mathrm{Sq}^1+\mathrm{Sq}^9)i_3+\mathrm{Sq}^5\,\mathrm{Sq}^2\, i_5$.

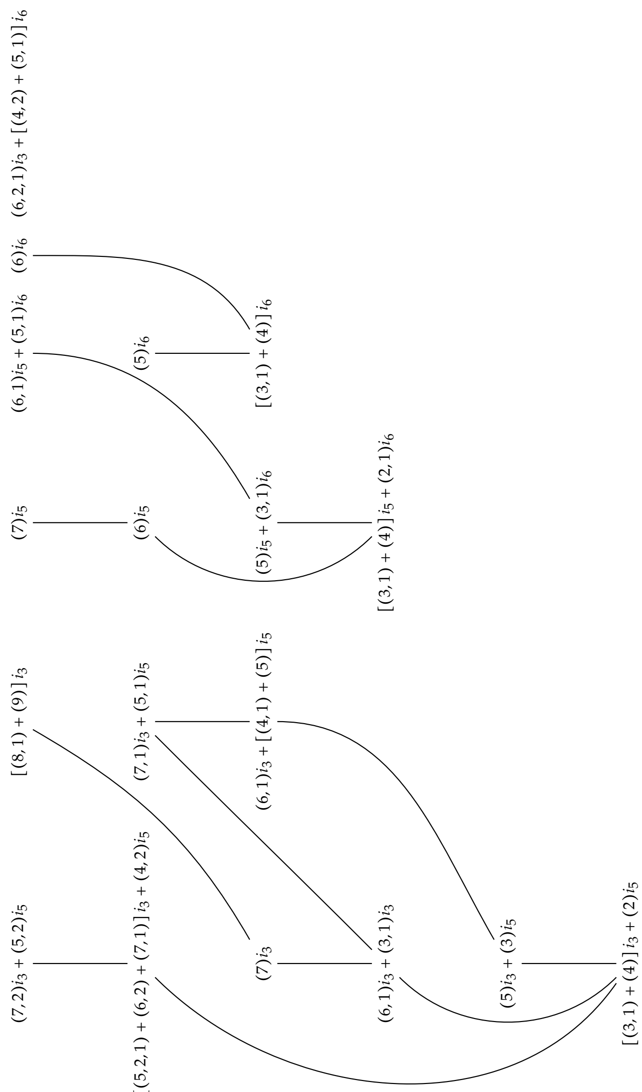


FIGURE 4.1. $K_0Z(0,0)$ through dimension 12

From this, we see that we can choose

$$P_1Z(0,0) = \Sigma^7 A \oplus \Sigma^9 A \oplus \Sigma^{10} A \oplus \Sigma^{12} A \oplus \bigoplus_{l_k > 12} \Sigma^{l_k} A$$

Let h_k denote the generator of $\Sigma^k A$ in $P_1 Z(0,0)$. We define $d_1\colon P_1 Z(0,0) \to P_0 Z(0,0)$ by

$$\begin{aligned} d_1(h_7) &= (\mathrm{Sq}^3\,\mathrm{Sq}^1 + \mathrm{Sq}^4) i_3 + \mathrm{Sq}^2\, i_5 \\ d_1(h_9) &= (\mathrm{Sq}^3\,\mathrm{Sq}^1 + \mathrm{Sq}^4) i_5 + \mathrm{Sq}^2\,\mathrm{Sq}^1\, i_6 \\ d_1(h_{10}) &= (\mathrm{Sq}^3\,\mathrm{Sq}^1 + \mathrm{Sq}^4) i_6 \\ d_1(h_{12}) &= \mathrm{Sq}^6\,\mathrm{Sq}^2\,\mathrm{Sq}^1\, i_3 + (\mathrm{Sq}^4\,\mathrm{Sq}^2 + \mathrm{Sq}^5\,\mathrm{Sq}^1) i_6 \end{aligned}$$

4.2. $K_0 Z(1,0)$ **and** $P_1 Z(1,0)$

Define $d_0\colon P_0 Z(1,0) \to H^* Z(1,0)$

$$\begin{aligned} d_0(i_3) &= \Sigma^{-1}\,\mathrm{Sq}^4 \\ d_0(i_5) &= \Sigma^{-1}(\mathrm{Sq}^4\,\mathrm{Sq}^2 + \mathrm{Sq}^5\,\mathrm{Sq}^1) \\ d_0(i_6) &= \Sigma^{-1}\,\mathrm{Sq}^4\,\mathrm{Sq}^2\,\mathrm{Sq}^1 \\ d_0(i_7) &= \Sigma^{-1}\,\mathrm{Sq}^8 \end{aligned}$$

It is clear that d_0 is surjective through dimension 12 by an examination of Figure 2.4.

Next, we will calculate $K_0 Z(1,0) = \ker(d_0\colon P_0 Z(1,0) \to H^* Z(1,0))$. We will make a table similar to that in Section 4.1:

TABLE 4.2. Calculation of $K_0 Z(1,0)$ through dimension 12

dimension	input	output	generator of $K_0 Z(1,0)$
3	i_3	$(4)+(3,1)$	

TABLE 4.2. (continued)

dimension	input	output	generator of $K_0 Z(1,0)$
4	$(1)i_3$	(5)	
5	$(2)i_3$	(6)	
	i_5	$(4,2)$	
6	$(3)i_3$	(7)	
	$(2,1)i_3$	$(6,1)$	
	$(1)i_5$	$(5,2)$	
	i_6	$(4,2,1)$	
7	$(4)i_3$	$(5,2,1)+(6,2)+(7,1)$	$[(3,1)+(4)]i_3+(2)i_5+(1)i_6$
	$(3,1)i_3$	$(7,1)$	
	$(2)i_5$	$(6,2)$	
	$(1)i_6$	$(5,2,1)$	
	i_7	(8)	
8	$(5)i_3$	$(7,2)$	$(5)i_3+(3)i_5$
	$(4,1)i_3$	$(7,2)+(8,1)+(9)$	
	$(3)i_5$	$(7,2)$	
	$(2,1)i_5$	$(6,3)$	
	$(2)i_6$	$(6,2,1)$	
	$(1)i_7$	(9)	
9	$(6)i_3$	$(6,3,1)+(7,3)$	$(6)i_3+(3,1)i_5+(2,1)i_6$
	$(5,1)i_3$	$(9,1)$	
	$(4,2)i_3$	$(8,2)+(10)$	$[(3,1)+(4)]i_5+(2,1)i_6$
	$(4)i_5$	$(6,3,1)+(7,3)$	

TABLE 4.2. (continued)

dimension	input	output	generator of $K_0Z(1,0)$
	$(3,1)i_5$	$(7,3)$	
	$(3)i_6$	$(7,2,1)$	
	$(2,1)i_6$	$(6,3,1)$	
	$(2)i_7$	$(9,1)+(10)$	
10	$(7)i_3$	$(7,3,1)$	$(7)i_3+(3,1)i_6$
	$(6,1)i_3$	$(8,3)+(9,2)$	$(6,1)i_3+[(4,1)+(5)]i_5$
	$(5,2)i_3$	$(9,2)+(11)$	
	$(4,2,1)i_3$	$(8,2,1)+(10,1)$	
	$(5)i_5$	$(7,3,1)$	$(5)i_5+(3,1)i_6$
	$(4,1)i_5$	$(7,3,1)+(8,3)+(9,2)$	$[(3,1)+(4)]i_6$
	$(4)i_6$	$(7,3,1)$	
	$(3,1)i_6$	$(7,3,1)$	
	$(3)i_7$	(11)	
	$(2,1)i_7$	$(10,1)$	
11	$(8)i_3$	$(8,4)+(8,3,1)$	
	$(7,1)i_3$	$(9,3)$	$(7,1)i_3+(5,1)i_5$
	$(6,2)i_3$	$(9,3)+(10,2)+(11,1)$	
	$(5,2,1)i_3$	$(11,2,1)$	$[(5,2,1)+(6,2)+(7,1)]i_3$
11	$(6)i_5$	zero	$+(4,2)i_5+(4,1)i_6$
	$(5,1)i_5$	$(9,3)$	$(6)i_5$
	$(4,2)i_5$	$(8,3,1)+(10,2)$	
	$(5)i_6$	zero	$(5)i_6$

TABLE 4.2. (continued)

dimension	input	output	generator of $K_0Z(1,0)$
	$(4,1)i_6$	$(8,3,1)+(9,2,1)$	
	$(4)i_7$	$(10,2)+(11,1)+(12)$	
	$(3,1)i_7$	$(11,1)$	
12	$(9)i_3$	$(9,4)+(9,3,1)$	$[(8,1)+(9)]i_3$
	$(8,1)i_3$	$(9,4)$	
	$(7,2)i_3$	$(11,2)$	$(7,2)i_3+(5,2)i_5+(5,1)i_6$
	$(6,3)i_3$	$(10,3)+(12,1)+(13)$	
	$(6,2,1)i_3$	$(9,3,1)+(10,2,1)$	
	$(7)i_5$	zero	$(7)i_5$
	$(6,1)i_5$	$(9,3,1)$	$(6,1)i_5+(5,1)i_6$
	$(5,2)i_5$	$(9,3,1)+(11,2)$	
	$(4,2,1)i_5$	$(8,4,1)+(9,4)+(10,3)$	
	$(6)i_6$	zero	$(6)i_6$
	$(5,1)i_6$	$(9,3,1)$	
	$(4,2)i_6$	$(10,2,1)$	$(6,2,1)i_3+[(4,2)+(5,1)]i_6$
	$(5)i_7$	$(11,2)+(13)$	
	$(4,1)i_7$	$(11,2)+(12,1)$	

The A-module structure of $K_0Z(1,0)$ through dimension 12 is pictured in Figure 4.2. The only Sq^{2^n} missing in this range is $\mathrm{Sq}^4(\mathrm{Sq}^5 i_3+\mathrm{Sq}^3 i_5) = (\mathrm{Sq}^7\mathrm{Sq}^2+\mathrm{Sq}^8\mathrm{Sq}^1+\mathrm{Sq}^9)i_3+\mathrm{Sq}^5\mathrm{Sq}^2 i_5$.

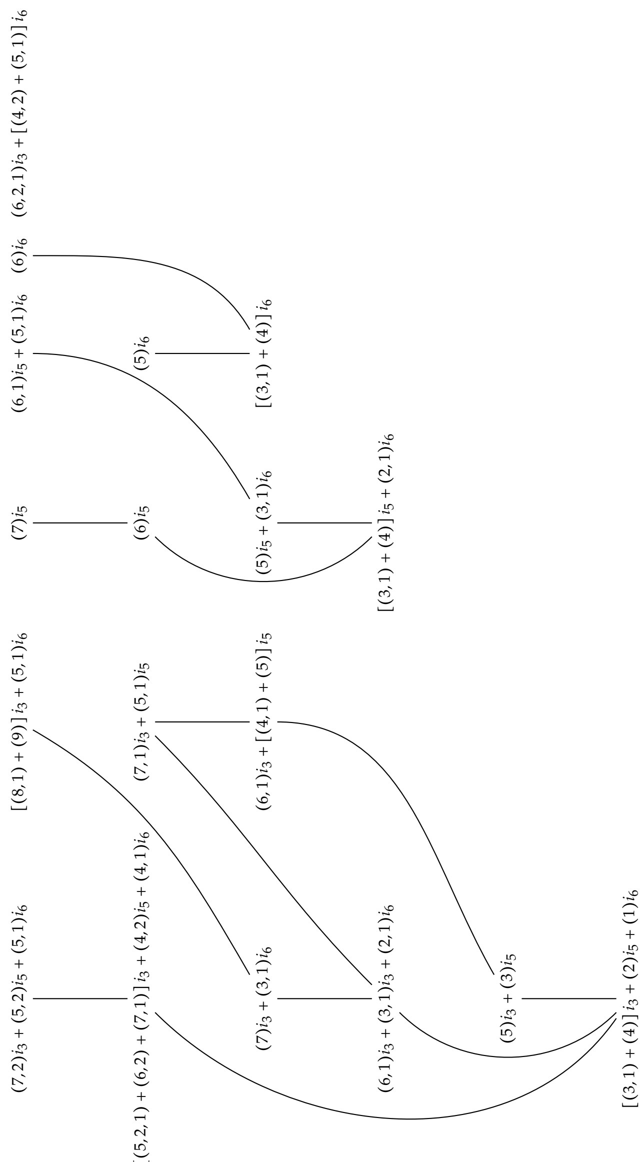


FIGURE 4.2. $K_0 Z(1,0)$ through dimension 12

From this, we see that we can choose

$$P_1 Z(1,0) = \Sigma^7 A \oplus \Sigma^9 A \oplus \Sigma^{10} A \oplus \Sigma^{12} A \oplus \bigoplus_{l_k > 12} \Sigma^{l_k} A$$

Let h_k denote the generator of $\Sigma^k A$ in $P_1Z(1,0)$. We define $d_1\colon P_1Z(1,0) \to P_0Z(1,0)$ by

$$
\begin{aligned}
d_1(h_7) &= (\mathrm{Sq}^3\,\mathrm{Sq}^1 + \mathrm{Sq}^4)i_3 + \mathrm{Sq}^2\, i_5 + 1i_6\\
d_1(h_9) &= (\mathrm{Sq}^3\,\mathrm{Sq}^1 + \mathrm{Sq}^4)i_5 + \mathrm{Sq}^2\,\mathrm{Sq}^1\, i_6\\
d_1(h_{10}) &= (\mathrm{Sq}^3\,\mathrm{Sq}^1 + \mathrm{Sq}^4)i_6\\
d_1(h_{12}) &= \mathrm{Sq}^6\,\mathrm{Sq}^2\,\mathrm{Sq}^1\, i_3 + (\mathrm{Sq}^4\,\mathrm{Sq}^2 + \mathrm{Sq}^5\,\mathrm{Sq}^1)i_6
\end{aligned}
$$

4.3. $K_0Z(0,1)$ **and** $P_1Z(0,1)$

Define $d_0\colon P_0Z(0,1) \to H^*Z(0,1)$

$$
\begin{aligned}
d_0(i_3) &= \Sigma^{-1}\,\mathrm{Sq}^4\\
d_0(i_5) &= \Sigma^{-1}(\mathrm{Sq}^4\,\mathrm{Sq}^2 + \mathrm{Sq}^5\,\mathrm{Sq}^1)\\
d_0(i_6) &= \Sigma^{-1}\,\mathrm{Sq}^4\,\mathrm{Sq}^2\,\mathrm{Sq}^1\\
d_0(i_7) &= \Sigma^{-1}\,\mathrm{Sq}^8
\end{aligned}
$$

It is clear that d_0 is surjective through dimension 12 by an examination of Figure 2.5.

Next, we will calculate $K_0Z(0,1) = \ker(d_0\colon P_0Z(0,1) \to H^*Z(0,1))$. We will make a table similar to that in Section 4.1:

TABLE 4.3. Calculation of $K_0Z(0,1)$ through dimension 12

dimension	input	output	generator of $K_0Z(0,1)$
3	i_3	(4)	

TABLE 4.3. (continued)

dimension	input	output	generator of $K_0Z(0,1)$
4	$(1)i_3$	(5)	
5	$(2)i_3$	$(5,1)+(6)$	
	i_5	$(4,2)+(5,1)$	
6	$(3)i_3$	(7)	
	$(2,1)i_3$	$(6,1)$	
	$(1)i_5$	$(5,2)$	
	i_6	$(4,2,1)$	
7	$(4)i_3$	$(6,2)+(7,1)$	$[(3,1)+(4)]i_3+(2)i_5$
	$(3,1)i_3$	$(7,1)$	
	$(2)i_5$	$(6,2)$	
	$(1)i_6$	$(5,2,1)$	
	i_7	(8)	
8	$(5)i_3$	$(7,2)$	$(5)i_3+(3)i_5$
	$(4,1)i_3$	$(7,2)+(8,1)+(9)$	
	$(3)i_5$	$(7,2)$	
	$(2,1)i_5$	$(6,3)$	
	$(2)i_6$	$(6,2,1)$	
	$(1)i_7$	(9)	
9	$(6)i_3$	$(7,3)$	$(6)i_3+(3,1)i_5$
	$(5,1)i_3$	$(9,1)$	
	$(4,2)i_3$	$(7,2,1)+(9,1)+(8,2)+(10)$	$(5,1)i_3+[(3,1)+(4)]i_5$
	$(4)i_5$	$(6,3,1)+(7,3)+$	$+[(2,1)+(3)]i_6$

TABLE 4.3. (continued)

dimension	input	output	generator of $K_0Z(0,1)$
		$(7,2,1)+(9,1)$	
	$(3,1)i_5$	$(7,3)$	
	$(3)i_6$	$(7,2,1)$	
	$(2,1)i_6$	$(6,3,1)$	
	$(2)i_7$	$(9,1)+(10)$	
10	$(7)i_3$	zero	$7i_3$
	$(6,1)i_3$	$(8,3)+(9,2)$	$(6,1)i_3+[(4,1)+(5)]i_5$
	$(5,2)i_3$	$(9,2)+(11)$	
	$(4,2,1)i_3$	$(8,2,1)+(10,1)$	
	$(5)i_5$	$(7,3,1)$	$(5)i_5+(3,1)i_6$
	$(4,1)i_5$	$(7,3,1)+(8,3)+(9,2)$	$[(3,1)+(4)]i_6$
	$(4)i_6$	$(7,3,1)$	
	$(3,1)i_6$	$(7,3,1)$	
	$(3)i_7$	(11)	
	$(2,1)i_7$	$(10,1)$	
11	$(8)i_3$	$(8,4)$	
	$(7,1)i_3$	$(9,3)$	$(7,1)i_3+(5,1)i_3$
	$(6,2)i_3$	$(8,3,1)+(9,2,1)+$	
		$(9,3)+(10,2)+(11,1)$	
	$(5,2,1)i_3$	$(9,2,1)+(11,1)$	$[(5,2,1)+(6,2)+(7,1)]i_3$
	$(6)i_5$	$(8,3,1)+(9,2,1)$	$+(4,2)i_5$
	$(5,1)i_5$	$(9,3)$	

TABLE 4.3. (continued)

dimension	input	output	generator of $K_0Z(0,1)$
	$(4,2)i_5$	$(8,3,1)+(10,2)$	
	$(5)i_6$	zero	$(5)i_6$
	$(4,1)i_6$	$(8,3,1)+(9,2,1)$	$(6)i_5+[(4,1)+(5)]i_6$
	$(4)i_7$	$(10,2)+(11,1)+(12)$	
	$(3,1)i_7$	$(11,1)$	
12	$(9)i_3$	$(9,4)$	$[(8,1)+(9)]i_3$
	$(8,1)i_3$	$(9,4)$	
	$(7,2)i_3$	$(9,3,1)+(11,2)$	$(7,2)i_3+(5,2)i_5$
	$(6,3)i_3$	$(10,3)+(12,1)+(13)$	
	$(6,2,1)i_3$	$(9,3,1)+(10,2,1)$	
	$(7)i_5$	$(9,3,1)$	$(7)i_5+(5,1)i_6$
	$(6,1)i_5$	$(9,3,1)$	$(6,1)i_5+(5,1)i_6$
	$(5,2)i_5$	$(9,3,1)+(11,2)$	
	$(4,2,1)i_5$	$(8,4,1)+(9,4)+(10,3)$	
	$(6)i_6$	zero	$(6)i_6$
	$(5,1)i_6$	$(9,3,1)$	
	$(4,2)i_6$	$(10,2,1)$	$(6,2,1)i_3+[(4,2)+(5,1)]i_6$
	$(5)i_7$	$(11,2)+(13)$	
	$(4,1)i_7$	$(11,2)+(12,1)$	

The A-module structure of $K_0Z(0,1)$ through dimension 12 is pictured in Figure 4.3. The only Sq^{2^n} missing in this range is $\mathrm{Sq}^4(\mathrm{Sq}^5 i_3 + \mathrm{Sq}^3 i_5) = (\mathrm{Sq}^7\mathrm{Sq}^2 + \mathrm{Sq}^8\mathrm{Sq}^1 + \mathrm{Sq}^9)i_3 + \mathrm{Sq}^5\mathrm{Sq}^2 i_5$.

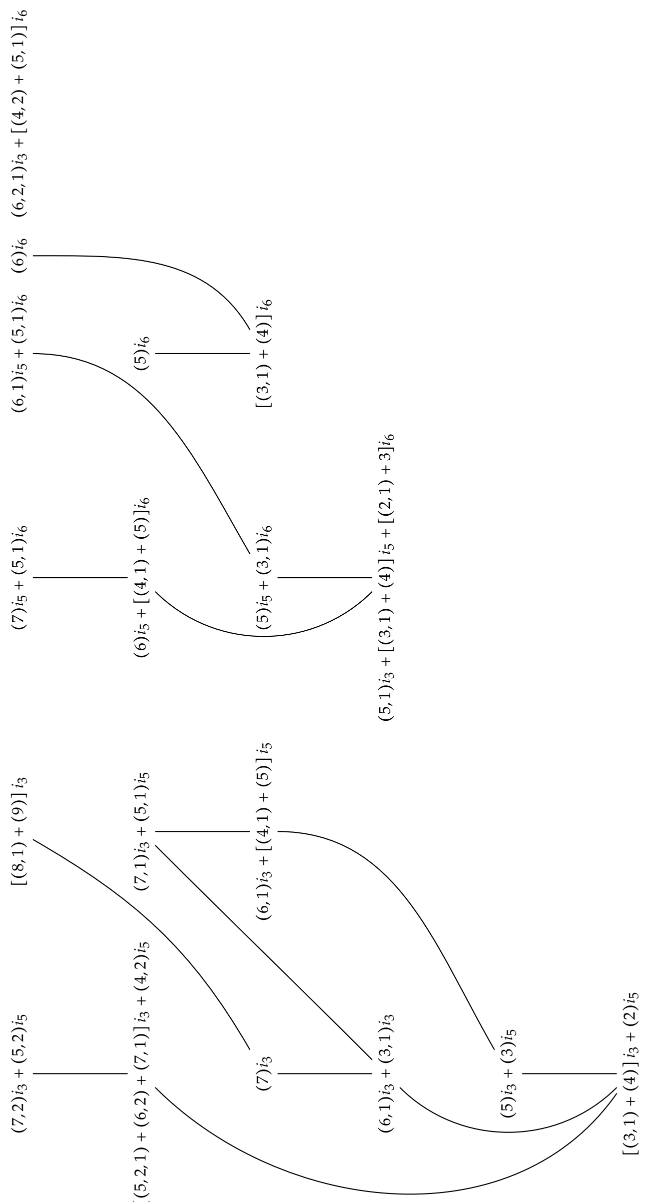


FIGURE 4.3. $K_0Z(0,1)$ through dimension 12

From this, we see that we can choose

$$P_1Z(0,1) = \Sigma^7 A \oplus \Sigma^9 A \oplus \Sigma^{10} A \oplus \Sigma^{12} A \oplus \bigoplus_{l_k > 12} \Sigma^{l_k} A$$

Let h_k denote the generator of $\Sigma^k A$ in $P_1Z(0,1)$. We define $d_1\colon P_1Z(0,1) \to P_0Z(0,1)$ by

$$\begin{aligned}
d_1(h_7) &= (\mathrm{Sq}^3\,\mathrm{Sq}^1 + \mathrm{Sq}^4)i_3 + \mathrm{Sq}^2\, i_5 \\
d_1(h_9) &= \mathrm{Sq}^5\,\mathrm{Sq}^1\, i_3 + (\mathrm{Sq}^3\,\mathrm{Sq}^1 + \mathrm{Sq}^4)i_5 + (\mathrm{Sq}^2\,\mathrm{Sq}^1 + \mathrm{Sq}^3)i_6 \\
d_1(h_{10}) &= (\mathrm{Sq}^3\,\mathrm{Sq}^1 + \mathrm{Sq}^4)i_6 \\
d_1(h_{12}) &= \mathrm{Sq}^6\,\mathrm{Sq}^2\,\mathrm{Sq}^1\, i_3 + (\mathrm{Sq}^4\,\mathrm{Sq}^2 + \mathrm{Sq}^5\,\mathrm{Sq}^1)i_6
\end{aligned}$$

4.4. $K_0Z(1,1)$ **and** $P_1Z(1,1)$

Define $d_0\colon P_0Z(1,1) \to H^*Z(1,1)$

$$\begin{aligned}
d_0(i_3) &= \Sigma^{-1}\,\mathrm{Sq}^4 + \mathrm{Sq}^3\,\mathrm{Sq}^1 \\
d_0(i_5) &= \Sigma^{-1}(\mathrm{Sq}^4\,\mathrm{Sq}^2 + \mathrm{Sq}^5\,\mathrm{Sq}^1) \\
d_0(i_6) &= \Sigma^{-1}\,\mathrm{Sq}^4\,\mathrm{Sq}^2\,\mathrm{Sq}^1 \\
d_0(i_7) &= \Sigma^{-1}\,\mathrm{Sq}^8
\end{aligned}$$

It is clear that d_0 is surjective through dimension 12 by an examination of Figure 2.6.

Next, we will calculate $K_0Z(1,1) = \ker(d_0\colon P_0Z(1,1) \to H^*Z(1,1))$. We will make a table similar to that in Section 4.1:

TABLE 4.4. Calculation of $K_0Z(1,1)$ through dimension 12

dimension	input	output	generator of $K_0Z(1,1)$
3	i_3	$(4)+(3,1)$	
4	$(1)i_3$	(5)	
5	$(2)i_3$	(6)	
	i_5	$(4,2)+(5,1)$	
6	$(3)i_3$	(7)	
	$(2,1)i_3$	$(6,1)$	
	$(1)i_5$	$(5,2)$	
	i_6	$(4,2,1)$	
7	$(4)i_3$	$(5,2,1)+(6,2)+(7,1)$	$[(3,1)+(4)]i_3+(2)i_5+1i_6$
	$(3,1)i_3$	$(7,1)$	
	$(2)i_5$	$(6,2)$	
	$(1)i_6$	$(5,2,1)$	
	i_7	(8)	
8	$(5)i_3$	$(7,2)$	$(5)i_3+(3)i_5$
	$(4,1)i_3$	$(7,2)+(8,1)+(9)$	
	$(3)i_5$	$(7,2)$	
	$(2,1)i_5$	$(6,3)$	
	$(2)i_6$	$(6,2,1)$	
	$(1)i_7$	(9)	
9	$(6)i_3$	$(6,3,1)+(7,3)$	$(6)i_3+(3,1)i_5+(2,1)i_6$
	$(5,1)i_3$	$(9,1)$	
	$(4,2)i_3$	$(8,2)+(10)$	$(5,1)i_3+[(3,1)+(4)]i_5$

TABLE 4.4. (continued)

dimension	input	output	generator of $K_0Z(1,1)$
	$(4)i_5$	$(6,3,1)+(7,2,1)+$	$+[(2,1)+(3)]i_6$
		$(7,3)+(9,1)$	
	$(3,1)i_5$	$(7,3)$	
	$(3)i_6$	$(7,2,1)$	
	$(2,1)i_6$	$(6,3,1)$	
	$(2)i_7$	$(9,1)+(10)$	
10	$(7)i_3$	$(7,3,1)$	$(7)i_3+(3,1)i_6$
	$(6,1)i_3$	$(8,3)+(9,2)$	$(6,1)i_3+[(4,1)+(5)]i_5$
	$(5,2)i_3$	$(9,2)+(11)$	
	$(4,2,1)i_3$	$(8,2,1)+(10,1)$	
	$(5)i_5$	$(7,3,1)$	$(5)i_5+(3,1)i_6$
	$(4,1)i_5$	$(7,3,1)+(8,3)+(9,2)$	$[(3,1)+(4)]i_6$
	$(4)i_6$	$(7,3,1)$	
	$(3,1)i_6$	$(7,3,1)$	
	$(3)i_7$	(11)	
	$(2,1)i_7$	$(10,1)$	
11	$(8)i_3$	$(8,4)+(8,3,1)$	
	$(7,1)i_3$	$(9,3)$	$(7,1)i_3+(5,1)i_3$
	$(6,2)i_3$	$(9,3)+(10,2)+(11,1)$	
	$(5,2,1)i_3$	$(9,2,1)+(11,1)$	$[(5,2,1)+(6,2)+(7,1)]i_3$
	$(6)i_5$	$(8,3,1)+(9,2,1)$	$+(4,2)i_5+(4,1)i_6$
	$(5,1)i_5$	$(9,3)$	$(6)i_5+[(4,1)+(5)]i_6$

TABLE 4.4. (continued)

dimension	input	output	generator of $K_0Z(1,1)$
	$(4,2)i_5$	$(8,3,1)+(10,2)$	
	$(5)i_6$	zero	$(5)i_6$
	$(4,1)i_6$	$(8,3,1)+(9,2,1)$	
	$(4)i_7$	$(10,2)+(11,1)+(12)$	
	$(3,1)i_7$	$(11,1)$	
12	$(9)i_3$	$(9,4)+(9,3,1)$	$[(8,1)+(9)]i_3+(5,1)i_6$
	$(8,1)i_3$	$(9,4)$	
	$(7,2)i_3$	$(11,2)$	$(7,2)i_3+(5,2)i_5+(5,1)i_6$
	$(6,3)i_3$	$(10,3)+(12,1)+(13)$	
	$(6,2,1)i_3$	$(9,3,1)+(10,2,1)$	
	$(7)i_5$	$(9,3,1)$	$(7)i_5+(5,1)i_6$
	$(6,1)i_5$	$(9,3,1)$	$(6,1)i_5+(5,1)i_6$
	$(5,2)i_5$	$(9,3,1)+(11,2)$	
	$(4,2,1)i_5$	$(8,4,1)+(9,4)+(10,3)$	
	$(6)i_6$	zero	$(6)i_6$
	$(5,1)i_6$	$(9,3,1)$	
	$(4,2)i_6$	$(10,2,1)$	$(6,2,1)i_3+[(4,2)+(5,1)]i_6$
	$(5)i_7$	$(11,2)+(13)$	
	$(4,1)i_7$	$(11,2)+(12,1)$	

The A-module structure of $K_0Z(1,1)$ through dimension 12 is pictured in Figure 4.4. The only Sq^{2^n} missing in this range is $Sq^4(Sq^5 i_3 + Sq^3 i_5) = (Sq^7 Sq^2 + Sq^8 Sq^1 + Sq^9)i_3 + Sq^5 Sq^2 i_5$.

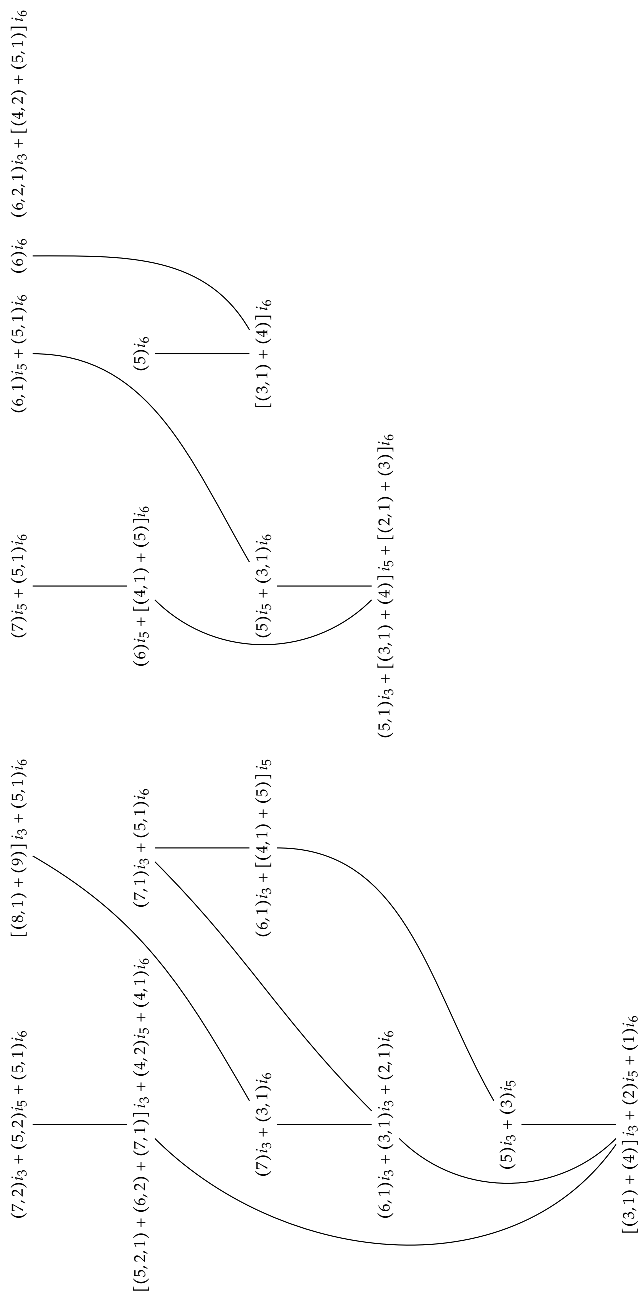


FIGURE 4.4. $K_0 Z(1,1)$ through dimension 12

From this, we see that we can choose

$$P_1 Z(1,1) = \Sigma^7 A \oplus \Sigma^9 A \oplus \Sigma^{10} A \oplus \Sigma^{12} A \oplus \bigoplus_{l_k > 12} \Sigma^{l_k} A$$

Let h_k denote the generator of $\Sigma^k A$ in $P_1 Z(1,1)$. We define $d_1 \colon P_1 Z(1,1) \to P_0 Z(1,1)$ by

$$\begin{aligned}
d_1(h_7) &= (\mathrm{Sq}^3\,\mathrm{Sq}^1 + \mathrm{Sq}^4) i_3 + \mathrm{Sq}^2\, i_5 + \mathrm{Sq}^1\, i_6 \\
d_1(h_9) &= \mathrm{Sq}^5\,\mathrm{Sq}^1\, i_3 + (\mathrm{Sq}^3\,\mathrm{Sq}^1 + \mathrm{Sq}^4) i_5 + (\mathrm{Sq}^2\,\mathrm{Sq}^1 + \mathrm{Sq}^3) i_6 \\
d_1(h_{10}) &= (\mathrm{Sq}^3\,\mathrm{Sq}^1 + \mathrm{Sq}^4) i_6 \\
d_1(h_{12}) &= \mathrm{Sq}^6\,\mathrm{Sq}^2\,\mathrm{Sq}^1\, i_3 + (\mathrm{Sq}^4\,\mathrm{Sq}^2 + \mathrm{Sq}^5\,\mathrm{Sq}^1) i_6
\end{aligned}$$

CHAPTER V

NONEXISTENCE OF U_2^1 ON $Z(\beta_1, \beta_2)$

We will now show that none of the complexes $Z(\beta_1, \beta_2)$ support a u_2^1-map. To do so, we will show that

$$\mathrm{Ext}_A^{1,6}(H^*Z(\beta_1, \beta_2), H^*Z(\beta_1, \beta_2)) = \mathbb{F}_2$$

for each choice of β_1, β_2. Then we will show that the generator of these groups, which is represented by a map

$$f\colon P_1Z(\beta_1, \beta_2) \to H^*\Sigma^6 Z(\beta_1, \beta_2)$$

cannot induce an isomorphism

$$H(K_0Z(\beta_1, \beta_2), x_2) \to H(H^*\Sigma^6 Z(\beta_1, \beta_2), x_2)$$

and therefore cannot be a u_2^1-map.

We first prove a lemma:

Lemma 5.1. *1.* $\mathrm{Ext}_A^{0,6}(H^*Z(\beta_1, \beta_2), \Sigma^{-1}A) = 0$

2. $\mathrm{Ext}_A^{1,6}(H^*Z(\beta_1, \beta_2), \Sigma^{-1}A) = 0$

Proof. 1. By Theorem 13.3.12 in [Mar83], A is self-injective. Therefore, if we apply $\mathrm{Hom}_A(-, \Sigma^5 A)$ to the short exact sequence

$$0 \to \Sigma^{-1}K_0(Y) \to \Sigma^{-1}A \to H^*\Sigma^{-1}Y \to 0$$

we get

$$0 \leftarrow \mathrm{Hom}_A(\Sigma^{-1}K_0(Y), \Sigma^5 A) \leftarrow \mathrm{Hom}_A(\Sigma^{-1}A, \Sigma^5 A)$$

$$\leftarrow \mathrm{Hom}_A(\Sigma^{-1}H^*Y, \Sigma^5 A) \leftarrow 0$$

The rightmost group is zero for dimensional reasons, so we have

$$\mathrm{Hom}_A(\Sigma^{-1}K_0(Y), \Sigma^5 A) \cong \mathrm{Hom}_A(\Sigma^{-1}A, \Sigma^5 A).$$

But the only degree 0 A-module map $\Sigma^{-1}A \to \Sigma^5 A$ is the zero map, so both of these groups are also zero.

Now, apply $\mathrm{Hom}_A(-, \Sigma^5 A)$ to

$$0 \to H^*Z(\beta_1, \beta_2) \to H^*Y_1 \to H^*\Sigma^2 Y \to 0$$

to get

$$0 \leftarrow \mathrm{Hom}_A(H^*Z(\beta_1, \beta_2), \Sigma^5 A) \leftarrow \mathrm{Hom}_A(H^*Y_1, \Sigma^5 A)$$

$$\leftarrow \mathrm{Hom}_A(H^*\Sigma^2 Y, \Sigma^5 A) \leftarrow 0$$

Recall that $H^*Y_1 = \Sigma^{-1}K_0(Y)$, and so the middle group is 0. Thus $\mathrm{Hom}_A(H^*Z(\beta_1, \beta_2), \Sigma^5 A) = \mathrm{Ext}_A^{0,6}(H^*Z(\beta_1, \beta_2), \Sigma^{-1}A) = 0$.

2. This follows directly from the self-injectivity of A (Theorem 13.3.12 in [Mar83]).

□

We will use this to prove the following:

Theorem 5.2.

$$\mathrm{Ext}_A^{1,6}(H^*Z(\beta_1,\beta_2),H^*Z(\beta_1,\beta_2)) = \mathbb{F}_2$$

for each choice of $\beta_1,\beta_2 \in \{0,1\}$*, and the only nonzero element of this group is not a* u_2^1*-map.*

The proof will be in four cases corresponding to the choices of β_1 and β_2, though the format of each case is quite similar. The general strategy used in each case is the following:

1. Apply the functor $\mathrm{Ext}_A^{*,6}(H^*Z(\beta_1,\beta_2),-)$ to the short exact sequences

$$0 \to H^*Y_1 \to \Sigma^{-1}A \to H^*\Sigma^{-1}Y \to 0$$

   and

$$0 \to H^*Z(\beta_1,\beta_2) \to H^*Y_1 \to H^*\Sigma^2 Y \to 0$$

2. Use the resulting long exact sequences to show that

$$\mathrm{Hom}_A(H^*Z(\beta_1,\beta_2),\Sigma^5 H^*Y) = \mathbb{F}_2 \oplus \mathbb{F}_2$$

3. Compute the connecting homomorphism

$$\delta_0\colon \mathrm{Hom}_A(H^*Z(\beta_1,\beta_2),\Sigma^5 H^*Y) \xrightarrow{\cong} \mathrm{Ext}_A^{1,6}(H^*Z(\beta_1,\beta_2),H^*Y_1)$$

   to identify two generators of $\mathrm{Ext}_A^{1,6}(H^*Z(\beta_1,\beta_2),H^*Y_1)$.

4. Examine the maps $\iota_*, \phi_* = (\phi_3 + \beta_1\phi_4 + \beta_2\phi_6)_*$ in the exact sequence below:

$$0 \to \mathrm{Ext}_A^{1,6}(H^*Z(\beta_1,\beta_2), H^*Z(\beta_1,\beta_2)) \xrightarrow{\iota_*} \mathrm{Ext}_A^{1,6} H^*Z(\beta_1,\beta_2), H^*Y_1)$$

$$\xrightarrow{\phi_*} \mathrm{Ext}_A^{1,6}(H^*Z(\beta_1,\beta_2), H^*\Sigma^2 Y) \to \dots$$

and show that, of the two generators in the previous step, one is not in $\ker(\phi_*)$, so it does not pull back to $\mathrm{Ext}_A^{1,6}(H^*Z(\beta_1,\beta_2), \mathrm{Ext}_A^{1,6}(H^*Z(\beta_1,\beta_2))$. The other generator does pull back to $\mathrm{Ext}_A^{1,6}(H^*Z(\beta_1,\beta_2), \mathrm{Ext}_A^{1,6}(H^*Z(\beta_1,\beta_2))$, but is the zero map through dimension 12, so in no case can it induce the desired isomorphism in Margolis homology. Thus, it cannot be a u_2^1-map.

Proof. $\underline{\beta_1 = 0, \beta_2 = 0}$:

We apply $\mathrm{Ext}_A^n(H^*Z(0,0), -)$ to the short exact sequence

$$0 \to H^*Y_1 \to \Sigma^{-1}A \to H^*\Sigma^{-1}Y \to 0$$

to get the long exact sequence

$$\begin{array}{l} 0 \to \mathrm{Ext}_A^{0,6}(H^*Z(0,0), H^*Y_1) \to \mathrm{Ext}_A^{0,6}(H^*Z(0,0), \Sigma^{-1}A) \to \mathrm{Ext}_A^{0,6}(H^*Z(0,0), \Sigma^{-1}H^*Y) \xrightarrow{\delta_0} \\ \mathrm{Ext}_A^{1,6}(H^*Z(0,0), H^*Y_1) \to \mathrm{Ext}_A^{1,6}(H^*Z(0,0), \Sigma^{-1}A) \to \mathrm{Ext}_A^{1,6}(H^*Z(0,0), \Sigma^{-1}H^*Y) \to \dots \end{array}$$

By Lemma 5.1, we have

$$\mathrm{Ext}_A^{0,6}(H^*Z(0,0),\Sigma^{-1}A) = \mathrm{Ext}_A^{1,6}(H^*Z(0,0),\Sigma^{-1}A) = 0.$$

Therefore we have an isomorphism

$$\mathrm{Ext}_A^{0,6}(H^*Z(0,0),\Sigma^{-1}H^*Y) \cong \mathrm{Ext}_A^{1,6}(H^*Z(0,0),H^*Y_1)$$

Lemma 5.3.

$$\mathrm{Ext}_A^{0,6}(H^*Z(0,0),\Sigma^{-1}H^*Y) \cong \mathrm{Hom}_A(H^*Z(0,0),H^*\Sigma^5 Y) \cong \mathbb{F}_2 \oplus \mathbb{F}_2$$

Proof. Note that $\Sigma^5 H^*Y$ is 0 except in dimensions 5, 6, 7, and 8, and so $\mathrm{Hom}_A(H^*Z(0,0),H^*\Sigma^5 Y) = 0$ except in dimensions 5, 6, 7, 8. We will show that there are no nonzero maps in dimensions 5 and 6, but there are nonzero maps in dimensions 7 and 8. Let $g \in \mathrm{Hom}_A(H^*Z(0,0),\Sigma^5 H^*Y)$.

Since $\Sigma^5 H^*Y = 0$ in dimension 4, we must have that $g(\Sigma^{-1}\mathrm{Sq}^4) = 0$. Therefore $g(\Sigma^{-1}\mathrm{Sq}^5) = 0$ (see the diagram in Section 2.3).

Then we must also have

$$\begin{aligned} 0 &= \mathrm{Sq}^4(g(\Sigma^{-1}(\mathrm{Sq}^4 + \mathrm{Sq}^3\mathrm{Sq}^1)) \\ &= g(\Sigma^{-1}\mathrm{Sq}^4(\mathrm{Sq}^4 + \mathrm{Sq}^3\mathrm{Sq}^1)) \\ &= g(\Sigma^{-1}\mathrm{Sq}^6\mathrm{Sq}^2 + \mathrm{Sq}^7\mathrm{Sq}^1)) \end{aligned}$$

Thus we have $g(\Sigma^{-1}\mathrm{Sq}^6\mathrm{Sq}^2) = g(\Sigma^{-1}\mathrm{Sq}^7\mathrm{Sq}^1)$. But we know that

$$\begin{aligned} 0 &= \mathrm{Sq}^3\mathrm{Sq}^1(g(\Sigma^{-1}(\mathrm{Sq}^4)) \\ &= g(\Sigma^{-1}\mathrm{Sq}^3\mathrm{Sq}^1\mathrm{Sq}^4) \\ &= g(\Sigma^{-1}\mathrm{Sq}^7\mathrm{Sq}^1) \end{aligned}$$

Therefore $g(\Sigma^{-1}\mathrm{Sq}^6\mathrm{Sq}^2) = g(\Sigma^{-1}\mathrm{Sq}^7\mathrm{Sq}^1) = 0$.

Therefore, $\mathrm{Hom}_A(H^*Z(0,0), H^*\Sigma^5 Y) = 0$ in dimension 6.

For dimensions 7 and 8, we have we have

$$g(\Sigma^{-1}\mathrm{Sq}^4\mathrm{Sq}^2\mathrm{Sq}^1) = \epsilon_1(\Sigma^5\alpha_1) \quad \text{and} \quad g(\Sigma^{-1}\mathrm{Sq}^8) = \epsilon_2(\Sigma^5\alpha_2)$$

for $\epsilon_1, \epsilon_2 \in \mathbb{F}_2$. Define two maps $f_1, f_2 \in \mathrm{Hom}_A(H^*Z(0,0), \Sigma^5 H^*Y)$ by setting $\epsilon_1 = 1, \epsilon_2 = 0$ for f_1, and $\epsilon_1 = 0$, $\epsilon_2 = 1$ for f_2. More explicitly, we have

$$\begin{aligned} f_1(\Sigma^{-1}\mathrm{Sq}^4\mathrm{Sq}^2\mathrm{Sq}^1) &= \Sigma^5\alpha_1 \\ f_1(\Sigma^{-1}\mathrm{Sq}^6\mathrm{Sq}^2\mathrm{Sq}^1) &= \Sigma^5\alpha_3 \\ f_1(\eta) &= 0 \text{ for all other elements } \eta \in H^*Z(0,0) \end{aligned}$$

and

$$\begin{aligned} f_2(\Sigma^{-1}\mathrm{Sq}^8) &= \Sigma^5\alpha_2 \\ f_2(\eta) &= 0 \text{ for all other elements } \eta \in H^*Z(1,1) \end{aligned}$$

Both of these elements are module maps in that they do not contradict any relations involving elements in lower dimensions. Therefore, these two elements are a basis of $\mathrm{Hom}_A(H^*Z(0,0),\Sigma^5 H^*Y)$ over A, so

$$\mathrm{Hom}_A(H^*Z(0,0),\Sigma^5 H^*Y) = \mathbb{F}_2 \oplus \mathbb{F}_2.$$

$\square$

Thus we have

$$\mathrm{Ext}_A^{0,6}(H^*Z(0,0),\Sigma^{-1}H^*Y) \cong \mathrm{Ext}_A^{1,6}(H^*Z(0,0),H^*Y_1) \cong \mathbb{F}_2 \oplus \mathbb{F}_2$$

We identified two generators $f_1, f_2 \in \mathrm{Hom}_A(H^*Z(0,0),\Sigma^5 H^*Y)$. We calculate their image under δ_0. In order to do so, we examine the following diagram:

$$\begin{array}{ccccccccc}
 & & \mathrm{Hom}_A(H^*Z(0,0),\Sigma^6 H^*Y_1) & \xrightarrow{i_*} & \mathrm{Hom}_A(H^*Z(0,0),\Sigma^5 A) & \xrightarrow{\pi_*} & \mathrm{Hom}_A(H^*Z(0,0),\Sigma^5 H^*Y) & & \\
 & & \downarrow d_0^* & & \downarrow d_0^* & & \downarrow d_0^* & & \\
0 & \to & \mathrm{Hom}_A(P_0(Z(0,0)),\Sigma^6 H^*Y_1) & \xrightarrow{i_*} & \mathrm{Hom}_A(P_0(Z(0,0)),\Sigma^5 A) & \xrightarrow{\pi_*} & \mathrm{Hom}_A(P_0(Z(0,0)),\Sigma^5 H^*Y) & \to & 0 \\
 & & \downarrow d_1^* & & \downarrow d_1^* & & \downarrow d_1^* & & \\
0 & \to & \mathrm{Hom}_A(P_1(Z(0,0)),\Sigma^6 H^*Y_1) & \xrightarrow{i_*} & \mathrm{Hom}_A(P_1(Z(0,0)),\Sigma^5 A) & \xrightarrow{\pi_*} & \mathrm{Hom}_A(P_1(Z(0,0)),\Sigma^5 H^*Y) & \to & 0
\end{array}$$

where the horizontal maps are coming from the sequence

$$0 \to H^*Y_1 \xrightarrow{i} \Sigma^{-1}A \xrightarrow{\pi} H^*\Sigma^{-1}Y \to 0$$

Starting with $f_1 \in \mathrm{Hom}_A(H^*Z(0,0),\Sigma^5 H^*Y)$, we calculate it d_0^* as an element of $\mathrm{Hom}_A(P_0(Z(0,0)),\Sigma^5 H^*Y)$.. The map $d_0^*(f_1) = (f_1 \circ d_0)$ is given on the generators of $P_0(Z(0,0))$ in dimensions 12 and below by:

$$\begin{aligned}(f_1 \circ d_0)(i_3) &= 0\\ (f_1 \circ d_0)(i_5) &= 0\\ (f_1 \circ d_0)(i_6) &= \Sigma^5 \alpha_1\\ (f_1 \circ d_0)(i_7) &= 0\end{aligned}$$

Now we need to lift $d_0^*(f_1)$ to a map in $\gamma_1 \in \mathrm{Hom}_A(P_1(Z(0,0)), \Sigma^5 A)$. We choose γ_1 to be given by $\gamma_1(i_6) = \Sigma^5 \mathrm{Sq}^1$ and $\gamma_1(i_k) = 0$ for $k \neq 6, k \leq 12$.

Next, we apply d_1^* to get a map $d_1^*(\gamma_1) \in \mathrm{Hom}_A(P_1(Z(0,0)), \Sigma^5 A)$. The map $d_1^*(\gamma_1) = (\gamma_1 \circ d_1)$ is given on the generators of $P_1(Z(0,0))$ by:

$$\begin{aligned}(\gamma_1 \circ d_1)(h_7) &= 0\\ (\gamma_1 \circ d_1)(h_9) &= 0\\ (\gamma_1 \circ d_1)(h_{10}) &= \Sigma^5 \mathrm{Sq}^4 \mathrm{Sq}^1\\ (\gamma_1 \circ d_1)(h_{12}) &= \Sigma^5 \mathrm{Sq}^4 \mathrm{Sq}^2 \mathrm{Sq}^1\end{aligned}$$

Last, we must lift $d_1^*(\gamma_1)$ to $\mathrm{Hom}_A(P_1(Z(0,0)), \Sigma^6 H^* Y_1)$. We use that $H^* Y_1 \subseteq \Sigma^{-1} A$ and that this is an isomorphism in dimensions where $d_1^*(\gamma_1) \neq 0$. Call this lift g_1; on the generators of $P_1(Z(0,0))$ through dimension 12, g_1 is given by the same

formula as $d_1^*(\gamma_1)$. Explicitly, we have:

$$\begin{aligned} g_1(h_7) &= 0 \\ g_1(h_9) &= 0 \\ g_1(h_{10}) &= \Sigma^5 \mathrm{Sq}^4 \mathrm{Sq}^1 \\ g_1(h_{12}) &= \Sigma^5 \mathrm{Sq}^4 \mathrm{Sq}^2 \mathrm{Sq}^1 \end{aligned}$$

It is evident that g_1 is a lift of $d_1^*(\gamma_1)$ since $H^*Y_1 = \Sigma^{-1}K_0(Y)$ is a submodule of $\Sigma^{-1}A$.

We have identified a representative of one of the two generators of $\mathrm{Ext}_A^{1,6}(H^*Z(0,0), H^*Y_1)$. We will show that it does not pull back to a generator of $\mathrm{Ext}_A^{1,6}(H^*Z(0,0), H^*Z(1,1))$ by examining the short exact sequence

$$0 \to \mathrm{Hom}_A(P_1(Z(0,0)), \Sigma^6 H^*Z(0,0)) \xrightarrow{\iota_*} \mathrm{Hom}_A(P_1(Z(0,0)), \Sigma^6 H^*Y_1)$$
$$\xrightarrow{(\phi_3)_*} \mathrm{Hom}_A(P_1(Z(0,0)), \Sigma^8 H^*Y) \to 0$$

We see that $(\phi_3)_*(g_1) = (\phi_3) \circ g_1$ is given by

$$\begin{aligned} ((\phi_3) \circ g_1)(h_7) &= 0 \\ ((\phi_3) \circ g_1)(h_9) &= 0 \\ ((\phi_3) \circ g_1)(h_{10}) &= \Sigma^8 \alpha_2 \\ ((\phi_3) \circ g_1)(h_{12}) &= 0 \end{aligned}$$

Thus $g_1 \notin \ker((\phi_3)_*) = \mathrm{im}(\iota_*)$. Since $\mathrm{Hom}_A(P_0(Z(0,0)), \Sigma^8 H^*Y) = 0$, we know that $\phi_3(g_1)$ is not a boundary. On the level of Ext groups, this

shows that the element $[g_1] \in \mathrm{Ext}_A^{1,6}(H^*Z(0,0), H^*Y_1)$ does not pull back under the inclusion $\mathrm{Ext}_A^{1,6}(H^*Z(0,0), H^*Z(0,0)) \hookrightarrow \mathrm{Ext}_A^{1,6}(H^*Z(0,0), H^*Y_1)$, so $\mathrm{Ext}_A^{1,6}(H^*Z(0,0), H^*Z(0,0))$ has at most rank 1.

We now investigate $g_2 = \delta_0(f_2)$.

We take $f_2 \in \mathrm{Hom}_A(H^*Z(0,0), \Sigma^5 H^*Y)$ and apply d_0^* to get $d_0^*(f_2) = f_2 \circ d_0 \in \mathrm{Hom}_A(P_0(Z(0,0)), \Sigma^5 H^*Y)$. We have

$$\begin{aligned}
(f_2 \circ d_0)(i_3) &= 0 \\
(f_2 \circ d_0)(i_5) &= 0 \\
(f_2 \circ d_0)(i_6) &= 0 \\
(f_2 \circ d_0)(i_7) &= \Sigma^5 \alpha_2
\end{aligned}$$

Now we need to lift $d_0^*(f_2)$ to a map in $\gamma_2 \in \mathrm{Hom}_A(P_1(Z(0,0)), \Sigma^5 A)$. We choose γ_2 so that $\gamma_1(i_7) = \Sigma^5 \mathrm{Sq}^2$ and $\gamma_2(i_k) = 0$ for $k \neq 7, k \leq 12$. Next, we apply d_1^* to get a map $d_1^*(\gamma_2) \in \mathrm{Hom}_A(P_1(Z(0,0)), \Sigma^5 A)$. The map $d_1^*(\gamma_2) = (\gamma_2 \circ d_1)$ is given on the generators of $P_1(Z(0,0))$ by:

$$\begin{aligned}
(\gamma_2 \circ d_1)(h_7) &= 0 \\
(\gamma_2 \circ d_1)(h_9) &= 0 \\
(\gamma_2 \circ d_1)(h_{10}) &= 0 \\
(\gamma_2 \circ d_1)(h_{12}) &= 0
\end{aligned}$$

The calculations here are simple from the information provided; essentially, this map is 0 through dimension 12 because $d_1(h_i)$ never involves an A-multiple of i_7 in dimensions below 12.

So we lift $d_1^*(\gamma_2)$ to $g_2 \in \mathrm{Hom}_A(P_1(Z(0,0)), H^*Y_1)$ with

$$\begin{aligned} g_2(h_7) &= 0 \\ g_2(h_9) &= 0 \\ g_2(h_{10}) &= 0 \\ g_2(h_{12}) &= 0 \end{aligned}$$

We have now identified a representative of the second generator of $\mathrm{Ext}_A^{1,6}(H^*Z(0,0), H^*Y_1)$. Unlike the first generator, we have $(\phi_3)_*(g_2) = 0$ in the short exact sequence

$$0 \to \mathrm{Hom}_A(P_1(Z(0,0)), \Sigma^6 H^*Z(0,0)) \xrightarrow{\iota_*} \mathrm{Hom}_A(P_1(Z(0,0)), \Sigma^6 H^*Y_1)$$
$$\xrightarrow{(\phi_3)_*} \mathrm{Hom}_A(P_1(Z(0,0)), \Sigma^8 H^*Y) \to 0$$

because $\Sigma^8 H^*Y$ is nonzero only in dimensions 8, 9, 10, and 11, and $(\phi_3)_*(g_2)$ is zero in those dimensions. This we can find a map $\tilde{g_2} \in \mathrm{Hom}_A(P_1(Z(0,0)), \Sigma^6 H^*Z(0,0))$; however, this map is zero through dimension 11. Since $H_*(K_0Z(0,0), x_2)$ is nonzero in dimension 11, $\tilde{g_2}$ cannot be a u_2^1-map.

<u>$\beta_1 = 1, \beta_2 = 0$:</u>

We apply $\mathrm{Ext}_A^n(H^*Z(1,0), -)$ to the short exact sequence

$$0 \to H^*Y_1 \to \Sigma^{-1}A \to H^*\Sigma^{-1}Y \to 0$$

to get the long exact sequence

$$0 \to \mathrm{Ext}_A^{0,6}(H^*Z(1,0),H^*Y_1) \to \mathrm{Ext}_A^{0,6}(H^*Z(1,0),\Sigma^{-1}A) \to \mathrm{Ext}_A^{0,6}(H^*Z(1,0),\Sigma^{-1}H^*Y) \xrightarrow{\delta_0} \mathrm{Ext}_A^{1,6}(H^*Z(1,0),H^*Y_1) \to \mathrm{Ext}_A^{1,6}(H^*Z(1,0),\Sigma^{-1}A) \to \mathrm{Ext}_A^{1,6}(H^*Z(1,0),\Sigma^{-1}H^*Y) \to \dots$$

By lemma 5.1, we have

$$\mathrm{Ext}_A^{0,6}(H^*Z(1,0),\Sigma^{-1}A) = \mathrm{Ext}_A^{1,6}(H^*Z(1,0),\Sigma^{-1}A) = 0.$$

Therefore we have an isomorphism

$$\mathrm{Ext}_A^{0,6}(H^*Z(1,0),\Sigma^{-1}H^*Y) \cong \mathrm{Ext}_A^{1,6}(H^*Z(1,0),H^*Y_1)$$

Lemma 5.4.

$$\mathrm{Ext}_A^{0,6}(H^*Z(1,0),\Sigma^{-1}H^*Y) \cong \mathrm{Hom}_A(H^*Z(1,0),H^*\Sigma^5 Y) \cong \mathbb{F}_2 \oplus \mathbb{F}_2$$

Proof. Note that $\Sigma^5 H^*Y$ is 0 except in dimensions 5, 6, 7, and 8, and so $\mathrm{Hom}_A(H^*Z(1,0),H^*\Sigma^5 Y) = 0$ except in dimensions 5, 6, 7, 8. We will show that there are no nonzero maps in dimensions 5 and 6, but there are nonzero maps in dimensions 7 and 8. Let $g \in \mathrm{Hom}_A(H^*Z(1,0),\Sigma^5 H^*Y)$.
Since $\Sigma^5 H^*Y = 0$ in dimension 4, we must have that $g(\Sigma^{-1}\mathrm{Sq}^4) = 0$. Therefore $g(\Sigma^{-1}\mathrm{Sq}^5) = 0$ (see the diagram in Section 2.3).
Then we must also have

$$\begin{aligned} 0 &= \mathrm{Sq}^4(g(\Sigma^{-1}(\mathrm{Sq}^4 + \mathrm{Sq}^3\mathrm{Sq}^1)) \\ &= g(\Sigma^{-1}\mathrm{Sq}^4(\mathrm{Sq}^4 + \mathrm{Sq}^3\mathrm{Sq}^1)) \\ &= g(\Sigma^{-1}(\mathrm{Sq}^5\mathrm{Sq}^2\mathrm{Sq}^1 + \mathrm{Sq}^6\mathrm{Sq}^2 + \mathrm{Sq}^7\mathrm{Sq}^1)) \end{aligned}$$

But we know that

$$\begin{aligned} 0 &= \mathrm{Sq}^3\mathrm{Sq}^1(g(\Sigma^{-1}(\mathrm{Sq}^4+\mathrm{Sq}^3\mathrm{Sq}^1)) \\ &= g(\Sigma^{-1}\mathrm{Sq}^3\mathrm{Sq}^1\mathrm{Sq}^4) \\ &= g(\Sigma^{-1}\mathrm{Sq}^7\mathrm{Sq}^1) \end{aligned}$$

Therefore $g(\Sigma^{-1}(\mathrm{Sq}^5\mathrm{Sq}^2\mathrm{Sq}^1) = g(\Sigma^{-1}(\mathrm{Sq}^6\mathrm{Sq}^2)$. We will show that both are zero. Suppose that $g(\Sigma^{-1}\mathrm{Sq}^4\mathrm{Sq}^2) = \Sigma^5\alpha_0$. Then $g(\Sigma^{-1}\mathrm{Sq}^5\mathrm{Sq}^2) = \Sigma^5\alpha_1$, and $g(\Sigma^{-1}\mathrm{Sq}^2\mathrm{Sq}^4\mathrm{Sq}^2) = \Sigma^5\alpha_2$. Then we would have $g(\Sigma^{-1}\mathrm{Sq}^5\mathrm{Sq}^2\mathrm{Sq}^1) = \Sigma^5\alpha_2$. But this is a contradiction, because $g(\Sigma^{-1}\mathrm{Sq}^4\mathrm{Sq}^2\mathrm{Sq}^1) = \epsilon\alpha_1$ for $\epsilon \in \{0,1\}$, but $g(\Sigma^{-1}\mathrm{Sq}^5\mathrm{Sq}^2\mathrm{Sq}^1) = \mathrm{Sq}^1 g(\Sigma^{-1}\mathrm{Sq}^4\mathrm{Sq}\,2\,\mathrm{Sq}^1) = \mathrm{Sq}^1\epsilon\alpha_1 = 0$ for $\epsilon \in \{0,1\}$. So we have $g(\Sigma^{-1}(\mathrm{Sq}^4\mathrm{Sq}^2) = 0$.

Therefore, $\mathrm{Hom}_A(H^*Z(1,0), H^*\Sigma^5 Y) = 0$ in dimension 6.

For dimensions 7 and 8, we have we have

$$g(\Sigma^{-1}\mathrm{Sq}^4\mathrm{Sq}^2\mathrm{Sq}^1) = \epsilon_1(\Sigma^5\alpha_1) \quad \text{and} \quad g(\Sigma^{-1}\mathrm{Sq}^8) = \epsilon_2(\Sigma^5\alpha_2)$$

for $\epsilon_1, \epsilon_2 \in \mathbb{F}_2$. Define two maps $f_1, f_2 \in \mathrm{Hom}_A(H^*Z(0,0), \Sigma^5 H^*Y)$ by setting $\epsilon_1 = 1, \epsilon_2 = 0$ for f_1, and $\epsilon_1 = 0$, $\epsilon_2 = 1$ for f_2. More explicitly, we have

$$\begin{aligned} f_1(\Sigma^{-1}\mathrm{Sq}^4\mathrm{Sq}^2\mathrm{Sq}^1) &= \Sigma^5\alpha_1 \\ f_1(\Sigma^{-1}\mathrm{Sq}^6\mathrm{Sq}^2\mathrm{Sq}^1) &= \Sigma^5\alpha_3 \\ f_1(\eta) &= 0 \text{ for all other elements } \eta \in H^*Z(1,0) \end{aligned}$$

and

$$
\begin{aligned}
f_2(\Sigma^{-1}\,\mathrm{Sq}^8) &= \Sigma^5\alpha_2 \\
f_2(\eta) &= 0 \text{ for all other elements } \eta \in H^*Z(1,0)
\end{aligned}
$$

Both of these elements are module maps in that they do not contradict any relations involving elements in lower dimensions. Therefore, these two elements are a basis of $\mathrm{Hom}_A(H^*Z(1,0),\Sigma^5 H^*Y)$ over A, so

$$\mathrm{Hom}_A(H^*Z(0,0),\Sigma^5 H^*Y) = \mathbb{F}_2 \oplus \mathbb{F}_2.$$

□

Thus we have

$$\mathrm{Ext}_A^{0,6}(H^*Z(1,0),\Sigma^{-1}H^*Y) \cong \mathrm{Ext}_A^{1,6}(H^*Z(1,0),H^*Y_1) \cong \mathbb{F}_2 \oplus \mathbb{F}_2$$

We identified two generators $f_1, f_2 \in \mathrm{Hom}_A(H^*Z(1,0),\Sigma^5 H^*Y)$. We calculate their imagine under δ_0. In order to do so, we examine the following diagram:

$$
\begin{array}{ccccccccc}
 & & \mathrm{Hom}_A(H^*Z(1,0),\Sigma^6 H^*Y_1) & \xrightarrow{i_*} & \mathrm{Hom}_A(H^*Z(1,0),\Sigma^5 A) & \xrightarrow{\pi_*} & \mathrm{Hom}_A(H^*Z(1,0),\Sigma^5 H^*Y) & & \\
 & & \downarrow{\scriptstyle d_0^*} & & \downarrow{\scriptstyle d_0^*} & & \downarrow{\scriptstyle d_0^*} & & \\
0 & \to & \mathrm{Hom}_A(P_0(Z(1,0)),\Sigma^6 H^*Y_1) & \xrightarrow{i_*} & \mathrm{Hom}_A(P_0(Z(1,0)),\Sigma^5 A) & \xrightarrow{\pi_*} & \mathrm{Hom}_A(P_0(Z(1,0)),\Sigma^5 H^*Y) & \to & 0 \\
 & & \downarrow{\scriptstyle d_1^*} & & \downarrow{\scriptstyle d_1^*} & & \downarrow{\scriptstyle d_1^*} & & \\
0 & \to & \mathrm{Hom}_A(P_1(Z(1,0)),\Sigma^6 H^*Y_1) & \xrightarrow{i_*} & \mathrm{Hom}_A(P_1(Z(1,0)),\Sigma^5 A) & \xrightarrow{\pi_*} & \mathrm{Hom}_A(P_1(Z(1,0)),\Sigma^5 H^*Y) & \to & 0
\end{array}
$$

where the horizontal maps are coming from the sequence

$$0 \to H^*Y_1 \xrightarrow{i} \Sigma^{-1}A \xrightarrow{\pi} H^*\Sigma^{-1}Y \to 0$$

Starting with $f_1 \in \mathrm{Hom}_A(H^*Z(1,0), \Sigma^5 H^*Y)$, we calculate it d_0^* as an element of $\mathrm{Hom}_A(P_0(Z(1,0)), \Sigma^5 H^*Y)$. The map $d_0^*(f_1) = (f_1 \circ d_0)$ is given on the generators of $P_0(Z(1,0))$ in dimensions 12 and below by:

$$
\begin{aligned}
(f_1 \circ d_0)(i_3) &= 0 \\
(f_1 \circ d_0)(i_5) &= 0 \\
(f_1 \circ d_0)(i_6) &= \Sigma^5 \alpha_1 \\
(f_1 \circ d_0)(i_7) &= 0
\end{aligned}
$$

Now we need to lift $d_0^*(f_1)$ to a map in $\gamma_1 \in \mathrm{Hom}_A(P_1(Z(1,0)), \Sigma^5 A)$. We choose γ_1 to be given by $\gamma_1(i_6) = \Sigma^5 \mathrm{Sq}^1$ and $\gamma_1(i_k) = 0$ for $k \neq 6, k \leq 12$.
Next, we apply d_1^* to get a map $d_1^*(\gamma_1) \in \mathrm{Hom}_A(P_1(Z(1,0)), \Sigma^5 A)$. The map $d_1^*(\gamma_1) = (\gamma_1 \circ d_1)$ is given on the generators of $P_1(Z(1,0))$ by:

$$
\begin{aligned}
(\gamma_1 \circ d_1)(h_7) &= 0 \\
(\gamma_1 \circ d_1)(h_9) &= 0 \\
(\gamma_1 \circ d_1)(h_{10}) &= \Sigma^5 \mathrm{Sq}^4 \mathrm{Sq}^1 \\
(\gamma_1 \circ d_1)(h_{12}) &= \Sigma^5 \mathrm{Sq}^4 \mathrm{Sq}^2 \mathrm{Sq}^1
\end{aligned}
$$

Last, we must lift $d_1^*(\gamma_1)$ to $\mathrm{Hom}_A(P_1(Z(1,0)), \Sigma^6 H^*Y_1)$. We use that $H^*Y_1 \subseteq \Sigma^{-1} A$ and that this is an isomorphism in dimensions where $d_1^*(\gamma_1) \neq 0$. Call this lift g_1; on the generators of $P_1(Z(1,0))$ through dimension 12, g_1 is given by the same

formula as $d_1^*(\gamma_1)$. Explicitly, we have:

$$\begin{aligned} g_1(h_7) &= 0 \\ g_1(h_9) &= 0 \\ g_1(h_{10}) &= \Sigma^5 \mathrm{Sq}^4 \mathrm{Sq}^1 \\ g_1(h_{12}) &= \Sigma^5 \mathrm{Sq}^4 \mathrm{Sq}^2 \mathrm{Sq}^1 \end{aligned}$$

It is evident that g_1 is a lift of $d_1^*(\gamma_1)$ since $H^*Y_1 = \Sigma^{-1}K_0(Y)$ is a submodule of $\Sigma^{-1}A$.

We have identified a representative of one of the two generators of $\mathrm{Ext}_A^{1,6}(H^*Z(1,0), H^*Y_1)$. We will show that it does not pull back to a generator of $\mathrm{Ext}_A^{1,6}(H^*Z(1,0), H^*Z(1,0))$ by examining the short exact sequence

$$0 \to \mathrm{Hom}_A(P_1(Z(1,0)), \Sigma^6 H^*Z(1,0)) \xrightarrow{\iota_*} \mathrm{Hom}_A(P_1(Z(1,0)), \Sigma^6 H^*Y_1)$$

$$\xrightarrow{(\phi_3+\phi_4)_*} \mathrm{Hom}_A(P_1(Z(1,0)), \Sigma^8 H^*Y) \to 0$$

We see that $(\phi_3 + \phi_4)_*(g_1) = (\phi_3 + \phi_4) \circ g_1$ is given by

$$\begin{aligned} ((\phi_3 + \phi_4) \circ g_1)(h_7) &= 0 \\ ((\phi_3 + \phi_4) \circ g_1)(h_9) &= 0 \\ ((\phi_3 + \phi_4) \circ g_1)(h_{10}) &= \Sigma^8 \alpha_2 \\ ((\phi_3 + \phi_4) \circ g_1)(h_{12}) &= 0 \end{aligned}$$

Thus $g_1 \notin \ker((\phi_3 + \phi_4 + \phi_6)_*) = \mathrm{im}(\iota_*)$. Since $\mathrm{Hom}_A(P_0(Z(1,0)), \Sigma^8 H^*Y) = 0$, we know that g_1 is not a boundary. On the level of Ext groups, this

shows that the element $[g_1] \in \mathrm{Ext}_A^{1,6}(H^*Z(1,0), H^*Y_1)$ does not pull back under the inclusion $\mathrm{Ext}_A^{1,6}(H^*Z(1,0), H^*Z(1,0)) \hookrightarrow \mathrm{Ext}_A^{1,6}(H^*Z(1,0), H^*Y_1)$, so $\mathrm{Ext}_A^{1,6}(H^*Z(1,0), H^*Z(1,0))$ has at most rank 1.

We now investigate $g_2 = \delta_0(f_2)$.

We take $f_2 \in \mathrm{Hom}_A(H^*Z(1,0), \Sigma^5 H^*Y)$ and apply d_0^* to get $d_0^*(f_2) = f_2 \circ d_0 \in \mathrm{Hom}_A(P_0(Z(1,0)), \Sigma^5 H^*Y)$. We have

$$\begin{aligned}
(f_2 \circ d_0)(i_3) &= 0 \\
(f_2 \circ d_0)(i_5) &= 0 \\
(f_2 \circ d_0)(i_6) &= 0 \\
(f_2 \circ d_0)(i_7) &= \Sigma^5 \alpha_2
\end{aligned}$$

Now we need to lift $d_0^*(f_2)$ to a map in $\gamma_2 \in \mathrm{Hom}_A(P_1(Z(1,0)), \Sigma^5 A)$. We choose γ_2 so that $\gamma_1(i_7) = \Sigma^5 \mathrm{Sq}^2$ and $\gamma_2(i_k) = 0$ for $k \neq 7, k \leq 12$. Next, we apply d_1^* to get a map $d_1^*(\gamma_2) \in \mathrm{Hom}_A(P_1(Z(1,0)), \Sigma^5 A)$. The map $d_1^*(\gamma_2) = (\gamma_2 \circ d_1)$ is given on the generators of $P_1(Z(1,0))$ by:

$$\begin{aligned}
(\gamma_2 \circ d_1)(h_7) &= 0 \\
(\gamma_2 \circ d_1)(h_9) &= 0 \\
(\gamma_2 \circ d_1)(h_{10}) &= 0 \\
(\gamma_2 \circ d_1)(h_{12}) &= 0
\end{aligned}$$

The calculations here are simple from the information provided; essentially, this map is 0 through dimension 12 because $d_1(h_i)$ never involves an A-multiple of i_7 in dimensions below 12.

So we lift $d_1^*(\gamma_2)$ to $g_2 \in \mathrm{Hom}_A(P_1(Z(1,0)), H^*Y_1)$ with

$$\begin{aligned} g_2(h_7) &= 0 \\ g_2(h_9) &= 0 \\ g_2(h_{10}) &= 0 \\ g_2(h_{12}) &= 0 \end{aligned}$$

We have now identified a representative of the second generator of $\mathrm{Ext}_A^{1,6}(H^*Z(1,0), H^*Y_1)$. Unlike the first generator, we have $(\phi_3+\phi_4)_*(g_2) = 0$ in the short exact sequence

$$0 \to \mathrm{Hom}_A(P_1(Z(1,0)), \Sigma^6 H^*Z(1,0)) \xrightarrow{\iota_*} \mathrm{Hom}_A(P_1(Z(1,0)), \Sigma^6 H^*Y_1)$$
$$\xrightarrow{(\phi_3+\phi_4)_*} \mathrm{Hom}_A(P_1(Z(1,0)), \Sigma^8 H^*Y) \to 0$$

because $\Sigma^8 H^*Y$ is nonzero only in dimensions 8, 9, 10, and 11, and $(\phi_3 + \phi_4)_*(g_2)$ is zero in those dimensions. This we can find a map $\tilde{g_2} \in \mathrm{Hom}_A(P_1(Z(1,0)), \Sigma^6 H^*Z(1,0))$; however, this map is zero through dimension 11. Since $H_*(K_0Z(1,0), x_2)$ is nonzero in dimension 11, $\tilde{g_2}$ cannot be a u_2^1-map.

$\underline{\beta_1 = 0, \beta_2 = 1:}$

We apply $\mathrm{Ext}_A^n(H^*Z(0,1), -)$ to the short exact sequence

$$0 \to H^*Y_1 \to \Sigma^{-1}A \to H^*\Sigma^{-1}Y \to 0$$

to get the long exact sequence

$$0 \to \mathrm{Ext}_A^{0,6}(H^*Z(0,1), H^*Y_1) \to \mathrm{Ext}_A^{0,6}(H^*Z(0,1), \Sigma^{-1}A) \to \mathrm{Ext}_A^{0,6}(H^*Z(0,1), \Sigma^{-1}H^*Y) \xrightarrow{\delta_0}$$
$$\mathrm{Ext}_A^{1,6}(H^*Z(0,1), H^*Y_1) \to \mathrm{Ext}_A^{1,6}(H^*Z(0,1), \Sigma^{-1}A) \to \mathrm{Ext}_A^{1,6}(H^*Z(0,1), \Sigma^{-1}H^*Y) \to \dots$$

By lemma 5.1, we have

$$\mathrm{Ext}_A^{0,6}(H^*Z(0,1), \Sigma^{-1}A) = \mathrm{Ext}_A^{1,6}(H^*Z(0,1), \Sigma^{-1}A) = 0.$$

Therefore we have an isomorphism

$$\mathrm{Ext}_A^{0,6}(H^*Z(0,1), \Sigma^{-1}H^*Y) \cong \mathrm{Ext}_A^{1,6}(H^*Z(0,1), H^*Y_1)$$

Lemma 5.5.

$$\mathrm{Ext}_A^{0,6}(H^*Z(0,1), \Sigma^{-1}H^*Y) \cong \mathrm{Hom}_A(H^*Z(0,1), H^*\Sigma^5 Y) \cong \mathbb{F}_2 \oplus \mathbb{F}_2$$

Proof. Note that $\Sigma^5 H^*Y$ is 0 except in dimensions 5, 6, 7, and 8, and so $\mathrm{Hom}_A(H^*Z(0,1), H^*\Sigma^5 Y) = 0$ except in dimensions 5, 6, 7, 8. We will show that there are no nonzero maps in dimensions 5 and 6, but there are nonzero maps in dimensions 7 and 8. Let $g \in \mathrm{Hom}_A(H^*Z(0,1), \Sigma^5 H^*Y)$.
Since $\Sigma^5 H^*Y = 0$ in dimension 4, we must have that $g(\Sigma^{-1}(\mathrm{Sq}^4) = 0$. Therefore $g(\Sigma^{-1}\mathrm{Sq}^5) = 0$ (see the diagram in Section 2.3).
Then we must also have

$$\begin{aligned} 0 &= \mathrm{Sq}^4(g(\Sigma^{-1}\mathrm{Sq}^4)) \\ &= g(\Sigma^{-1}\mathrm{Sq}^4\mathrm{Sq}^4) \\ &= g(\Sigma^{-1}(\mathrm{Sq}^6\mathrm{Sq}^2 + \mathrm{Sq}^7\mathrm{Sq}^1)) \end{aligned}$$

So we have $g(\Sigma^{-1}\,\mathrm{Sq}^6\,\mathrm{Sq}^2) = g(\Sigma^{-1}\,\mathrm{Sq}^7\,\mathrm{Sq}^1)$. But we also have

$$\begin{aligned} 0 &= \mathrm{Sq}^3\,\mathrm{Sq}^1(g(\Sigma^{-1}\,\mathrm{Sq}^4)) \\ &= g(\Sigma^{-1}\,\mathrm{Sq}^3\,\mathrm{Sq}^1\,\mathrm{Sq}^4) \\ &= g(\Sigma^{-1}\,\mathrm{Sq}^7\,\mathrm{Sq}^1) \end{aligned}$$

Thus we have $0 = g(\Sigma^{-1}\,\mathrm{Sq}^6\,\mathrm{Sq}^2) = g(\Sigma^{-1}\,\mathrm{Sq}^7\,\mathrm{Sq}^1)$.

So we have

$$\begin{aligned} 0 &= g(\mathrm{Sq}^6\,\mathrm{Sq}^2) \\ &= g(\mathrm{Sq}^2(\mathrm{Sq}^4\,\mathrm{Sq}^2 + \mathrm{Sq}^5\,\mathrm{Sq}^1)) \\ &= \mathrm{Sq}^2\,g(\mathrm{Sq}^4\,\mathrm{Sq}^2 + \mathrm{Sq}^5\,\mathrm{Sq}^1) \end{aligned}$$

But $g(\mathrm{Sq}^4\,\mathrm{Sq}^2 + \mathrm{Sq}^5\,\mathrm{Sq}^1) = \epsilon\alpha_0$ and $\mathrm{Sq}^2\,\epsilon\alpha_0 = \epsilon\alpha_2$, so $\epsilon = 0$.

So we have that $g(\Sigma^{-1}\,\mathrm{Sq}^4) = 0$ and $g(\Sigma^{-1}(\mathrm{Sq}^4\,\mathrm{Sq}^2 + \mathrm{Sq}^5\,\mathrm{Sq}^1) = 0$. Then

$$g(\Sigma^{-1}\,\mathrm{Sq}^4\,\mathrm{Sq}^2\,\mathrm{Sq}^1) = \epsilon_1(\Sigma^5\alpha_1) \quad \text{and} \quad g(\Sigma^{-1}\,\mathrm{Sq}^8) = \epsilon_2(\Sigma^5\alpha_2)$$

for $\epsilon_1, \epsilon_2 \in \mathbb{F}_2$. Define two maps $f_1, f_2 \in \mathrm{Hom}_A(H^*Z(0,1), \Sigma^5 H^*Y)$ by setting $\epsilon_1 = 1, \epsilon_2 = 0$ for f_1, and $\epsilon_1 = 0$, $\epsilon_2 = 1$ for f_2. More explicitly, we have

$$\begin{aligned} f_1(\Sigma^{-1}\,\mathrm{Sq}^4\,\mathrm{Sq}^2\,\mathrm{Sq}^1) &= \Sigma^5\alpha_1 \\ f_1(\Sigma^{-1}\,\mathrm{Sq}^6\,\mathrm{Sq}^2\,\mathrm{Sq}^1) &= \Sigma^6\alpha_3 \\ f_1(\eta) &= 0 \text{ for all other elements } \eta \in H^*Z(0,1) \end{aligned}$$

and

$$
\begin{aligned}
f_2(\Sigma^{-1}\mathrm{Sq}^8) &= \Sigma^5\alpha_2 \\
f_2(\eta) &= 0 \text{ for all other elements } \eta \in H^*Z(0,1)
\end{aligned}
$$

These two elements are a basis of $\mathrm{Hom}_A(H^*Z(0,1),\Sigma^5H^*Y)$ over A, so

$$
\mathrm{Hom}_A(H^*Z(0,1),\Sigma^5H^*Y) = \mathbb{F}_2 \oplus \mathbb{F}_2.
$$

□

Thus we have

$$
\mathrm{Ext}_A^{0,6}(H^*Z(0,1),\Sigma^{-1}H^*Y) \cong \mathrm{Ext}_A^{1,6}(H^*Z(0,1),H^*Y_1) \cong \mathbb{F}_2 \oplus \mathbb{F}_2
$$

We identified two maps $f_1, f_2 \in \mathrm{Hom}_A(H^*Z(0,1),\Sigma^5H^*Y)$. Starting with f_1, we calculate its image under δ_0. In order to do so, we examine the following diagram:

$$
\begin{array}{ccccccccc}
 & & \mathrm{Hom}_A(H^*Z(0,1),\Sigma^6H^*Y_1) & \xrightarrow{i_*} & \mathrm{Hom}_A(H^*Z(0,1),\Sigma^5A) & \xrightarrow{\pi_*} & \mathrm{Hom}_A(H^*Z(0,1),\Sigma^5H^*Y) & & \\
 & & \downarrow d_0^* & & \downarrow d_0^* & & \downarrow d_0^* & & \\
0 & \to & \mathrm{Hom}_A(P_0(Z(0,1)),\Sigma^6H^*Y_1) & \xrightarrow{i_*} & \mathrm{Hom}_A(P_0(Z(0,1)),\Sigma^5A) & \xrightarrow{\pi_*} & \mathrm{Hom}_A(P_0(Z(0,1)),\Sigma^5H^*Y) & \to & 0 \\
 & & \downarrow d_1^* & & \downarrow d_1^* & & \downarrow d_1^* & & \\
0 & \to & \mathrm{Hom}_A(P_1(Z(0,1)),\Sigma^6H^*Y_1) & \xrightarrow{i_*} & \mathrm{Hom}_A(P_1(Z(0,1)),\Sigma^5A) & \xrightarrow{\pi_*} & \mathrm{Hom}_A(P_1(Z(0,1)),\Sigma^5H^*Y) & \to & 0
\end{array}
$$

where the horizontal maps are coming from the sequence

$$
0 \to H^*Y_1 \xrightarrow{i} \Sigma^{-1}A \xrightarrow{\pi} H^*\Sigma^{-1}Y \to 0
$$

Starting with $f_1 \in \mathrm{Hom}_A(H^*Z(0,1),\Sigma^5H^*Y)$, we calculate it d_0^* as an element of $\mathrm{Hom}_A(P_0(Z(0,1)),\Sigma^5H^*Y)$. The map $d_0^*(f_1) = (f_1 \circ d_0)$ is given on the generators of $P_0(Z(0,1))$ in dimensions 12 and below by:

$$\begin{aligned}(f_1 \circ d_0)(i_3) &= 0\\(f_1 \circ d_0)(i_5) &= 0\\(f_1 \circ d_0)(i_6) &= \Sigma^5 \alpha_1\\(f_1 \circ d_0)(i_7) &= 0\end{aligned}$$

Now we need to lift $d_0^*(f_1)$ to a map in $\gamma_1 \in \mathrm{Hom}_A(P_1(Z(0,1)), \Sigma^5 A)$. We choose γ_1 to be given by $\gamma_1(i_6) = \Sigma^5 \mathrm{Sq}^1$ and $\gamma_1(i_k) = 0$ for $k \neq 6, k \leq 12$.

Next, we apply d_1^* to get a map $d_1^*(\gamma_1) \in \mathrm{Hom}_A(P_1(Z(0,1)), \Sigma^5 A)$. The map $d_1^*(\gamma_1) = (\gamma_1 \circ d_1)$ is given on the generators of $P_1(Z(0,1))$ by:

$$\begin{aligned}(\gamma_1 \circ d_1)(h_7) &= 0\\(\gamma_1 \circ d_1)(h_9) &= \Sigma^5 \mathrm{Sq}^3 \mathrm{Sq}^1\\(\gamma_1 \circ d_1)(h_{10}) &= \Sigma^5 \mathrm{Sq}^4 \mathrm{Sq}^1\\(\gamma_1 \circ d_1)(h_{12}) &= \Sigma^5 \mathrm{Sq}^4 \mathrm{Sq}^2 \mathrm{Sq}^1\end{aligned}$$

Last, we must lift $d_1^*(\gamma_1)$ to $\mathrm{Hom}_A(P_1(Z(0,1)), \Sigma^6 H^* Y_1)$. We use that $H^* Y_1 \subseteq \Sigma^{-1} A$ and that this is an isomorphism in dimensions where $d_1^*(\gamma_1) \neq 0$. Call this lift g_1; on the generators of $P_1(Z(0,0))$ through dimension 12, g_1 is given by the same

formula as $d_1^*(\gamma_1)$. Explicitly, we have:

$$\begin{aligned} g_1(h_7) &= 0 \\ g_1(h_9) &= \Sigma^5 \mathrm{Sq}^3 \mathrm{Sq}^1 \\ g_1(h_{10}) &= \Sigma^5 \mathrm{Sq}^4 \mathrm{Sq}^1 \\ g_1(h_{12}) &= \Sigma^5 \mathrm{Sq}^4 \mathrm{Sq}^2 \mathrm{Sq}^1 \end{aligned}$$

It is evident that g_1 is a lift of $d_1^*(\gamma_1)$ since $H^*Y_1 = \Sigma^{-1}K_0(Y)$ is a submodule of $\Sigma^{-1}A$.

We have identified a representative of one of the two generators of $\mathrm{Ext}_A^{1,6}(H^*Z(0,1), H^*Y_1)$. We will show that it does not pull back to a generator of $\mathrm{Ext}_A^{1,6}(H^*Z(0,1), H^*Z(0,1))$ by examining the short exact sequence

$$0 \to \mathrm{Hom}_A(P_1(Z(0,1)), \Sigma^6 H^*Z(0,1)) \xrightarrow{\iota_*} \mathrm{Hom}_A(P_1(Z(0,1)), \Sigma^6 H^*Y_1)$$
$$\xrightarrow{(\phi_3+\phi_6)_*} \mathrm{Hom}_A(P_1(Z(0,1)), \Sigma^8 H^*Y) \to 0$$

We see that $(\phi_3 + \phi_6)_*(g_1) = (\phi_3 + \phi_6) \circ g_1$ is given by

$$\begin{aligned} ((\phi_3 + \phi_6) \circ g_1)(h_7) &= 0 \\ ((\phi_3 + \phi_6) \circ g_1)(h_9) &= \Sigma^8 \alpha_1 \\ ((\phi_3 + \phi_6) \circ g_1)(h_{10}) &= \Sigma^8 \alpha_2 \\ ((\phi_3 + \phi_6) \circ g_1)(h_{12}) &= 0 \end{aligned}$$

Thus $g_1 \notin \ker((\phi_3 + \phi_4 + \phi_6)_*) = \mathrm{im}(\iota_*)$. Since $\mathrm{Hom}_A(P_0(Z(0,1)), \Sigma^8 H^*Y) = 0$, we know that g_1 is not a boundary. On the level of Ext groups, this

shows that the element $[g_1] \in \mathrm{Ext}_A^{1,6}(H^*Z(0,1), H^*Y_1)$ does not pull back under the inclusion $\mathrm{Ext}_A^{1,6}(H^*Z(0,1), H^*Z(0,1)) \hookrightarrow \mathrm{Ext}_A^{1,6}(H^*Z(0,1), H^*Y_1)$, so $\mathrm{Ext}_A^{1,6}(H^*Z(0,1), H^*Z(0,1))$ has at most rank 1.

We now investigate $g_2 = \delta_0(f_2)$.

We take $f_2 \in \mathrm{Hom}_A(H^*Z(0,1), \Sigma^5 H^*Y)$ and apply d_0^* to get $d_0^*(f_2) = f_2 \circ d_0 \in \mathrm{Hom}_A(P_0(Z(0,1)), \Sigma^5 H^*Y)$. We have

$$\begin{aligned} (f_2 \circ d_0)(i_3) &= 0 \\ (f_2 \circ d_0)(i_5) &= 0 \\ (f_2 \circ d_0)(i_6) &= 0 \\ (f_2 \circ d_0)(i_7) &= \Sigma^5 \alpha_2 \end{aligned}$$

Now we need to lift $d_0^*(f_2)$ to a map in $\gamma_2 \in \mathrm{Hom}_A(P_1(Z(0,1)), \Sigma^5 A)$. We choose γ_2 so that $\gamma_1(i_7) = \Sigma^5 \mathrm{Sq}^2$ and $\gamma_2(i_k) = 0$ for $k \neq 7, k \leq 12$. Next, we apply d_1^* to get a map $d_1^*(\gamma_2) \in \mathrm{Hom}_A(P_1(Z(0,1)), \Sigma^5 A)$. The map $d_1^*(\gamma_2) = (\gamma_2 \circ d_1)$ is given on the generators of $P_1(Z(0,1))$ by:

$$\begin{aligned} (\gamma_2 \circ d_1)(h_7) &= 0 \\ (\gamma_2 \circ d_1)(h_9) &= 0 \\ (\gamma_2 \circ d_1)(h_{10}) &= 0 \\ (\gamma_2 \circ d_1)(h_{12}) &= 0 \end{aligned}$$

The calculations here are simple from the information provided; essentially, this map is 0 through dimension 12 because $d_1(h_i)$ never involves an A-multiple of i_7 in dimensions below 12.

So we lift $d_1^*(\gamma_2)$ to $g_2 \in \mathrm{Hom}_A(P_1(Z(0,1)), H^*Y_1)$ with

$$\begin{aligned} g_2(h_7) &= 0 \\ g_2(h_9) &= 0 \\ g_2(h_{10}) &= 0 \\ g_2(h_{12}) &= 0 \end{aligned}$$

We have now identified a representative of the second generator of $\mathrm{Ext}_A^{1,6}(H^*Z(0,1), H^*Y_1)$. Unlike the first generator, we have $(\phi_3 + \phi_6)_*(g_2) = 0$ in the short exact sequence

$$0 \to \mathrm{Hom}_A(P_1(Z(0,1)), \Sigma^6 H^*Z(1,0)) \xrightarrow{\iota_*} \mathrm{Hom}_A(P_1(Z(0,1)), \Sigma^6 H^*Y_1)$$
$$\xrightarrow{(\phi_3+\phi_6)_*} \mathrm{Hom}_A(P_1(Z(0,1)), \Sigma^8 H^*Y) \to 0$$

because $\Sigma^8 H^*Y$ is nonzero only in dimensions 8, 9, 10, and 11, and $(\phi_3 + \phi_6)_*(g_2)$ is zero in those dimensions. This we can find a map $\tilde{g_2} \in \mathrm{Hom}_A(P_1(Z(0,1)), \Sigma^6 H^*Z(0,1))$; however, this map is zero through dimension 12. Since $H_*(K_0 Z(0,1), x_2)$ is nonzero in dimension 12, $\tilde{g_2}$ cannot be a u_2^1-map.

$\underline{\beta_1 = 1, \beta_2 = 1}$

We apply $\mathrm{Ext}_A^n(H^*Z(1,1), -)$ to the short exact sequence

$$0 \to H^*Y_1 \to \Sigma^{-1}A \to H^*\Sigma^{-1}Y \to 0$$

to get the long exact sequence

$$
\begin{array}{l}
0 \to \mathrm{Ext}_A^{0,6}(H^*Z(1,1), H^*Y_1) \to \mathrm{Ext}_A^{0,6}(H^*Z(1,1), \Sigma^{-1}A) \to \mathrm{Ext}_A^{0,6}(H^*Z(1,1), \Sigma^{-1}H^*Y) \xrightarrow{\delta_0} \\
\mathrm{Ext}_A^{1,6}(H^*Z(1,1), H^*Y_1) \to \mathrm{Ext}_A^{1,6}(H^*Z(1,1), \Sigma^{-1}A) \to \mathrm{Ext}_A^{1,6}(H^*Z(1,1), \Sigma^{-1}H^*Y) \to \dots
\end{array}
$$

By lemma 5.1, we have

$$\mathrm{Ext}_A^{0,6}(H^*Z(1,1), \Sigma^{-1}A) = \mathrm{Ext}_A^{1,6}(H^*Z(1,1), \Sigma^{-1}A) = 0.$$

Therefore we have an isomorphism

$$\mathrm{Ext}_A^{0,6}(H^*Z(1,1), \Sigma^{-1}H^*Y) \cong \mathrm{Ext}_A^{1,6}(H^*Z(1,1), H^*Y_1)$$

Lemma 5.6.

$$\mathrm{Ext}_A^{0,6}(H^*Z(1,1), \Sigma^{-1}H^*Y) \cong \mathrm{Hom}_A(H^*Z(1,1), H^*\Sigma^5 Y) \cong \mathbb{F}_2 \oplus \mathbb{F}_2$$

Proof. Note that $\Sigma^5 H^*Y$ is 0 except in dimensions 5, 6, 7, and 8, and so $\mathrm{Hom}_A(H^*Z(0,0), H^*\Sigma^5 Y) = 0$ except in dimensions 5, 6, 7, 8. We will show that there are no nonzero maps in dimensions 5 and 6, but there are nonzero maps in dimensions 7 and 8. Let $g \in \mathrm{Hom}_A(H^*Z(1,1), \Sigma^5 H^*Y)$.
Since $\Sigma^5 H^*Y = 0$ in dimension 4, we must have that $g(\Sigma^{-1}(\mathrm{Sq}^4) = 0$. Therefore $g(\Sigma^{-1}\mathrm{Sq}^5) = 0$ (see the diagram in Section 2.3).

Then we must also have

$$\begin{aligned} 0 &= \mathrm{Sq}^4(g(\Sigma^{-1}(\mathrm{Sq}^4+\mathrm{Sq}^3\,\mathrm{Sq}^1)) \\ &= g(\Sigma^{-1}\,\mathrm{Sq}^4(\mathrm{Sq}^4+\mathrm{Sq}^3\,\mathrm{Sq}^1)) \\ &= g(\Sigma^{-1}\,\mathrm{Sq}^6\,\mathrm{Sq}^2+\mathrm{Sq}^7\,\mathrm{Sq}^1)) \end{aligned}$$

Thus we have $g(\Sigma^{-1}\,\mathrm{Sq}^6\,\mathrm{Sq}^2) = g(\Sigma^{-1}\,\mathrm{Sq}^7\,\mathrm{Sq}^1)$.

But we know that

$$\begin{aligned} 0 &= \mathrm{Sq}^3\,\mathrm{Sq}^1(g(\Sigma^{-1}(\mathrm{Sq}^4)) \\ &= g(\Sigma^{-1}\,\mathrm{Sq}^3\,\mathrm{Sq}^1\,\mathrm{Sq}^4) \\ &= g(\Sigma^{-1}\,\mathrm{Sq}^7\,\mathrm{Sq}^1) \end{aligned}$$

Therefore $g(\Sigma^{-1}\,\mathrm{Sq}^6\,\mathrm{Sq}^2) = g(\Sigma^{-1}\,\mathrm{Sq}^7\,\mathrm{Sq}^1) = 0$. Therefore, $\mathrm{Hom}_A(H^*Z(1,1), H^*\Sigma^5 Y) = 0$ in dimension 6.

For dimensions 7 and 8, we have we have

$$g(\Sigma^{-1}\,\mathrm{Sq}^4\,\mathrm{Sq}^2\,\mathrm{Sq}^1) = \epsilon_1(\Sigma^5\alpha_1) \quad \text{and} \quad g(\Sigma^{-1}\,\mathrm{Sq}^8) = \epsilon_2(\Sigma^5\alpha_2)$$

for $\epsilon_1, \epsilon_2 \in \mathbb{F}_2$. Define two maps $f_1, f_2 \in \mathrm{Hom}_A(H^*Z(1,1), \Sigma^5 H^*Y)$ by setting $\epsilon_1 = 1, \epsilon_2 = 0$ for f_1, and $\epsilon_1 = 0$, $\epsilon_2 = 1$ for f_2. More explicitly, we have

$$\begin{aligned} f_1(\Sigma^{-1}\,\mathrm{Sq}^4\,\mathrm{Sq}^2\,\mathrm{Sq}^1) &= \Sigma^5\alpha_1 \\ f_1(\Sigma^{-1}\,\mathrm{Sq}^6\,\mathrm{Sq}^2\,\mathrm{Sq}^1) &= \Sigma^5\alpha_3 \\ f_1(\eta) &= 0 \text{ for all other elements } \eta \in H^*Z(0,0) \end{aligned}$$

and

$$
\begin{aligned}
f_2(\Sigma^{-1}\,\mathrm{Sq}^8) &= \Sigma^5\alpha_2 \\
f_2(\eta) &= 0 \text{ for all other elements } \eta \in H^*Z(1,1)
\end{aligned}
$$

Both of these elements are module maps in that they do not contradict any relations involving elements in lower dimensions. Therefore, these two elements are a basis of $\mathrm{Hom}_A(H^*Z(1,1), \Sigma^5 H^*Y)$ over A, so

$$\mathrm{Hom}_A(H^*Z(1,1), \Sigma^5 H^*Y) = \mathbb{F}_2 \oplus \mathbb{F}_2.$$

□

Thus we have

$$\mathrm{Ext}_A^{0,6}(H^*Z(1,1), \Sigma^{-1}H^*Y) \cong \mathrm{Ext}_A^{1,6}(H^*Z(1,1), H^*Y_1) \cong \mathbb{F}_2 \oplus \mathbb{F}_2$$

We identified two maps $f_1, f_2 \in \mathrm{Hom}_A(H^*Z(1,1), \Sigma^5 H^*Y)$. Starting with f_1, we calculate its image under δ_0. In order to do so, we examine the following diagram:

$$
\begin{array}{ccccccccc}
 & & \mathrm{Hom}_A(H^*Z(1,1),\Sigma^6 H^*Y_1) & \xrightarrow{i_*} & \mathrm{Hom}_A(H^*Z(1,1),\Sigma^5 A) & \xrightarrow{\pi_*} & \mathrm{Hom}_A(H^*Z(1,1),\Sigma^5 H^*Y) & & \\
 & & \downarrow d_0^* & & \downarrow d_0^* & & \downarrow d_0^* & & \\
0 & \to & \mathrm{Hom}_A(P_0(Z(1,1)),\Sigma^6 H^*Y_1) & \xrightarrow{i_*} & \mathrm{Hom}_A(P_0(Z(1,1)),\Sigma^5 A) & \xrightarrow{\pi_*} & \mathrm{Hom}_A(P_0(Z(1,1)),\Sigma^5 H^*Y) & \to & 0 \\
 & & \downarrow d_1^* & & \downarrow d_1^* & & \downarrow d_1^* & & \\
0 & \to & \mathrm{Hom}_A(P_1(Z(1,1)),\Sigma^6 H^*Y_1) & \xrightarrow{i_*} & \mathrm{Hom}_A(P_1(Z(1,1)),\Sigma^5 A) & \xrightarrow{\pi_*} & \mathrm{Hom}_A(P_1(Z(1,1)),\Sigma^5 H^*Y) & \to & 0
\end{array}
$$

where the horizontal maps are coming from the sequence

$$0 \to H^*Y_1 \xrightarrow{i} \Sigma^{-1}A \xrightarrow{\pi} H^*\Sigma^{-1}Y \to 0$$

Starting with $f_1 \in \mathrm{Hom}_A(H^*Z(1,1), \Sigma^5 H^*Y)$, we calculate it d_0^* as an element of $\mathrm{Hom}_A(P_0(Z(1,1)), \Sigma^5 H^*Y)$. The map $d_0^*(f_1) = (f_1 \circ d_0)$ is given on the generators of $P_0(Z(1,1))$ in dimensions 12 and below by:

$$\begin{aligned}
(f_1 \circ d_0)(i_3) &= 0 \\
(f_1 \circ d_0)(i_5) &= 0 \\
(f_1 \circ d_0)(i_6) &= \Sigma^5 \alpha_1 \\
(f_1 \circ d_0)(i_7) &= 0
\end{aligned}$$

Now we need to lift $d_0^*(f_1)$ to a map in $\gamma_1 \in \mathrm{Hom}_A(P_1(Z(1,1)), \Sigma^5 A)$. We choose γ_1 to be given by $\gamma_1(i_6) = \Sigma^5 \mathrm{Sq}^1$ and $\gamma_1(i_k) = 0$ for $k \neq 6, k \leq 12$.

Next, we apply d_1^* to get a map $d_1^*(\gamma_1) \in \mathrm{Hom}_A(P_1(Z(1,1)), \Sigma^5 A)$. The map $d_1^*(\gamma_1) = (\gamma_1 \circ d_1)$ is given on the generators of $P_1(Z(1,1))$ by:

$$\begin{aligned}
(\gamma_1 \circ d_1)(h_7) &= 0 \\
(\gamma_1 \circ d_1)(h_9) &= \Sigma^5 \mathrm{Sq}^3 \mathrm{Sq}^1 \\
(\gamma_1 \circ d_1)(h_{10}) &= \Sigma^5 \mathrm{Sq}^4 \mathrm{Sq}^1 \\
(\gamma_1 \circ d_1)(h_{12}) &= \Sigma^5 \mathrm{Sq}^4 \mathrm{Sq}^2 \mathrm{Sq}^1
\end{aligned}$$

Last, we must lift $d_1^*(\gamma_1)$ to $\mathrm{Hom}_A(P_1(Z(1,1)), \Sigma^6 H^*Y_1)$. We use that $H^*Y_1 \subseteq \Sigma^{-1} A$ and that this is an isomorphism in dimensions where $d_1^*(\gamma_1) \neq 0$. Call this lift g_1; on the generators of $P_1(Z(1,1))$ through dimension 12, g_1 is given by the same

formula as $d_1^*(\gamma_1)$. Explicitly, we have:

$$\begin{aligned} g_1(h_7) &= 0 \\ g_1(h_9) &= \Sigma^5 \mathrm{Sq}^3 \mathrm{Sq}^1 \\ g_1(h_{10}) &= \Sigma^5 \mathrm{Sq}^4 \mathrm{Sq}^1 \\ g_1(h_{12}) &= \Sigma^5 \mathrm{Sq}^4 \mathrm{Sq}^2 \mathrm{Sq}^1 \end{aligned}$$

It is evident that g_1 is a lift of $d_1^*(\gamma_1)$ since $H^*Y_1 = \Sigma^{-1}K_0(Y)$ is a submodule of $\Sigma^{-1}A$.

We have identified a representative of one of the two generators of $\mathrm{Ext}_A^{1,6}(H^*Z(1,1), H^*Y_1)$. We will show that it does not pull back to a generator of $\mathrm{Ext}_A^{1,6}(H^*Z(1,1), H^*Z(1,1))$ by examining the short exact sequence

$$0 \to \mathrm{Hom}_A(P_1(Z(1,1)), \Sigma^6 H^*Z(1,0)) \xrightarrow{\iota_*} \mathrm{Hom}_A(P_1(Z(1,1)), \Sigma^6 H^*Y_1)$$
$$\xrightarrow{(\phi_3+\phi_4+\phi_6)_*} \mathrm{Hom}_A(P_1(Z(1,1)), \Sigma^8 H^*Y) \to 0$$

We see that $(\phi_3 + \phi_4)_*(g_1) = (\phi_3 + \phi_6) \circ g_1$ is given by

$$\begin{aligned} ((\phi_3+\phi_4+\phi_6)\circ g_1)(h_7) &= 0 \\ ((\phi_3+\phi_4+\phi_6)\circ g_1)(h_9) &= \Sigma^8 \alpha_1 \\ ((\phi_3+\phi_4+\phi_6)\circ g_1)(h_{10}) &= \Sigma^8 \alpha_2 \\ ((\phi_3+\phi_4+\phi_6)\circ g_1)(h_{12}) &= 0 \end{aligned}$$

Thus $g_1 \notin \ker((\phi_3 + \phi_4 + \phi_6)_*) = \mathrm{im}(\iota_*)$. Since $\mathrm{Hom}_A(P_0(Z(1,1)), \Sigma^8 H^*Y) = 0$, we know that g_1 is not a boundary. On the level of Ext groups, this

shows that the element $[g_1] \in \mathrm{Ext}_A^{1,6}(H^*Z(1,1), H^*Y_1)$ does not pull back under the inclusion $\mathrm{Ext}_A^{1,6}(H^*Z(1,1), H^*Z(1,1)) \hookrightarrow \mathrm{Ext}_A^{1,6}(H^*Z(1,1), H^*Y_1)$, so $\mathrm{Ext}_A^{1,6}(H^*Z(1,1), H^*Z(1,1))$ has at most rank 1.

We now investigate $g_2 = \delta_0(f_2)$.

We take $f_2 \in \mathrm{Hom}_A(H^*Z(1,1), \Sigma^5 H^*Y)$ and apply d_0^* to get $d_0^*(f_2) = f_2 \circ d_0 \in \mathrm{Hom}_A(P_0(Z(1,1)), \Sigma^5 H^*Y)$. We have

$$\begin{aligned}(f_2 \circ d_0)(i_3) &= 0\\ (f_2 \circ d_0)(i_5) &= 0\\ (f_2 \circ d_0)(i_6) &= 0\\ (f_2 \circ d_0)(i_7) &= \Sigma^5 \alpha_2\end{aligned}$$

Now we need to lift $d_0^*(f_2)$ to a map in $\gamma_2 \in \mathrm{Hom}_A(P_1(Z(1,1)), \Sigma^5 A)$. We choose γ_2 so that $\gamma_1(i_7) = \Sigma^5 \mathrm{Sq}^2$ and $\gamma_2(i_k) = 0$ for $k \neq 7, k \leq 12$. Next, we apply d_1^* to get a map $d_1^*(\gamma_2) \in \mathrm{Hom}_A(P_1(Z(1,1)), \Sigma^5 A)$. The map $d_1^*(\gamma_2) = (\gamma_2 \circ d_1)$ is given on the generators of $P_1(Z(1,1))$ by:

$$\begin{aligned}(\gamma_2 \circ d_1)(h_7) &= 0\\ (\gamma_2 \circ d_1)(h_9) &= 0\\ (\gamma_2 \circ d_1)(h_{10}) &= 0\\ (\gamma_2 \circ d_1)(h_{12}) &= 0\end{aligned}$$

The calculations here are simple from the information provided; essentially, this map is 0 through dimension 12 because $d_1(h_i)$ never involves an A-multiple of i_7 in dimensions below 12.

So we lift $d_1^*(\gamma_2)$ to $g_2 \in \mathrm{Hom}_A(P_1(Z(1,1)), H^*Y_1)$ with

$$\begin{aligned} g_2(h_7) &= 0 \\ g_2(h_9) &= 0 \\ g_2(h_{10}) &= 0 \\ g_2(h_{12}) &= 0 \end{aligned}$$

We have now identified a representative of the second generator of $\mathrm{Ext}_A^{1,6}(H^*Z(1,1), H^*Y_1)$. Unlike the first generator, we have $(\phi_3 + \phi_4 + \phi_6)_*(g_2) = 0$ in the short exact sequence

$$0 \to \mathrm{Hom}_A(P_1(Z(1,1)), \Sigma^6 H^*Z(1,0)) \xrightarrow{\iota_*} \mathrm{Hom}_A(P_1(Z(1,1)), \Sigma^6 H^*Y_1)$$
$$\xrightarrow{(\phi_3+\phi_4+\phi_6)_*} \mathrm{Hom}_A(P_1(Z(1,1)), \Sigma^8 H^*Y) \to 0$$

because $\Sigma^8 H^*Y$ is nonzero only in dimensions 8, 9, 10, and 11, and $(\phi_3 + \phi_4 + \phi_6)_*(g_2)$ is zero in those dimensions. This we can find a map $\tilde{g}_2 \in \mathrm{Hom}_A(P_1(Z(1,1)), \Sigma^6 H^*Z(1,1))$; however, this map is zero through dimension 11. Since $H_*(K_0Z(1,1), x_2)$ is nonzero in dimension 11, $\tilde{g}_2$ cannot be a u_2^1-map.

□

CHAPTER VI

NONEXISTENCE OF U_2^2 ON $Z(0,0)$

6.1. Calculation of $P_\bullet H^*Z(0,0)$

We ask whether there is an element of $\mathrm{Ext}_A^{2,12}(H^*Z(0,0), H^*Z(0,0))$ corresponding to u_2^2, and so we must identify the elements of this group explicitly. We outline the procedure for doing this here. Throughout, we use Z for $Z(0,0)$ for brevity.

1. Calculate H^*Z through dimension 17;

2. Calculate $P_\bullet Z$ through dimension 17 for $\bullet = 0,1,2$;

3. Examine the long exact sequence

$$\begin{array}{l} 0 \to \mathrm{Ext}_A^{0,12}(H^*Z,H^*Z) \to \mathrm{Ext}_A^{0,12}(H^*Z,H^*Y_1) \to \mathrm{Ext}_A^{0,12}(H^*Z,H^*\Sigma^2 Y) \xrightarrow{\delta_0} \\ \mathrm{Ext}_A^{1,12}(H^*Z,H^*Z) \to \mathrm{Ext}_A^{1,12}(H^*Z,H^*Y_1) \to \mathrm{Ext}_A^{1,12}(H^*Z,H^*\Sigma^2 Y)) \xrightarrow{\delta_1} \\ \mathrm{Ext}_A^{2,12}(H^*Z,H^*Z) \to \mathrm{Ext}_A^{2,12}(H^*Z,H^*Y_1) \to \mathrm{Ext}_A^{2,12}(H^*Z,H^*\Sigma^2 Y)) \to \dots \end{array}$$

   and use this to compute $\mathrm{Ext}_A^{2,12}(H^*Z,H^*Z)$.

Once we have representatives for elements of $\mathrm{Ext}_A^{2,12}(H^*Z,H^*Z)$, which will be given as maps $P_2(Z) \to \Sigma^{12}H^*Z$, then we can use Theorem 2.4 from [Rei17] to check whether or not they are u_2 maps. This will involve calculating $H(K_1(Z), x_2)$, which will be similar to the computation of $H(K_0(Z), x_2)$. Recall that $H^*Z = \ker(\phi_3)$ from Section 2.3., and $K_iZ = \ker(d_i\colon P_iZ \to P_{i-1}Z)$ where P_iZ is the i^{th} stage in a projective resolution of H^*Z. H^*Z is described in Figures 6.1., 6.2., 6.3., and 6.4.

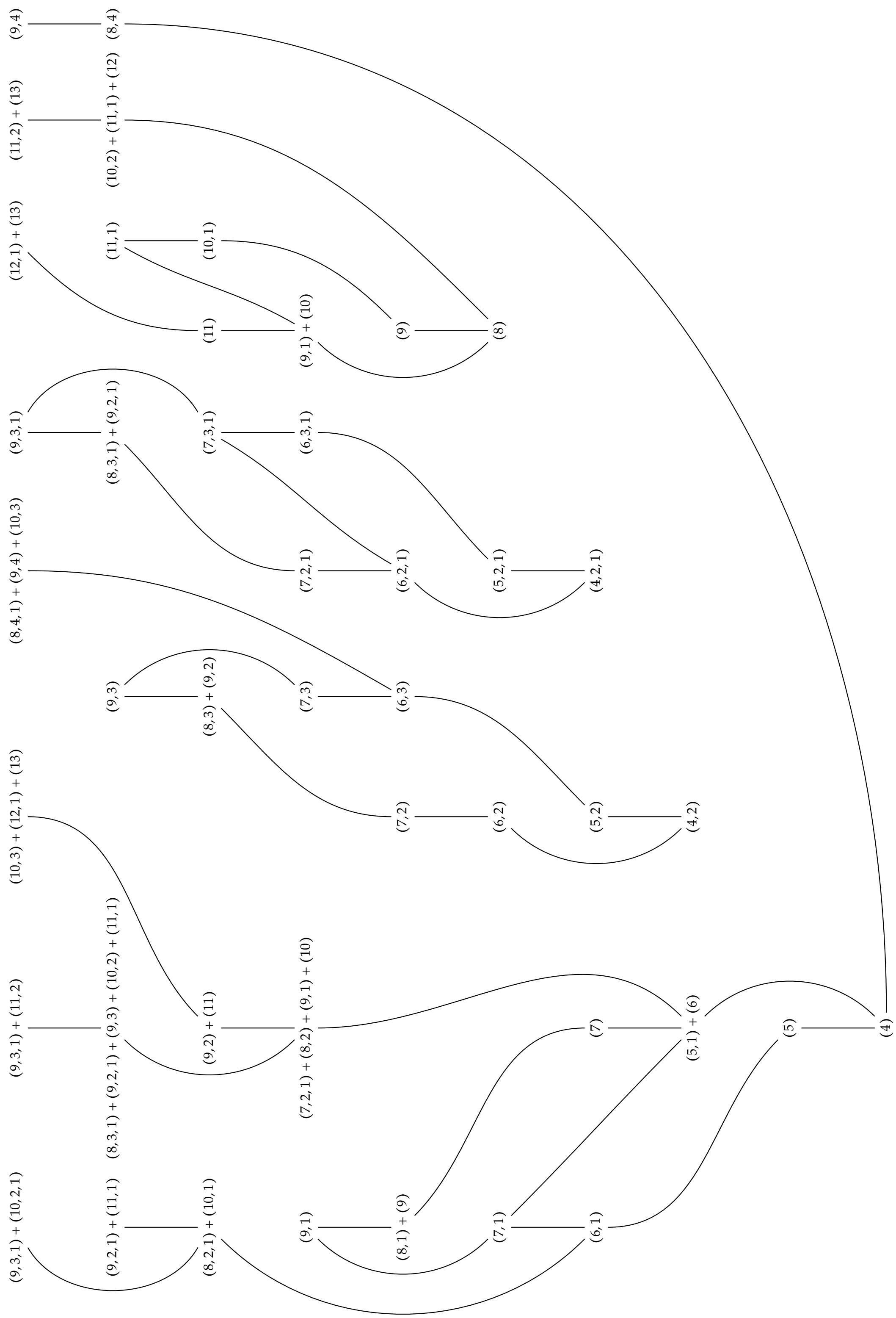


FIGURE 6.1. H^*Z through dimension 17, Part 1

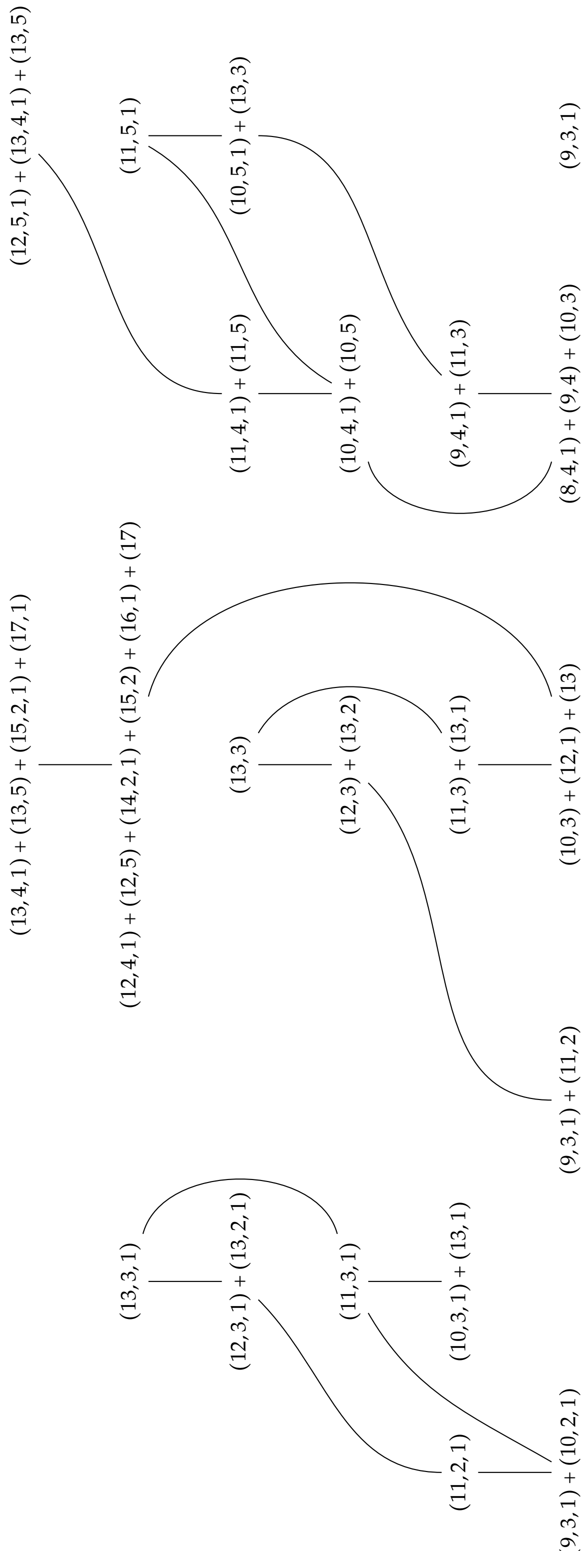


FIGURE 6.2. H^*Z through dimension 17, Part 2

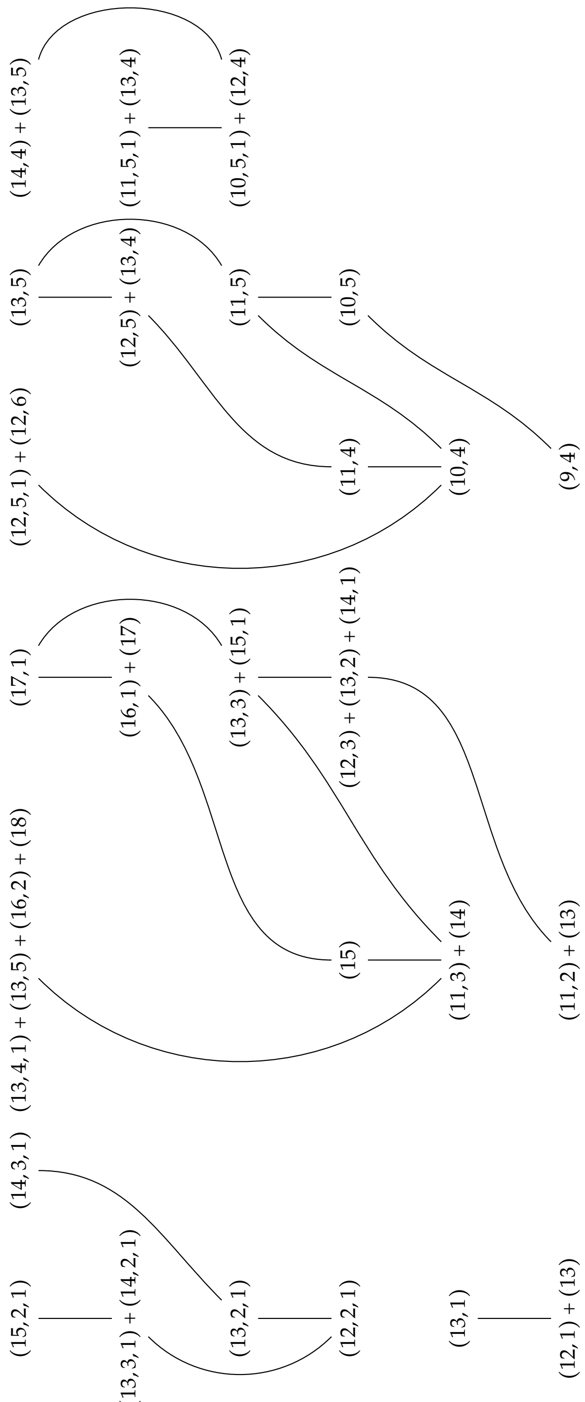


FIGURE 6.3. H^*Z through dimension 17, Part 3

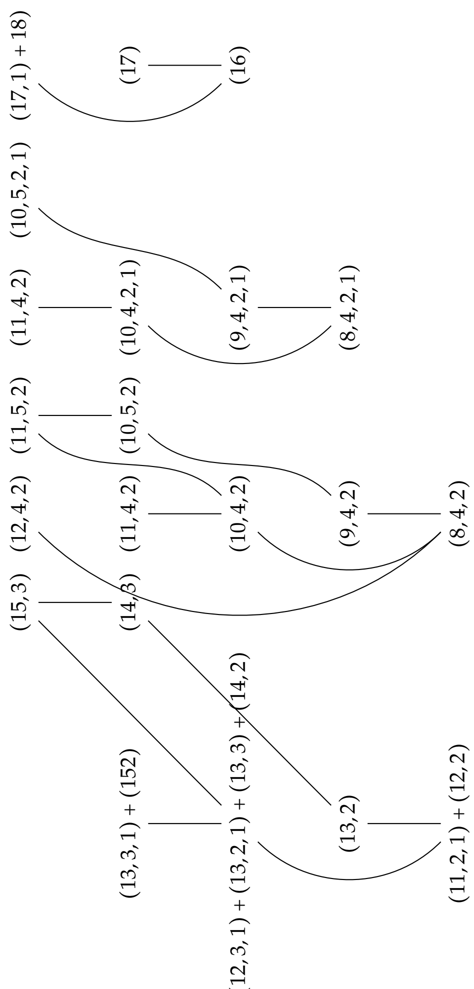


FIGURE 6.4. H^*Z through dimension 17, Part 4

The above images can be glued together along the matching entries in dimension 13, with the final image beginning in dimension 14. Though some elements may appear to be new generators, they are in fact connected to lower-dimensional elements via Sq^2, Sq^4, or Sq^8. The only generators in this range occur in dimensions 3, 5,6, 7, and 15.

6.1.1. Calculation of $P_\bullet(Z)$ through dimension 17

We observe that there are generators of H^*Z in dimensions 3, 5, 6, 7, and 15. So we choose

$$P_0(Z) = \Sigma^3 A \oplus \Sigma^5 A \oplus \Sigma^6 A \oplus \Sigma^7 A \oplus \Sigma^{15} A \oplus \bigoplus_{l_k > 17} \Sigma^{l_k} A$$

Let i_k stand for the generator of $\Sigma^k A$ in $P_0(Z)$. In order to calculate $P_1(Z)$, we must find $K_0(Z) = \ker(d_0\colon P_0(Z) \to H^*Z)$. In Table 6.1., we calculate d_0 in each dimension. In the "input" column, the elements are expressed as linear combinations of Serre-Cartan basis elements acting on the generators of $P_0(Z)$ so as to give a basis for $P_0 Z$ in the appropriate dimension. For example, $(3,1)i_3$ means $\mathrm{Sq}^3\mathrm{Sq}^1 i_3$. In the "output" column, we express elements of H^*Z as Serre-Cartan basis elements in $\Sigma^{-1}A$, though the desuspension is omitted for brevity.

TABLE 6.1. Calculation of $d_0\colon P_0 Z \to H^*Z$ through dimension 17

dimension	input	output
3	i_3	(4)
4	$(1)i_3$	(5)
5	$(2)i_3$	$(5,1)+(6)$
	i_5	$(4,2)$
6	$(7)i_3$	(7)
	$(2,1)i_3$	$(6,1)$
	$(1)i_5$	$(5,2)$
	i_6	$(4,2,1)$
7	$(3,1)i_3$	$(7,1)$
	$(4)i_3$	$(6,2)+(7,1)$

TABLE 6.1. (continued)

dimension	input	output
	$(2)i_5$	$(6,2)$
	$(1)i_6$	$(5,2,1)$
	i_7	8
8	$(5)i_3$	$(7,2)$
	$(4,1)i_3$	$(7,2)+(8,1)+(9)$
	$(3)i_5$	$(7,2)$
	$(2,1)i_5$	$(6,3)$
	$(2)i_6$	$(6,2,1)$
	$(1)i_7$	(9)
9	$(6)i_3$	$(7,3)$
	$(5,1)i_3$	$(9,1)$
	$(4,2)i_3$	$(7,2,1)+(8,2)+(9,1)+(10)$
	$(4)i_5$	$(6,3,1)+(7,3)$
	$(3,1)i_5$	$(7,3)$
	$(3)i_6$	$(7,2,1)$
	$(2,1)i_6$	$(6,3,1)$
	$(2)i_7$	$(9,1)+(10)$
10	$(7)i_3$	zero
	$(6,1)i_3$	$(8,3)+(9,2)$
	$(5,2)i_3$	$(9,2)+(11)$
	$(4,2,1)i_3$	$(8,2,1)+(10,1)$
	$(5)i_5$	$(7,3,1)$

TABLE 6.1. (continued)

dimension	input	output
	$(4,1)i_5$	$(7,3,1)+(8,3)+(9,2)$
	$(4)i_6$	$(7,3,1)$
	$(3,1)i_6$	$(7,3,1)$
	$(3)i_7$	(11)
	$(2,1)i_7$	$(10,1)$
11	$(8)i_3$	$(8,4)$
	$(7,1)i_3$	$(9,3)$
	$(6,2)i_3$	$(8,3,1)+(9,2,1)+(9,3)+(10,2)+(11,1)$
	$(5,2,1)i_3$	$(9,2,1)+(11,1)$
	$(6)i_5$	zero
	$(5,1)i_5$	$(9,3)$
	$(4,2)i_5$	$(8,3,1)+(10,2)$
	$(5)i_6$	zero
	$(4,1)i_6$	$(8,3,1)+(9,2,1)$
	$(4)i_7$	$(10,2)+(11,1)+(12)$
	$(3,1)i_7$	$(11,1)$
12	$(9)i_3$	$(9,4)$
	$(8,1)i_3$	$(9,4)$
	$(7,2)i_3$	$(9,3,1)+(11,2)$
	$(6,3)i_3$	$(10,3)+(12,1)+(13)$
	$(6,2,1)i_3$	$(9,3,1)+(10,2,1)$
	$(7)i_5$	zero

TABLE 6.1. (continued)

dimension	input	output
	$(6,1)i_5$	$(9,3,1)$
	$(5,2)i_5$	$(9,3,1)+(11,2)$
	$(4,2,1)i_5$	$(8,4,1)+(9,4)+(10,3)$
	$(6)i_6$	zero
	$(5,1)i_6$	$(9,3,1)$
	$(4,2)i_6$	$(10,2,1)$
	$(5)i_7$	$(11,2)+(13)$
	$(4,1)i_7$	$(11,2)+(12,1)$
13	$(10)i_3$	$(10,4)$
	$(9,1)i_3$	zero
	$(8,2)i_3$	$(9,4,1)+(10,4)+(11,3)$
	$(7,3)i_3$	$(11,3)+(13,1)$
	$(7,2,1)i_3$	$(11,2,1)$
	$(6,3,1)i_3$	$(10,3,1)+(13,1)$
	$(8)i_5$	$(8,4,2)$
	$(7,1)i_5$	zero
	$(6,2)i_5$	$(10,3,1)+(11,3)$
	$(5,2,1)i_5$	$(9,4,1)+(11,3)$
	$(7)i_6$	zero
	$(6,1)i_6$	zero
	$(5,2)i_6$	$(11,2,1)$
	$(4,2,1)i_6$	$(9,4,1)+(10,3,1)$

TABLE 6.1. (continued)

dimension	input	output
	$(6)i_7$	$(11,3)+(13,1)+(14)$
	$(5,1)i_7$	$(13,1)$
	$(4,2)i_7$	$(11,2,1)+(12,2)$
14	$(11)i_3$	$(11,4)$
	$(10,1)i_3$	$(10,5)$
	$(9,2)i_3$	$(11,4)$
	$(8,3)i_3$	$(11,4)+(12,3)+(13,2)$
	$(8,2,1)i_3$	$(10,4,1)+(11,3,1)$
	$(7,3,1)i_3$	$(11,3,1)$
	$(9)i_5$	$(9,4,2)$
	$(8,1)i_5$	$(9,4,2)$
	$(7,2)i_5$	$(11,3,1)$
	$(6,3)i_5$	$(12,3)+(13,2)$
	$(6,2,1)i_5$	$(10,4,1)+(10,5)$
	$(8)i_6$	$(8,4,2,1)$
	$(7,1)i_6$	zero
	$(6,2)i_6$	$(11,3,1)$
	$(5,2,1)i_6$	$(11,3,1)$
	$(7)i_7$	(15)
	$(6,1)i_7$	$(12,3)+(13,2)+(14,1)$
	$(5,2)i_7$	$(13,2)$
	$(4,2,1)i_7$	$(12,2,1)$

TABLE 6.1. (continued)

dimension	input	output
15	$(12)i_3$	$(12,4)$
	$(11,1)i_3$	$(11,5)$
	$(10,2)i_3$	$(10,5,1)+(11,5)$
	$(9,3)i_3$	$(13,3)$
	$(9,2,1)i_3$	$(11,4,1)$
	$(8,4)i_3$	$(10,4,2)+(11,4,1)+(12,3,1)+(13,2,1)$
	$(8,3,1)i_3$	$(11,4,1)+(12,3,1)+(13,2,1)$
	$(10)i_5$	$(10,4,2)$
	$(9,1)i_5$	zero
	$(8,2)i_5$	$(10,4,2)$
	$(7,3)i_5$	$(13,3)$
	$(7,2,1)i_5$	$(11,4,1)+(11,5)$
	$(6,3,1)i_5$	$(10,5,1)+(13,3)$
	$(9)i_6$	$(9,4,2,1)$
	$(8,1)i_6$	$(9,4,2,1)$
	$(7,2)i_6$	zero
	$(6,3)i_6$	$(12,3,1)+(13,2,1)$
	$(6,2,1)i_6$	$(10,5,1)$
	$(8)i_7$	$(12,4)+(14,2)+(15,1)$
	$(7,1)i_7$	$(13,3)+(15,1)$
	$(6,2)i_7$	$(12,3,1)+(13,2,1)+(13,3)+(14,2)$
	$(5,2,1)i_7$	$(13,2,1)$

TABLE 6.1. (continued)

dimension	input	output
	i_{15}	(16)
16	$(13)i_3$	$(13,4)$
	$(12,1)i_3$	$(12,5)$
	$(11,2)i_3$	$(11,5,1)$
	$(10,3)i_3$	$(12,5)+(13,4)$
	$(10,2,1)i_3$	$(11,5,1)$
	$(9,4)i_3$	$(11,4,2)+(13,3,1)$
	$(9,3,1)i_3$	$(13,3,1)$
	$(8,4,1)i_3$	$(11,4,2)+(12,4,1)+(13,3,1)+$
		$(13,4)+(14,2,1)+(15,2)+(16,1)+(17)$
	$(11)i_5$	$(11,4,2)$
	$(10,1)i_5$	$(10,5,2)$
	$(9,2)i_5$	$(11,4,2)$
	$(8,3)i_5$	$(11,4,2)+(13,3,1)$
	$(8,2,1)i_5$	$(10,5,2)+(11,5,1)$
	$(7,3,1)i_5$	$(11,5,1)$
	$(10)i_6$	$(10,4,2,1)$
	$(9,1)i_6$	zero
	$(8,2)i_6$	$(10,4,2,1)$
	$(7,3)i_6$	$(13,3,1)$
	$(7,2,1)i_6$	$(11,5,1)$
	$(6,3,1)i_6$	$(13,3,1)$

TABLE 6.1. (continued)

dimension	input	output
	$(9)i_7$	$(13,4)+(15,2)$
	$(8,1)i_7$	$(13,4)+(15,2)+(16,1)+(17)$
	$(7,2)i_7$	$(13,3,1)+(15,2)$
	$(6,3)i_7$	$(14,3)$
	$(6,2,1)i_7$	$(13,3,1)+(14,2,1)$
	$(1)i_{15}$	(17)
17	$(14)i_3$	$(14,4)$
	$(13,1)i_3$	$(13,5)$
	$(12,2)i_3$	$(12,5,1)+(12,6)$
	$(11,3)i_3$	$(13,5)$
	$(11,2,1)i_3$	zero
	$(10,4)i_3$	$(11,5,2)+(12,5,1)+(13,4,1)$
	$(10,3,1)i_3$	$(12,5,1)+(13,4,1)$
	$(9,4,1)i_3$	$(13,4,1)+(15,2,1)+(17,1)$
	$(8,4,2)i_3$	$(11,4,2,1)+(12,4,2)+(13,4,1)+(14,3,1)+$
		$(14,4)+(15,2,1)+(17,1)+(16,2)+(18)$
	$(12)i_5$	$(12,4,2)$
	$(11,1)i_5$	$(11,5,2)$
	$(10,2)i_5$	$(11,5,2)$
	$(9,3)i_5$	zero
	$(9,2,1)i_5$	$(11,5,2)$
	$(8,4)i_5$	$(10,5,2,1)+(11,5,2)+$

TABLE 6.1. (continued)

dimension	input	output
		$(12,5,1)+(13,4,1)+(13,5)$
	$(8,3,1)i_5$	$(11,5,2)+(12,5,1)+(13,4,1)+(13,5)$
	$(11)i_6$	$(11,4,2,1)$
	$(10,1)i_6$	$(10,5,2,1)$
	$(9,2)i_6$	$(11,4,2,1)$
	$(8,3)i_6$	$(11,4,2,1)$
	$(8,2,1)i_6$	$(10,5,2,1)$
	$(7,3,1)i_6$	zero
	$(10)i_7$	$(13,5)+(14,4)+(15,3)$
	$(9,1)i_7$	$(17,1)$
	$(8,2)i_7$	$(13,4,1)+(15,2,1)+(17,1)+(14,4)+$
		$(15,3)+(16,2)+(18)$
	$(7,2,1)i_7$	$(15,2,1)$
	$(6,3,1)i_7$	$(14,3,1)$
	$(2)i_{15}$	$(17,1)+(18)$

Now that we know $d_0\colon P_0(Z) \to H^*Z$ explicitly, we can write down elements of $K_0(Z) = \ker d_0$. Observe that $K_0(Z)$ is zero below dimension 7. Table 6.2. lists the generators of $K_0(Z)$ in dimensions 7 through 17, and its Steenrod algebra structure is given by the Figures 6.5., 6.6., 6.7..

TABLE 6.2. Generators of $K_0 Z$ through dimension 17

dimension	generators of $K_0(Z)$
7	$[(3,1)+(4)]i_3+(2)i_5$
8	$(5)i_3+(3)i_5$
9	$(6)i_3+(3,1)i_5$
	$[(3,1)+(4)]i_5+(2,1)i_6$
10	$(7)i_3$
	$(6,1)i_3+[(4,1)+(5)]i_5$
	$(5)i_5+(3,1)i_6$
	$[(3,1)+(4)]i_6$
11	$(7,1)i_3+(5,1)i_5$
	$[(5,2,1)+(6,2)+(7,1)]i_3+(4,2)i_5$
	$(6)i_5$
	$(5)i_6$
12	$[(8,1)+(9)]i_3$
	$(7,2)i_3+(5,2)i_5$
	$(7)i_5$
	$(6,1)i_5+(5,1)i_6$
	$(6)i_6$
	$(6,2,1)i_3+[(4,2)+(5,1)]i_6$
13	$[(8,2)+(10)]i_3+(5,2,1)i_5$
	$(9,1)i_3$
	$[(6,3,1)+(7,3)+(9,1)]i_3+(6,2)i_5$
	$[(5,2,1)+(6,2)+(7,1)]i_5+(4,2,1)i_6$

TABLE 6.2. (continued)

dimension	generators of $K_0(Z)$
	$(7,1)i_5$
	$(7)i_6$
	$(6,1)i_6$
	$(7,2,1)i_3+(6,2)i_6$
14	$[(9,2)+(11)]i_3$
	$[(8,2,1)+(10,1)]i_3+[(6,2,1)+(7,2)+(8,1)+(9)]i_5$
	$(7,3,1)i_3+(7,2)i_5$
	$[(8,3)+(9,2)]i_3+(6,3)i_5$
	$(7,2)i_5+(5,2,1)i_6$
	$[(8,1)+(9)]i_5$
	$(7,1)i_6$
	$(7,3,1)i_3+(6,2)i_6$
15	$[(8,3,1)+(8,4)]i_3+(8,2)i_5$
	$[(9,3)+(10,2)+(11,1)]i_3+(6,3,1)i_5$
	$[(9,2,1)+(11,1)]i_3+[(7,2,1)+(9,1)]i_5$
	$(9,3)i_3+(7,3)i_5$
	$[(6,3,1)+(7,3)+(9,1)]i_5+(6,2,1)i_6$
	$[(8,2)+(10)]i_5$
	$(9,1)i_5$
	$[(8,1)+(9)]i_6$
	$(7,2)i_6$
	$[(8,3,1)+(9,2,1)]i_3+(6,3)i_6$

TABLE 6.2. (continued)

dimension	generators of $K_0(Z)$
	$(12)i_3 + (6,3)i_6 + [(6,2) + (7,1) + (8)]i_7$
16	$[(9,3,1) + (9,4)]i_3 + (9,2)i_5$
	$(11,2,1)i_3 + (7,3,1)i_5$
	$[(10,3) + (12,1) + (13)]i_3$
	$[(9,3,1) + (10,2,1)]i_3 + [(7,3,1) + (8,3) + (9,2)]i_5$
	$(9,3,1) + [(8,3) + (9,2)]i_5$
	$(7,3,1)i_5 + (7,2,1)i_6$
	$[(8,3) + (9,2)]i_5 + (6,3,1)i_6$
	$[(9,2) + (11)]i_5$
	$[(8,2,1) + (10,1)]i_5 + [(7,2,1) + (9,1)]i_6$
	$[(8,2) + (10)]i_6$
	$(9,1)i_6$
	$(9,3,1)i_3 + (7,3)i_6$
	$(13)i_3 + (7,3)i_6 + [(7,2) + (9)]i_7$
17	$[(10,3,1) + (10,4)]i_3 + [(9,3) + (10,2)]i_5$
	$[(11,3) + (13,1)]i_3$
	$(11,2,1)i_3 + (9,3)i_5$
	$[(10,3,1) + (13,1)]i_3 + [(8,3,1) + (9,2,1)]i_5$
	$(9,3)i_5$
	$(9,3)i_5 + (7,3,1)i_6$
	$[(9,3) + (10,2) + (11,1)]i_5$
	$[(9,2,1) + (11,1)]i_5$

TABLE 6.2. (continued)

dimension	generators of $K_0(Z)$
	$[(8,3,1)+(8,4)]i_5+(8,2,1)i_6$
	$[(9,2)+(11)]i_6$
	$[(8,2,1)+(10,1)]i_6$
	$[(8,3)+(9,2)]i_6$
	$[(13,1)+(14)]i_3+[(7,3)+(10)]i_7$
	$(8,4,2)i_3+(12)i_5+(9,2)i_6+[(6,3,1)+(7,3)+(8,2)]i_7$

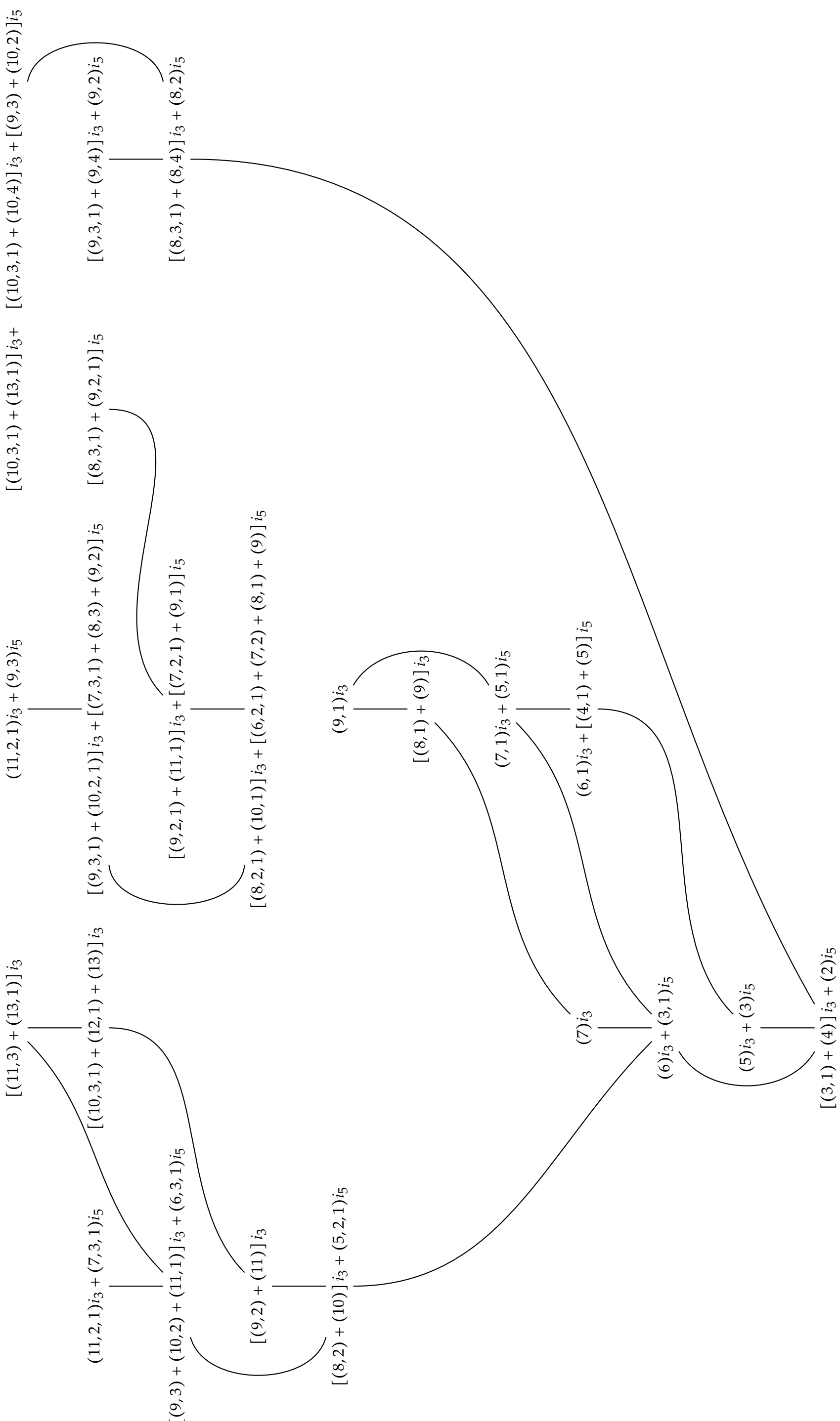


FIGURE 6.5. $K_0 Z$ through dimension 17, Part 1

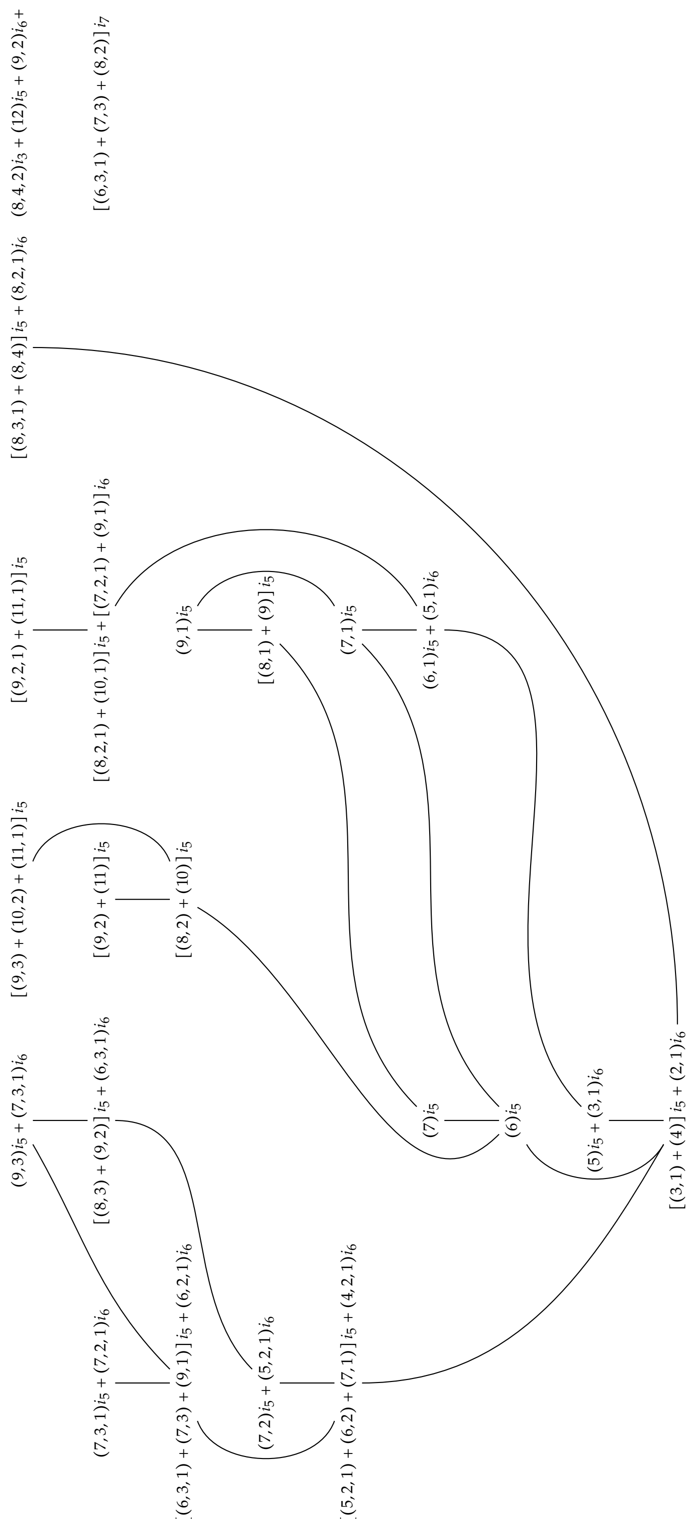


FIGURE 6.6. K_0Z through dimension 17, Part 2

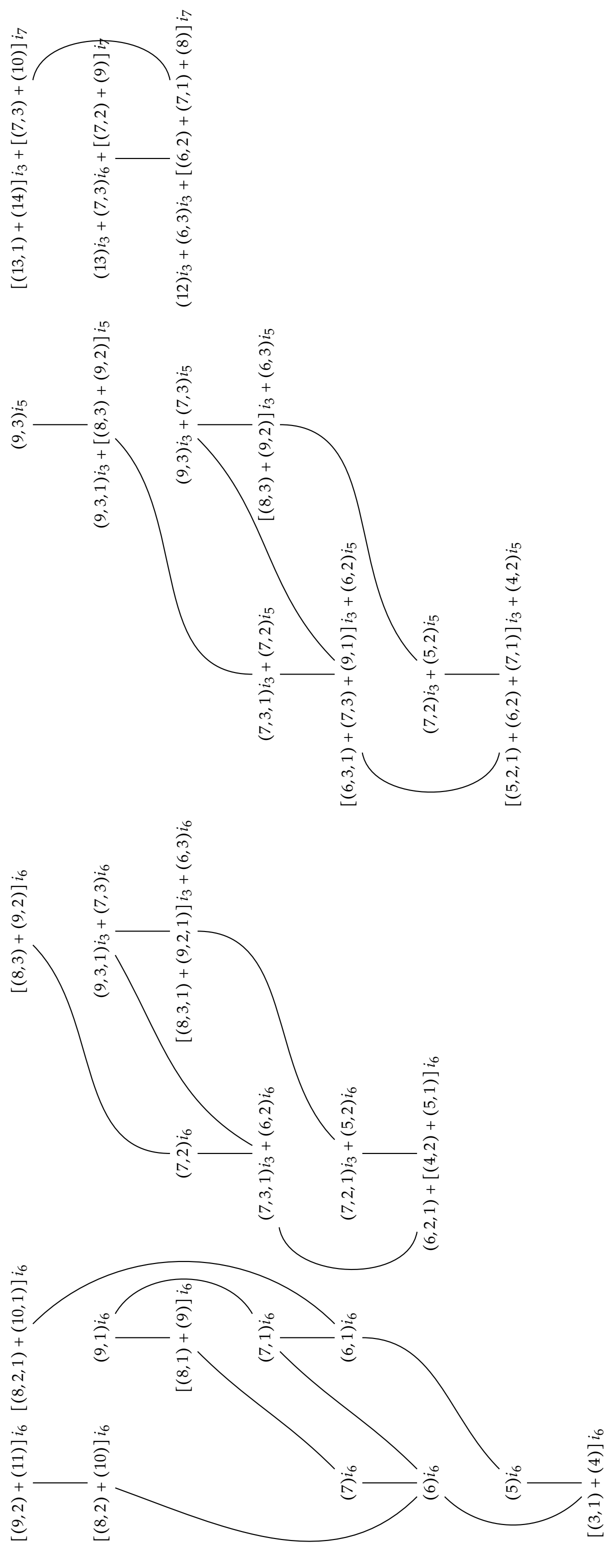


FIGURE 6.7. K_0Z through dimension 17, Part 3

Thus, we choose

$$P_1(Z) = \Sigma^7 A \oplus \Sigma^9 A \oplus \Sigma^{10} \oplus \Sigma^{12} A \oplus \Sigma^{15} A \oplus \Sigma^{17} A$$

Let h_k denote the generator of $\Sigma^k A$ in $P_1(Z)$. The map $d_1 \colon P_1(Z) \to P_0(Z)$ by

$$\begin{aligned}
d_1(h_7) &= (\mathrm{Sq}^3\,\mathrm{Sq}^1 + \mathrm{Sq}^4) i_3 + \mathrm{Sq}^2\, i_5 \\
d_1(h_9) &= (\mathrm{Sq}^3\,\mathrm{Sq}^1 + \mathrm{Sq}^4) i_5 + \mathrm{Sq}^2\,\mathrm{Sq}^1\, i_6 \\
d_1(h_{10}) &= (\mathrm{Sq}^3\,\mathrm{Sq}^1 + \mathrm{Sq}^4) i_6 \\
d_1(h_{12}) &= \mathrm{Sq}^6\,\mathrm{Sq}^2\,\mathrm{Sq}^1\, i_3 + (\mathrm{Sq}^4\,\mathrm{Sq}^2 + \mathrm{Sq}^5\,\mathrm{Sq}^1) i_6 \\
d_1(h_{15}) &= \mathrm{Sq}^{12}\, i_3 + \mathrm{Sq}^6\,\mathrm{Sq}^3\, i_6 + (\mathrm{Sq}^6\,\mathrm{Sq}^2 + \mathrm{Sq}^7\,\mathrm{Sq}^1 + \mathrm{Sq}^8) i_7 \\
d_1(h_{17}) &= \mathrm{Sq}^8\,\mathrm{Sq}^4\,\mathrm{Sq}^2\, i_3 + \mathrm{Sq}^{12}\, i_5 + \mathrm{Sq}^9\,\mathrm{Sq}^2\, i_6 + (\mathrm{Sq}^6\,\mathrm{Sq}^3\,\mathrm{Sq}^1 + \mathrm{Sq}^7\,\mathrm{Sq}^3 + \mathrm{Sq}^8\,\mathrm{Sq}^2) i_7
\end{aligned}$$

In order to find $P_2(Z)$, we must calculate $K_1(Z) = \ker(d_1 \colon P_1(Z) \to P_0(Z))$. We calculate d_1 in a manner similar to our calculation of d_0 above.

TABLE 6.3. Calculation of $d_1 \colon P_1 Z \to P_0 Z$ through dimension 17

dimension	input	output
7	h_7	$[(3,1)+(4)]i_3 + (2)i_5$
8	$(1)h_7$	$(5)i_3 + (3)i_5$
9	$(2)h_7$	$(6)i_3 + (3,1)i_5$
	h_9	$[(3,1)+(4)]i_5 + (2,1)i_5$
10	$(3)h_7$	$(7)i_3$
	$(2,1)h_7$	$(6,1)i_3 + [(4,1)+(5)]i_5$

TABLE 6.3. (continued)

dimension	input	output
	$(1)h_9$	$(5)i_5+(3,1)i_6$
	h_{10}	$[(3,1)+(4)]i_6$
11	$(4)h_7$	$[(5,2,1)+(6,2)+(7,1)]i_3+(4,2)i_5$
	$(3,1)h_7$	$(7,1)i_3+(5,1)i_5$
	$(2)h_9$	$(6)i_5$
	$(1)h_{10}$	$(5)i_6$
12	$(5)h_7$	$(7,2)i_3+(5,2)i_5$
	$(4,1)h_7$	$[(7,2)+(8,1)+(9)]i_3+(5,2)i_5$
	$(3)h_9$	$(7)i_5$
	$(2,1)h_9$	$(6,1)i_5+(5,1)i_6$
	$(2)h_{10}$	$(6)i_6$
	h_{12}	$(6,2,1)i_3+[(4,2)+(5,1)]i_6$
13	$(6)h_7$	$[(6,3,1)+(7,3)]i_3+(6,2)i_5$
	$(5,1)h_7$	$(9,1)i_3$
	$(4,2)h_7$	$[(8,2)+(10)]i_3+(5,2,1)i_5$
	$(4)h_9$	$[(5,2,1)+(6,2)+(7,1)]i_5+(4,2,1)i_6$
	$(3,1)h_9$	$(7,1)i_5$
	$(3)h_{10}$	$(7)i_6$
	$(2,1)h_{10}$	$(6,1)i_6$
	$(1)h_{12}$	$(7,2,1)i_3+(5,2)i_6$
14	$(7)h_7$	$(7,3,1)i_3+(7,2)i_5$
	$(6,1)h_7$	$[(8,3)+(9,2)]i_3+(6,3)i_5$

TABLE 6.3. (continued)

dimension	input	output
	$(5,2)h_7$	$[(9,2)+(11)]i_3$
	$(4,2,1)h_7$	$[(8,2,1)+(10,1)]i_3+[(6,2,1)+(7,2)+(8,1)+(9)]i_5$
	$(5)h_9$	$(7,2)i_5+(5,2,1)i_6$
	$(4,1)h_9$	$[(7,2)+(8,1)+(9)]i_5+(5,2,1)i_6$
	$(4)h_{10}$	$[(5,2,1)+(6,2)+(7,1)]i_6$
	$(3,1)h_{10}$	$(7,1)i_6$
	$(2)h_{12}$	$(7,3,1)i_3+(6,2)i_6$
15	$(8)h_7$	$[(8,3,1)+(8,4)]i_3+(8,2)i_5$
	$(7,1)h_7$	$(9,3)i_3+(7,3)i_5$
	$(6,2)h_7$	$[(9,3)+(10,2)+(11,1)]i_3+(6,3,1)i_5$
	$(5,2,1)h_7$	$[(9,2,1)+(11,1)]i_3+[(7,2,1)+(9,1)]i_5$
	$(6)h_9$	$[(6,3,1)+(7,3)]i_5+(6,2,1)i_6$
	$(5,1)h_9$	$(9,1)i_6$
	$(4,2)h_9$	$[(8,2)+(10)]i_5$
	$(5)h_{10}$	$(7,2)i_6$
	$(4,1)h_{10}$	$[(7,2)+(8,1)+(9)]i_6$
	$(3)h_{12}$	$(7,2)i_6$
	$(2,1)h_{12}$	$[(8,3,1)+(9,2,1)]i_3+(6,3)i_6$
	h_{15}	$(12)i_3+(6,3)i_6+[(6,2)+(7,1)+(8)]i_7$
16	$(9)h_7$	$[(9,3,1)+(9,4)]i_3+(9,2)i_5$
	$(8,1)h_7$	$(9,4)i_3+(8,3)i_5$
	$(7,2)h_7$	$(11,2)i_3+(7,3,1)i_5$

TABLE 6.3. (continued)

dimension	input	output
	$(6,3)h_7$	$[(10,3)+(12,1)+(13)]i_3$
	$(6,2,1)h_7$	$[(9,3,1)+(10,2,1)]i_3+[(7,3,1)+(8,3)+(9,2)]i_5$
	$(7)h_9$	$(7,3,1)i_5+(7,2,1)i_6$
	$(6,1)h_9$	$[(8,3)+(9,2)]i_5+(6,3,1)i_6$
	$(5,2)h_9$	$[(9,2)+(11)]i_5$
	$(4,2,1)h_9$	$[(8,2,1)+(10,1)]i_5+[(7,2,1)+(9,1)]i_6$
	$(6)h_{10}$	$[(6,3,1)+(7,3)]i_6$
	$(5,1)h_{10}$	$(9,1)i_6$
	$(4,2)h_{10}$	$[(8,2)+(10)]i_6$
	$(4)h_{12}$	$(10,2,1)i_3+[(6,3,1)+(7,2,1)+(7,3)+(9,1)]i_6$
	$(3,1)h_{12}$	$(9,3,1)i_3+(7,3)i_6$
	$(1)h_{15}$	$(13)i_3+(7,3)i_6+[(7,2)+(9)]i_7$
17	$(10)h_7$	$[(10,3,1)+(10,4)]i_3+(10,2)i_5$
	$(9,1)h_7$	$(9,3)i_5$
	$(8,2)h_7$	$[(10,4)+(11,3)]i_3+(8,3,1)i_5$
	$(7,3)h_7$	$[(11,3)+(13,1)]i_3$
	$(7,2,1)h_7$	$(11,2,1)i_3+(9,3)i_5$
	$(8)h_9$	$[(8,3,1)+(8,4)]i_5+(2,1)i_6$
	$(7,1)h_9$	$(9,3)i_5+(7,3,1)i_6$
	$(6,2)h_9$	$[(9,3)+(10,2)+(11,1)]i_5$
	$(5,2,1)h_9$	$[(9,2,1)+(11,1)]i_5$
	$(7)h_{10}$	$(7,3,1)i_6$

TABLE 6.3. (continued)

dimension	input	output
	$(6,1)h_{10}$	$[(8,3)+(9,2)]i_6$
	$(5,2)h_{10}$	$[(9,2)+(11)]i_6$
	$(4,2,1)h_{10}$	$[(8,2,1)+(10,1)]i_6$
	$(5)h_{12}$	$(11,2,1)i_3+(7,3,1)i_6$
	$(4,1)h_{12}$	$(11,2,1)i_3+[(7,3,1)+(8,3)+(9,2)]i_6$
	$(2)h_{15}$	$[(13,1)+(14)]i_3+[(7,3)+(10)]i_7$
	h_{17}	$(8,4,2)i_3+(12)i_5+(9,2)i_6+[(6,3,1)+(7,3)+(8,2)]i_7$

Now we calculate $K_1(Z)$. The generators of $K_1(Z)$, and their Steenrod algebra structure, is given below.

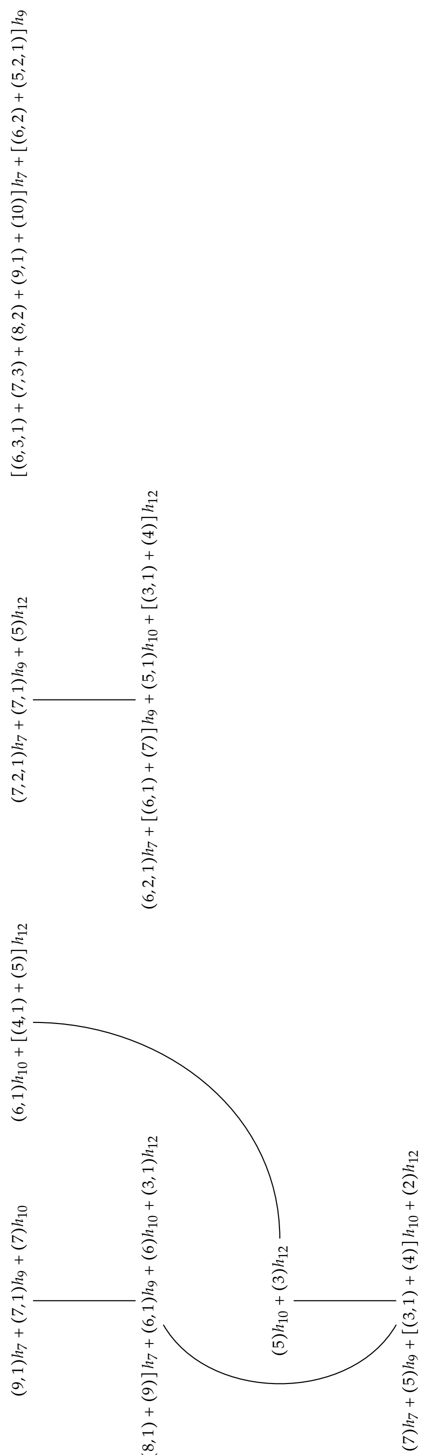


FIGURE 6.8. K_1Z through dimension 17

From this, we see that we can choose

$$P_2(Z) = \Sigma^{14} Z \oplus \Sigma^{16} A \oplus \Sigma^{17} A \oplus \bigoplus_{l_k > 17} \Sigma^{l_k} A$$

Let j_k be the generator of $\Sigma^k A$ in $P_2(Z)$. Then $d_2 \colon P_2(Z) \to P_1(Z)$ is given by

$$\begin{aligned}
d_2(j_{14}) &= \mathrm{Sq}^7 h_7 + \mathrm{Sq}^5 h_9 + (\mathrm{Sq}^3 \mathrm{Sq}^1 + \mathrm{Sq}^4) h_{10} + \mathrm{Sq}^2 h_{12} \\
d_2(j_{16}) &= \mathrm{Sq}^6 \mathrm{Sq}^2 \mathrm{Sq}^1 h_7 + (\mathrm{Sq}^6 \mathrm{Sq}^1 + \mathrm{Sq}^7) h_9 + \mathrm{Sq}^5 \mathrm{Sq}^1 h_{10} + (\mathrm{Sq}^3 \mathrm{Sq}^1 + \mathrm{Sq}^4) h_{12} \\
d_2(j_{17}) &= (\mathrm{Sq}^6 \mathrm{Sq}^3 \mathrm{Sq}^1 + \mathrm{Sq}^7 \mathrm{Sq}^3 + \mathrm{Sq}^8 \mathrm{Sq}^2 + \mathrm{Sq}^9 \mathrm{Sq}^1 + \mathrm{Sq}^{10}) h_7 \\
&= +(\mathrm{Sq}^5 \mathrm{Sq}^2 \mathrm{Sq}^1 + \mathrm{Sq}^6 \mathrm{Sq}^2) h_9
\end{aligned}$$

This calculation will be sufficient to show that there is no u_2^2-map on Z.

6.2. Long Exact Sequences

From the short exact sequence

$$0 \to H^* Y_1 \to \Sigma^{-1} A \to H^* \Sigma^{-1} Y$$

we get the long exact sequence

$$\begin{aligned}
0 \to \mathrm{Ext}_A^{0,12}(H^*Z, H^*Y_1) \to \mathrm{Ext}_A^{0,12}(H^*Z, \Sigma^{-1}A) \to \mathrm{Ext}_A^{0,12}(H^*Z, H^*\Sigma^{-1}Y) \\
\xrightarrow{\delta_0} \mathrm{Ext}_A^{1,12}(H^*Z, H^*Y_1) \to \mathrm{Ext}_A^{1,12}(H^*Z, \Sigma^{-1}A) \to \mathrm{Ext}_A^{1,12}(H^*Z, \Sigma^{-1}H^*Y)) \to \dots
\end{aligned}$$

Lemma 6.1. $\mathrm{Ext}_A^{0,12}(H^*Z, \Sigma^{-1}A) = \mathrm{Ext}_A^{1,12}(H^*Z, \Sigma^{-1}A) = 0$

Proof. Note that A is self-injective, and apply $\mathrm{Hom}_A(-,\Sigma^{11}A)$ the short exact sequence

$$0 \to \Sigma^{-1}K_0(Y) \to \Sigma^{-1}A \to H^*\Sigma^{-1}Y \to 0$$

to get

$$0 \leftarrow \mathrm{Hom}_A(\Sigma^{-1}K_0(Y),\Sigma^{11}A) \leftarrow \mathrm{Hom}_A(\Sigma^{-1}A,\Sigma^{11}A) \leftarrow \mathrm{Hom}_A(H^*\Sigma^{-1}Y,\Sigma^{11}A) \leftarrow 0$$

The rightmost group is zero for dimensional reasons, so we have

$$\mathrm{Hom}_A(\Sigma^{-1}K_0(Y),\Sigma^{11}A) \cong \mathrm{Hom}_A(\Sigma^{-1}A,\Sigma^{11}A).$$

But there are no degree 0 A-module maps $\Sigma^{-1}A \to \Sigma^{11}A$, so both of these groups are also zero.

Now, apply $\mathrm{Hom}_A(-,\Sigma^{11}A)$ to

$$0 \to H^*Z \to H^*Y_1 \to H^*\Sigma^2 Y \to 0$$

to get

$$0 \leftarrow \mathrm{Hom}_A(H^*Z,\Sigma^{11}A) \leftarrow \mathrm{Hom}_A(H^*Y_1,\Sigma^{11}A) \leftarrow \mathrm{Hom}_A(H^*\Sigma^2 Y,\Sigma^{11}A) \leftarrow 0$$

Recall that $H^*Y_1 = \Sigma^{-1}K_0(Y)$, and so the middle group is 0 since K_0Y has no generators in dimension 10. Thus $\mathrm{Hom}_A(H^*Z,\Sigma^{11}A) = \mathrm{Ext}_A^{0,12}(H^*Z,\Sigma^{-1}A) = 0$.

$\mathrm{Ext}_A^{1,12}(H^*Z, \Sigma^{-1}A) = 0$ since A is self-injective.

□

and so we have

$$\mathrm{Ext}_A^{0,12}(H^*Z, H^*\Sigma^{-1}Y) \cong \mathrm{Ext}_A^{1,12}(H^*Z, H^*Y_1)$$

The lefthand side of this is $\mathrm{Ext}_A^{0,12}(H^*Z, H^*\Sigma^{-1}Y) \cong \mathrm{Hom}_A(H^*Z, \Sigma^{11}H^*Y)$. We have that $\mathrm{Hom}_A(H^*Z, \Sigma^{11}H^*Y) = 0$ because H^*Z has no generators in dimensions 11, 12, 13, 14, whereas $\Sigma^{11}H^*Y$ is nonzero only in those dimensions. So we have that

$$\mathrm{Ext}_A^{1,12}(H^*Z, H^*Y_1) = 0.$$

Thus, from the short exact sequence

$$0 \to H^*Z \to H^*Y_1 \to \Sigma^2 H^*Y \to 0$$

we have the long exact sequence

$$\begin{array}{l}0 \to \mathrm{Ext}_A^{0,12}(H^*Z, H^*Z) \to \mathrm{Ext}_A^{0,12}(H^*Z, H^*Y_1) \to \mathrm{Ext}_A^{0,12}(H^*Z, H^*\Sigma^2 Y) \xrightarrow{\delta_0} \\ \mathrm{Ext}_A^{1,12}(H^*Z, H^*Z) \to \mathrm{Ext}_A^{1,12}(H^*Z, H^*Y_1) \to \mathrm{Ext}_A^{1,12}(H^*Z, H^*\Sigma^2 Y) \xrightarrow{\delta_1} \\ \mathrm{Ext}_A^{2,12}(H^*Z, H^*Z) \to \mathrm{Ext}_A^{2,12}(H^*Z, H^*Y_1) \to \mathrm{Ext}_A^{2,12}(H^*Z, H^*\Sigma^2 Y) \to \dots\end{array}$$

which simplifies to

$$\begin{array}{l}\dots \longrightarrow 0 \longrightarrow \mathrm{Ext}_A^{1,12}(H^*Z, H^*\Sigma^2 Y) \xrightarrow{\delta_1} \\ \mathrm{Ext}_A^{2,12}(H^*Z, H^*Z) \to \mathrm{Ext}_A^{2,12}(H^*Z, H^*Y_1) \to \mathrm{Ext}_A^{2,12}(H^*Z, H^*\Sigma^2 Y) \to \dots\end{array}$$

Next, we examine

$$\operatorname{Ext}_A^{2,12}(H^*Z, H^*Y_1)$$

In order to examine this, we again look at the long exact sequence

$$\cdots \to \operatorname{Ext}_A^{1,12}(H^*Z, H^*Y_1) \to \operatorname{Ext}_A^{1,12}(H^*Z, \Sigma^{-1}A) \to \operatorname{Ext}_A^{1,12}(H^*Z, H^*\Sigma^{-1}Y) \xrightarrow{\delta_1}$$
$$\operatorname{Ext}_A^{2,12}(H^*Z, H^*Y_1) \to \operatorname{Ext}_A^{2,12}(H^*Z, \Sigma^{-1}A) \to \operatorname{Ext}_A^{2,12}(H^*Z, \Sigma^{-1}H^*Y) \to \cdots$$

Again, since $\operatorname{Ext}_A^{1,12}(H^*Z, \Sigma^{-1}A) = \operatorname{Ext}_A^{2,12}(H^*Z, \Sigma^{-1}A) = 0$, we get

$$\operatorname{Ext}_A^{1,12}(H^*Z, H^*\Sigma^{-1}Y) \cong \operatorname{Ext}_A^{2,12}(H^*Z, H^*Y_1)$$

The lefthand side is a subquotient of

$$\operatorname{Hom}_A(P_1(Z), \Sigma^{11}H^*Y)$$

There is only one possible map here, which is f_{12} given by

$$f_{12}(h_k) = \begin{cases} \Sigma^{11}\alpha_1 & \text{if } k = 12 \\ 0 & \text{if } k \neq 12 \end{cases}$$

We claim this is not an element of Ext. Examining the long exact sequence

$$0 \to \operatorname{Hom}^{12}(P_0(Z), H^*\Sigma^{-1}Y) \to \operatorname{Hom}^{12}(P_1(Z), H^*\Sigma^{-1}Y)$$

$$\to \operatorname{Hom}^{12}(P_2(Z), H^*\Sigma^{-1}Y) \to \cdots$$

and particularly the map

$$\mathrm{Hom}^{12}_A(P_1(Z), H^*\Sigma^{-1}Y) \xrightarrow{d_2^*} \mathrm{Hom}^{12}_A(P_2(Z), H^*\Sigma^{-1}Y)$$

we see that $d_2^*(f_{12}) = f_{12} \circ d_2$ is given on the generators of $P_2(Z)$ by

$$\begin{aligned}(f_{12} \circ d_2)(h_{14}) &= f_{12}(\mathrm{Sq}^7 h_7 + \mathrm{Sq}^5 h_9 + (\mathrm{Sq}^3 \mathrm{Sq}^1 + \mathrm{Sq}^4)h_{10} + \mathrm{Sq}^2 h_{12}) \\ &= \mathrm{Sq}^2 \Sigma^{11} \alpha_1 \\ &= \Sigma^{11} \alpha_3\end{aligned}$$

So $f_{12} \notin \ker d_2^*$, and therefore does not represent a nonzero element of $\mathrm{Ext}^{1,12}_A(H^*Z, H^*\Sigma^{-1}Y)$. Thus, $\mathrm{Ext}^{1,12}_A(H^*Z, H^*\Sigma^{-1}Y) = 0$ and so $\mathrm{Ext}^{2,12}_A(H^*Z, H^*Y_1) = 0$.

Returning to the long exact sequence associated to

$$0 \to H^*Z \to H^*Y_1 \to \Sigma^2 H^*Y \to 0$$

we have an isomorphism

$$\mathrm{Ext}^{1,12}_A(H^*Z, H^*\Sigma^2 Y) \cong \mathrm{Ext}^{2,12}_A(H^*Z, H^*Z)$$

which is given by the connecting homomorphism.

In order to compute the connecting homomorphism, we identify representatives for elements of $\mathrm{Ext}^{1,12}_A(H^*Z, H^*\Sigma^2 Y)$. Such representatives are given as elements of $\mathrm{Hom}_A(P_1(Z), \Sigma^{14} H^*Y)$. Since $\Sigma^{14}H^*Y$ is nonzero only in dimensions 14, 15, 16, 17, a basis for this vector space is given by f_1 and f_2, where

$$f_1(h_k) = \begin{cases} \Sigma^{14}\alpha_1 & \text{if } k = 15 \\ 0 & \text{if } k \neq 15 \end{cases}$$

$$f_2(h_k) = \begin{cases} \Sigma^{14}\alpha_3 & \text{if } k = 17 \\ 0 & \text{if } k \neq 17 \end{cases}$$

We compute the connecting homomorphisms by looking at the diagram

$$\begin{array}{ccccccccc} 0 \to & \mathrm{Hom}_A(P_1(Z),\Sigma^{12}H^*Z) & \xrightarrow{i_*} & \mathrm{Hom}_A(P_1(Z),\Sigma^{12}H^*Y_1) & \xrightarrow{(\phi_3)_*} & \mathrm{Hom}_A(P_1(Z),\Sigma^{14}H^*Y) & \to 0 \\ & \downarrow{\scriptstyle d_2^*} & & \downarrow{\scriptstyle d_2^*} & & \downarrow{\scriptstyle d_2^*} & \\ 0 \to & \mathrm{Hom}_A(P_2(Z),\Sigma^{12}H^*Z) & \xrightarrow{i_*} & \mathrm{Hom}_A(P_2(Z),\Sigma^{12}H^*Y_1) & \xrightarrow{(\phi_3)_*} & \mathrm{Hom}_A(P_2(Z),\Sigma^{14}H^*Y) & \to 0 \end{array}$$

where the horizontal maps are coming from the sequence

$$0 \to H^*Z \to H^*Y_1 \to \Sigma^2 H^*Y \to 0$$

We begin with $f_1 \in \mathrm{Hom}_A(P_1(Z),\Sigma^{14}H^*Y)$. This pulls back to $\gamma_1 \in \mathrm{Hom}_A(P_1(Z),\Sigma^{12}H^*Y_1)$ so that $\gamma_1(h_{15}) = \Sigma^{11}\,\mathrm{Sq}^3\,\mathrm{Sq}^1$ and $\gamma_1(h_k) = 0$ for $k \neq 15$. Composing with d_2^*, we see that $d_2^*(\gamma_1) = \gamma_1 \circ d_2$ is given by

$$\begin{aligned} (\gamma_1 \circ d_2)(h_{14}) &= \gamma_1(\mathrm{Sq}^7 h_7 + \mathrm{Sq}^5 h_9 + (\mathrm{Sq}^3\,\mathrm{Sq}^1 + \mathrm{Sq}^4)h_{10} + \mathrm{Sq}^2 h_{12}) \\ &= 0 \end{aligned}$$

$$
\begin{aligned}
(\gamma_1 \circ d_2)(h_{16}) &= \gamma_1(\mathrm{Sq}^6\,\mathrm{Sq}^2\,\mathrm{Sq}^1\,h_7 + (\mathrm{Sq}^6\,\mathrm{Sq}^1 + \mathrm{Sq}^7)h_9 \\
&= +\mathrm{Sq}^5\,\mathrm{Sq}^1\,h_{10} + (\mathrm{Sq}^3\,\mathrm{Sq}^1 + \mathrm{Sq}^4)h_{12}) \\
&= 0 \\
(\gamma_1 \circ d_2)(h_{17}) &= \gamma_1(\mathrm{Sq}^6\,\mathrm{Sq}^3\,\mathrm{Sq}^1 + \mathrm{Sq}^7\,\mathrm{Sq}^3 + \mathrm{Sq}^8\,\mathrm{Sq}^2 + \mathrm{Sq}^9\,\mathrm{Sq}^1 + \mathrm{Sq}^{10}\,h_7 \\
&= +(\mathrm{Sq}^5\,\mathrm{Sq}^2\,\mathrm{Sq}^1 + \mathrm{Sq}^6\,\mathrm{Sq}^2)h_9 \\
&= 0
\end{aligned}
$$

So, the map $d_2^*(\gamma_1) \in \mathrm{Hom}_A(P_2(Z), \Sigma^{12}H^*Y_1)$ is zero through dimensions 17. Therefore, the pullback in $\mathrm{Hom}_A(P_2(Z), \Sigma^{12}H^*Z)$ is also zero through dimension 17.

Next, we examine $f_2 \in \mathrm{Hom}_A(P_1(Z), \Sigma^{14}H^*Y)$. This pulls back to $\gamma_2 \in \mathrm{Hom}_A(P_1(Z), \Sigma^{12}H^*Y_1)$ so that $\gamma_2(h_{17}) = \Sigma^{11}\,\mathrm{Sq}^5\,\mathrm{Sq}^1$ and $\gamma_1(h_k) = 0$ for $k \neq 17$. Composing with d_2^*, we see that $d_2^*(\gamma_2) = \gamma_2 \circ d_2$ is given by

$$
\begin{aligned}
(\gamma_2 \circ d_2)(h_{14}) &= \gamma_2(\mathrm{Sq}^7\,h_7 + \mathrm{Sq}^5\,h_9 + (\mathrm{Sq}^3\,\mathrm{Sq}^1 + \mathrm{Sq}^4)h_{10} + \mathrm{Sq}^2\,h_{12}) \\
&= 0
\end{aligned}
$$

$$
\begin{aligned}
(\gamma_2 \circ d_2)(h_{16} &= \gamma_2(\mathrm{Sq}^6\,\mathrm{Sq}^2\,\mathrm{Sq}^1\,h_7 + (\mathrm{Sq}^6\,\mathrm{Sq}^1 + \mathrm{Sq}^7)h_9 \\
&= +\mathrm{Sq}^5\,\mathrm{Sq}^1\,h_{10} + (\mathrm{Sq}^3\,\mathrm{Sq}^1 + \mathrm{Sq}^4)h_{12}) \\
&= 0
\end{aligned}
$$

$$
\begin{aligned}
(\gamma_2 \circ d_2)(h_{17}) &= \gamma_2(\mathrm{Sq}^6\,\mathrm{Sq}^3\,\mathrm{Sq}^1 + \mathrm{Sq}^7\,\mathrm{Sq}^3 + \mathrm{Sq}^8\,\mathrm{Sq}^2 + \mathrm{Sq}^9\,\mathrm{Sq}^1 + \mathrm{Sq}^{10}\,h_7 \\
&= +(\mathrm{Sq}^5\,\mathrm{Sq}^2\,\mathrm{Sq}^1 + \mathrm{Sq}^6\,\mathrm{Sq}^2)h_9 \\
&= 0
\end{aligned}
$$

So, the map $d_2^*(\gamma_2) \in \mathrm{Hom}_A(P_2(Z), \Sigma^{12}H^*Y_1)$ is zero through dimensions 17. Therefore, the pullback in $\mathrm{Hom}_A(P_2(Z), \Sigma^{12}H^*Z)$ is also zero through dimension 17.

We claim that this is sufficient to show that there is no u_2^2 on Z. We saw before that H^*Z is nonzero only in dimensions 5, 6, 7, 8, 9, 10, 11. $H_*(K_0(Z), x_2)$ is nonzero only in dimensions 11, 12, 13, 14, 15, 16, 17. Using the short exact sequence

$$
0 \to K_1(Z) \to P_1(Z) \to K_0(Z) \to 0
$$

we see that $H_*(K_1(Z), x_2)$ in nonzero only in dimensions 17, 18, 19, 20, 21, 22, 23. By [Rei17] an element $[f] \in \mathrm{Ext}_A^{2,12}(H^*Z, H^*Z)$ that is a u_2^2 map must induce an isomorphism $f\colon K_0(Z) \to \Sigma^{12}H^*Z$. But since the images of $f_1, f_2 \in \mathrm{Hom}_A(P_2(Z), \Sigma^{12}H^*Z)$ are zero through dimension 17, they cannot be isomorphisms in dimension 17. Therefore, they cannot represent u_2^2-maps.

REFERENCES CITED

[BE16] Prasit Bhattacharya and Philip Egger. A class of 2-local finite spectra with admit a v_2^1-self-map. Preprint available at https://arxiv.org/abs/1608.06250, 2016.

[BHHM08] Mark Behrens, Michael Hill, Michael J. Hopkins, and Mark Mahowald. On the existence of a v_2^{32}-self map on $m(1,4)$ at the prime 2. *Homology, Homotopy and Applications*, 10(3):45–84, 2008.

[DHS88] Ethan S. Devinatz, Michael J. Hopkins, and Jeffrey H. Smith. Nilpotence and stable homotopy theory i. *Annals of Mathematics*, 128(2):207–241, 1988.

[DM81] Donald M. Davis and Mark Mahowald. v_1- and v_2-periodicty in stable homotopy theory. *American Journal of Mathematics*, 103:615–659, 1981.

[HS98] Michael J. Hopkins and Jeffrey H. Smith. Nilpotence and stable homotopy theory ii. *Annals of Mathematics*, 148(2):1–49, 1998.

[Mar83] H. R. Margolis. *Spectra and the Steenrod Algebra*. North Holland Publishers, 1983.

[Mil58] John Milnor. The steenrod algebra and its dual. *Annals of Mathematics*, 67(1):150–171, January 1958.

[MM65] John W. Milnor and John C. Moore. On the structure of hopf algebras. *Annals of Mathematics*, 81(2):211–264, March 1965.

[PS94] John H. Palmieri and Hal Sadofsky. Sef-maps of spectra, a theorem of j. smith, and margolis's killing construction. *Mathematics Zeitschrift*, 215:477–490, 1994.

[Rav86] Douglas Ravenel. *Complex Cobordism and the Stable Homotopy Groups of Spheres*, volume 121. Academic Press, 1986.

[Rav92] Douglas Ravenel. *Nilpotence and Periodicity in Stable Homotopy Theory*. Princeton University Press, 1992.

[Rei17] Benjamin William Reid. *Constructing a v_2^1 self map at $p = 3$*. PhD thesis, University of Oregon, 2017.

[Swi02] Robert M. Switzer. *Algebraic Topology - Homology and Homotopy*. Springer, 2002.