Stable Noisy K-state Markov Sunspots*

George W. Evans
University of Oregon

Bruce McGough
Oregon State University

July 18, 2002

Abstract

We consider a linear stochastic univariate rational expectations model, with a predetermined variable, and consider solutions driven by an extraneous finite state Markov process as well as by the fundamental noise. We obtain conditions for existence of noisy k-state sunspot equilibria (noisy k-SSEs) and, for the case $k = 2$, of noisy k-state dependent sunspot equilibria (noisy k-*SDSs). k-*SDSs are driven by a finite state sunspot but have an infinite range of values even in the nonstochastic model. Stability under econometric learning is analyzed using representations that nest both types of solution. For the case $k = 2$, we find that noisy 2-SSEs and noisy 2-*SDSs are learnable for parameters in proper subsets of the regions of their existence.

1 Introduction

The existence of "self-fulfilling" solutions, driven by extraneous stochastic processes known as "sunspots," was initially demonstrated by (Shell 1977), (Azariadis 1981), (Cass and Shell 1983), (Azariadis and Guesnerie 1986) and (Guesnerie 1986) in simple stylized models, such as the Overlapping Generations model of money. More recently the existence of such solutions in linearized versions of Real Business Cycle models with distortions has emphasized the possibility that sunspot equilibria may provide a way of accounting

*Financial support from the National Science Foundation is gratefully acknowledged. The authors would like to thank Brad Shelton for valuable discussions.

for macroeconomic fluctuations. For the recent literature see (Guesnerie and Woodford 1992), (Farmer 1999) and (Benhabib and Farmer 1999).

A question that has arisen in this literature concerns the attainability of sunspot equilibria. That sunspot solutions could be stable under adaptive learning was demonstrated for the basic Overlapping Generations model by (Woodford 1990), and conditions for local stability under adaptive learning were provided in (Evans and Honkapohja 1994b) and (Evans and Honkapohja 2002) for one-step forward looking univariate nonlinear models.[1] The solutions considered in these papers take the form of a finite state Markov process, a type of solution that is prominent in the theoretical literature and described at length, for example, in (Chiappori, Geoffard, and Guesnerie 1992)

For linear models with predetermined variables, (Evans and Honkapohja 1994a) considered the stability under learning of sunspot solutions taking an autoregressive-moving average form, but their analysis did not take up the stability of finite state Markov solutions. Indeed, until the work of (Dávila 1997), it was not generally recognized that finite state Markov sunspot solutions could exist in models with predetermined variables.[2] (Dávila 1997) and (Dávila and Guesnerie 2001) give conditions for existence of finite state Markov solutions in both linear and nonlinear nonstochastic models with memory. It is not at all obvious whether analogous solutions exist in stochastic models with memory, and these solutions have not been analyzed for stability under adaptive leaning. By restricting attention to linear models we here are able to examine both of these issues.

In the current paper we show how to extend Dávila's existence result to stochastic models with a predetermined variable, and we analyze the stability under learning of noisy sunspot equilibria driven by k-state Markov processes. In the process of obtaining our results we uncover another class of solutions that has not previously been noted. These solutions, like the finite state Markov solutions, depend on an extraneous exogenous k-state Markov process, but do not, even in the absence of intrinsic noise, take on a finite number of values. We characterize these solutions and also analyze their stability under adaptive learning.

[1](Desgranges and Negroni 2001) have obtained conditions for eductive stability of two-state Markov stationary sunspot equilibria in an overlapping generations model.

[2]An exception is (Howitt and McAfee 1992). However this model relied on a nonlinear model that produced multiple steady states. (Evans, Honkapohja, and Romer 1998) also relied on finite state Markov sunspot equilibria near distinct steady states.

We begin by defining the notion of a noisy k-state Markov stationary sunspot equilibrium (noisy k-SSE); this is the natural generalization of a k-state Markov sunspot to a model with intrinsic noise. It is then straightforward to show that such equilibria exist provided k-state sunspot equilibria exist in the associated non-stochastic model, and to show that this existence result obtains for parameters in a strict subset of the region in parameter space corresponding to indeterminacy in the model.

To analyze stability under learning of these sunspot equilibria, we begin by showing that each of these rational expectations equilibria (REE) can be obtained as a solution to a member of a certain class of linear difference equations; we call these equations *representations* of the equilibria. Representations are most easily characterized as fixed points of a map T, which takes agents' perceived law of motion to the corresponding actual law of motion. This T-map, and hence the corresponding fixed points, will depend on the transition probabilities π of the associated Markov process.

It turns out that even for parameter regions in which noisy k-SSEs exist, the corresponding transition probabilities π must satisfy certain constraints. On the other hand, whether or not π satisfies these conditions, a T-map is still defined, and to every non-trivial fixed point of this T-map corresponds at least one associated REE that depends explicitly on the Markov process. These observations lead us to the following definition. We call any solution to our representations *noisy k-state dependant sunspot equilibria* (noisy k-SDSs) and label by noisy k-*SDS those solutions which are not noisy k-SSEs.

To obtain specific results about the existence of noisy k-*SDSs and to analyze the stability under learning of noisy k-*SDSs and noisy k-SSEs, we consider in detail the case $k = 2$. We find that, provided the model is indeterminate, noisy 2-*SDSs exist. If agents use the functional form of these representations as their perceived law of motion (i.e. their regression model) then provided the parameters are appropriately restricted to be in certain proper subsets of the region of existence, agents can learn the true form of the representation, thus implying that the associated sunspot equilibria, be they noisy 2-SSEs or noisy 2-*SDSs as determined by the model's parameter values and the transition array, are stable under learning.

We illustrate these results for a modified version of Cagan's model and for Sargent's extension of the Lucas-Prescott model of investment under uncertainty to incorporate tax distortions and externalities.

2 The Model

We consider the following model:

$$y_t = \beta E_t y_{t+1} + \delta y_{t-1} + v_t, \tag{1}$$

where v_t is a white noise exogenous process and $E_t y_{t+1}$ denotes the mathematical expectation of y_{t+1} conditional on information available at t. We assume throughout that $\beta \neq 0$, $\delta \neq 0$ and $\beta + \delta \neq 0$. Our focus will be on stochastic versions of the model in which v_t is nontrivial, but we will also at times need to refer to the nonstochastic (homogeneous) model

$$y_t = \beta E_t y_{t+1} + \delta y_{t-1}, \tag{2}$$

which appears frequently in the literature. For simplicity of presentation we have omitted a constant intercept from the model. If instead, say, $y_t = \mu + \beta E_t y_{t+1} + \delta y_{t-1} + v_t$, then the model can be rewritten in the form (1) where y_t is reinterpreted as its deviation from $\hat{y} = \mu/(1 - \beta - \delta)$.

(Dávila 1997) showed that finite state Markov solutions to the nonstochastic model (2) could exist provided the Markov process is second-order. We begin by describing these solutions. For $n \leq k$, let S_n be the n^{th}-coordinate vector of unit length in $\mathbb{R}^k$, and let ΔS_k be the set of all convex combinations of the vectors S_n. Notice that ΔS_k is the $k-1$ unit simplex and thus elements of ΔS_k represent probability distributions over the "states" S_n. A second order k-state Markov process (with states S_n) is a sequence of random variables s_t and matrix of probabilities $\pi \in \Pi_{ij} \Delta S_k$ such that for all $i, j, n \in \{1, \cdots, k\}$,

$$prob\{s_{t+1} = S_n | s_{t-1} = S_i, s_t = S_j\} = \pi_{ij}(n).$$

We identify a k-state Markov process with its transition array π. A *k-state Markov stationary sunspot equilibrium* (k-SSE) is a pair $(\pi, \overline{y})$, where π is a k-state second order Markov process and $\overline{y} \in \mathbb{R}^k$ with $\overline{y}_i \neq \overline{y}_j$ for $i \neq j$, is such that

$$y_t = \overline{y}_i \Leftrightarrow s_t = S_i \tag{3}$$

satisfies (2). We will also sometimes refer to y_t as a k-state Markov sunspot, since y_t itself follows a second-order k-state Markov process.

By explicitly considering the restrictions imposed by the model, we can obtain a set of linear equations, any solution to which yields a k-state Markov sunspot of the nonstochastic model. For each m and n write $\pi_{mn} \in \Delta S_k$ as a column vector. If y_t satisfies (3) then

$$E_t y_{t+1} = \pi'_{mn}\overline{y} \Leftrightarrow s_{t-1} = S_m \text{ and } s_t = S_n.$$

We conclude that the pair $(\pi, \overline{y})$ is a k-SSE if and only if

$$\overline{y}_n - \delta\overline{y}_m = \beta\pi'_{mn}\overline{y} \quad \forall n, m \in \{1, \cdots, k\}. \tag{4}$$

(4) represents a homogeneous system of k^2 linear equations. Thus $\overline{y}_i = 0$ for all i is always a solution; this trivial sunspot coincides with the solution $y_t = 0$ to the homogeneous model. Existence of non-trivial solutions requires the system of equations to be dependant; this requirement imposes restrictions on the possible values of the parameters. Further restrictions are imposed by the requirements that the $\overline{y}_i$ be distinct (so that the k-state sunspot is not degenerate) and that the transition array represents legitimate probability distributions. (Dávila 1997) and (Dávila and Guesnerie 2001) demonstrated existence for a subset of the parameter space specified in the next section.

It is not obvious that when k-SSEs exist there can also exist analogous solutions to the stochastic model (1). Because of the dependence of y_t on y_{t-1} it seems that the white noise disturbance, which will make any value of y_{t-1} possible, might disturb the careful balance required for k-SSEs. Our first task is to show that this intuition is not correct. Analogues of k-SSEs do exist in the stochastic model and they exist whenever k-SSEs exist in the nonstochastic model.

3 Existence of Noisy K-state Markov Sunspots

A rational expectations equilibrium (REE) of the stochastic model is any process y_t which solves (1). We restrict attention to doubly infinite, covariance stationary solutions. For a complete characterization of such solutions see (Evans and McGough 2002); there it is also shown that the norm of a doubly infinite, covariance stationary solution will be uniformly bounded in both conditional and unconditional expectation.

Let z_t be a stationary solution to the stochastic model (1) that does not depend on extrinsic noise; in this paper we will refer to z_t as a "fundamentals solution". It is well known that there exist such solutions to the noisy model (1) if and only if the associated quadratic $\beta a^2 - a + \delta$ has at least one root with norm less than one. (In the next section we will give an explicit expression for z_t). Let ζ_t be a solution to the homogeneous model (2) and let $y_t \equiv z_t + \zeta_t$. Since $E_t y_{t+1} = E_t z_{t+1} + E_t \zeta_{t+1}$ and $y_{t-1} = z_{t-1} + \zeta_{t-1}$, it is immediate that y_t is a solution to (1). If ζ_t is a k-SSE we call y_t a *noisy k-state Markov stationary sunspot equilibrium* (noisy k-SSE).

Finally, Woodford's conjecture, that the model must be indeterminate for stationary sunspot equilibria to exist, holds in both the stochastic and nonstochastic model, and indeterminacy in either model obtains when both roots of the above quadratic lie inside the unit circle; see e.g. (Evans and McGough 2002) for details. Thus if the model's parameters are such that k-SSEs exist in the homogeneous model then a stationary fundamentals solution of the noisy model exists. Combining this observation with the definition of a noisy k-state Markov sunspot and the restrictions on the model's parameters obtained by Dávila, we have the following:

Proposition 1 *A noisy k-SSE of (1) exists if and only if*

1. $-1 < \frac{1-\delta}{\beta} < 1$
2. $-1 < \frac{1+\delta}{\beta} < 1$

For given parameter values satisfying these conditions, the system (4) can be used to construct a k-SSE; in Section 6 we construct k-SSEs for the case $k = 2$. Note further that since the system (4) is homogeneous, the existence of one non-trivial solution implies the existence of a continuum of non-trivial solutions.[3]

Finally, we remark that the region specified in Proposition 1 is a proper subset of the indeterminacy region. The latter is given by the union of the two regions (i) $\beta + \delta > 1$ with $|\delta| < |\beta|$ and (ii) $\beta + \delta < -1$ with $|\delta| < |\beta|$.

4 Representations

We now distinguish between a rational expectations equilibrium (REE) and a rational expectations equilibrium representation (REER). As mentioned

[3] The system (4) can be obtained from Dávila's paper by noting that the solution set to the ij^{th}-equation in (4) is equivalent to his manifold V_{ij}.

above, an REE of the noisy model is any stochastic process y_t which satisfies the associated expectational difference equation (1). An REER of the noisy model is a linear difference equation, any solution to which is an REE. The importance of this distinction stems from the fact that the analysis of stability under econometric learning requires the specification of an REER; in fact, most accurately, it is the representation, not the REE that is or is not stable under learning. Furthermore, the stability of a particular REE may depend on the associated representation induced by the perceived law of motion; for a detailed analysis of these topics, see (Evans and McGough 2002). In this section, we develop representations of noisy k-SSEs that can be used to analyze their stability under learning.

Representations can be obtained as fixed points of a T-map, and thus we begin with its construction, which will be greatly facilitated by the following lemma:

Lemma 2 *Let A be a $k \times k$ matrix. Then there exists matrix $B = B(A)$, depending on π, such that*

$$E_t s_t' A s_{t+1} = s_{t-1}' B s_t.$$

We include the proof here, in the body of the paper, because it is constructive in nature and we will require the construction again in Section 6. We employ the following notation: if A is an $n_1 \times m_1$ matrix and B is $n_2 \times m_2$ then their direct sum is an $(n_1 + n_2) \times (m_1 + m_2)$ matrix given by

$$A \oplus B = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}.$$

Proof of Lemma 2. Let $\pi(i)$ be the matrix $(\pi_{nm}(i))$ and observe that if λ is a $k \times 1$ vector whose entries are each one, then

$$E_t s_{t+1} = \left(\oplus_i s_{t-1}' \pi(i) s_t\right) \lambda. \tag{5}$$

Then

$$\begin{aligned} E_t s_t' A s_{t+1} &= s_t' A \left(\oplus_i s_{t-1}' \pi(i) s_t\right) \lambda \\ &= \lambda' \left(\oplus_i s_{t-1}' \pi(i) s_t\right) A' s_t \\ &= s_{t-1}' \left[\pi(1) s_t, \cdots, \pi(k) s_t\right] A' s_t. \end{aligned} \tag{6}$$

Now let C^n be the $k \times k$ matrix whose m^{th}-column is the n^{th}-column of $\pi(m)$ and B be the matrix whose n^{th}-column is the n^{th}-column of $C^n A'$. Then for all possible s_t and s_{t-1} we have that the right hand side of (6) is equal to $s'_{t-1} B s_t$.[4] ■

To construct the T-map, we begin by specifying a perceived law of motion (PLM), that is, a functional form of the representation in terms of parameters (coefficients) and observables; the input of the T-map is the collection of coefficients. Agents are assumed to form their expectations using this PLM. We now interpret the model (1) as holding outside of an REE, so that

$$y_t = \beta E_t^* y_{t+1} + \delta y_{t-1} + v_t, \tag{1'}$$

where $E_t^* y_{t+1}$ denotes the forecast corresponding to the PLM. Inserting these expectations into the reduced form model (1') yields the actual coefficients on the observables; the associated difference equation is called the actual law of motion (ALM), and its set of coefficients is the output of the T-map.

Using the above lemma as a guide, we take our PLM to be

$$y_t = a y_{t-1} + s'_{t-1} A s_t + b v_t. \tag{7}$$

For any real number x, denote by $I_k(x)$ the $k \times k$ matrix with x in each diagonal entry and zeros elsewhere. Now notice that

$$\begin{aligned} E_t^* y_{t+1} &= a^2 y_{t-1} + a E_t s'_{t-1} A s_t + s'_{t-1} B(A) s_t + abv_t \\ &= a^2 y_{t-1} + s'_{t-1} \left(I_k(a) A + B(A) \right) s_t + abv_t. \end{aligned}$$

Inserting this into the reduced form equation (1) yields the actual law of motion (ALM) and thus determines the output of the T-map.[5] Set

$$T_1(a) = \beta a^2 + \delta \tag{8}$$

$$T_2(a, A) = I_k(\beta) \left(I_k(a) A + B(A) \right) \tag{9}$$

$$T_3(a, b) = \beta ab + 1. \tag{10}$$

[4] For an explicit expression for $B(A)$ see the proof of Proposition 3.

[5] Implicitly we are assuming that when expectations are formed the information set includes s_t, s_{t-1}, v_t and y_{t-1} but not y_t. See (Evans and McGough 2002) for further details and a discussion of the case in which y_t is also included in the information set.

Then the ALM can be written

$$y_t = T_1(a)y_{t-1} + s'_{t-1}T_2(a, A)s_t + T_3(a, b)v_t.$$

Notice that a fixed point of the T-map determines a representation of an REE. We denote by $\Omega(\pi) \subset \mathbb{R} \times \mathbb{R}^{k\times k} \times \mathbb{R}$ the collection of fixed points of T; the index π reflects the fact that the T-map, and hence the set of fixed points, depends on the matrix of transition probabilities π.

The form of the PLM (7) restricts the set of representable noisy k-SSEs in the following way. Recall that a noisy k-SSE y_t has the form $z_t + \zeta_t$ where z_t is a fundamentals solution to the stochastic model. If y_t has a representation of the form (7) then we can take z_t to have a representation of the form

$$z_t = az_{t-1} + bv_t. \tag{11}$$

Assuming the model is indeterminate and that the roots of the associated quadratic are real, it can be shown that there are precisely two stationary solutions of the form (11). These are given by

$$z_t^i = (1 - a_iL)^{-1}(1 - \beta a_i)^{-1}v_t = (1 - \beta a_i)^{-1}\sum_{j=0}^{\infty} a_i^j v_{t-j}, \tag{12}$$

for $i = 1, 2$, where the a_i are roots of $\beta a^2 - a + \delta$; see (Evans and McGough 2002) for details[6]. Here L denotes the lag operator defined by $Ly_t = y_{t-1}$ and we note that the indicated sum can be shown to converge in mean square. Because of their parsimonious representation, these REE are often called minimum state variable (MSV) solutions. We have the following result.

Proposition 3 *Assume the parameters of the model are such that noisy k-SSEs exist and the a_i are real. Let y_t be a stationary rational expectations equilibrium. If $y_t = z_t^i + \zeta_t$ is a noisy k-SSE with associated transition array π, then there exists a point $(a, A, b) \in \Omega(\pi)$ such that $y_t = ay_{t-1} + s'_{t-1}As_t + bv_t$.*

Proof. See Appendix.

Note that $(a_i, 0, (1 - \beta a_i)^{-1}) \in \Omega(\pi)$ so that $\Omega(\pi)$ is not empty. We say that $\Omega(\pi)$ is non-trivial if it contains points other than $(a_i, 0, (1 - \beta a_i)^{-1})$.

[6] The quadratic has real roots if and only if $\beta\delta \leq \frac{1}{4}$. If the roots are nonreal there still exists a solution, driven by v_t, taking the form $\hat{z}_t = (1 - \beta^{-1}L + \beta^{-1}\delta L^2)^{-1}(1 - \beta L)v_t$.

The above result verifies that there are non-trivial fixed points to the T-map. Furthermore, it shows that any noisy k-SSE whose particular solution is an MSV solution can be represented as a fixed point of the T-map.

In the following sections we will use the T-map to analyze stability under learning of noisy k-SSEs, using the E-stability principle. This will enable us to provide additional model parameter restrictions required for learning stability. However, the above proposition also raises an interesting question: Are there REE having representations of the form (7) that are not themselves noisy k-SSEs? To address this question more formally we make the following definition: A *noisy k-state dependant sunspot equilibrium* (noisy k-SDS) is any process y_t satisfying (7) for some transition array π and associated fixed point (a, A, b). Obviously, a noisy k-SSE is a noisy k-SDS; thus, to provide a distinction, we label with "noisy k-*SDS" the noisy k-SDSs that are not themselves noisy k-SSEs. The following natural questions arise.

1. Do k-*SDSs exist?
2. If so,
   (a) are they stable under learning?
   (b) if the transition array π corresponds to a noisy k-SSE, then do noisy k-*SDSs exist with respect to this π?

The relevance of question 2, b, is the following: if no such k-*SDSs exist, then, when we show stability under learning, we can be confident that our agents are learning a noisy k-SSE, and not a noisy k-*SDS. These questions appear difficult to address in the general case. For the case $k = 2$ we will find the answers to be "yes", "sometimes", and "no", respectively.

5 E-stability

Write $\theta = (a, A, b)$ and $T(\theta) = (T_1(a), T_2(a, A), T_3(a, b))$. Note that T maps $\mathbb{R} \times \mathbb{R}^{k \times k} \times \mathbb{R}$ into itself. Let θ^* be a fixed point of the T-map. We say θ^* (and the associated REER) is *E-stable* (or "expectationally stable") provided the differential equation

$$\frac{d\theta}{d\tau} = T(\theta) - \theta \tag{13}$$

is locally asymptotically stable at θ^*. The *E-stability Principle* says that if the REER is E-stable then it is learnable by a reasonable adaptive algorithm. This principle is known to be valid for least squares and closely related statistical learning rules in a wide variety of models. For a thorough discussion see (Evans and Honkapohja 2001).[7]

The definition of expectational stability just given is inadequate when there is a non-trivial connected set of rest points of the differential equation (13), as is the case for our model; if $\Omega(\pi)$ is locally connected then no point in Ω is locally asymptotically stable. In this context we restate the notion of E-stability as follows: we say that a set of fixed points, Q, is E-stable provided there is a neighborhood U of Q so that for any $\theta_0 \in U$ the trajectory of θ determined by the differential system (13) converges to a point in Q. A necessary condition for E-stability of Q is that for all $q \in Q$, the non-zero eigenvalues of the derivative $T(\theta) - \theta$ evaluated at q have negative real part. Sufficient conditions are more difficult to obtain because of the presence of zero eigenvalues.[8]

Assume the parameters of the model are such that sunspots exist and the roots of the associated quadratic are real. To analyze the stability of $\Omega(\pi)$ in this case, begin by noticing that these real roots are the fixed points of T_1 and are given by

$$a_1 = \frac{1 - \sqrt{1 - 4\beta\delta}}{2\beta} \quad \text{and} \quad a_2 = \frac{1 + \sqrt{1 - 4\beta\delta}}{2\beta}.$$

It follows that $\Omega(\pi) = \Omega_1(\pi) \cup \Omega_2(\pi)$ where

$$\Omega_i(\pi) = \{(a_i, A, b_i) \in \Omega(\pi) \,|\, b_i = (1 - \beta a_i)^{-1}\}.$$

Since T_1 is decoupled from T_2 and T_3, we can analyze its stability independently. We have that $DT_1(a) = 2\beta a$, which immediately implies that the subsystem in T_1 is locally asymptotically stable if and only if $a = a_1$. Thus we have the following proposition:

Proposition 4 *The set $\Omega_2(\pi)$ is not E-stable.*

[7]The connection between statistical learning and E-stability is established using convergence results from the stochastic approximation literature. This technique is described in (Marcet and Sargent 1989), (Woodford 1990) and Chapters 6 and 7 of (Evans and Honkapohja 2001).

[8]For more details and a related conjecture, see p. 245 of (Evans and Honkapohja 2001).

Note that $DT_1(a_1) < 1$, which gives us hope that $\Omega_1(\pi)$ may be E-stable. Furthermore, $DT_3(a_1, b_1) < 1$ so that to show $\Omega_1(\pi)$ satisfies the necessary condition for E-stability, it suffices to show that $(a_1, A, b_1) \in \Omega_1(\pi)$ implies that the eigenvalues of $DT_2(a_1, A)$ have real part less than one. In general this appears to be difficult to demonstrate. In the following Section we analyze the case $k = 2$.

6 2-State Sunspots

To obtain explicit results concerning the existence and representations of noisy k-*SDSs, as well as to derive specific stability results for both noisy k-*SDSs and noisy k-SSEs, we now use the theory set out above to consider in detail the case $k = 2$. In the process we obtain a simple method by which both 2-SSE's and 2-*SDSs can be constructed. Some of the details presented below are contained in (Dávila and Guesnerie 2001); we include them here for completeness.

6.1 Existence of 2-SSEs

The pair $(\pi, \overline{y})$ is a 2-SSE for the model (2) if and only if

$$\begin{aligned}
(\pi_{11}(1) + \beta^{-1}\delta)\overline{y}_1 + \pi_{11}(2)\overline{y}_2 &= \beta^{-1}\overline{y}_1, \\
\pi_{22}(1)\overline{y}_1 + (\pi_{22}(2) + \beta^{-1}\delta)\overline{y}_2 &= \beta^{-1}\overline{y}_2, \\
\pi_{21}(1)\overline{y}_1 + (\pi_{21}(2) + \beta^{-1}\delta)\overline{y}_2 &= \beta^{-1}\overline{y}_1, \\
(\pi_{12}(1) + \beta^{-1}\delta)\overline{y}_1 + \pi_{12}(2)\overline{y}_2 &= \beta^{-1}\overline{y}_2.
\end{aligned}$$

The following restrictions are thus implied:

$$\begin{aligned}
\pi_{11}(1) + \pi_{22}(2) &= 1 + \beta^{-1}(1 - \delta), && (14) \\
\pi_{21}(1) + \pi_{12}(2) &= 1 + \beta^{-1}(1 + \delta), && (15) \\
\pi_{22}(1)\overline{y}_1 + \pi_{11}(2)\overline{y}_2 &= 0, && (16) \\
(1 + \delta\pi_{21}(1) - \pi_{12}(2))\overline{y}_1 &= (\pi_{21}(1) - (1 + \delta\pi_{12}(2)))\overline{y}_2. && (17)
\end{aligned}$$

For the sunspot to be non-trivial, it must be that $\overline{y}_1 \neq \overline{y}_2$, which, by restrictions (16) and (17), implies

$$\frac{\pi_{22}(1)}{1 + \delta\pi_{21}(1) - \pi_{12}(2)} = \frac{\pi_{11}(2)}{1 + \delta\pi_{12}(2) - \pi_{21}(1)}. \tag{18}$$

Restrictions (14),(15), and (18) can be combined to determine the transition array up to one degree of freedom. Equation (16) or (17) can then be used to determine the ratio of the states.

According to the preceding arguments, 2-SSEs exist provided transition arrays satisfying (14),(15), and (18) exist. The restrictions on the transition probabilities can be rewritten as

$$\begin{aligned} \pi_{11} &= \frac{1-\delta}{\beta} + \pi_{22}, \\ \pi_{12} &= -\frac{\delta}{\beta} + \pi_{22}, \\ \pi_{21} &= \frac{1}{\beta} + \pi_{22}. \end{aligned} \tag{19}$$

where, for notational simplicity, we write $\pi_{ij} = \pi_{ij}(1)$. Set

$$\begin{aligned} L(\beta,\delta) &= \max\{\frac{\delta-1}{\beta}, \frac{\delta}{\beta}, -\frac{1}{\beta}\}, \\ U(\beta,\delta) &= \min\{1-\frac{1-\delta}{\beta}, 1+\frac{\delta}{\beta}, 1-\frac{1}{\beta}\}. \end{aligned}$$

Imposing $\pi_{ij} \in (0,1)$ yields the following: A 2-SSE exists if and only if

$$(0,1) \cap (L(\beta,\delta), U(\beta,\delta)) \neq \emptyset,$$

where we say $(L,U) = \emptyset$ if $U \leq L$. A straightforward argument then shows that this set is non-empty if and only if β and δ satisfy the restrictions in Proposition 1.

FIGURES 1 AND 2 ABOUT HERE

In Figures 1 and 2, the regions of parameters corresponding to existence of 2-SSEs and, simultaneously, real roots of the quadratic, are denoted by A_1 and B_1. Regions A_2 and B_2 denote those parts of the indeterminacy regions in which there are real roots but 2-SSEs do not exist.

6.2 Representations

The T-map can be explicitly computed as

$$\begin{aligned}
T_1(a) &= \beta a^2 + \delta \\
T_2(a, A) &= \begin{bmatrix} \beta(aA_{11} + \pi_{11}(1)A_{11} + \pi_{11}(2)A_{12}) & \beta(aA_{12} + \pi_{12}(1)A_{21} + \pi_{12}(2)A_{22}) \\ \beta(aA_{21} + \pi_{21}(1)A_{11} + \pi_{21}(2)A_{12}) & \beta(aA_{22} + \pi_{22}(1)A_{21} + \pi_{22}(2)A_{22}) \end{bmatrix} \\
T_3(a, b) &= \beta ab + 1.
\end{aligned}$$

The fixed points of T_1 are $a = \frac{1 \pm \sqrt{1-4\beta\delta}}{2\beta}$, and the fixed point of T_3 is $(1-\beta a)^{-1}$. The fixed points of T_2 are determined by the following four equations:

$$\begin{aligned}
(1/\beta - a)A_{11} &= \pi_{11}A_{11} + (1 - \pi_{11})A_{12} \\
(1/\beta - a)A_{12} &= \pi_{12}A_{21} + (1 - \pi_{12})A_{22} \\
(1/\beta - a)A_{21} &= \pi_{21}A_{11} + (1 - \pi_{21})A_{12} \\
(1/\beta - a)A_{22} &= \pi_{22}A_{21} + (1 - \pi_{22})A_{22}
\end{aligned}$$

where we employ the convention $\pi_{ij} = \pi_{ij}(1)$. This linear homogeneous system has non-trivial solutions only if linear dependence is exhibited.

To investigate this, notice that we can write $A_{ij} = K_{ij}A_{12}$ where

$$\begin{aligned}
K_{11} &= \frac{1 - \pi_{11}}{1/\beta - a - \pi_{11}}, \\
K_{21} &= \frac{\pi_{21}K_{11} + 1 - \pi_{21}}{1/\beta - a}, \\
K_{22} &= \frac{\pi_{22}K_{21}}{1/\beta - a - (1 - \pi_{22})}.
\end{aligned}$$

We conclude that a non-trivial solution exists if and only if the following equation holds:

$$1/\beta - a = \pi_{12}K_{21} + (1 - \pi_{12})K_{22}. \tag{20}$$

Recall it was shown that 2-state sunspots exist if and only if the transition array satisfies (19). Algebra (Mathematica) shows that imposing these restrictions on π implies that equation (20) holds. Notice that, in this case, there exists a one-dimensional continuum of representations; the choice of A_{12} is free, but once made, the remaining A_{ij} are pinned down. Further, to construct a noisy 2-SSE, simply choose $\pi_{22} \in (L(\beta, \delta), U(\beta, \delta))$ and pick A_{12} arbitrarily. The above equations can then be used to determine the remaining parameter values.

6.3 Existence of Noisy 2-*SDSs

In Section 4 we wondered, in question 2, b, for π satisfying (19), whether fixed points determining representations of k-SDSs existed.

Proposition 5 *Suppose β and δ are such that 2-SSEs exist and the roots of the associated quadratic are real. Suppose a denotes a root of this quadratic and $b = (1-\beta a)^{-1}$. Let π satisfy the existence restrictions (19). If $T_2(a, A) = A$ and*

$$y_t = ay_{t-1} + s'_{t-1}As_t + v_t,$$

then y_t is a noisy 2-SSE.

The proof of this Proposition uses the following Lemma.

Lemma 6 *Let π satisfy the existence restrictions (19). If $T_2(a, A) = A$ then*

$$A_{ij} = \frac{A_{jj}}{1-a} - \frac{aA_{ii}}{1-a}.$$

Proof. Let $A_{12} = 1$. Since A is a fixed point, we have that $A_{ij} = K_{ij}A_{12}$. Thus it suffices to show $K_{ij} = \frac{K_{jj}}{1-a} - \frac{aK_{ii}}{1-a}$. This can be shown using Maple. ■

Proof of Proposition 5: For each realization of the process s_t there is a map $\tau : \mathbb{Z} \to \{1, 2\}$ such that $s_t = S_i \Leftrightarrow \tau(t) = i$. Thus we can write (7) as

$$(1 - aL)y_t = A_{\tau(t-1)\tau(t)} + bv_t. \tag{21}$$

Now let $\hat{A} = vec(A)$ and for reasons to be clear later, index $\hat{A}$ starting with 0, that is, $\hat{A} = (\hat{A}_0, \hat{A}_1, \hat{A}_2, \hat{A}_3)'$. Let $\sigma : \mathbb{Z} \to \{0, 1, 2, 3\}$ be defined by

$$\sigma(t) = \hat{\tau}(t-1) + \hat{\tau}(t-1)\hat{\tau}(t) + \hat{\tau}(t) + \hat{\tau}(t-1)(1 - \hat{\tau}(t)),$$

where $\hat{\tau} = \tau - 1$. Then $\hat{A}_{\sigma(t)} = A_{\tau(t)\tau(t-1)}$. This allows us to write equation (21) as

$$(1 - aL)y_t = \hat{A}_{\sigma(t)} + bv_t. \tag{22}$$

The Lemma implies

$$\hat{A}_{\sigma(t)} = A_{\tau(t-1)\tau(t)} = \frac{1}{1-a}(1 - aL)A_{\tau(t)\tau(t)}.$$

Combining this with (22) yields

$$y_t = \frac{1}{1-a} A_{\tau(t)\tau(t)} + \frac{b}{1-aL} v_t,$$

which is a 2-SSE. ■

Proposition 5 implies that if π satisfies (19), and if $\Omega_1(\pi)$ is E-stable, then agents will necessarily be learning a noisy 2-SSE, and not possibly a noisy 2-*SDS.

Next we have the following result concerning the existence of 2-*SDSs.

Proposition 7 *If (β, δ) is in the indeterminate region and the roots of the usual quadratic are real then there exists transition arrays π such that $\Omega(\pi)$ is non-trivial. Furthermore, π can be chosen to violate (19).*

Proof. We provide the proof here because it is constructive in nature and we will require construction of 2-*SDSs when we consider stability analysis. Label the real roots as a_i and notice that $1/\beta - a_i = a_j \equiv \alpha$ for $i, j = 1, 2$ with $i \neq j$. Thus we can write the conditions for a fixed point of the T-map as

$$\alpha A_{11} = \pi_{11} A_{11} + (1 - \pi_{11}) A_{12} \quad (23)$$
$$\alpha A_{12} = \pi_{12} A_{21} + (1 - \pi_{12}) A_{22} \quad (24)$$
$$\alpha A_{21} = \pi_{21} A_{11} + (1 - \pi_{21}) A_{12} \quad (25)$$
$$\alpha A_{22} = \pi_{22} A_{21} + (1 - \pi_{22}) A_{22} \quad (26)$$

where $|\alpha| < 1$ by indeterminacy. We proceed as follows: fix α and show that we can choose A_{ij} and π_{ij} so that the above conditions are satisfied. The key observations are that, for each equation, the left hand side is a convex combination of the A_{ij} on the right hand side, and that the π_{ij} are independent across equations.

Case 1: $\alpha \geq 0$. Choose A_{ij} so that

$$A_{22} < A_{12} < 0 < A_{21} < A_{11}. \quad (27)$$

Case 2: $\alpha < 0$. Choose A_{ij} so that

$$-A_{12} < A_{21} < \alpha A_{12} < A_{11} < 0 < A_{22} < -A_{21} < A_{12}. \quad (28)$$

Case 1 implies

$$\begin{aligned}\alpha A_{11} &\in (A_{12}, A_{11}) \\ \alpha A_{12} &\in (A_{22}, A_{21}) \\ \alpha A_{21} &\in (A_{12}, A_{11}) \\ \alpha A_{22} &\in (A_{22}, A_{21}),\end{aligned}$$

and Case 2 implies similar set membership except that the endpoints of each of the intervals are reversed. Equations (23) - (26) follow immediately from the implied choice of π. It remains to show that if $(\beta, \delta) \in A_1$ then there exist π not satisfying (19) such that $\Omega_1(\pi)$ is not trivial. Given the choices of A_{ij}, the associated transition array can be constructed as follows:

$$\begin{aligned}\pi_{11} &= (A_{11} - A_{12})^{-1}(\alpha A_{11} - A_{12}) \\ \pi_{12} &= (A_{21} - A_{22})^{-1}(\alpha A_{12} - A_{22}) \\ \pi_{21} &= (A_{11} - A_{12})^{-1}(\alpha A_{21} - A_{12}) \\ \pi_{22} &= (A_{21} - A_{22})^{-1}(\alpha A_{22} - A_{22})\end{aligned}$$

Begin by noticing that, according to (19), the set of π to which correspond 2-SSEs is one dimensional, being pinned down by the choice of π_{22}. Now notice the choice of A_{21} and A_{22} determines π_{22}. On the other hand, in both case 1 (case 2), for given choice of A_{21} and A_{22} there are multiple A_{12} and A_{11} which satisfy the restriction (27) ((28)). In particular, the choice of A_{21} and A_{22} does not pin down the values of π_{11}, π_{12} and π_{21}. This shows the set of π to which correspond non-trivial $\Omega_1(\pi)$ has dimension greater than one, thus completing the proof. ■

The result implies that provided the model is indeterminate, there exist 2-*SDSs. Thus, in Figures 1 and 2, 2-*SDSs exist throughout regions A_1, A_2, B_1 and B_2.

We conjecture that the above results hold for $k > 2$. Also notice that while the transition array π need not satisfy (19) for noisy 2-SDSs to exist, and, in fact, cannot satisfy (19) for noisy 2-*SDSs to exist, there may still be restrictions on its values. In fact we have the following result on the dimensionality of noisy 2-*SDSs.

Proposition 8 *Let I be the unit cube in $\mathbb{R}^4$. Let $R = \{\pi \in I$ such that (19) holds$\}$ and $\hat{R} = \{\pi \in I$ such that $\Omega(\pi)$ is nontrivial$\}$. Then (i) R is homeomorphic to $\mathbb{R}$ and (ii) there is a subset of $\hat{R}$ that is homeomorphic to $\mathbb{R}^3$, but $\hat{R}$ has Lesbesgue measure zero as a subset of I.*

Proof. See Appendix.

Notice that $R \subset \hat{R}$, and for $\pi \in R$ there exist noisy 2-SSEs while if $\pi \in \hat{R} \setminus R$ there exist noisy 2-*SDSs. Thus this proposition shows that the set of transition arrays for which noisy 2-*SDSs exist is much bigger than the set for which noisy 2-SSEs exist. However, $\hat{R}$ is still very restrictive since it has measure zero as a subset of I.

We can also characterize the range of 2-*SDSs. Here it is of interest to focus on solutions to the nonstochastic model (2) taking the form

$$y_t = ay_{t-1} + s'_{t-1}As_t.$$

(Equivalently we are considering the component of a noisy 2-*SDS that solves the homogeneous equation). We have

Proposition 9 *Given a and A, assume that y_t is a 2-*SDS. The range of possible values for y_t is infinite. Furthermore, with positive probability the stochastic process y_t takes on infinitely many values.*

Proof. See Appendix.

Thus 2-*SDSs are qualitatively very different from 2-SSEs. Both types of solution are driven by second-order 2-state Markov processes, but 2-*SDSs can take infinite many values, while 2-SSEs take only two values. The distinction between 2-SSEs and 2-*SDSs does not appear in the purely forward looking model and arises specifically because of the dependence of y_t on its previous value.

6.4 E-stability

We can now use the explicit form of the T-map to analyze E-stability. As observed in Section 5, to show that $\Omega_1(\pi)$ satisfies the necessary condition for E-stability, it suffices to analyze the subsystem T_2 for $a = a_1$. We have that

$$\frac{\partial vec(T_2)}{\partial vec(A)} = \begin{bmatrix} \beta(a_1 + \pi_{11}) & \beta(1-\pi_{11}) & 0 & 0 \\ 0 & \beta a_1 & \beta\pi_{12} & \beta(1-\pi_{12}) \\ \beta\pi_{21} & \beta(1-\pi_{21}) & \beta a_1 & 0 \\ 0 & 0 & \beta\pi_{22} & \beta(a_1 + 1 - \pi_{22}) \end{bmatrix}. \quad (29)$$

E-stability requires that the real parts of the eigenvalues of this derivative must be less than one when evaluated at points in Ω_1. Unfortunately, the complex nature of the eigenvalues algebraic representation makes formal analysis difficult; thus we proceed numerically. We analyze the stability of noisy 2-SSEs and noisy 2-*SDSs separately.

6.4.1 E-stability of Noisy 2-SSEs

First, notice that the above derivative is invariant over the set $\Omega_1(\pi)$. Also, provided the transition array π satisfies the restrictions (19), it can be verified algebraically that the eigenvalues are independent of the chosen transition array. Thus, numerical analysis of the eigenvalues of DT_2 requires only varying β and δ. We further restrict the parameter space to guarantee existence of representable 2-SSEs. Specifically, we assume that parameters satisfy the conditions in Proposition 1, and further that $\beta\delta < 1/4$ so that the associated roots are real. Finally, we consider negative and positive values of β separately, labeling the relevant regions "Area A_1" and "Area B_1" respectively: see Figures 1 and 2.

Matlab was used to plot level curves of the function yielding the maximum value of the real part of the eigenvalues not equal to one. As indicated by Figures 1 and 2, $\Omega_1(\pi)$ appears to satisfy the necessary conditions for E-stability throughout Area A_1, and to fail to satisfy these conditions throughout Area B_1. In particular we have:

Proposition 10 *There exist parameter values in Area A_1 such that provided π satisfies (19), the set $\Omega_1(\pi)$ satisfies the necessary condition for E-stability.*

Conjecture 11

1. *The set Ω_1 is E-stable for parameters in Area A_1.*
2. *The set Ω_1 is unstable for parameters in Area B_1.*

Using simulations we verify below that in Area A_1 there exist noisy k-SSEs that are stable under learning.

6.4.2 E-stability of Noisy 2-*SDSs

To assess the E-stability of noisy k-*SDSs, we again analyze the eigenvalues of (29). However, the transition arrays necessarily do not satisfy (19), and, in fact, these eigenvalues may depend on the values of the probabilities π. This has the unfortunate consequence of increasing the dimension of the parameter space. Specifically, we must now consider stability of 2-*SDSs for different

β, δ, and π. As we intend only to establish existence of stable 2-*SDSs, we proceed as follows: for values of (β, δ) in each of the four regions A_i, B_i, $(i = 1, 2)$, we use the method described in Section 6.3 to choose a value of π to which corresponds a 2-*SDS, and then we analyze the stability of the associated set of fixed points $\Omega_1(\pi)$ by numerically computing the eigenvalues of (29). We obtain the following result.

Proposition 12 *For $i = 1, 2$, there exist $(\beta, \delta) \in A_i$ and transition array π violating (19) so that $\Omega_1(\pi)$ is non-trivial and satisfies the necessary condition for E-stability.*

This result implies that there exist stable 2-*SDSs for values (β, δ) such that 2-SSEs do not even exist.

Our numerical analysis was more promising than is perhaps indicated by the above Proposition. The stability of $\Omega_1(\pi)$ obtained for all values of $(\beta, \delta) \in A_i$ and for all transition arrays π we tested. Based on this we make the following conjecture.

Conjecture 13 *If $(\beta, \delta) \in A = A_1 \cup A_2$ and if π is such that $\Omega_1(\pi)$ is non-trivial then $\Omega_1(\pi)$ satisfies the necessary condition for E-stability.*

Strong instability results are more difficult to obtain. Using the same method described above we obtain the following somewhat unsatisfying result.

Proposition 14 *For $i = 1, 2$, there exist $(\beta, \delta) \in B_i$ and transition array π violating (19) so that $\Omega_1(\pi)$ is not E-stable.*

Again, for all values of $(\beta, \delta) \in B_i$ and π tested, we found instability obtained, which leads us to the following conjecture.

Conjecture 15 *If $(\beta, \delta) \in B = B_1 \cup B_2$ and if π is such that $\Omega_1(\pi) \neq \{0\}$ then $\Omega_1(\pi)$ is not E-stable.*

6.5 Simulations

Thus far we have only be able to show that the key necessary conditions for expectational stability hold for parameters in a subset of Area A. We conjecture that these conditions are in fact sufficient. In this section we provide simulations that support this conjecture, and which furthermore suggest that the E-stability principle holds.

6.5.1 E-stability

Recall that the representations determined by the set of fixed points $\Omega_1(\pi)$ are E-stable provided that solutions to the differential equation

$$\frac{d\theta}{d\tau} = T(\theta) - \theta$$

originating near $\Omega_1(\pi)$ converge to points in $\Omega_1(\pi)$. We are unaware of conditions on $T(\theta) - \theta$ sufficient to guarantee such convergence, due to the presence of zero eigenvalues, and so we proceed as follows. Begin with stability analysis of 2-SSEs. For various values of β and δ in Area A_1, and initial conditions θ_0 selected randomly within a small neighborhood of the set $\Omega_1(\pi)$, the differential equation $d\theta/d\tau = T(\theta) - \theta$ was solved using an ODE solver in Matlab. The corresponding trajectories were plotted for 400 time periods and the time-series representing the distance between $\theta(\tau)$ and the set $\Omega_1(\pi)$ was computed.[9] We found that convergence appears to obtain for the selected values of β and δ, that is, the paths appear to converge to the set $\Omega_1(\pi)$[10]. Stability analysis of 2-*SDSs was done in a similar fashion and analogous results obtained.

FIGURES 3 AND 4 ABOUT HERE

See Figures 3 and 4 for the time paths obtained when $\beta = -3$ and $\delta = 1$.[11] Graphs obtained using other values of β and δ were qualitatively similar.

6.5.2 Real-Time Learning

The E-stability principle says that if an REER is E-stable then it is learnable by a reasonable learning algorithm. However, the E-stability principle is only known to formally apply to REER, which are isolated rest points of the differential equation (13). To support our focus on E-stability we now simulate least squares learning of the REER.

[9] We take the distance between a point and a set to be the infimum of the set of distances between the point and points in the set.

[10] The Figures suggest that convergence to the set $\Omega_1(\pi)$ obtains, whereas our definition of E-stability requires convergence to a point in the set. Graphs (not displayed) of the evolution of the point in the set $\Omega_1(\pi)$ nearest to the current value of $\theta(\tau)$ suggest convergence to a point in fact obtains. Analogous statements hold in our real time learning analysis presented below.

[11] Subsequent Figures also use this choice of β and δ to illustrate our results.

Agents are assumed to have the PLM (7), reproduced here for convenience,

$$y_t = ay_{t-1} + s'_{t-1}As_t + bv_t,$$

and are assumed to use OLS (ordinary least squares) to estimate the parameters of the model. Set

$$X_t = [y_{t-1}, s_{t-1}(1)s_t(1), s_{t-1}(1)s_t(2), s_{t-1}(2)s_t(1), s_{t-1}(2)s_t(2), v_t]',$$

where $s_t(i)$, for $i = 1, 2$, denotes the components of s_t, and write $\theta = [a, \text{vec}(A)', b]'$, where vec$(A)$ is the operator that stacks in order the columns of A into a column vector. (Earlier we defined $\theta = (a, A, b)$ but it is now convenient to rewrite θ as a column vector). This allows us to write the stochastic process for the estimators recursively[12] as

$$\begin{aligned} \theta_t &= \theta_{t-1} + \frac{1}{t}R_t^{-1}X_t(y_t - \theta'_{t-1}X_t) \\ R_t &= R_{t-1} + \frac{1}{t}(X_tX'_t - R_{t-1}) \end{aligned}$$

where

$$y_t = T_1(a_{t-1})y_{t-1} + s'_{t-1}T_2(A_{t-1})s_t + b_{t-1}v_t.$$

The term t^{-1} in the recursive algorithm is called the "gain sequence." This or closely related gain sequences appear in least squares and other statistical estimators and have the role of making possible the convergence of parameter estimates. The behavior of this algorithm was analyzed via simulations. The algorithm was initialized by choosing points at random within a given neighborhood of the set $\Omega_1(\pi)$.

Analytic results implying convergence with probability one typically require amending the algorithm with a projection facility. Alternatively, one can adjust the gain of the algorithm to obtain convergence with probability approaching one. For the simulations produced here we scale the gain of the RLS algorithm by 1/25, thus increasing the probability of convergence.

Our results on E-stability indicate that convergence to a noisy 2-SSE obtains only if the model's parameter values are chosen to lie in Area A_1.

[12] See, for example, (Marcet and Sargent 1989) or pp. 32-3 of (Evans and Honkapohja 2001).

We chose several different parameter pairs in this area and for each pair ran several simulations. For each pair of values we found that with positive probability, that is for a positive proportion of the simulations, convergence to $\Omega_1(\pi)$ appears to obtain. (For all simulations in which convergence did not appear to obtain, the norms of the estimates appear to diverge to infinity.)

FIGURES 5 AND 6 ABOUT HERE

See Figure 5 for a representative simulation showing convergence. Note that while the qualitative shape of the path giving the distance to $\Omega_1(\pi)$ is similar to that for the E-stability simulation shown in Figure 3, the path is now irregular and the time scale of convergence is longer. These characteristics reflect the stochastic dynamics and the decreasing gain feature of the model under real time econometric learning. In contrast, for parameter values in B_1 all simulations failed to converge to $\Omega_1(\pi)$. Figure 6 shows a representative simulation showing convergence to a 2-*SDS.

FIGURES 7 AND 8 ABOUT HERE

Figures 7 and 8 show portions of the time series for y_t for a simulation in which there is convergence to a 2-SSE or noisy 2-SSE. Here REE denotes the path under fully rational expectations, while RTL denotes the path (for the same sequence of random shocks) under real time learning.[13] Note that while the path of the non-noisy SSE, Figure 7, is implausibly regular, with strong negative serial correlation, the noisy 2-SSE shows the kind of irregular fluctuations typical in macroeconomic data.

Similar results obtain when 2-*SDSs were analyzed and are illustrated in Figures 9, 10 and 11.

FIGURES 9, 10 AND 11 ABOUT HERE

Figures 9 and 11 give the paths time series for y_t of convergence to a 2-*SDS for two different choices of π. Figure 10 shows convergence to a noisy 2-*SDS. The time series of the stable non-noisy 2-*SDSs in Figures 9 and 11 are particularly intriguing because they show complicated dynamics even though

[13]The REE path illustrated in the Figures is obtained using the parameter vector $\theta \in \Omega_1(\pi)$ that is closest, in the Euclidean metric, to the terminal simulation value for θ_t under RTL.

there is no intrinsic noise and the process is driven entirely by an exogenous two state Markov process. This shows the clear potential of stationary sunspot equilibria, driven by finite state Markov processes, for explaining complex economic fluctuations. When intrinsic random noise is added to the model, as in Figure 10, the paths appear even more realistic.

7 Examples

In this section we consider two examples that illustrate the application of the theory developed above. We first consider a modified Cagan model, which can be indeterminate, yet in which neither noisy k-SSEs nor stable noisy k-*SDSs exist. We then consider the (Sargent 1987) extension of the (Lucas and Prescott 1971) model of investment under uncertainty to allow for taxes and externalities. For this model there are parameter values that yield both stable k-SSEs and stable k-*SDSs.

7.1 Cagan's Model

The discrete form of the Cagan model can be given as

$$p_t = \beta E_t p_{t+1} + \alpha m_t, \tag{30}$$

where β lies in the unit interval. Assume a money supply rule of the form

$$m_t = \overline{m} + \xi p_{t-1} + u_t, \tag{31}$$

where u_t is white noise. Combining the money rule with equation (30) yields the following reduced form:

$$p_t = \alpha\overline{m} + \beta E_t p_{t+1} + \alpha\xi p_{t-1} + \alpha u_t. \tag{32}$$

Let $\delta = \alpha\xi$. Since $\beta > 0$, this model is indeterminate provided $\delta > 1 - \beta$ and $\delta < \beta$. With the value of δ unrestricted, it follows that for $\frac{1}{2} < \beta < 1$ and appropriate δ, the model is indeterminate. However, with $0 < \beta < 1$ only the B_2 region of indeterminacy is feasible. Thus in this model noisy k-SSEs do not exist. Noisy 2-*SDSs do exist, but will not be stable under learning.[14]

This example shows clearly that the requirement that sunspot equilibria be stable under learning can be a demanding test.

[14]If the Cagan model is interpreted as obtained from a linearized overlapping generations model of money then $\beta < 0$ is possible. We do not pursue these cases here.

7.2 Investment under Uncertainty

Our second example is based on Sargent's extension of the (Lucas and Prescott 1971) model of investment under uncertainty to allow for dynamic market distortions due to taxes and externalities; see Ch. XIV of (Sargent 1987).[15]

Consider a competitive industry with N identical firms. Output x_t of the representative firm at t is given by

$$x_t = x_0 + f_0 k_t + f_1 K_t + f_2 K_{t-1},$$

where k_t is the capital stock of the individual firm and $K_t = N k_t$ denotes the aggregate capital stock. The presence of the two terms in K_t reflect contemporaneous and lagged external effects. These may be positive or negative, so we do not restrict the signs of f_1 or f_2, but we assume $f_0 > 0$ and $x_0 > 0$. Taxes are levied on firms on capital in place. The rate itself is assumed to depend on current and lagged aggregate capita stock, so that $\tau_t = g_0 + g_1 K_t + g_2 K_{t-1}$. Total output is given by $X_t = N x_t$, and market demand is

$$p_t = D - A X_t + u_t,$$

where u_t is white noise. We require $p_t \geq 0$.

The firm chooses k_t to maximize

$$E_0 \sum_{t=0}^{\infty} B^t \{p_t(x_0 + f_0 k_t + f_1 K_t + f_2 K_{t-1}) - w k_t - \tau_t k_t - \frac{C}{2}(k_t - k_{t-1})^2\},$$

where k_{-1} is given and w, the rental on capital goods, is for convenience assumed to be constant. $C > 0$ reflects adjustment costs for changing k_t. The Euler equation for this problem can be written

$$p_t f_0 - (w + \tau_t) + BCE_t^* k_{t+1} - C(1+B)k_t + Ck_{t-1} = 0 \tag{33}$$

for $t \geq 0$. For an optimum solution for the firm we also require that $k_t \geq 0, x_t \geq 0$, and that the transversality condition is met.

[15] For further details of the temporary equilibrium set-up see (Evans and McGough 2002). Stability under learning of the MSV solutions was examined in Section 8.6.2 of (Evans and Honkapohja 2001).

In order to define the temporary equilibrium and study learning, we need to be careful about the information structure. We assume that firms use observations of lagged capital stock, the current intrinsic exogenous shocks u_t and the current extrinsic exogenous variable s_t to make forecasts $E_t^* k_{t+1}$. Given these forecasts, firms choose their demands for capital k_t, conditional on p_t and τ_t, to satisfy (33). The temporary equilibrium is then given by the market clearing values of p_t, τ_t and k_t. Using the identical agent assumption, and combining equations, we obtain the reduced form

$$k_t = \mu + \beta E_t^* k_{t+1} + \delta k_{t-1} + \gamma u_t,$$

where $\beta = BC\Omega^{-1}$, $\delta = -(f_0 A f_2 N^2 + g_2 N - C)\Omega^{-1}$, $\Omega = f_0 AN(f_0 + f_1 N) + g_1 N + C(1+B)$ and $\gamma = \Omega^{-1}$.

This form differs from our stochastic model only by the presence of a constant term. Incorporating the constant term into the above theory is straightforward and the details are left to the reader. The only issue concerns E-stability: the PLM must be modified to include a constant thus creating an addition component of the T-map. This component is not coupled with the system T_2 and analysis of its derivative is elementary. It can be shown that if the model without the constant exhibits stable k-SSEs, then the model with the intercept does as well.

With no externalities or taxes there is a unique stationary REE. However, in general the parameters β and δ are unrestricted. In particular, for some parameter regions the associated quadratic has both real roots inside the unit circle, and there are multiple stationary solutions, including stable noisy k-state Markov sunspots. This model is easy to study numerically. As already noted, when externality or tax distortions are present the indeterminacy case is possible. Furthermore, both k-SSEs and k-*SDSs that are stable under learning arise in some regions of the parameter space. For example, normalizing with $N = 1$, the parameter values $A = 1, B = 0.95, C = 0.46, g_1 = -1, g_2 = 0.3, f_0 = 1, f_1 = -1, f_2 = 0.3$ leads to $\beta = -4.24$ and $\delta = 1.36$, which is in region A_1.

8 Conclusion

Finite state Markov stationary sunspot equilibria, in forward-looking models, played a central role in the early literature on expectations driven fluctuations. More recently they have received renewed interest because of their stability under learning in a substantial region of the parameter space. In this paper we have laid forth a theory which allows for the analysis of such equilibria in stochastic linear models with a predetermined variable. We obtained existence for parameters in a proper subset of the region of indeterminacy. We have also shown existence of a related but distinct class of sunspot equilibria, namely those that are driven by finite-state Markov processes, but which take on infinitely many values (even in nonstochastic models).

To analyze stability under learning we developed representations compatible with both classes of sunspot solutions, which allowed us to establish the existence of stable sunspot equilibria for parameter values in a proper subset of the regions of existence. These theoretical results were supported by simulations which suggest that the E-stability principle holds for this model when agents learn using least squares estimators. The results of this paper indicate that noisy finite state Markov sunspot equilibria, and noisy finite state dependent sunspot equilibria, can arise quite generally. Extension of these results to higher order and to multivariate models would be of considerable importance to applied macroeconomic models that incorporate both expectations and predetermined variables.

Appendix

Proof of Proposition 3: It suffices to prove that if $\zeta_t = \overline{\zeta}_i \Leftrightarrow s_t = S_i$ is a solution to the homogeneous model then

$$\zeta_t = a\zeta_{t-1} + s'_{t-1}As_t \tag{34}$$

for some fixed point A of T_2, where a is a fixed point of T_1. Set $A_{ij} = \overline{\zeta}_j - a\overline{\zeta}_i$. Explicit computation shows $\zeta_t = \overline{\zeta}_i \Leftrightarrow s_t = S_i$ satisfies (34), so it remains to show A_{ij} is a fixed point of T_2. This requires an explicit formula for the T-map. The verbal description of the matrix $B(A)$ given in Lemma 2 yields the following form for the ij^{th} component of $T_2(A)$:

$$T_2(A)_{ij} = \beta(aA_{ij} + \sum_{m=1}^{k} \pi_{ij}(m)A_{jm}). \tag{35}$$

Thus A is a fixed point if and only if

$$(\frac{1}{\beta} - a)A_{ij} = \sum_{m=1}^{k} \pi_{ij}(m)A_{jm}. \tag{36}$$

This set of k^2 linear homogeneous equations always has zero as a solutions; nonzero solutions exist only in case of linear dependency, which will not hold in general. However, because $\overline{\zeta}_t$ is a sunspot equilibrium, we have that equations (4) hold, that is

$$\overline{\zeta}_j - \delta\overline{\zeta}_i = \beta \sum_{m=1}^{k} \pi_{ij}(m)\overline{\zeta}_m. \tag{37}$$

We proceed to show that if $A_{ij} = \overline{\zeta}_j - a\overline{\zeta}_i$ then (37) implies (36). We have (37)

$$\begin{aligned}
&\Leftrightarrow \quad \delta(\overline{\zeta}_j - a\overline{\zeta}_i) = (\delta - a)\overline{\zeta}_j + \beta a \sum \pi_{ij}(m)\overline{\zeta}_m \\
&\Leftrightarrow \quad \delta(\overline{\zeta}_j - a\overline{\zeta}_i) = -\beta a^2\overline{\zeta}_j + \beta a \sum \pi_{ij}(m)\overline{\zeta}_m \\
&\Leftrightarrow \quad \delta(\overline{\zeta}_j - a\overline{\zeta}_i) = \beta a \sum \pi_{ij}(m)(\overline{\zeta}_m - a\overline{\zeta}_j) \\
&\Leftrightarrow \quad \delta A_{ij} = \beta a \sum \pi_{ij}(m)A_{jm} \\
&\Leftrightarrow \quad (a - a^2\beta)A_{ij} = \beta a \sum \pi_{ij}(m)A_{jm},
\end{aligned}$$

and this last line holds if and only if (36) holds.
Proof of Proposition 8. Clearly R, the set of π satisfying (19), is a line segment and is therefore homeomorphic to $\mathbb{R}$. Let $\Omega(\pi)$ be the set of fixed points of the T-map. If $(a, A, b) \in \Omega(\pi)$ and $A \neq 0$, then the process

$$y_t = ay_{t-1} + s'_{t-1}As_t + bv_t$$

is a 2-SDS, and if the associated $\pi \notin R$ then this process is a 2-*SDS. Let $\hat{R}$ be the set of all $\pi \in I$ so that $\Omega(\pi)$ is non-trivial, that is, so that $\Omega(\pi)$ contains points other than $(a, 0, b)$. Every 2-SSE is a 2-SDS, thus $R \subset \hat{R}$. Let a be a fixed point of T_1, $\alpha = 1/\beta - a$, and define a map $M : I \to \mathbb{R}^{4\times4}$ by

$$M(\pi) = \begin{bmatrix} \alpha - \pi_{11} & 0 & \pi_{11} - 1 & 0 \\ -\pi_{21} & \alpha & \pi_{21} - 1 & 0 \\ 0 & -\pi_{12} & \alpha & \pi_{12} - 1 \\ 0 & -\pi_{22} & 0 & \alpha + \pi_{22} - 1 \end{bmatrix},$$

and set $\Gamma : I \to \mathbb{R}$ by $\Gamma = \det \circ M$. Note that A is a fixed point of T_2 if and only if $M \cdot vec(A) = 0$. This may be seen using equations (23)-(26) in the paper. Thus

$$\hat{R} = \{\pi \in I : \Gamma(\pi) = 0\}.$$

Now notice that Γ is a polynomial in π_{ij}. The gradient of this polynomial can be explicitly computed and shown not to vanish on $\hat{R}$. The implicit function theorem then applies to show that about any point at which the gradient does not vanish there is a neighborhood homeomorphic to $\mathbb{R}^3$. That $\hat{R}$ has measure zero in I follows from the fact that it is the zero-set of a non-zero polynomial. ■

Proof of Proposition 9. Assume $\pi \in \hat{R} \setminus R$. It is straightforward to show that all non-trivial fixed points of the T-map have the property that there are at least two distinct values among the A_{ij}, and we write

$$y_t = \sum_{n=0}^{\infty} a^n \hat{A}_{\sigma(t-n)}.$$

We would like to know about the possible values obtained by y_t; denote this set by Y. We proceed as follows: fix t and consider realizations of the process

s_t up to time t. Note that each realization identifies a sequence $\{\hat{A}_n\}_{n=-\infty}^{t}$, and any such sequence is possible. It is immediate that Y contains infinitely many points. For example, set

$$z_k = \sum_{n \neq k}^{\infty} a^n \hat{A}_1 + a^k \hat{A}_2,$$

where we are assuming $\hat{A}_1 \neq \hat{A}_2$. Then $z_k \in Y$, and $i \neq j \Rightarrow z_i \neq z_j$. It is somewhat more difficult to show that y_t takes on infinity many values with positive probability. We now turn to this problem.

Let I be the unit interval in $\mathbb{R}$ and express all elements of I in base 4. Specifically, for $\gamma \in I$, write

$$\gamma = \sum_{k=1}^{\infty} \gamma_k \left(\frac{1}{4}\right)^k,$$

where $\gamma_k \in \{0, 1, 2, 3\}$. Define $f : I \to \mathbb{R}$ by

$$f(\gamma) = \sum_{n=1}^{\infty} a^{n-1} \hat{A}_{\gamma_n}.$$

Note that $Y = f(I)$.

Lemma 16 *The function f is continuous.*

Proof: Let $\varepsilon > 0$ and $\varepsilon' = (1/4)^n$. Let $A^s = max_i\{|\hat{A}_i|\}$. Notice that if $|\gamma - \gamma'| < \varepsilon'$ then $\gamma_k = \gamma'_k$ for $k < n$. Thus

$$|f(\gamma) - f(\gamma')| \leq 2 \sum_{m=n+1}^{\infty} |a|^m A^s.$$

Note that rearrangement of the sums is legitimate because the series are absolutely convergent. The right hand side goes to zero as $n \to \infty$ and so can be made smaller than ε. ■

We know that continuous functions send connected sets to connected sets, and that connected subsets of $\mathbb{R}$ are intervals. Noting that $f(i/4) \neq f(j/4)$ if $\hat{A}_i \neq \hat{A}_j$ shows that Y contains an interval. It follows that there are uncountably many possible values of y_t.

The next step is to note that the Markov process s_t induces a probability measure μ on the interval I. Specifically, each realization $\hat{s}$ of the process $\{s_t\}$ induces a function $\sigma(\hat{s})$ which may, in turn be thought of as a realization of a four state first order Markov process.[16] To each σ is identified a real number $\gamma \in I$ whose base four expansion has $\sigma(t-k+1)$ as the k^{th} coefficient γ_k. Now let E be any Borel set in I. Then $\mu(E) = prob(\sigma(\hat{s}) \in E)$, or, in words, the measure of the set E is the probability that the realization $\sigma(\hat{s})$ is identified with some element of E. Notice that, by construction, if $E \subset Y$ then $prob(y_t \in E) = \mu(f^{-1}(E))$ where

$$f^{-1}(E) = \{x \in I | f(x) \in E\}$$

denotes the pre-image of E under f.

Lemma 17 *If $U \subset I$ is open then $\mu(U) > 0$.*

Proof: Let U be open. Then U contain an interval (α, β). Within this interval it is straightforward to construct an interval $J = (\hat{\alpha}, \hat{\beta})$ so that there exists n with $\hat{\alpha}_m = 0$ and $\hat{\beta}_m = 0$ for $m > n$, where $\hat{\alpha}_m$ is the m^th component of the base four expansion of α. We claim that $\mu(J) > 0$. To see this, begin with the simple case that $m = 1$ and $J = (1/4, 3/4)$. Then $\mu(J)$ is the unconditional probability that $\sigma(t) \in \{1, 2, 3\}$. Since all transition arrays have full support, this probability is non-zero. Now consider the general case. Set

$$P = \{(\delta_1, \cdots, \delta_n) | \delta_i \in \{\hat{\alpha}_i, \cdots, \hat{\beta}_i\}\}.$$

Now notice that $\mu(J)$ is the unconditional probability that

$$(\sigma(t), \cdots, \sigma(t-n)) \in P.$$

This probability is non-zero because the transition arrays have full support. ■

We now prove that with positive probability y_t takes infinitely many values. Suppose not. Then there exists $F \subset Y$ such that F is finite and $\mu(f^{-1}(F)) = 1$. Recall there is an interval I_Y in Y. Since F is closed, $O = (\mathbb{R} \setminus F) \cap I_Y$ is open and non-empty. Let $U = f^{-1}(O) \subset I$. By Lemma 1, U is open and by Lemma 2, $\mu(U) > 0$. But $U \subset f^{-1}(\mathbb{R} \setminus F)$ and $\mu(f^{-1}(\mathbb{R} \setminus F)) = 0$. Thus we reach the required contradiction. ■

[16]Note that all relevant transition arrays have full support. Thus, given any state, each state is reachable with positive probability.

References

AZARIADIS, C. (1981): "Self-Fulfilling Prophecies," *Journal of Economic Theory*, 25, 380–396.

AZARIADIS, C., AND R. GUESNERIE (1986): "Sunspots and Cycles," *Review of Economic Studies*, 53, 725–737.

BENHABIB, J., AND R. E. FARMER (1999): "Indeterminacy and Sunspots in Macroeconomics," in (Taylor and Woodford 1999), chap. 6, pp. 387–448.

CASS, D., AND K. SHELL (1983): "Do Sunspots Matter?," *Journal of Political Economy*, 91, 193–227.

CHIAPPORI, P. A., P.-Y. GEOFFARD, AND R. GUESNERIE (1992): "Sunspot Fluctuations around a Steady State: The Case of Multidimensional, One-Step Forward Looking Economic Models," *Econometrica*, 60, 1097–1126.

DESGRANGES, G., AND G. NEGRONI (2001): "Expectations Coordination on a Sunspot Equilibrium: an Eductive Approach," *Macroeconomic Dynamics, forthcoming.*

DÁVILA, J. (1997): "Sunspot Fluctuations in Dynamics with Predetermined Variables," *Economic Theory*, 10, 483–495.

DÁVILA, J., AND R. GUESNERIE (2001): "Sunspot Equilibria in Systems with Memory: An introductory Presentation," in (Guesnerie 2001), chap. 9, pp. 255–280.

EVANS, G. W., AND S. HONKAPOHJA (1994a): "Learning, Convergence, and Stability with Multiple Rational Expectations Equilibria," *European Economic Review*, 38, 1071–1098.

——— (1994b): "On the Local Stability of Sunspot Equilibria under Adaptive Learning Rules," *Journal of Economic Theory*, 64, 142–161.

——— (2001): *Learning and Expectations in Macroeconomics.* Princeton University Press, Princeton, New Jersey.

——— (2002): "Existence of Adaptively Stable Sunspot Equilibria near an Indeterminate Steady State," *Journal of Economic Theory, forthcoming.*

Evans, G. W., S. Honkapohja, and P. Romer (1998): "Growth Cycles," *American Economic Review*, 88, 495–515.

Evans, G. W., and B. McGough (2002): "Stable Sunspot Solutions in Models with Predetermined Variables," mimeo.

Farmer, R. E. (1999): *The Economics of Self-Fulfilling Prophecies, Second edition*. MIT Press, Cambridge, Mass.

Guesnerie, R. (1986): "Stationary Sunspot Equilibria in an N-commodity World," *Journal of Economic Theory*, 40, 103–128.

——— (2001): *Assessing Rational Expectations: Sunspot Multiplicity and Economic Fluctuations*. MIT Press, Cambridge, MA.

Guesnerie, R., and M. Woodford (1992): "Endogenous Fluctuations," in (Laffont 1992), chap. 6, pp. 289–412.

Howitt, P., and R. P. McAfee (1992): "Animal Spirits," *American Economic Review*, 82, 493–507.

Laffont, J.-J. (ed.) (1992): *Advances in Economic Theory: Sixth World Congress. Volume 2.* Cambridge University Press, Cambridge, UK.

Lucas, Jr., R. E., and E. C. Prescott (1971): "Investment under Uncertainty," *Econometrica*, 39, 659–681.

Marcet, A., and T. J. Sargent (1989): "Convergence of Least-Squares Learning Mechanisms in Self-Referential Linear Stochastic Models," *Journal of Economic Theory*, 48, 337–368.

Sargent, T. J. (1987): *Macroeconomic Theory, Second edition*. Academic Press, New York.

Shell, K. (1977): "Monnaie et Allocation Intertemporelle," Working paper, CNRS Seminaire de E.Malinvaud, Paris.

Taylor, J., and M. Woodford (eds.) (1999): *Handbook of Macroeconomics, Volume 1.* Elsevier, Amsterdam.

Woodford, M. (1990): "Learning to Believe in Sunspots," *Econometrica*, 58, 277–307.

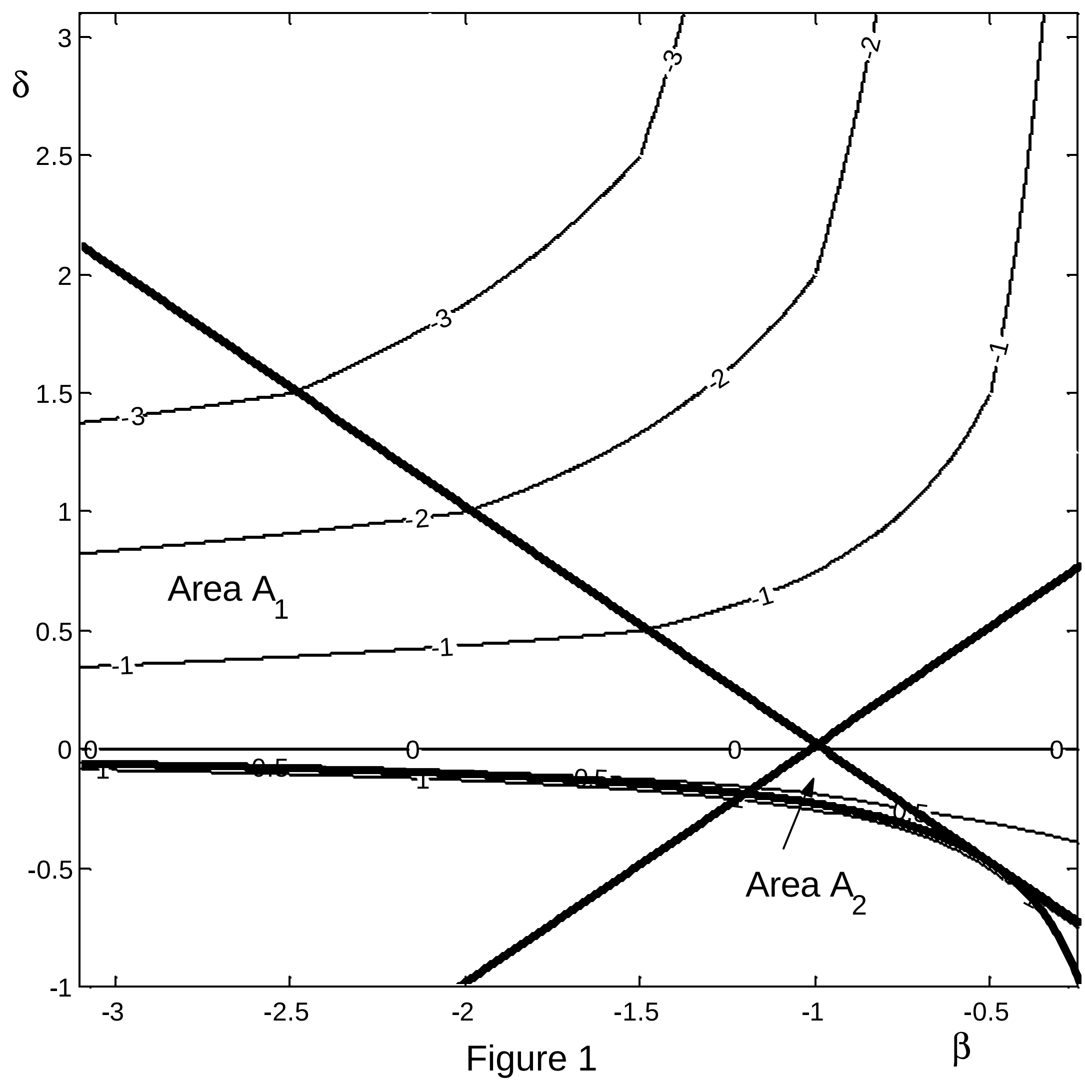


Figure 1

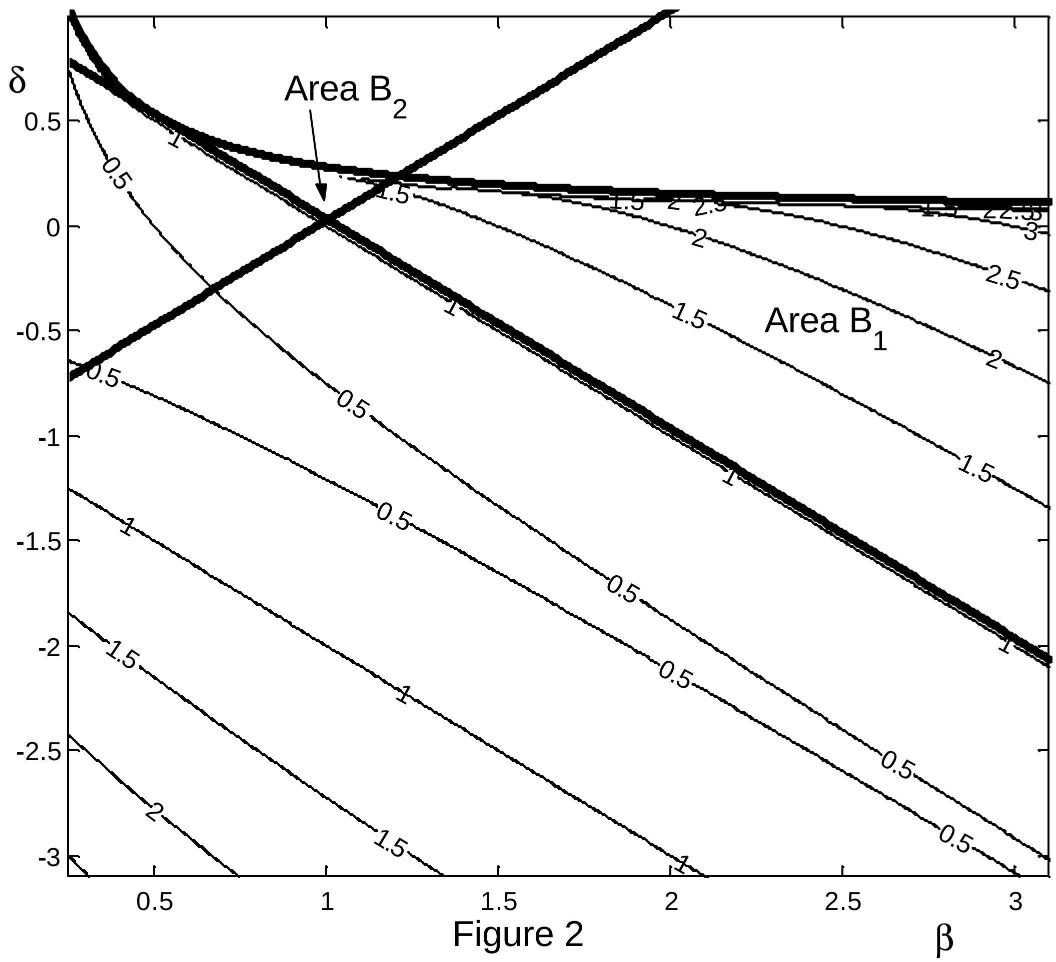


Figure 2

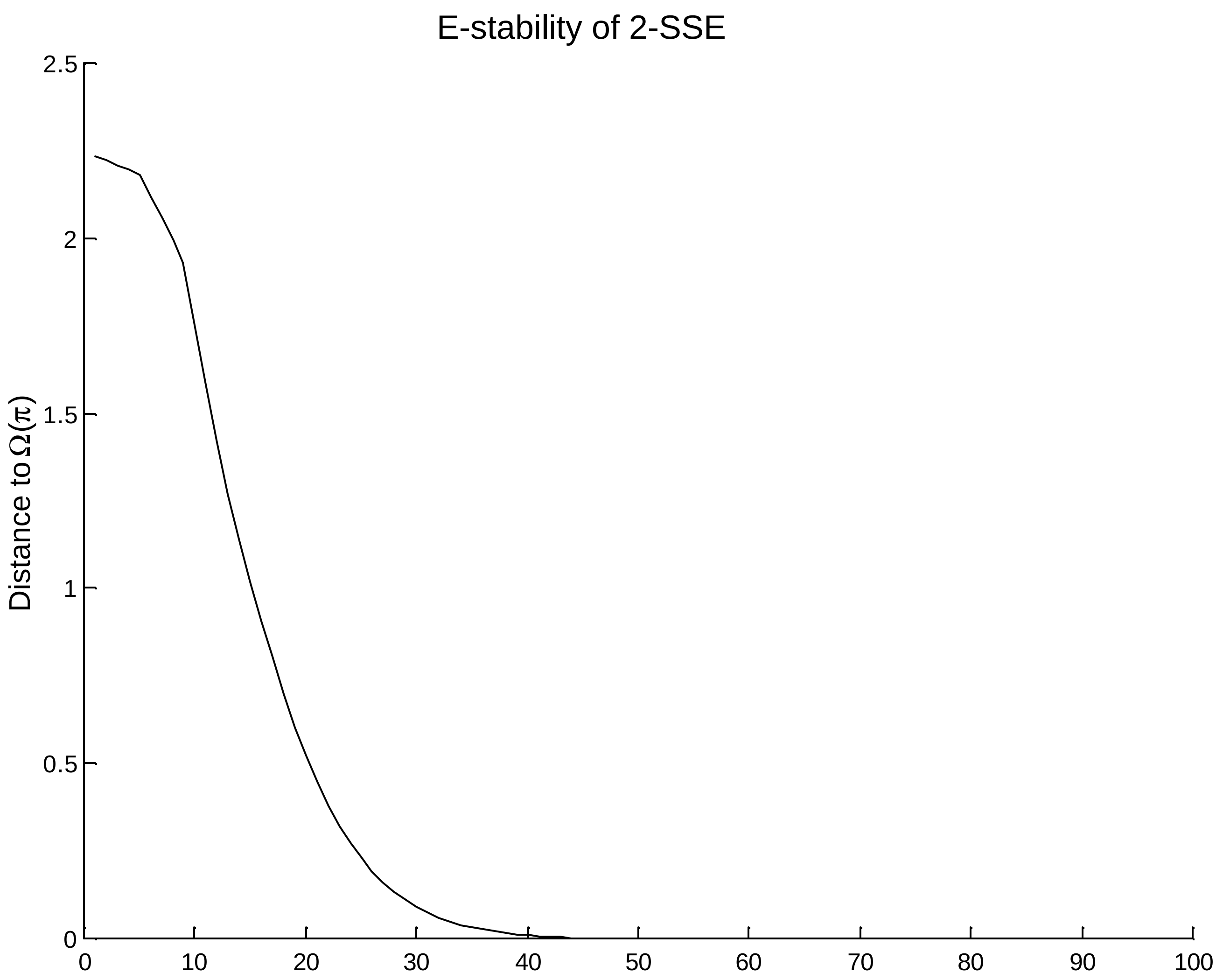

E-stability of 2-SSE
Distance to Ω(π)
2.5
2
1.5
1
0.5
0
0
10
20
30
40
50
60
70
80
90
100
Time

Figure 3

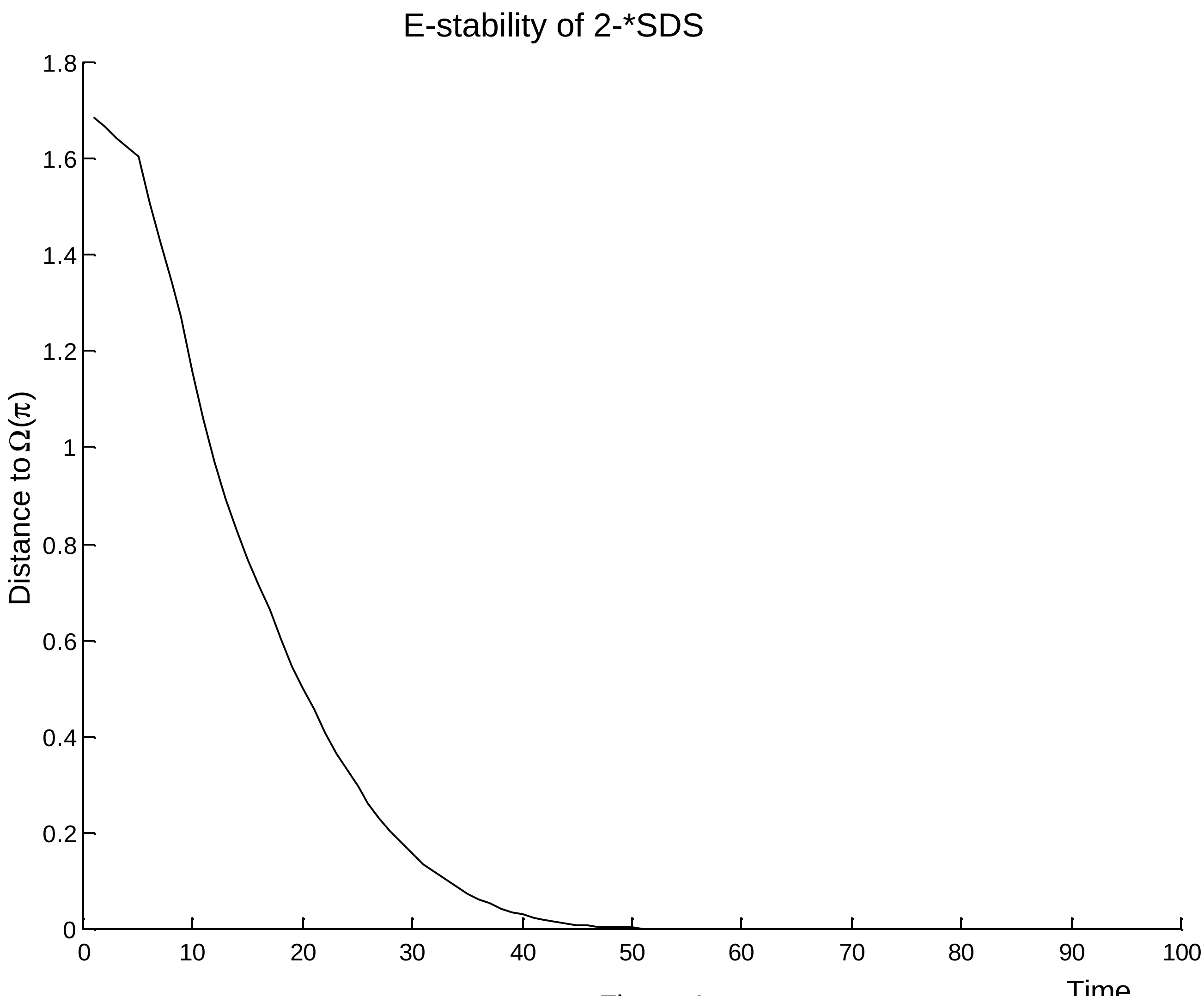
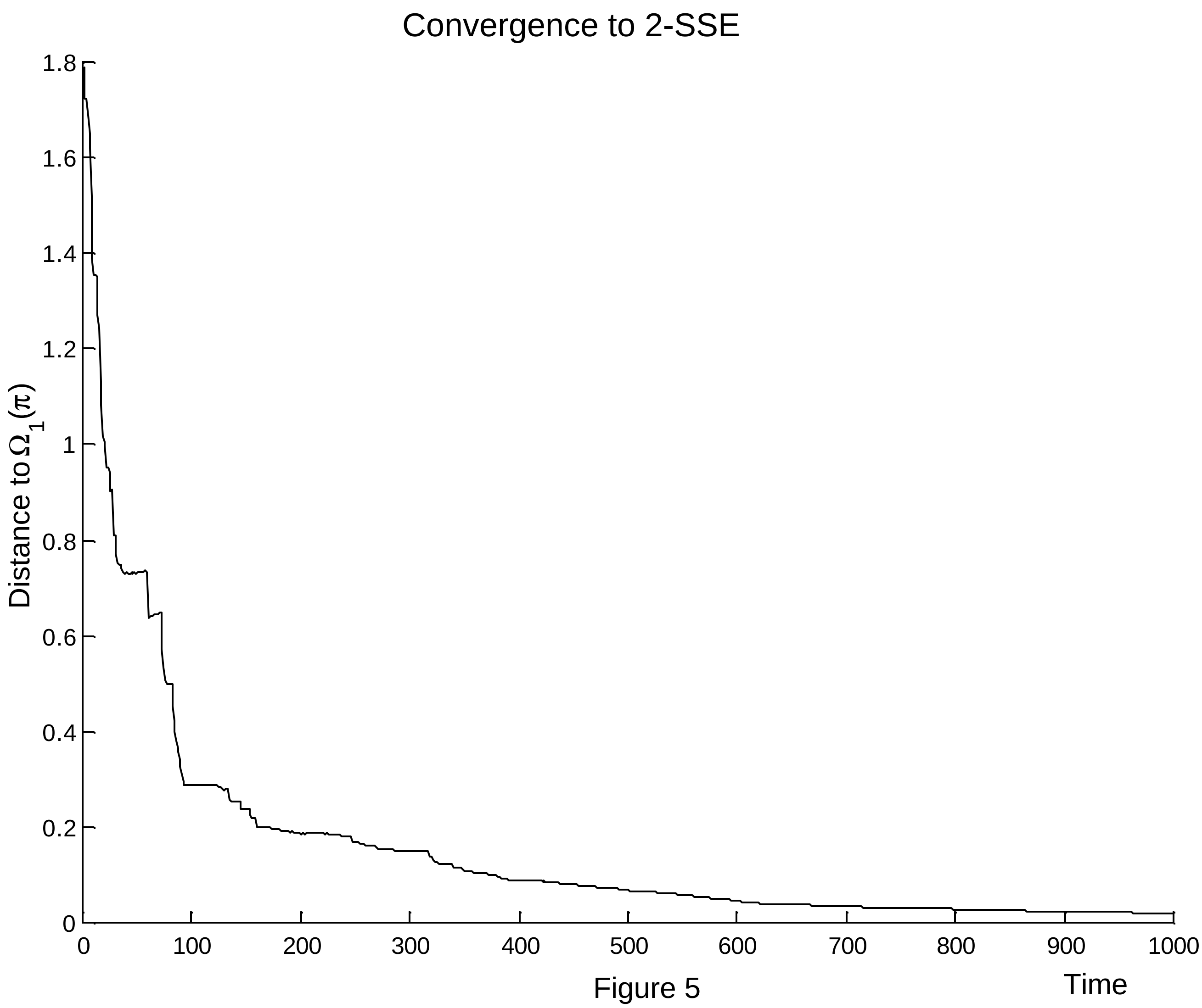

E-stability of 2-*SDS
Distance to Ω(π)
1.8
1.6
1.4
1.2
1
0.8
0.6
0.4
0.2
0
0
10
20
30
40
50
60
70
80
90
100
Time

Figure 4



Convergence to 2-SSE
1.8
1.6
1.4
1.2
1
0.8
0.6
0.4
0.2
0
Distance to $\Omega_1(\pi)$
0
100
200
300
400
500
600
700
800
900
1000
Time

Figure 5

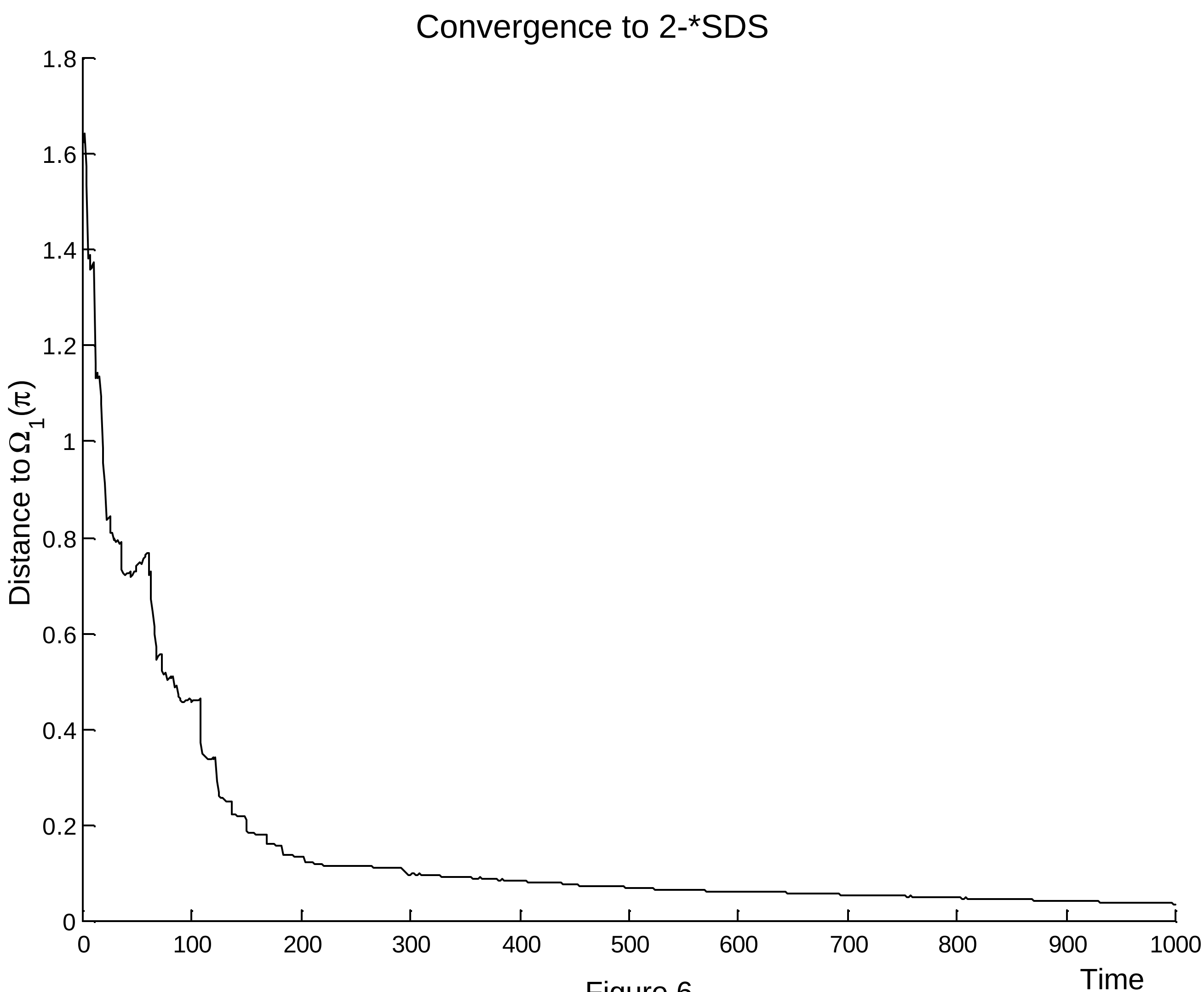

Convergence to 2-*SDS
Distance to $\Omega_1(\pi)$
Time
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8
0
100
200
300
400
500
600
700
800
900
1000

Figure 6

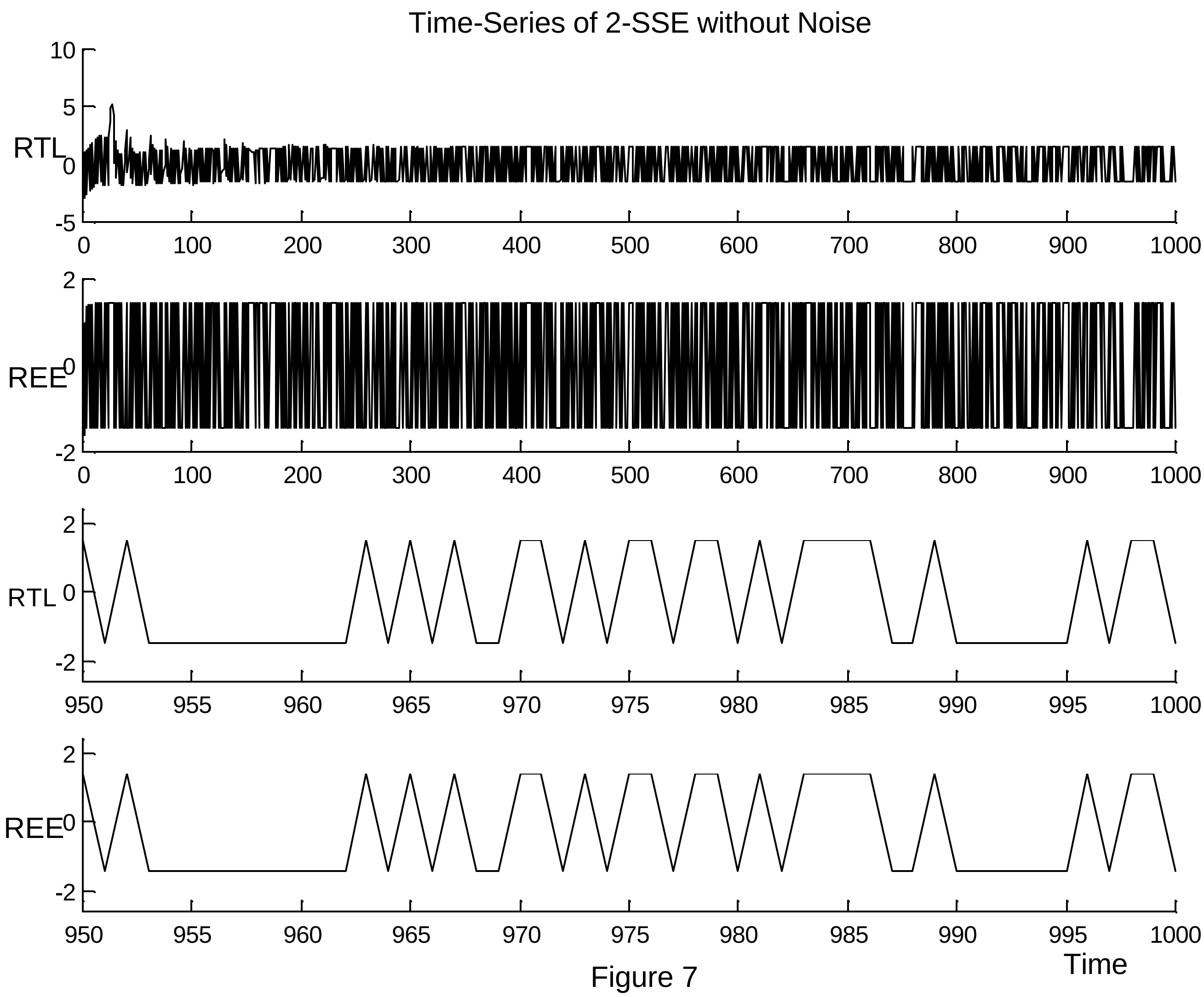


Figure 7

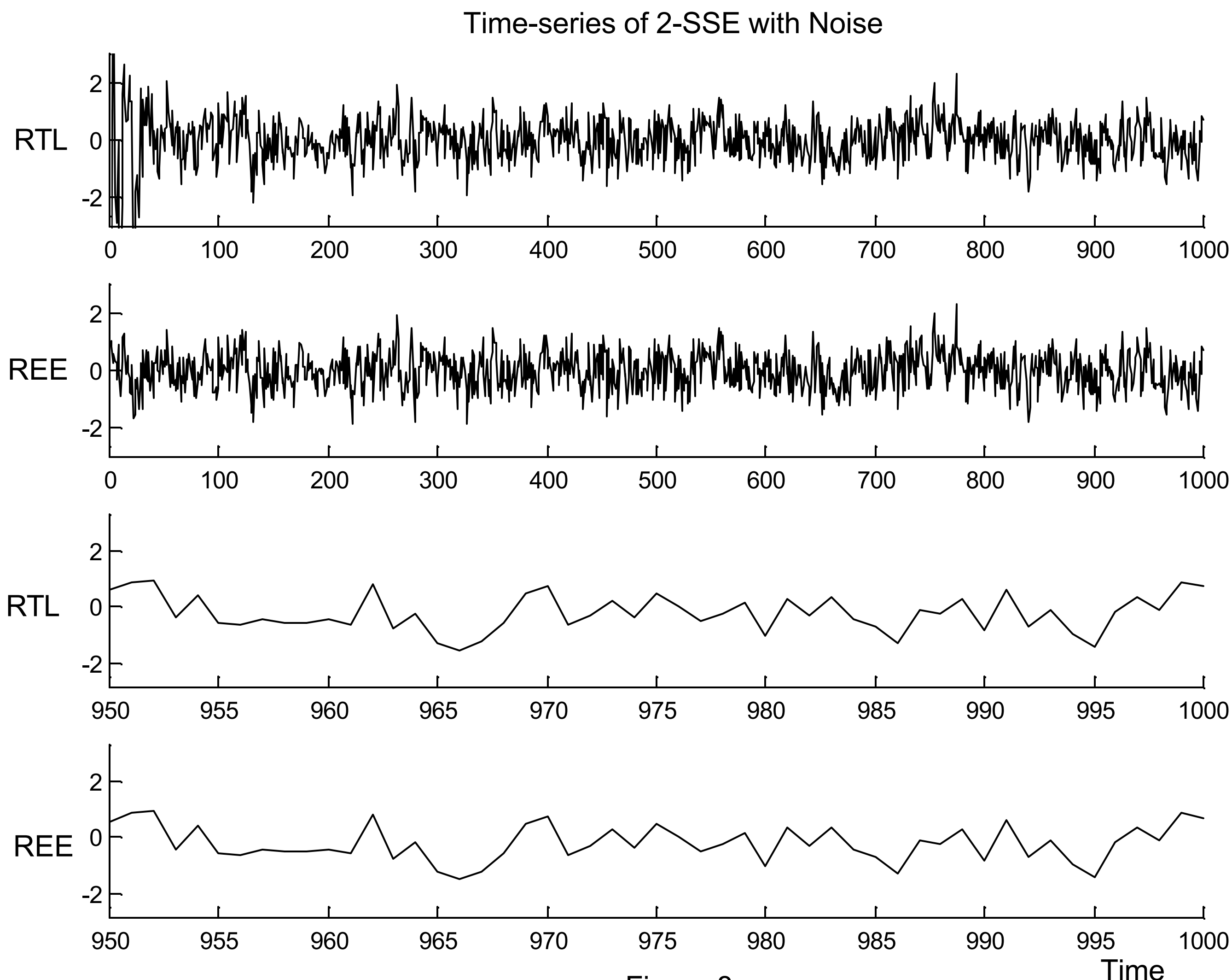

Time-series of 2-SSE with Noise
RTL
REE
RTL
REE
2
0
-2
0 100 200 300 400 500 600 700 800 900 1000
950 955 960 965 970 975 980 985 990 995 1000
Time

Figure 8

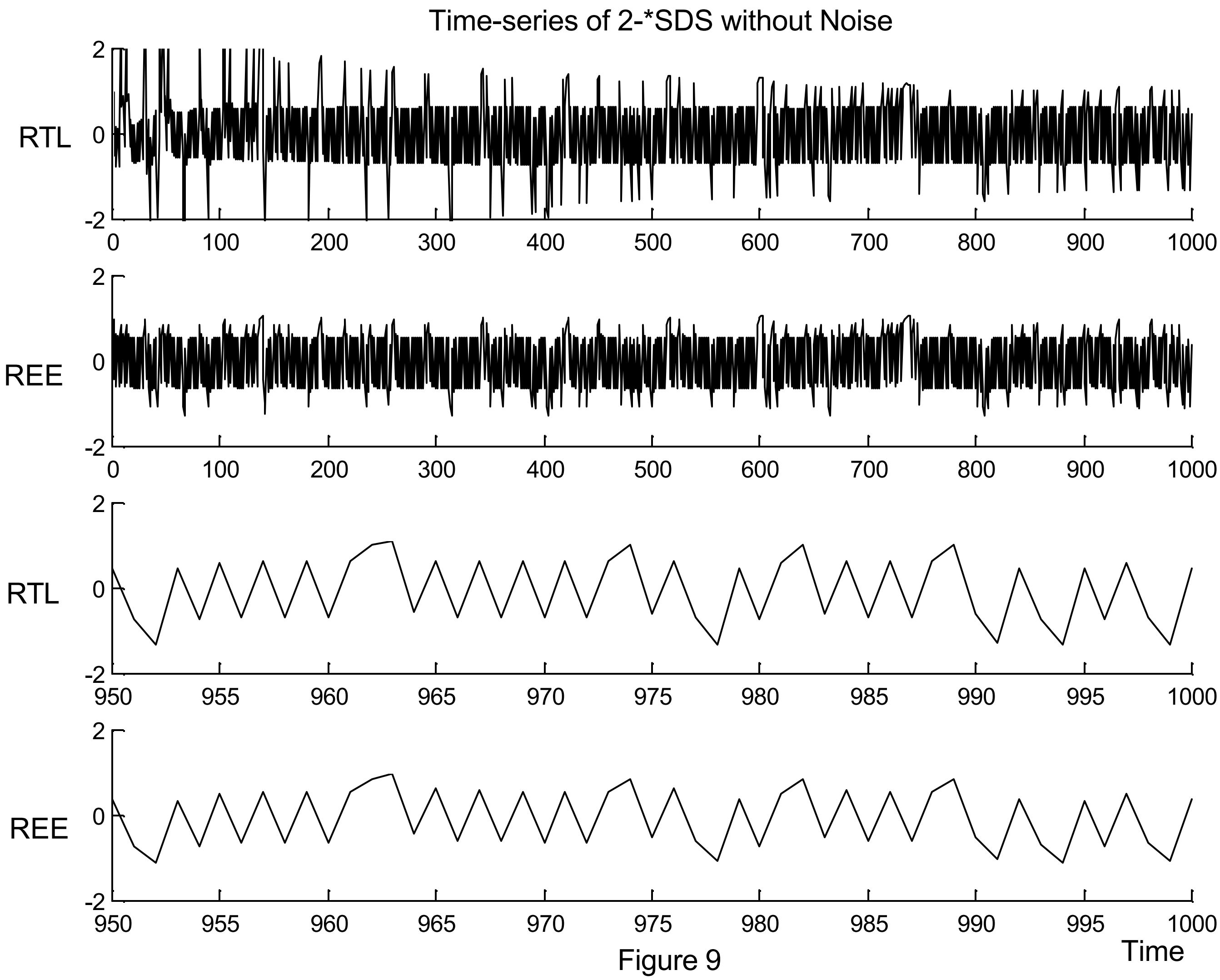


Figure 9

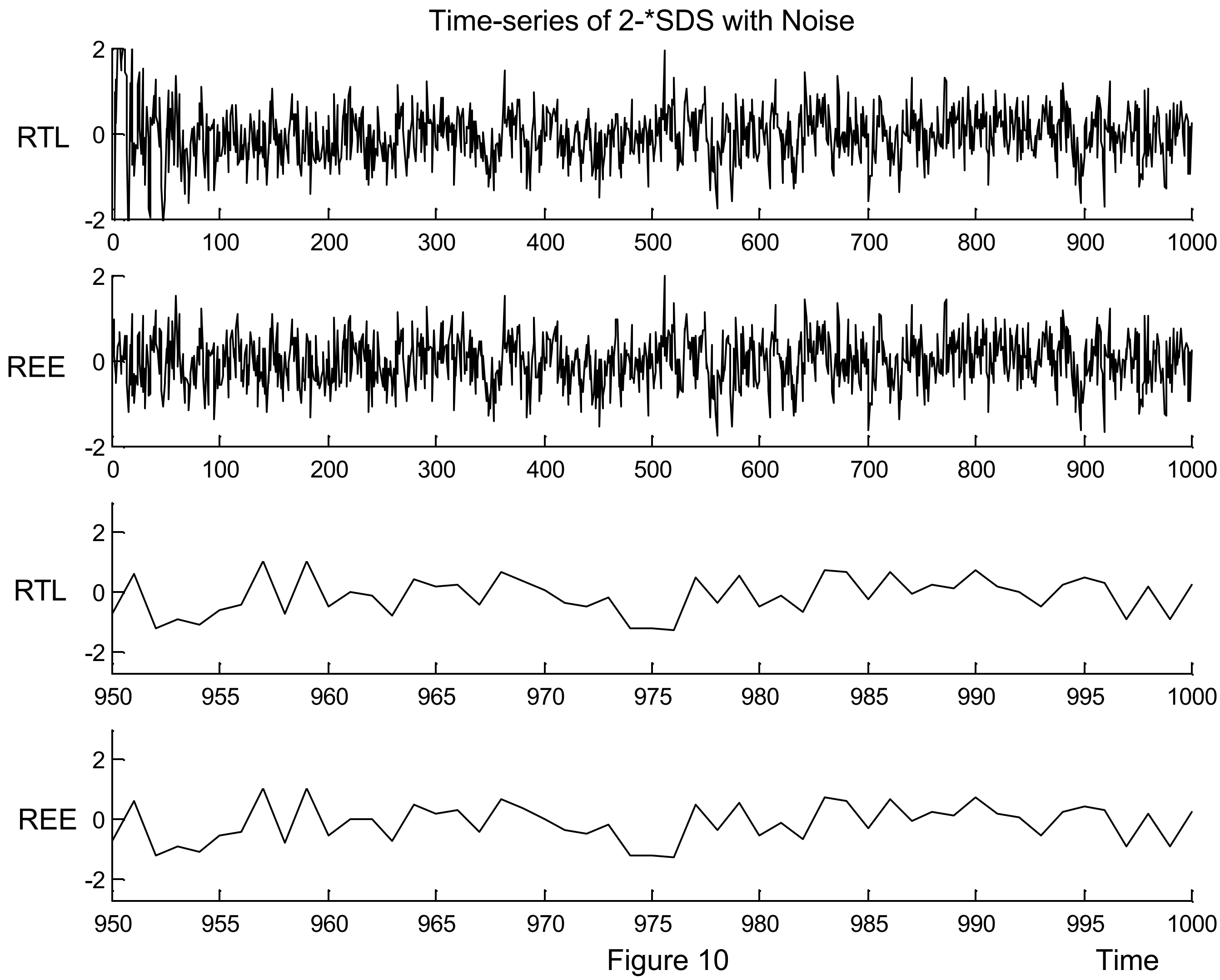

Time-series of 2-*SDS with Noise
RTL
REE
RTL
REE
2
0
-2
0
100
200
300
400
500
600
700
800
900
1000
950
955
960
965
970
975
980
985
990
995
1000
Time

Figure 10

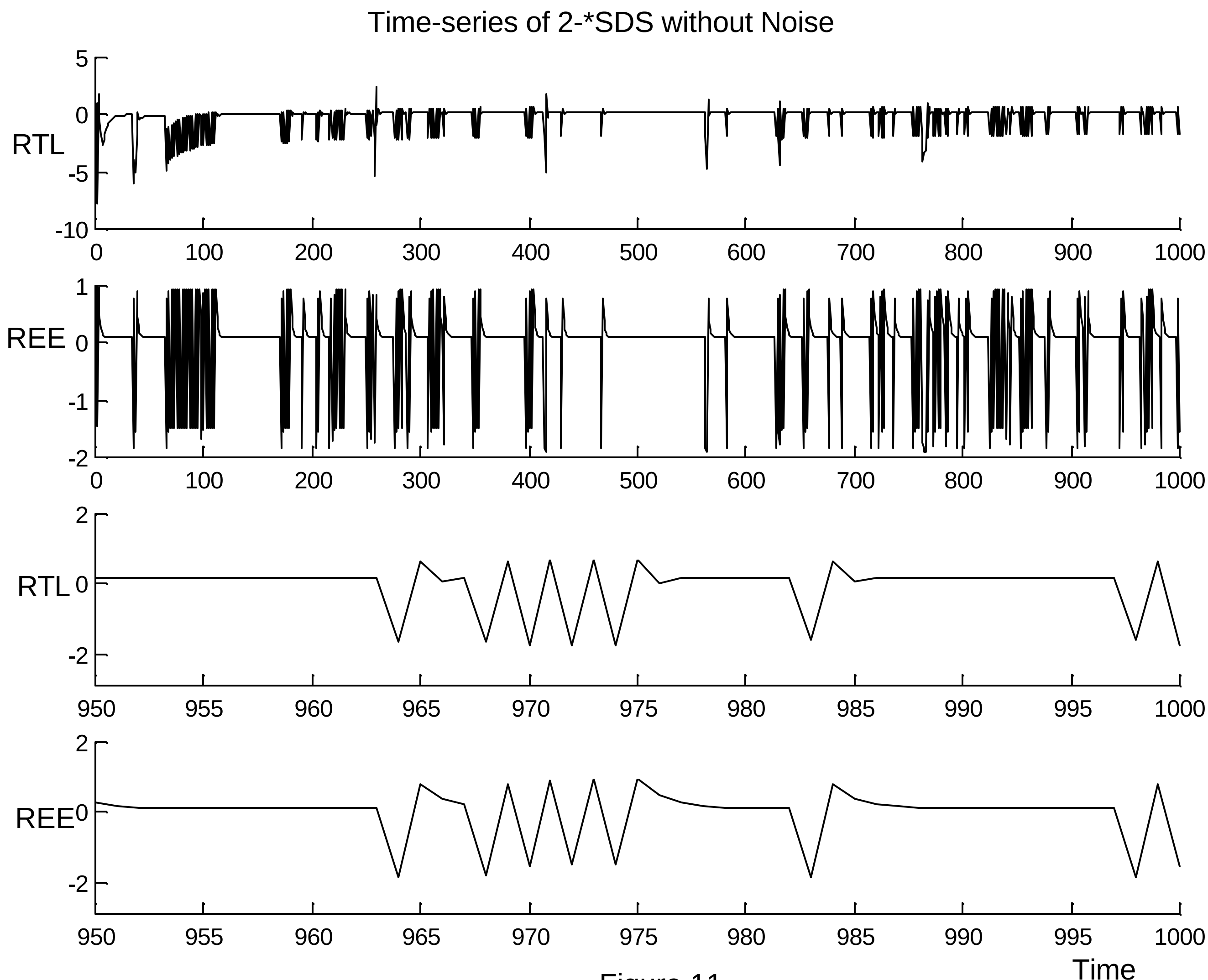


Figure 11